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TECHNIQUES OF RESONANCE ANALYSIS†

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The technique of isolating resonant behavior is described using an example. The significance of stochastic layers in enhancing diffusion is discussed. Diffusion along resonance layers, that arises in more than two degrees of freedom, e.g., Arnol'd diffusion, and modulation diffusion is described and illustrated with an example.

I. INTRODUCTION

It is well known that Hamiltonian systems with one degree of freedom $H(p, q)$ are integrable. For two degrees of freedom $H(p_1, p_2, q_1, q_2)$, integrability is exceptional. In general, resonances between the two degrees of freedom lead to the formation of resonance layers in the action space. Within each resonance layer, chaotic motion appears. Energy conservation prevents large excursions of the motion along the layer. Only motion across the layer is important. For an integrable system with a weak perturbation, the chaotic layers are isolated by Kolmogorov–Arnol'd–Moser (KAM) surfaces. Thus motion from one layer to another is forbidden. For strong perturbations, resonance layers can overlap, the intervening KAM surfaces being destroyed. A more global chaotic motion then develops, leading to large excursions in both actions over long times.

For three or more degrees of freedom, strong perturbations also lead to overlap of resonance layers and globally chaotic motion. However, for weak perturbations, two new effects appear:

1. Resonance layers are no longer isolated by KAM surfaces. Generically, the layers intersect, forming a connected web dense in the action space.
2. Conservation of energy no longer prevents large chaotic motions of the actions along the layers over long times. As a result, large, long-time excursions of the actions along resonance layers are generic in systems with three or more degrees of freedom. The interconnection of the dense set of layers ensures that the chaotic motion, stepping from layer to layer, can carry the system arbitrarily close to any region of the phase space consistent with energy conservation.

The basic technique for analyzing these phenomena starts with the isolation of a single resonance. A transformation is made to action-angle variables for which one angle is slowly varying near the resonance. The method of averaging is then employed to average over the fast variables, reducing the problem to a single

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degree of freedom, for which the solution is obtainable. By reintroducing the higher-order terms, the process can be repeated to isolate a second-order resonance between the primary motion and the slow oscillation about the resonance. Since the resonances are derived from Fourier expansions, all resonances have the same form, when expressed in appropriate variables. The process may be called resonance renormalization.

We consider first the technique of resonance renormalization with an example. We then indicate the phenomenon of diffusion along resonance layers (Arnol'd diffusion) in systems with more than two degrees of freedom, again illustrating the behavior with an example. The treatment in the following sections will be quite brief, with references given for the longer calculations. For simplicity the references will refer to the monograph *Regular and Stochastic Motion,*[1] rather than to the original sources which are referenced there.

II. RESONANCE RENORMALIZATION AND RESONANCE OVERLAP

We use the technique of resonance renormalization to calculate the transition at which the intermingling of the stochastic layers of neighboring resonances creates a band of stochasticity. We use as an example a toroidal magnetic field which is perturbed by a helical perturbation. The toroidal coordinates are shown in Fig. 1. The equations of a field line are defined by

$$\frac{dr}{B_r}=\frac{r\,d\theta}{B_\theta}=\frac{R\,d\phi}{B_\phi}, \tag{1}$$

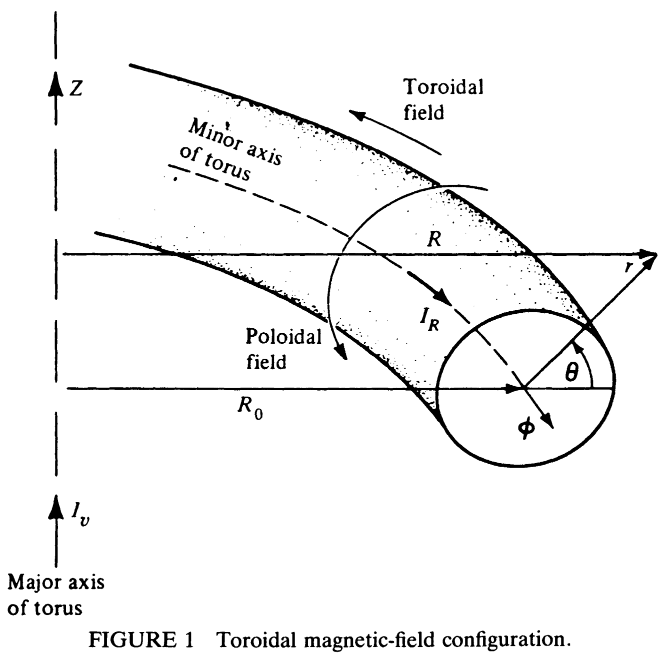

FIGURE 1 Toroidal magnetic-field configuration.

where $R = R_0 + r\cos\theta$ and, for a tokamak, the field-strength variation from outside to inside the torus is also proportional to the inverse aspect ratio r/R.

The Hamiltonian for a field line is

$$H = H_0(J) + H_1(J, \theta, \phi), \tag{2}$$

where

$$H_0(J) = \int_0^J t(J)\, dJ, \qquad H_1 \approx \sum_k J^{|k/2|} e^{ik\theta} \sum_{m,n} A_{mn} e^{i(m\theta - n\phi)}, \tag{3}$$

with $t(J)$ the rotational transform, $t = 1/q$ (q is the safety factor), and, to lowest order, $J = r^2/2$; the summation over k is the expansion of the toroidal correction, and the double summation over m, n is the expansion of the perturbing field.

We shall first assume that the main island amplitude is determined by the lowest-order perturbation term $m = n = 1$. (The methods to be described below are taken from Ref. 1, Chapters 2 and 4.) We transform to new variables in a rotating coordinate system:

$$\begin{aligned} \hat{\theta} &= \theta - \phi, \\ \hat{J} &= J \end{aligned} \tag{4}$$

and expand about J_0, where $t(J_0) = 1$, $J = J_0 + \Delta J$, giving the Hamiltonian for the perturbation:

$$\begin{aligned} \Delta\hat{H} \approx & \left.\frac{dt}{dJ}\right|_{J=J_0} \frac{(\Delta J)^2}{2} + A_{11}\cos\hat{\theta} \\ & + \sum_k \left(\frac{r_0}{R_0}\right)^{|k|} e^{k(\theta+\phi)} \sum_{\substack{m,n \\ m,n \neq \pm 1 \text{ for } k=0}} A_{mn} e^{i[m\theta + (m-n)\phi]}, \end{aligned} \tag{5}$$

where we have added the ± 1 terms (assumed equal to obtain a pendulum in lowest order) and evaluated the perturbation at $J_0 = r_0^2/2$. Here and in the following calculations, we ignore small corrections to the dominant terms, which is justified by the lack of detailed knowledge of the Fourier coefficients A_{mn}. In the same spirit, we do not distinguish the symmetry so that exponentials and sinusoids are taken to be interchangeable.

Averaging over the fast variable, ϕ, we obtain the pendulum Hamiltonian with the amplitude on the separatrix:

$$\Delta\hat{J}_M = 2(F/G)^{1/2}, \tag{6}$$

and the frequency at the fixed point:

$$\omega_0 = (FG)^{1/2}, \tag{7}$$

where

$$F = A_{11}, \qquad G = -\frac{dt}{dJ}. \tag{8}$$

We transform to action-angle variables, I, θ, in the pendulum Hamiltonian frame, to exhibit the second-order resonances (islands). Reintroducing the rapidly varying terms in the new variables, we have a Hamiltonian of the form

$$K(I, \theta, \phi) = K_0(I) + \sum_{k,m,n} \left(\frac{r_0}{R_0}\right)^{|k|} A_{mn} \sum_l V_l e^{i[l\theta+(m+k-n)\phi]}, \tag{9}$$

where the V_l are the coefficients of the Fourier expansion in θ. Since $r_0/R_0 \ll 1$, we look at small k. Since $\dot{\theta} \ll \dot{\phi}$, we only consider $m+k-n=\pm 1$. For harmonics with $q=1$ helical symmetry, $m=n$ for which we take $k=\pm 1$. In addition, the $m=2$, $n=1$ mode can be important because it is resonant with $k=0$.

Considering these modes only, we obtain the second-order islands by expanding about the resonant action $I=I_0$, for which

$$l\omega(I_0) = 1. \tag{10}$$

Setting $\hat{\theta} = l\theta - \phi$, $\hat{I} = I/l$, and, as previously, averaging over the fast variables, we obtain the pendulum Hamiltonian for second-order islands:

$$G_s \frac{(\Delta \hat{I})^2}{2} + F_s \cos\theta, \tag{11}$$

where the largest forcing terms can be collected as

$$F_s \approx \frac{r_0}{R_0} V_l \sum_m A_{mm} + V_l A_{21}, \qquad G_s = -l^2\, \partial^2 K_0/\partial I^2. \tag{12}$$

For the second-order islands we then have for the amplitude and the frequency:

$$\Delta \hat{I}_M = 2(F_S/G_s)^{1/2}, \qquad \hat{\omega}_0 = (F_s G_s)^{1/2}. \tag{13}$$

In second order, we have the universal relations for island "overlap," i.e., no KAM surfaces between the l and $l+1$ islands (Ref. 1, Chapter 4):

$$\frac{\Delta I_{Ml} + \Delta I_{Ml+1}}{|I_{0l} - I_{0l+1}|} \geqslant \frac{2}{3}. \tag{14}$$

For nearly equal-size islands, substituting from the above, we obtain the condition for stochasticity joining two island chains:

$$36 F_s G_s / FG \geqslant 1. \tag{15}$$

The overlap of second-order resonances determines the thickness of the stochastic layer that exists generically around the separatrices of each resonance.

The diffusion rate across the resonance layer can be calculated by first modifying the phase space so that it looks locally like a standard mapping (see Ref. 1, Sections 3.5 and 4.1) and then using the long-time diffusion calculation in the presence of correlations and islands (see Ref. 1, Sections 5.4d and 5.5). This procedure has been shown to give a reasonably accurate picture of the local, short-time, diffusion rate. Of course KAM barriers exist to inhibit long-time diffusion to distant parts of the phase space.

III. ARNOL'D DIFFUSION

We consider the geometry of stochastic layers in the $2N$-dimensional phase space. The layers, defined by Eq. (3), are surfaces having dimension $2N-1$. The KAM surfaces, being perturbed tori, are N-dimensional. The interconnection of resonance layers into the Arnol'd web can then be understood geometrically. For $N \geqslant 3$, the $(2N-1)$-dimensional resonance surfaces cannot be isolated from one another by N-dimensional KAM surfaces. The "Arnol'd" diffusion along the separatrix layers of the interconnected resonances is generic to systems with more than two degrees of freedom.

A simple example of a system illustrating Arnol'd diffusion is that of a ball bouncing back and forth between a smooth wall at $z=h$ and a fixed wall that is rippled in two dimensions, x and y, at $z=0$. The surface of section is given in terms of the ball positions in the x_n and y_n directions and the trajectory angles $\alpha_n = \tan^{-1} v_x/v_z$ and $\beta_n = \tan^{-1} v_y/v_z$, just before the nth collision with the rippled wall. The ball motion is shown schematically, and variables in the x, z-plane are defined in Fig. 2. Assuming that the ripple is small, the rippled wall may be replaced by a flat wall at $z=0$ whose normal vector is a function of x and y, analogous to the idea of a Fresnel mirror. The simplified difference equations exhibit the general features of the exact equations and may be written in explicit form

$$\begin{aligned}
\alpha_{n+1} &= \alpha_n - 2a_x k_x \sin k_x x + \mu k_x \gamma_c, \\
x_{n+1} &= x_n + 2h \tan \alpha_{n+1}, \\
\beta_{n+1} &= \beta_n - 2a_y k_y \sin k_y y + \mu k_c, \\
y_{n+1} &= y_n + 2h \tan \beta_{n+1},
\end{aligned}$$

where $\gamma_c = \sin(k_x x + k_y y)$, a_x and a_y are the amplitudes of the ripple in the x and y directions, respectively, and μ is the amplitude of the diagonal ripple and represents the coupling between the x and y motions.

A typical numerical calculation showing Arnol'd diffusion in the coupled system is given in Fig. 3. The surface of section for the system is four-dimensional (α, x, β, y), which we represent in the form of a pair of two-dimensional plots

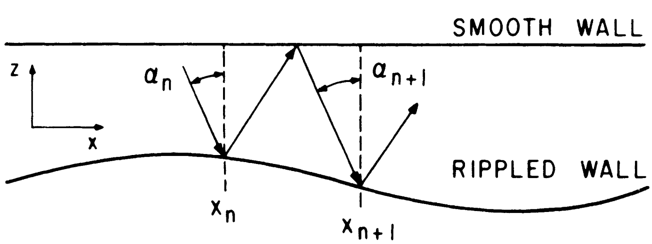

FIGURE 2 Motion in two degrees of freedom, illustrating the definition of the trajectory angle α_n and bounce position x_n just before the nth collision with the rippled ball.

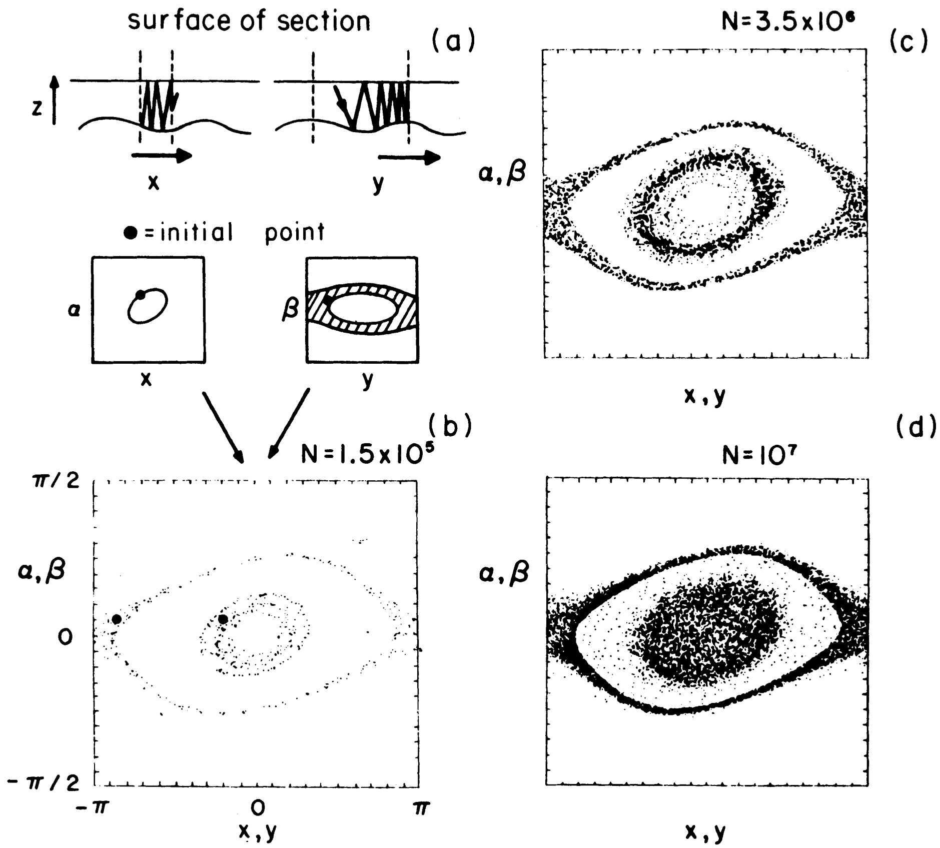

FIGURE 3 Thin-layer diffusion. Initial conditions are close to the central resonance in the $\alpha - x$ space and within the separatrix stochastic layer in the $\beta - y$ space. Parameters are $\mu/h = 0.004$; $\lambda_x : h : a_x$ and $\lambda_y : h : a_y$ as 100:10:2.

(α, x) and (β, y). Thus, two points, one in (α, x) and one in (β, y), are required to specify a point in the four-dimensional section. In Fig. 3 the two plots are superimposed for convenience, and x and y have been normalized to their respective wavelengths, $2\pi/k_x$ and $2\pi/k_y$. The initial condition (Fig. 3a) has been chosen on an island encircling the central resonance in x, and within the thin separatrix layer for y. This corresponds to an initial adiabatic motion in x, well confined in the valley, while in y the motion just reaches or passes over a hill. We observe numerically that the y motion is confined to its separatrix layer until the x motion reaches its own separatrix layer. The successive stages of the diffusion of the $\alpha - x$ motion are shown in Figs 3b, 3c, and 3d, respectively. In the absence of coupling ($\mu = 0$), the motion in the $\alpha - x$ plane should be confined to a smooth closed curve encircling the central resonance. For a finite coupling, α and x diffuse slowly because of the small randomizing influence of the stochastic $\beta - y$ motion. The $\alpha - x$ diffusion is motion along the $\beta - y$ stochastic layer; that is, it is the Arnol'd diffusion. The diffusion is shown for 1.5×10^5, 3.5×10^6, and 10^7

iterations of the mapping. At this time the $\alpha - x$ motion has diffused out to its own thin separatrix layer. Continued iteration of the mapping shows that the trajectory point diffuses over the entire $\alpha - x$ plane. In particular, the change of direction from diffusion along the $\beta - y$ separatrix layer to diffusion along the $\alpha - x$ separatrix layer has been observed numerically.

The rate of diffusion for this example has been calculated, using the stochastic pump model, and compared with the numerical results giving good agreement (Ref. 1, Section 6.2). The diffusion rates can be very sensitive, however, to resonance positions and thickness, so that accurate diffusion calculations are dfficult in most practical cases.

There is a an interesting, related case in which the frequency of one of the two interacting degrees of feedom is slowly modulated in time. For a single modulated degree of freedom the slow variation is adiabatic, and KAM surfaces exist. Remarkably, with the addition of the second degree of freedom a multiplet layer is formed which can overlap, leading to diffusion across the layer and enhanced diffusion along the layer, known as modulational diffusion. The onset of the overlap depends on the slowness of the modulation, which is in contradistinction to the normal adiabatic problem. The diffusion along a multiplet layer can be calculated and is often strong compared to the diffusion along the separatrix layer of a single resonance.

REFERENCE

1. A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).