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TRACKING CODES IN ACCELERATORS: TYPES AND LIMITATIONS†

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The motivation for tracking simulations of single-particle motion in circular accelerators is outlined. Three mathematical models, which are employed in many tracking programs describing charged-particle behavior, are presented, and the advantages and disadvantages of each model are discussed.

I. INTRODUCTION

The goal of "tracking" is to simulate directly single-particle motion in circular accelerators, in the presence of nonlinear magnetic fields and errors of various types, to determine those regions of phase space that are stable. "Stability" has two meanings in this context. The first is absolute physical stability within the accelerator (which must be determined). The second is particle stability within the mathematical model employed by a particular tracking program. Such studies assume that if a test particle with initial condition at a specified location in phase space remains within a bounded region (respresenting a beam pipe) for a specified interval of simulated time, then all real particles in the vicinity of the simulated particle will be absolutely stable in the physical accelerator. All relevant physical effects must be included in the model, and the single-particle motion must not exhibit pathological features (such as extensive stochasticity) for this assumption to be valid.

The geometry commonly employed in tracking simulations is as follows. A "cut" is made at a convenient location along the accelerator, and a test particle is launched into the machine at that point using initial conditions which deviate from the phase-space location of an ideal particle, which is assumed to remain at the origin of phase space as it circulates within the accelerator. The motion of the test particle is then simulated by solving Hamilton's equations for the position and momentum of the test particle, relative to the ideal particle, after one turn about the machine. (The deviations of test-particle motion from the motion of the ideal particle are small, so that approximation schemes may be used.) The long-term behavior of the test particle may be simulated by iterating this process, using as initial conditions for each iterate the test-particle, phase-space location at the end of the preceding iterate.

The key feature of the process is the Hamiltonian nature of the motion. The location of the test particle at the end of one turn may thereby be viewed as the

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image of the initial condition under a nonlinear canonical transformation called a "transfer map,"[1] which completely specifies the motion of the test particle for an entire turn around the accelerator. Particle stability over many turns is thus governed by the stability of high powers of the transfer map. If the one-turn transfer map is denoted by **M**, the particle motion for n turns is governed by $\mathbf{M}^n$. The accelerator itself is then viewed as an analog computer, which evaluates the action of the map **M** on the single-particle phase space. Tracking codes are distinguished only by how they represent and evaluate the nonlinear transfer map.

Alternate representations of the transfer map prove more or less convenient for inclusion of different physical effects. We shall outline the types of nonlinear effects that influence particle stability in accelerators and that are thus of interest in tracking simulations and then discuss three models employed to describe such effects. In each model it is assumed that proper care has been taken in the choice of Hamiltonian and canonical variables and that the models are obtained through a consistent set of approximations.

II. PHYSICAL EFFECTS OF INTEREST IN TRACKING SIMULATIONS

At least two effects are of interest in tracking simulations. The first is the influence of the nonlinear spatial dependence of magnetic fields within the accelerator upon particle stability. In conventional accelerators, the primary sources of such fields are the chromatic-correction sextupoles, within which a particle experiences a magnetic field (and hence, a magnetic force) that varies quadratically with displacement. In superconducting accelerators, the major sources of magnetic nonlinearities are systematic and random magnetic multipoles within the superconducting magnets. In contrast to conventional magnets (which have very weak high-order nonlinear field components), superconducting magnets possess a rich dependence of magnetic field upon position. In either case, the dependence of magnetic field upon position is a major source of nonlinear effects and is thus of great interest.

A second effect of importance in tracking simulations is that of energy oscillations. Although the energy variation of a particle within an accelerator is usually well described by linear mathematics, this 'synchrotron" oscillation couples strongly to the nonlinear motion of a particle. As the particle energy varies, the natural frequency of transverse oscillations is slowly modulated. As a consequence, the structure (in time) of the nonlinear transverse force applied to a particle varies, and the particle may repeatedly cross isolated or multiple resonances, causing instability. Treatment of energy oscillations is thus needed.

III. A THIN-LENS MODEL FOR THE TRANSFER MAP

The simplest method employed in tracking programs to describe a transfer map is referred to as a thin-lens or "kick" model. In this model, a magnetic element with nonlinear fields is treated in the impulsive approximation. The motion in all other elements is assumed to be linear. Mathematically, the accelerator is then modeled by a sequence of interleaved linear matrix and nonlinear impulsive transformations. The nonlinear transforms have the following explicitly canonical form in one degree of freedom (q, p_q are canonical conjugates):

$$\begin{aligned} \bar{q} &= q, \\ \bar{p}_q &= p_q + f(q). \end{aligned} \tag{1}$$

Equations (1) model the effect of a thin magnetic lens on a particle, hence the name 'thin-lens model." The method is simple and the transformations are easily evaluated. Moreover, nonlinearities of any order may be treated. However, the impulse approximation must be valid, so the nonlinear elements under consideration must be short in length and high in field. The required numerical simulation time will increase linearly with the number of nonlinear elements, and the accuracy will be degraded as the nonlinearities weaken. Little analytic information is provided, but this method is ideal for treating accelerators with a moderate number of high-field, strongly nonlinear elements.

It is possible to treat energy oscillations in a straightforward manner within this model either by using a four-dimensional single-particle phase space (while adjusting the transformations used for simulation to account for the variation of magnetic bending with energy) or by using a full six-dimensional phase space and simulating the effect of the rf system using an additional family of transformations of the form of Eq. (1).

An example of a widely distributed impulse code is the program PATRICIA.[2]

IV. A TAYLOR'S-SERIES EXPANSION TO MODEL THE TRANSFER MAP

A Taylor's series may be used for a second representation of the transfer map. In this method, each position or momentum-vector component of a test particle after any element is viewed as a function of all components upon entering the element. This function is written as a Taylor's expansion and truncated at some order. Thus, in a four-dimensional phase space we may write the result for a transverse displacement (relative to an ideal particle), accurate to second order, as follows;

$$\begin{aligned} \bar{x} = {} & M(x;x)x + M(x;p_x)p_x + M(x;y)y + M(x;p_y)p_y \\ & + T(x;x,x)x^2 + T(x;x,p_x)xp_x + T(x;x,y)xy + \cdots + T(x;p_y,p_y)p_xp_y. \end{aligned} \tag{2}$$

The "matrix elements" M and T may be computed using the equations of motion.[3] Relations such as Eq. (2) may therefore be used to model the transfer map by carrying the test-particle initial condition at the beginning of a turn in sequence through each element of the accelerator. However, Eq. (2) is a truncated series, and so it is not canonical. This problem may be overcome by replacing Eq. (2) by a transformation of the following form:

$$\bar{x} = \partial F(x, y, \bar{p}_x, \bar{p}_y)/\partial \bar{p}_x; \qquad \bar{y} = \partial F(x, y, \bar{p}_x, \bar{p}_y)/\partial \bar{p}_y;$$
$$p_x = \partial F(x, y, \bar{p}_x, \bar{p}_y)/\partial x; \qquad p_y = \partial F(x, y, \bar{p}_x, \bar{p}_y)/\partial y. \tag{3}$$

The generating function F is chosen to duplicate the Taylor's series through the desired order of accuracy [order 2 in Eq. (2)] and may be simply written in terms of M and T. Equation (3) is canonical in all orders, thereby providing a Hamiltonian representation of the transfer map. Unfortunately, Eq. (3) is implicit; $\bar{x}, \bar{p}_x, \ldots$ may only be obtained numerically in most cases.

This representation may be extended to second or third order in most applications. It is therefore restricted to the description of low-order nonlinearities. However, a full six-dimensional phase space may be treated, and there are no restrictions to either "short" or "strong" elements (an improvement over the impulse approximation). Moreover, as a specific series is provided for every element in the accelerator, collections of transformations may be combined to provide a single transform that describes (approximately) a collection of nonlinear elements. If only low-order nonlinearities are relevant, a reduction in the number of transformations required to represent the single-turn transfer map is thereby achieved, and a corresponding increase in tracking speed is obtained.

Energy oscillations may be introduced in a natural manner by treating rf cavities as an additional type of element in the accelerator. As an explicit nonlinear expression [such as Eq. (2)] is utilized, significant analytical information is provided. The program DIMAT[4] is one tracking code which employs a Taylor's-series representation of the transfer map.

V. A LIE ALGEBRAIC REPRESENTATION FOR THE TRANSFER MAP

The final transfer-map model we consider makes use of certain group properties of Hamiltonian-generated canonical transformations. In this model, the transfer map is represented by use of a product of "Lie transformations."[5] We denote this map as follows:

$$\mathbf{M} = \exp :f_2: \exp :f_3: \exp :f_4: \cdots \exp :f_n:. \tag{4}$$

Here, $\mathbf{M}$ is the transfer map for an element, collection of elements, or even an entire turn. The f_m are homogeneous polynomials of degree m in the canonical variables. The location and momentum of a particle after entering the element or beam line described by the map Eq. (4) are obtained by evaluating the action of $\mathbf{M}$ on the phase-space location of the test particle upon entering the element or beam line. Here, as elsewhere, the transfer map is to be regarded as an active canonical transformation carrying the test particle through the accelerator.

In this formalism, the polynomials f_m are uniquely specified if the Hamiltonian for motion through each element in the accelerator is given. The representation of Eq. (4) is explicitly canonical and accurate to order $n-1$. In practice, third-order ($n=4$) descriptions are available, and fourth-order ($n=5$) descriptions may be possible. The formalism is thus limited to low-order nonlinearities, but it allows the description of finite-length nonlinear elements and is often of higher accuracy (order 3 vs. order 2) than the Taylor's-series method. As the formalism is based on Eq. (4), considerable analytic information is available, and it appears possible in some cases to describe a fairly large portion of an accelerator with a single transfer map. Numerical calculations with this formalism may be carried out with considerable speed in such cases. However, firm guidelines for when this is possible do not as yet exist.

In practice, Eq. (4) is not evaluated exactly (which would require summing an infinite series) but is expanded in powers of the operators $:f_m:$. The resulting (noncanonical) transform is treated in the same manner as the matrix transform Eq. (2); a generating-function representation is obtained and evaluated. A final strength of the Lie algebraic formalism is that it, like the Taylor's series formalism, is explicitly six-dimensional and thereby allows a natural description of energy oscillations. Additional power is provided by the analytic capabilities of the formalism, which emphasize the equivalence of all directions in six-dimensional phase space. No distinction is made between longitudinal and tranverse coordinates (a critical feature when simulating large-amplitude energy oscillations). One tracking program making use of the Lie algebraic formalism is the program MARYLIE.[6]

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