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CUMULATIVE BEAM BREAKUP IN RF LINACS†

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I. INTRODUCTION

The phenomenon of cumulative beam breakup has been widely observed in rf electron linacs[1] and in induction linacs.[2] Briefly stated, the mechanism responsible for this instability is the interaction of the beam with a deflecting mode (one having cos ϕ or sin ϕ azimuthal dependence) of the accelerating structure. Any beam offset or structure misalignment will cause the beam to drive such a mode, which will then cause subsequent portions of the beam to be deflected. This deflected beam will drive the same (or a similar) mode in the next accelerating unit, which will further deflect the later portions of the beam, which will drive the mode in the next unit even more effectively, and so on. Ultimately the beam transverse amplitude may be driven to the point where the beam will intercept the walls of the accelerating structure and be lost.

A previous analysis[3] of this instability did not treat the rf structure of the beam, an omission that will not give the correct steady-state solution. As linac pulse lengths increase and as *cw* operation becomes desirable, a detailed knowledge of the steady-state beam behavior, as well as details of the transient behavior, becomes necessary. This paper treats first the coasting beam with an initial offset. An exact (though non-transparent) solution is derived, various limits are investigated, and numerical examples are given. Included is the extension of the analysis to accelerating and decelerating beams.

II. ANALYSIS OF BEAM CAVITY INTERACTION

We model the accelerator as consisting of identical accelerating units (cavities) placed along the axis with a periodicity L. The accelerating units are not coupled to each other. The beam is considered to be a series of delta-function pulses traveling with speed c separated in time by an amount τ and each containing charge $N_p e$. We consider beam displacements in only one plane, and denote the displacement from the axis of pulse M at cavity N by $\xi_M{}^N$. Both the cavities and the bunches are enumerated starting at 0. The displacement of a given bunch is assumed not to change within a given cavity, and it is further assumed that a bunch is not deflected by the fields it generates in a given cavity (i.e., single-bunch wake-field effects are not considered). We further consider that a single mode is responsible for the instability (identified, for example, as the HEM_{11}

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mode in SLAC, and the TM_{130} mode in the ETA [cf. Ref. 2]). The formulation in this section is essentially that of Helm and Loew.[4]

The vector potential of the electromagnetic field is expanded, following Condon,[5] as

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\lambda} q_\lambda(t)\mathbf{A}_\lambda(\mathbf{r})$$

where the normal-mode eigenvectors $\mathbf{A}_\lambda$ satisfy the Helmholtz equation $\nabla^2\mathbf{A} + (\omega_\lambda/c)^2\mathbf{A} = 0$ with $\nabla \cdot \mathbf{A} = 0$. The $\mathbf{A}_\lambda$ form an orthogonal set of functions and as a result, the time-dependent coefficients q satisfy the differential equation

$$\ddot{q}_\lambda + \frac{\omega_\lambda}{Q_\lambda}\dot{q}_\lambda + \omega_\lambda{}^2 q_\lambda = \frac{1}{\epsilon_0}\frac{\int \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{A}_\lambda(\mathbf{r})\, dV}{\int \mathbf{A}_\lambda{}^2\, dV}. \tag{1}$$

(Rationalized MKS units are used throughout this work). In the following we will drop the subscript λ, since we are considering only a single mode. In cavity number N, we take the current density $\mathbf{j}(\mathbf{r}, t)$ arising from bunch number M to be

$$\mathbf{j}(\mathbf{r}, t) = \hat{z} N_p ec\, \delta(x - \xi_M{}^N)\, \delta(y)\, \delta(z - ct) \tag{2}$$

The coordinates x, y, and z are local to the cavity under consideration, with the time being that appropriate to the presence of the bunch. Considering then that the cavity extends from $-g/2$ to $g/2$ in the z direction, the integral on the right side of Eq. (1) is given by

$$\int \mathbf{j}(\mathbf{r}, t) \cdot \mathbf{A}(\mathbf{r})\, dV = N_p ec A_z(\xi_M{}^N, 0, ct) \qquad -g/2c < t < g/2c \tag{3}$$

Expanding A_z in a Taylor series in x, noting that $A_z = 0$ on axis and retaining only the linear term, then the solution of Eq. (1) resulting from bunch M alone is, once the bunch has left the cavity,

$$q = \frac{N_p ec}{\varepsilon_0 \omega \int \mathbf{A}^2\, dV}\, \xi_M{}^N \int_{-g/2c}^{g/2c} \frac{\partial A_z}{\partial x}(0, 0, ct') \sin \omega(t - t')\, e^{-\omega(t-t')/2Q}\, dt' \tag{4}$$

where the integral is taken over the time the bunch is inside the cavity. If we define

$$I_A \equiv \int_{\text{length of cavity}} \frac{\partial A_z}{\partial x}(0, 0, z)\, e^{-i\omega/cz}\, dz, \tag{5}$$

then, using superposition, we can write the time-dependent amplitude of cavity number N, after bunch number $M - 1$ has passed through it, as

$$q_{M-1}^N(t) = \frac{N_p e}{\epsilon_0 \omega \int \mathbf{A}^2\, dV} \operatorname{Im}\left\{ I_A \sum_{l=0}^{M-1} \xi_l{}^N\, e^{-\omega(t-l\tau)/2Q}\, e^{i\omega(t-l\tau)} \right\} \tag{6}$$

The next bunch through the cavity, bunch number M, will have its transverse momentum changed by an amount

$$\Delta p_x = -ec \int B_y(\mathbf{r}, t)\, dt = ec \int q_{M-1}^N(t) \frac{\partial A_z}{\partial x} (\xi_M{}^N, 0, z = ct')\, dt', \tag{7}$$

where $t' = t - M\tau$. Again ignoring the change in the damping term during the transit of the cavity, and observing that $\partial A_z/\partial x$ does not depend, to first order, on displacement from the axis, we find

$$\Delta p_x = \frac{N_p e^2}{\epsilon_0 \omega \int \mathbf{A}^2\, dV} \operatorname{Im}\left\{ I_A \sum_{l=0}^{M-1} \xi_l{}^N e^{-(M-l)\omega\tau/2Q} e^{i(M-l)\omega\tau} \times \int e^{i\omega t'} \frac{\partial A_z}{\partial x}(0, 0, z = ct')\, d(ct') \right\}$$

$$= \frac{N_p e^2 I_A I_A^*}{\epsilon_0 \omega \int \mathbf{A}^2\, dV} \sum_{l=0}^{M-1} \xi_l{}^N e^{-(M-l)\omega\tau/2Q} \sin(M-l)\omega\tau. \tag{8}$$

Note that if I_A is a real quantity, as for a mode with even z-dependence in a symmetric cavity with $z = 0$ at the cavity center, this last equation can be written as $\Delta p_x = eI_A q_{M-1}^N(M\tau)$.

Now the transverse coupling impedance, corrected for transit-time effects and divided by the mode-quality factor, is defined by

$$\frac{Z_\perp T^2}{Q} = \frac{2}{\omega\epsilon_0} \frac{\left| \int e^{-i\omega z/c} \frac{\partial A_z}{\partial x}(0, 0, z)\, dz \right|^2}{\int \mathbf{A}^2\, dV} = \frac{2 I_A I_A^*}{\omega\epsilon_0 \int \mathbf{A}^2\, dV} \quad (\text{ohm/m}^2). \tag{9}$$

Thus cavity number N changes the transverse momentum of each particle in bunch number M by the amount

$$\Delta p_x = \frac{1}{2} N_p e^2 \frac{Z_\perp T^2}{Q} \sum_{l=0}^{M-1} s_{M-l}\, \xi_l{}^N, \tag{10}$$

where

$$s_k \equiv e^{-\frac{k\omega\tau}{2Q}} \sin k\omega\tau. \tag{11}$$

III. THE BEAM-BREAKUP DIFFERENCE EQUATIONS

We now consider tracking the bunches through the accelerator. Subject to the assumptions made in the previous section, we can write the relationships relating beam parameters before and after the action of the cavity as

$$\xi_M{}^{N+} = \xi_M{}^N, \tag{12}$$

$$\theta_M{}^{N+} \equiv \frac{p_x{}^+}{p_z{}^+} = \frac{p_x{}^-}{p_z{}^+} + \frac{\frac{1}{2}N_p e^2(Z_\perp T^2/Q)}{p_z{}^+} \sum_{l=0}^{M-1} s_{M-l}\xi_l{}^N,$$

so that

$$\theta_M{}^{N+} = \frac{\gamma_N}{\gamma_{N+1}} \theta_M{}^N + \frac{R}{\gamma_{N+1}} \sum_{l=0}^{M-1} s_{M-l}\xi_l{}^N, \tag{13}$$

where

$$R \equiv \frac{N_p e^2(Z_\perp T^2/Q)}{2m_0 c}, \tag{14}$$

and the + superscript indicates the value after the cavity action. The quantity γ_N is the value of the relativistic energy parameter upon entry to cavity number N. If we consider the cavity action as having taken place at the cavity midplane, then we transport the beam from cavity center to cavity center by a transport matrix

$$\begin{pmatrix} \xi_M{}^{N+1} \\ \theta_M{}^{N+1} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \xi_M{}^{N+} \\ \theta_M{}^{N+} \end{pmatrix}. \tag{15}$$

In the following analysis we will assume that this transport matrix is a constant, i.e., it does not depend on N.

Combining Eqs. (12), (13), and (15) gives the final difference equations which must be solved, given $\xi_M{}^0$ and $\theta_M{}^0$:

$$\xi_M{}^{N+1} = M_{11}\xi_M{}^N + M_{12}\frac{\gamma_N}{\gamma_{N+1}}\theta_M{}^N + \frac{M_{12}R}{\gamma_{N+1}} \sum_{l=0}^{M-1} s_{M-l}\xi_l{}^N \tag{16}$$

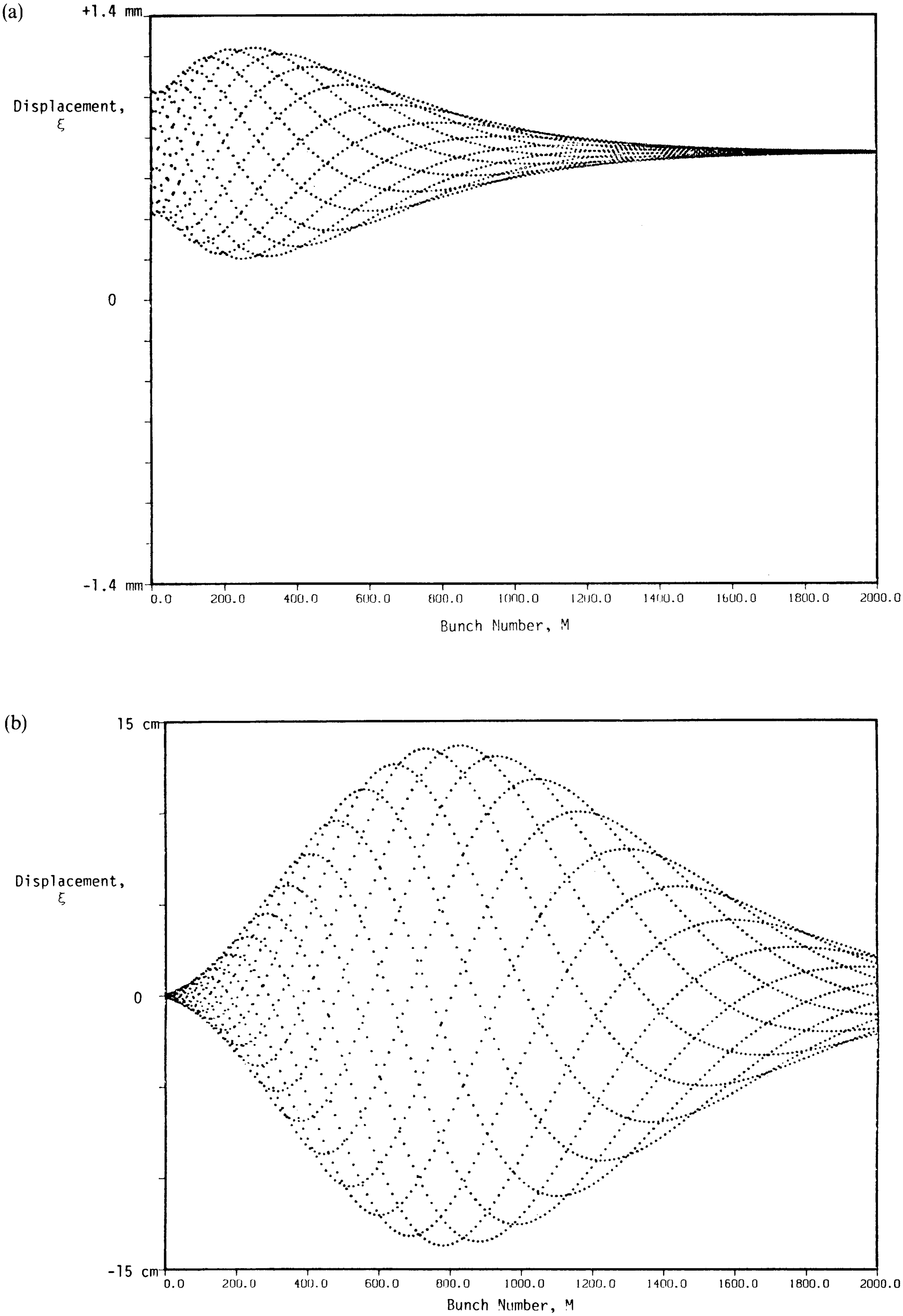
$$\theta_M{}^{N+1} = M_{21}\xi_M{}^N + M_{22}\frac{\gamma_N}{\gamma_{N+1}}\theta_M{}^N + \frac{M_{22}R}{\gamma_{N+1}} \sum_{l=0}^{M-1} s_{M-l}\xi_l{}^N \tag{17}$$

In the next section, these difference equations are solved for a coasting beam (γ = constant). Equations (16) and (17) serve as the basis for numerical simulations of the beam-breakup problem, and allow for arbitrary initial displacements, acceleration, deceleration, and varied focusing.

Figures 1a–d show the solution of Eqs. (16) and (17) for parameters appropriate to a 1300-MHz standing-wave rf linear accelerator structure with a coasting 6.5-A beam of $\gamma = 6$. The parameter $M_{12}R/\gamma = 2.88 \times 10^{-3}$, $\omega\tau/2\pi = 1.846$, and $Q = 1000$. Figure 1a shows the displacement of the first 2000 bunches as they leave the 15th cavity, all bunches having entered the first cavity offset from the axis by 1 mm. One should note that there is a steady-state displacement of the beam, and that individual bunches have displacements that are centered about the steady-state value. Figure 1b shows the

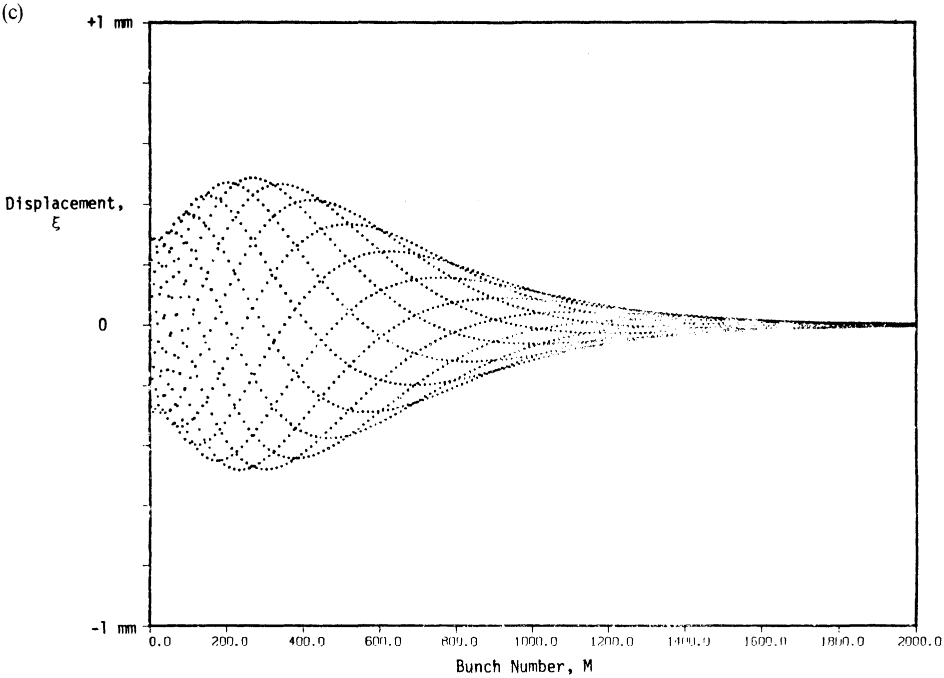
(a)

(b)



FIGURES 1a–d Displacement at fixed cavity number as a function of bunch number. Figures 1a and 1b for cavities number 15 and 30, respectively, result from a beam in which all bunches are initially offset by 1 mm. Figures 1c and 1d result from a beam in which only the first bunch is offset by 1 mm.

(c)



(d)

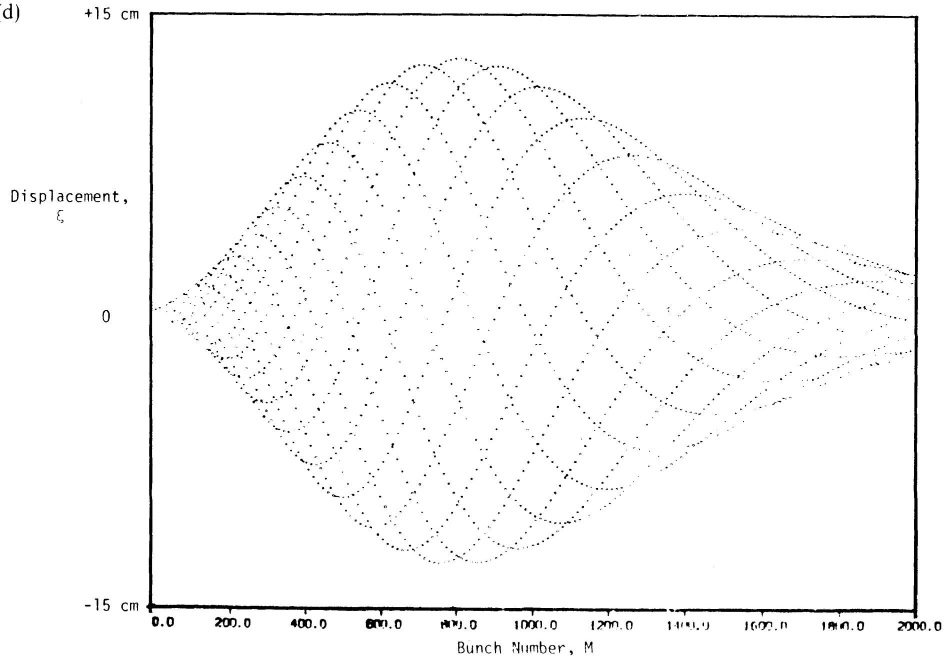


FIGURE 1 (*Cont.*)

displacement of the first 2000 bunches after the 30th cavity has been traversed. The same observations as in Fig. 1a apply.

Figure 1c shows the displacement of the first 2000 bunches as they leave the 15th cavity for the case in which only the first bunch entering the structure was offset by 1 mm, the rest of the bunches entering along the axis. Note that the steady-state displacement is zero since the first cavity is driven only by the first pulse, and the cavity excitation then decays to zero. Figure 1d shows bunch displacement after 30 cavities.

The structure evident in these figures is a result of the fact that the ratio of the beam breakup frequency to the accelerating frequency is 24/13.

The various features of these solutions are calculated in subsequent sections of this paper.

IV. SOLUTION OF DIFFERENCE EQUATIONS, COASTING BEAM

We rewrite the equations for the displacement and angle of the Mth bunch as it enters cavity N in a form convenient for subsequent analysis, (γ = constant),

$$\xi(N+1,M) = M_{11}\xi(N,M) + M_{12}\theta(N,M) + \frac{M_{12}R}{\gamma}\sum_{l=0}^{M-1} s_{M-l}\xi(N,l) \tag{18}$$

$$\theta(N+1,M) = M_{21}\xi(N,M) + M_{22}\theta(N,M) + \frac{M_{22}R}{\gamma}\sum_{l=0}^{M-1} s_{M-l}\xi(N,l) \tag{19}$$

In analogy to using the Laplace transform to solve a differential equation, let us transform these difference equations by multiplying by e^{-yN} and summing from $N = 0$ to ∞. If we define

$$\bar{\xi}(y, M) = \sum_{N=0}^{\infty} e^{-yN}\xi(N, M) \tag{20}$$

$$\bar{\theta}(y, M) = \sum_{N=0}^{\infty} e^{-yN}\theta(N, M) \tag{21}$$

we can write

$$\sum_{N=0}^{\infty} e^{-yN}\xi(N+1, M) = e^{y}[\bar{\xi}(y, M) - \xi(0, M)] \tag{22}$$

$$\sum_{N=0}^{\infty} e^{-yN}\theta(N+1, M) = e^{y}[\bar{\theta}(y, M) - \theta(0, M)] \tag{23}$$

leading to

$$(1 - e^{-y}M_{11})\bar{\xi}(y, M) - e^{-y}M_{12}\bar{\theta}(y, M) - e^{-y}\frac{M_{12}R}{\gamma}\sum_{l=0}^{M-1} s_{M-l}\bar{\xi}(y, l) = \xi(0, M) \tag{24}$$

$$-e^{-y}M_{21}\bar{\xi}(y, M) + (1 - e^{-y}M_{22})\bar{\theta}(y, M) - e^{-y}\frac{M_{22}R}{\gamma}\sum_{l=0}^{M-1} s_{M-l}\bar{\xi}(y, l) = \theta(0, M) \tag{25}$$

Equations (24) and (25) can be written in matrix form as

$$A\bar{\xi} + B\bar{\theta} = \xi_0 \tag{26}$$

$$C\bar{\xi} + D\bar{\theta} = \theta_0 \tag{27}$$

where

$$\begin{aligned} A &= (1 - e^{-y}M_{11})I - e^{-y}\frac{M_{12}R}{\gamma}S, \\ B &= -e^{-y}M_{12}I, \\ C &= -e^{-y}M_{21}I - e^{-y}\frac{M_{22}R}{\gamma}S, \quad \text{and} \\ D &= (1 - e^{-y}M_{22})I. \end{aligned} \tag{28}$$

Here I is the identity matrix and

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ s_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ s_2 & s_1 & 0 & 0 & \cdot & \cdot & \cdot \\ s_3 & s_2 & s_1 & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{29}$$

Since A, B, C, D all commute with one another, we can multiply Eq. (26) by D, Eq. (27) by B and subtract, obtaining

$$(AD - BC)\bar{\xi} = D\xi_0 - B\theta_0. \tag{30}$$

The matrix $AD - BC$ can be written as

$$AD - BC = (1 - 2\cos\mu\, e^{-y} + e^{-2y})I - e^{-y}\frac{M_{12}R}{\gamma}S \tag{31}$$

where

$$\cos\mu = \frac{M_{11} + M_{22}}{2}, \quad M_{11}M_{22} - M_{12}M_{21} = 1.$$

It is useful to write the matrix inverse of $AD - BC$ as an infinite sum. Specifically

$$(AD - BC)^{-1} = \sum_{j=0}^{\infty} \frac{e^{-jy}\left(\dfrac{M_{12}R}{\gamma}\right)^j S^j}{(1 - 2\cos\mu\, e^{-y} + e^{-2y})^{j+1}}, \tag{32}$$

and we can now write the solution for $\bar{\xi}$ as

$$\bar{\xi} = \sum_{j=0}^{\infty} \frac{e^{-jy}\left(\dfrac{M_{12}R}{\gamma}\right)^j}{(1 - 2\cos\mu\, e^{-y} + e^{-2y})^{j+1}} [S^j(1 - e^{-y}M_{22})\xi_0 - S^j e^{-y} M_{12}\theta_0] \quad (33)$$

We can now obtain $\xi(N, M)$ by equating it, according to Eq. (20), to the coefficient of e^{-Ny} in an expansion in powers of e^{-y}. In order to do this explicitly, we use the generating function for Gegenbauer polynomials (see Appendix A), namely

$$\frac{1}{(1 - 2e^{-y}\cos\mu + e^{-2y})^m} = \sum_{n=0}^{\infty} e^{-ny} C_n^{\,m}(\cos\mu). \quad (34)$$

We thereby obtain

$$\begin{aligned} \xi(N, M) = \sum_{M'=0}^{M} \sum_{j=0}^{\infty} (S^j)_{MM'} \left(\frac{M_{12}R}{\gamma}\right)^j \\ \times \big[(C_{N-j}^{j+1}(\cos\mu) - M_{22} C_{N-j-1}^{j+1}(\cos\mu))\xi(0, M') \\ + M_{12} C_{N-j-1}^{j+1}(\cos\mu)\, \theta(0, M')\big]. \end{aligned} \quad (35)$$

where we have used the explicit dependence of the matrices and vectors on M and M'.

We can recover a familiar result at this point if we let $R \to 0$ (i.e., let the charge per bunch become negligible). Then only the $j = 0$ term remains. Recognizing that S^j is the identity matrix when $j = 0$, only the term $M' = M$ contributes to the first summation, leaving

$$\xi(N, M) = [C_N^{\,1}(\cos\mu) - M_{22} C_{N-1}^1(\cos\mu)]\xi(0, M) + M_{12} C_{N-1}^1(\cos\mu)\, \theta(0, M) \quad (36)$$

If we use the relation (Eq. (A-14) in Appendix A)

$$C_N^{\,1}(\cos\mu) = \frac{\sin(N + 1)\mu}{\sin\mu}, \quad (37)$$

and the Courant-Snyder parameters

$$M_{22} = \cos\mu - \alpha\sin\mu, \quad (38)$$

$$M_{12} = \beta\sin\mu, \quad (39)$$

then we obtain

$$\xi(N, M) = (\cos N\mu + \alpha\sin N\mu)\, \xi(0, M) + \beta\sin N\mu\, \theta(0, M), \quad (40)$$

which is the standard transport result.

Returning to the solution Eq. (35) and again employing the Courant-Snyder parameters, together with the recurrence relation [see Appendix A, Eq. (A-6)]

$$C_{N-j}^{j+1}(\cos\mu) - \cos\mu\, C_{N-j-1}^{j+1}(\cos\mu) = \frac{N+j}{2j}\, C_{N-j}^{j}(\cos\mu), \tag{41}$$

we can write Eq. (35) in the form

$$\xi(N, M) = \sum_{M'=0}^{M} \sum_{j=0}^{\infty} (S^j)_{MM'} \left(\frac{M_{12}R}{\gamma}\right)^j \times \left[\left(\frac{N+j}{2j}\right) C_{N-j}^{j}(\cos\mu)\xi(0, M') + \sin\mu\, C_{N-j-1}^{j+1}(\cos\mu)\eta(0, M')\right], \tag{42}$$

where

$$\eta = (\beta\theta + \alpha\xi). \tag{43}$$

In Appendix C, it is shown that $(S^j)_{MM'}$ can be written as a Gegenbauer polynomial. Specifically, in Eq. (C-5) we show that

$$(S^j)_{MM'} = e^{-(M-M')\omega\tau/2Q} \sin^j \omega\tau\, C_{M-M'-j}^{j}(\cos\omega\tau). \tag{44}$$

Our final result for $\xi(N, M)$ is therefore†

$$\xi(N, M) = \sum_{m=0}^{M} e^{-m(\omega\tau/2Q)} \sum_{j=0}^{\infty} \left(\frac{M_{12}R\sin\omega\tau}{\gamma}\right)^j C_{m-j}^{j}(\cos\omega\tau) \times \left[\left(\frac{N+j}{2j}\right) C_{N-j}^{j}(\cos\mu)\xi(0, M-m) + \sin\mu\, C_{N-j-1}^{j+1}(\cos\mu)\eta(0, M-m)\right], \tag{45}$$

where we have changed the running index from M' to $m = M - M'$. Note that for $j = 0$, since $S^0 = I$, the $C_m{}^0$ $(\cos\omega\tau)$ terms become δ_{m0}, while the term multiplying $\xi(0, M - m)$, i.e., $((N + j)/2j)C_{N-j}^{j}(\cos\mu)$, becomes $\cos N\mu$, as discussed immediately after Eq. (35).

Another previously established result can be recovered from this last equation, namely the displacement of a continuous beam with a delta-function displacement at the head of the beam, given by $\xi(0, t) = d\delta(t)/\omega$. This problem was treated in Ref. 3; the result obtained there can be gotten from Eq. (45) as follows. We take $\alpha = 0$, $\theta(0, M) = 0$, and $\xi(0, M) = 0$ if $M \neq 0$. That is, only $\xi(0, 0)$ is different from zero. Then the solution Eq. (45) is, retaining only the current-dependent part,

$$\xi(N, M) = \xi(0, 0)e^{-M\omega\tau/2Q} \sum_{l=1}^{\infty} \left(\frac{M_{12}R}{\gamma}\right)^l \frac{N+l}{2l} C_{N-l}^{l}(\cos\mu) \sin^l \omega\tau\, C_{M-l}^{l}(\cos\omega\tau). \tag{46}$$

† Since C_{k-j}^{j} vanishes for $k < j$, it is clear that there is no contribution to the sum over j from terms with either $j > m$ or $j > N$ in Eq. (45).

Now we must take $\tau \to 0$, $M \to \infty$ such that $M\tau \to t$, the time measured from the head of the beam. At the same time we observe that the summation truncates at $l = N$ because of the $C^l_{N-l}(\cos \mu)$ term. Now as τ goes to zero, as is shown in Appendix A,

$$\sin^l \omega\tau\, C^l_{M-l}(\cos \omega\tau)_{\tau \to 0} \frac{(M + l - 1)!(M\omega\tau) j_{l-1}(M\omega\tau)}{(M - l)!M^l 2^{l-1}(l - 1)!},$$

where j_{l-1} is a spherical Bessel function.

For large M we take $(M + l - 1)!/(M - l)! = M^{2l-1}$ so that Eq. (46) becomes

$$\xi(N,t) = e^{-\omega t/2Q}\xi(0,0)\sum_{l=1}^{N}\left(\frac{M_{12}RM}{2\gamma}\right)^l \frac{(N + l)}{l!}\omega\tau C^l_{N-l}(\cos \mu) j_{l-1}(\omega\tau) \tag{47}$$

The parameter $M_{12}RM/2\gamma$ can be written, using $e^2 = (4\pi/Z_0)r_e m_0 c$ where r_e is the classical particle radius and Z_0 is the characteristic impedance of free space

$$\begin{aligned}\frac{M_{12}RM}{2\gamma} &= \frac{M_{12}}{4\gamma}\frac{t}{\tau}N_p e \frac{r_e}{e}\left(\frac{4\pi}{Z_0}\right)\left(\frac{Z_\perp T^2}{Q}\right),\\ &= \frac{I}{I_A}\frac{M_{12}}{4}\frac{4\pi}{Z_0}\frac{Z_\perp T^2}{Q}ct,\end{aligned}$$

where $I = N_p e/\tau$ and $I_A = \gamma ec/r_e$. To convert this expression to the notation of Ref. 3, $4\pi/Z_0 \to c$, $M_{12} \to 2\rho \sin \theta$, and $(Z_\perp T^2/Q) \to 2\omega k/c^2$, so that

$$\frac{M_{12}RM}{2\gamma} \to \frac{I}{I_A}k\rho \sin \theta\, \omega t, \tag{48}$$

and observing that $\omega\tau\xi(0, 0)$ equals the parameter d of Ref. 3, Eq. (47) is identical with Eq. (4.22) of Ref. 3.

V. INTEGRAL REPRESENTATION

It is possible to express Eq. (45) in integral form by using the integral representation for $C^j_{n-j}(x)$ given in (A-2), Appendix A. After considerable manipulation, which is outlined in Appendix B, we obtain

$$\xi(N, M) = \sum_{m=0}^{M} e^{-m\omega\tau/2Q}\left[\xi(0, M - m)G(x, y) + \sin \mu\, \eta(0, M - m)H(x, y)\right] \tag{49}$$

where

$$x = \cos \omega\tau, \qquad y = \cos \mu, \tag{50}$$

$$G(x, y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta}\left[\cosh N\sigma + J(\theta)\frac{\sinh N\sigma}{\sinh \sigma}\right], \tag{51}$$

$$H(x, y) = \frac{-i}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta}\sinh N\sigma. \tag{52}$$

Here

$$\cosh \sigma = \cos \mu + J(\theta) \tag{53}$$

$$J(\theta) = \frac{\Delta}{\cos \theta - \cos \omega\tau} \tag{54}$$

$$\Delta = \frac{M_{12} R \sin \omega\tau}{4\gamma} \tag{55}$$

Thus we have reduced our final expression for the displacement to a sum over initial pulse displacements and angles and a single integral over θ. The result in Eq. (49) is exact and serves as the starting point for analysis of the steady-state result, the single-pulse result and the approach to steady-state.

VI. STEADY-STATE SOLUTION

A steady state appropriate to a beam that is misaligned upon entry† to the accelerator will be produced by setting all $\xi(0, M') = \xi_0$, $\eta(0, M') = \eta_0$ for $M' \geq 0$ and evaluating the sum in Eq. (49) for $M \to \infty$. Using the expressions for $G(x, y)$ and $H(x, y)$ in Eq. (B-13) and (B-14), the sum over m becomes the geometric series

$$\sum_{m=0}^{\infty} \frac{e^{-m\omega\tau/2Q}}{u^{m+1}} = \frac{1}{u - e^{-\omega\tau/2Q}}, \tag{56}$$

convergent as long as $|u| > e^{-\omega\tau/2Q}$. At the same time, we must have $|u| < 1$ in order that we pass inside the singularities in the integrand in Eq. (B-1), as discussed in Appendix B. The resulting expression for $\xi(N, \infty)$ is

$$\xi(N, \infty) = \xi_0 \cosh N\sigma + (J\xi_0 + \eta_0 \sin \mu) \frac{\sinh N\sigma}{\sinh \sigma} \tag{57}$$

where we have taken the residue at $u = e^{-\omega\tau/2Q}$, since this is the only pole remaining within the contour in Eqs. (B-13) and (B-14). Specifically we have

$$\cosh \sigma = \cos \mu + J \tag{58}$$

with J in Eq. (54) being written as

$$J = \frac{\Delta}{\cosh \dfrac{\omega\tau}{2Q} - \cos \omega\tau} = \left(\frac{M_{12} R}{8\gamma}\right) \left(\frac{\sin \omega\tau}{\sinh^2 \dfrac{\omega\tau}{4Q} + \sin^2 \dfrac{\omega\tau}{2}}\right). \tag{59}$$

The result in Eqs. (57)–(59) can also be obtained from Eqs. (1) and (2) by requiring $\xi(N, M)$, $\theta(N, M)$ to be independent of M as $M \to \infty$, and solving the

† The case in which the beam displacement and angle are periodically modulated can also be treated analytically (see Appendix F).

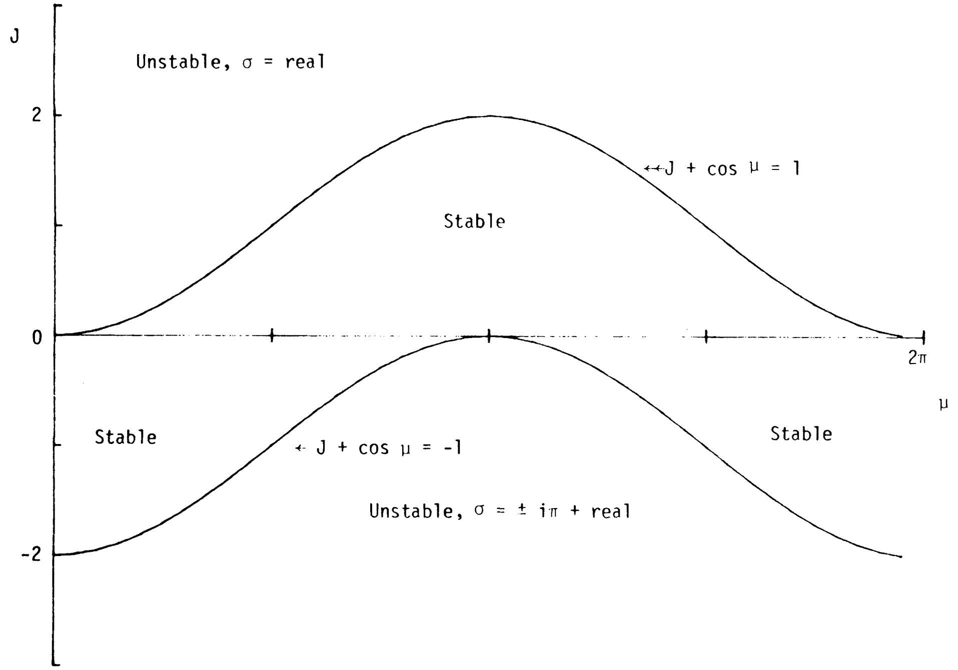


FIGURE 2 Relating transport phase advance per cell to rf parameter J.

resulting first order difference equations in $\xi(N)$ and $\theta(N)$ vs N by simple diagonalization.

The result in Eq. (57) is exact, and indicates that the solution will either oscillate or grow exponentially as N increases, depending on whether $|\cos \mu + J|$ is less than or greater than 1.

Equations (58) and (59) show the intimate connection between the beam breakup frequency, the pulse repetition frequency, and the focusing. Figure 2 shows a plot of J vs μ and indicates the region of stability, i.e., that region for which the steady state beam displacement remains bounded as $N \to \infty$. Note that for $\mu = 0$ the sign of J is all important; negative J is stable while positive J gives unstable solutions. This result means that $\sin \omega\tau$ should be negative for stable steady-state displacements. A focusing phase advance of $\pi/2$ per cell clearly gives maximum latitude in the value of J, although, since $|J| \ll 1$ in the usual case, a practical focusing system could have a much lower phase advance per cell and still tolerate positive and negative J values.

The exponent corresponding to exponential growth is

$$e = N\sigma = N \ln [\cos \mu + J + (J^2 + 2J \cos \mu - \sin^2 \mu)^{1/2}] \tag{60}$$

In the absence of external focusing, $\mu = 0$ in Eq. (58), and we obtain exponential growth with exponent

$$e_0 = N\sigma = N\, ln\, [1 + J + (2J + J^2)^{1/2}] \simeq N(2J)^{1/2} \tag{61}$$

where the last form is valid for $|J| \ll 1$.

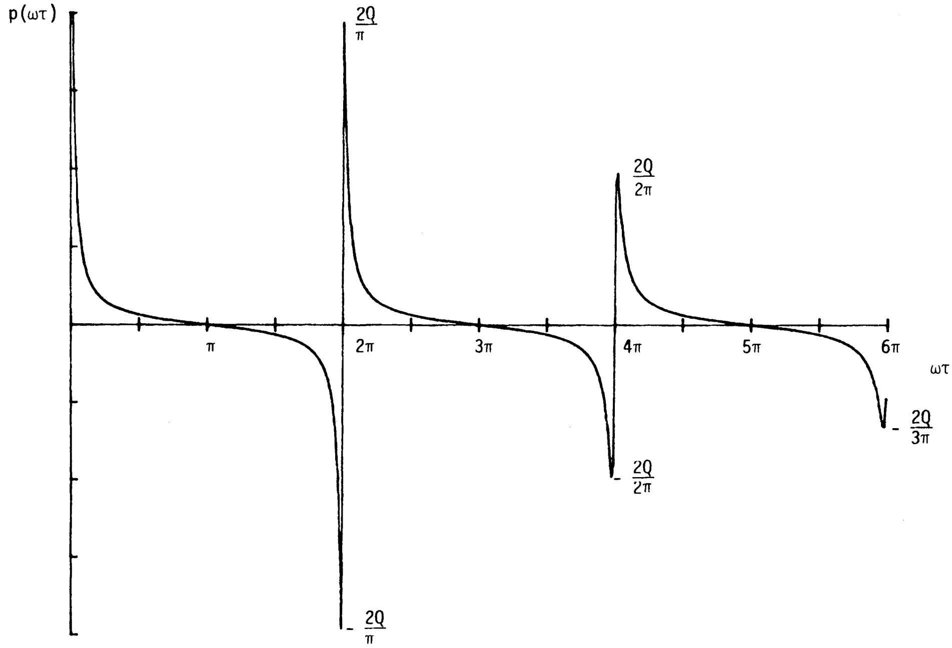


FIGURE 3 The resonance function $p(\omega\tau)$, plotted for $Q = 100$.

The effect of resonance is contained in Eq. (59). Figure 3 contains a plot of

$$p(\omega\tau) = \frac{\sin \omega\tau}{\sinh^2 \dfrac{\omega\tau}{4Q} + \sin^2 \dfrac{\omega\tau}{2}} \tag{62}$$

vs $\omega\tau$ for large Q, with the resonant peaks occurring at

$$\omega\tau = 2n\pi\left(1 \pm \frac{1}{2Q}\right) \tag{63a}$$

with peak values given by

$$|p_{\max}| = \frac{2Q}{n\pi}. \tag{63b}$$

The size of J is then the product of $p(\omega\tau)$ and the small parameter $M_{12}R/8\gamma$.

The maximum growth of displacement with cavity number occurs then for $\omega\tau = 2\pi(1 \pm 1/2Q)$, with the growth exponent [cf. Eq. (61)] being given, for $|J| \ll 1$, by

$$e_0 = N\left(\frac{M_{12}N_p e^2 Z_\perp T^2}{\gamma m_0 c}\right)^{1/2}. \tag{64}$$

VII. SOLUTION FOR A SINGLE PULSE

Further analysis will be carried out for the case where $\eta(0, M) = 0$, and where there is no external transverse focusing, so that

$$y = \cos\mu = 1 \quad \text{and} \quad M_{12} = \beta \sin\mu = L,$$

where L is the center-to-center separation of adjacent cavities. We then have from Eqs. (49) and (51)

$$\frac{\xi(N, M)}{\xi_0} = \frac{e^{-M\omega\tau/2Q}}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-iM\theta}\left(\cosh N\sigma + \frac{J(\theta)}{\sinh\sigma} \sinh N\sigma\right), \tag{65}$$

where

$$\cosh\sigma = 1 + J, \tag{66}$$

and where we have set $\xi(0, 0) \equiv \xi_0$, and $\xi(0, M) = 0$ for $M > 0$. We will assume, and later confirm, that for the parameters of interest, $|J| \ll 1$, in which case we can write

$$\sigma \simeq \sqrt{2J} \tag{67}$$

and obtain

$$\frac{\xi(N, M)}{\xi_0} \simeq \frac{e^{-M\omega\tau/2Q}}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-iM\theta}\left(\frac{e^{N\sqrt{2J}} + e^{-N\sqrt{2J}}}{2}\right) \tag{68}$$

A saddle-point calculation carried out in Appendix D leads to the result given in Eqs. (D-7) and (D-8), namely

$$\frac{\xi(N, M)}{\xi_0} \simeq \frac{\sqrt{E}}{2M\sqrt{6\pi}} e^{-M\omega\tau/2Q - iM\omega\tau + 3E/2} + \text{complex conjugate} \tag{69}$$

where

$$E = \left(\frac{4\Delta N^2 M}{\sin\omega\tau}\right)^{1/3} e^{\pi i/6} = \left(\frac{M_{12}R}{\gamma} N^2 M\right)^{1/3} e^{\pi i/6} \tag{70}$$

Equation (69) contains the behavior for large values of M and N of the amplitude of pulse number M at cavity number N for a single initially offset pulse (pulse number 0). If N is fixed, the amplitude reaches a maximum where the real part of the exponent,

$$-\frac{M\omega\tau}{2Q} + \frac{3\sqrt{3}}{4}\left(\frac{M_{12}R}{\gamma}\right)^{1/3} N^{2/3} M^{1/3} \tag{71}$$

reaches a maximum. This maximum occurs at†

$$M_{\max} = N \frac{3^{3/4}}{2^{3/2}} \left(\frac{Q}{\omega\tau}\right)^{3/2} \left(\frac{M_{12}R}{\gamma}\right)^{1/2}, \tag{72}$$

in which case

$$\frac{|\xi_{\max}|}{\xi_0} \simeq \left(\frac{4}{3}\right)^{1/4} \left(\frac{\omega\tau}{6\pi M_{\max} Q}\right)^{1/2} e^{M_{\max}\omega\tau/Q}. \tag{73}$$

Using $M_{\max} = 1042$ for the parameters of Fig. 1d, Eq. (73) predicts a ratio of 147, which is to be compared with the ratio of 128 obtained in the simulation.

VIII. APPROACH TO STEADY STATE

The approach to steady state can now be worked out quite readily. If we set all $\xi(0, M') = \xi_0$, $\eta(0, M') = \eta_0$(for $M' \geq 0$), we can write Eq. (49) as

$$\xi(N, M) - \xi(N, \infty) = - \sum_{m=M+1}^{\infty} e^{-m\omega\tau/2Q} [\xi_0 G(x, y) + \sin\mu\, \eta_0 H(x, y)]. \tag{74}$$

The sum over m is therefore modified from Eq. (56) to

$$- \sum_{m=M+1}^{\infty} \frac{e^{-m\omega\tau/2Q}}{u^{m+1}} = - \frac{e^{-(M+1)\omega\tau/2Q}}{u^{M+1}} \frac{1}{u - e^{-\omega\tau/2Q}}. \tag{75}$$

It can be seen from Eq. (65), with $u = e^{i\theta}$ that the M dependence for the case of the single pulse is contained only in the factor.

$$\frac{e^{-M\omega\tau/2Q}}{u^M}.$$

Equation (75) therefore suggests that we multiply the integrand in our single pulse expression by

$$K = - \frac{e^{-\omega\tau/2Q}}{u(u - e^{-\omega\tau/2Q})}, \tag{76}$$

† The value of M at which ξ reaches a maximum will be shifted because the nonexponential terms in Eq. (69) also contain M dependence. Furthermore, the result in Eq. (69) is only the first term in an asymptotic series in inverse powers of M. The order of magnitude of the shift can be estimated from the $M^{-5/6}$ behavior of the external factor in Eq. (69). It leads to a predicted shift of the form

$$\bar{M}_{\max} \simeq M_{\max} - \frac{5Q}{2\omega\tau} \tag{72a}$$

where $M_{\max}$ is given in Eq. (72). The original value of $M_{\max}$ is to be used in Eq. (73). For the parameters appropriate to Fig. 1d, $M_{\max} = 1042$, $5Q/2\omega\tau = 216$, and $\bar{M}_{\max} = 826$. The simulation in Fig. 1d shows that the maximum amplitude is reached for $M = 810$.

which is independent of M and N. Since our single pulse result comes from a saddle point calculation, the result for $\xi(N, M) - \xi(N, \infty)$ can be obtained from the single pulse result in Eq. (D-7) by multiplying by the value of K in Eq. (76) evaluated at the saddle given by Eq. (D-4), where

$$u_s = e^{i\theta_s} \simeq e^{i\omega\tau}. \tag{77}$$

We therefore find, for the parameters used in the preceding section

$$\frac{\xi(N, M) - \xi(N, \infty)}{\xi_0} \simeq \frac{K\sqrt{E}}{2M\sqrt{6\pi}}\, e^{-M\omega\tau/2Q - iM\omega\tau + 3E/2} + \text{complex conjugate} \tag{78}$$

Equation (78) indicates that $\xi(N, M)$ oscillates rapidly and symmetrically around $\xi(N, \infty)$, with the amplitude of oscillation reaching a maximum at $M_{\max}$ given in Eq. (72), whose value is

$$\frac{|\xi(N, M) - \xi(N, \infty)|_{\max}}{\xi_0} \simeq |K| \left(\frac{4}{3}\right)^{1/4} \left(\frac{\omega\tau}{6\pi M_{\max} Q}\right)^{1/2} e^{M_{\max}\omega\tau/Q}. \tag{79}$$

Once again our result is enhanced by resonant behavior. Specifically

$$|K| \simeq \frac{1}{2}\left(\sinh^2 \frac{\omega\tau}{4Q} + \sin^2 \frac{\omega\tau}{2}\right)^{-1/2}, \tag{80}$$

reaching a maximum value

$$|K|_{\max} \simeq \frac{Q}{n\pi} \tag{81}$$

for $\omega\tau = 2n\pi$. The resonant width is again of order $1/2Q$ compared with the resonant frequency.

IX. ACCELERATING AND DECELERATING BEAMS

The exact analytic solution of Eqs. (18) and (19) for γ and/or the matrix elements M_{ij} varying with N is beyond the scope of this work. Nevertheless it is possible to determine the generalization of the exponential growth behavior for the steady-state solution in Eq. (57) and for the single pulse behavior in Eq. (69) from consideration of the differential equation equivalent to Eqs. (18) and (19).

The generalization of the steady-state exponent $N\sigma$ in Eq. (57) is clear; by analogy with the WKB solution of a differential equation with slowly varying parameters, it is

$$e_{\text{s.s.}} = N\sigma \rightarrow \int \sigma\, dN \tag{82}$$

where σ is given in Eqs. (58) and (59), allowing for variation of γ and M_{ij} with N. For example, for $\mu = 0$ and $J \ll 1$, we find

$$e_{\text{s.s.}} = \int \sigma\, dN \simeq \int \sqrt{2J}\, dN = \int \left(\frac{\sin \omega\tau}{\sinh^2 \dfrac{\omega\tau}{4Q} + \sin^2 \dfrac{\omega\tau}{2}} \right)^{1/2} \left(\frac{M_{12}R}{4\gamma} \right)^{1/2} dN. \quad (83)$$

Here it is possible to take into account arbitrary dependence of γ, M_{12} (separation between cavities), R, and Q on N.

The generalization of the single-pulse exponent in Eq. (69) is somewhat more complicated, and is presented in Appendix E. The final result, given in Eqs. (E-14) and (E-15) for $\mu \simeq 0$, $\omega\tau \ll 1$, is that Eqs. (69) and (70) are valid, provided

$$\left(\frac{M_{12}R}{\gamma} N^2 M \right)^{1/3} \quad (84a)$$

in Eq. (70) is replaced by

$$M^{1/3} \left[\int \left(\frac{M_{12}R}{\gamma} \right)^{1/2} dN \right]^{2/3}. \quad (84b)$$

This replacement also applies to Eq. (72) which now becomes

$$M_{\max} = \frac{3^{3/4}}{2^{3/2}} \left(\frac{Q}{\omega\tau} \right)^{3/2} \left[\int \left(\frac{M_{12}R}{\gamma} \right)^{1/2} dN \right]. \quad (85)$$

For the case of constant M_{12} and R, and γ depending linearly on N, one can show that the replacement in Eq. (84b) is equivalent to

$$\left(\frac{M_{12}RN^2M}{\gamma} \right)^{1/3} \rightarrow (M_{12}RN^2M)^{1/2} \left(\frac{\sqrt{\gamma_N} + \sqrt{\gamma_0}}{2} \right)^{-2/3}. \quad (86)$$

Thus γ in Eq. (70) is to be replaced by the square of the average value of $\gamma^{1/2}$.

X. COMPARISON WITH EARLIER WORK

The beam-breakup phenomenon can be characterized by three regimes, which can be identified in Fig. 1. The first regime is characterized by an exponential increase of bunch displacement with bunch number (i.e., time). The second regime is that at which the displacement is maximum, while the third regime is the steady-state regime.

In the exponential-growth regime, the solutions Eqs. (69) and (78) are dominated by the 3E/2 term in the exponential. In the notation of Ref. 4, the growth exponent $F_e = 3E/2$. Comparison with Ref. 4 is facilitated by making the substitutions (for no focusing) $M_{12} = L$, $NL = z$, $(Z_\perp T^2/Q) = (2\pi/\lambda)^2 R_\perp/Q$, $N_p e/\tau = I_0$, $M\tau = t$, $\gamma m_0 c^2 = eV_0$, all of which leads to the exponent for a coasting beam

$$F_e(t) = \frac{3^{3/2}}{2^{5/3}} \left(\frac{z^2 I_0 ct\pi^2 R_\perp}{LV_0\lambda^2 Q} \right)^{1/3} \quad (87)$$

From Eq. (86) we deduce that for an accelerating beam with constant accelerating gradient V', the parameter V_0 in Eq. (87) should be replaced by $V'z/4$ when the particle energy at z is much greater than the initial energy. Thus for an accelerating beam

$$F_e(t) = \frac{3^{3/2}}{2}\left(\frac{zI_0 ct\pi^2 R_\perp}{LV'\lambda^2 Q}\right)^{1/3}, \tag{88}$$

which is identical to the result of Ref. 4, as quoted in Ref. 7.

The maximum steady-state displacement arising from an initially modulated beam has a growth exponent given by Eq. (F-10), which can be rewritten for a coasting beam with no focusing as

$$F_e(cw) = \frac{3^{3/4}}{2^{3/2}}\left(\frac{z^2\pi I_0 R_\perp}{LV_0\lambda}\right)^{1/2} \tag{89}$$

For an accelerating beam with energy much greater than the initial energy,

$$F_e(cw) = 3^{3/4}\left(\frac{z\pi I_0 R_\perp}{2LV'\lambda}\right)^{1/2} \tag{90}$$

in agreement with Ref. 7.

XI. SUMMARY AND CONCLUSIONS

Starting with the difference equations for a coasting beam in the variables bunch number M and cavity number N [Eqs. (18), (19)], we have obtained an analytic solution for the transverse displacement of bunch M at cavity N for arbitrary initial displacement [Eq. (45)]. This solution, expressed as a sum over Gegenbauer polynomials, includes the effect of a bunched beam, and external focusing. We have also obtained an integral representation for the solution [Eq. (49)]. If one takes the limit in which M is a continuous variable, our results are in agreement with previous results for a continuous beam.[3]

The steady-state solution is obtained [Eq. (51)] directly from the integral representation, and the effect of a resonance between the transverse mode frequency and the bunch frequency is evaluated [Eqs. (62) and (63)] and displayed (Fig. 3). Steady-state results are also obtained for an incoming beam with periodically modulated displacement and angle (Appendix F).

Approximate solutions are obtained in the absence of external focusing for the transient for a single incoming displaced bunch [Eqs. (69), (72), (73)], which can lead to much larger displacements than for the steady state. The transient approach to the steady state is also evaluated [Eq. (69)] and shown to be closely related to the behavior for a single displaced bunch. These results are then shown to be in close agreement with numerical results obtained from a simulation using the difference equations [Eqs. (18) and (19)] for parameters appropriate to the electron linac used at Los Alamos in the free electron laser.

Finally, we show how to modify the results for an adiabatic change of energy [Eqs. (85) and (86)].

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APPENDIX A

Some useful relations for the Gegenbauer polynomials.

Generating Function

$$\frac{1}{(1-2hx+h^2)^M} = \sum_{N=0}^{\infty} h^N C_N{}^M(x), \quad -1 \leq x \leq 1, \quad |h| < 1 \tag{A-1}$$

Integral Representation

$$C_N{}^M(x) = \frac{1}{2\pi i}\oint \frac{dh}{h^{N+1}}\frac{1}{(1-2hx+h^2)^M} \tag{A-2}$$

where the contour in the h plane encircles the pole at $h = 0$, and where $|h| < 1$ everywhere on the contour. This equation can be obtained directly by multiplying Eq. (A-1) by h^{-N-1} and integrating.

Series Expansions

Expansion of Eq. (A-2) in powers of x, and term-by-term integration leads to

$$C_N{}^M(x) = \sum_{r=0}^{\frac{N}{2},\frac{N-1}{2}} \frac{(M+N-r-1)!}{(M-1)!}\frac{(-1)^r}{(N-2r)!r!}(2x)^{N-2r} \tag{A-3}$$

an expression useful for small values of x. On the other hand, expansion of Eq. (A-2) in powers of $1 - x$ and term by term integration leads to

$$C_N{}^M(x) = \frac{2^{1-2M}}{(M-1)!}\sum_{r=0}^{\frac{N}{2},\frac{N-1}{2}} \frac{(2M+N+r-1)!\Gamma(\frac{1}{2})(-1)^r}{(N-r)!\Gamma(M+r+\frac{1}{2})}\left(\frac{1-x}{2}\right)^r, \tag{A-4}$$

an expression useful for values of x near 1.

See for example Magnus, Oberhettinger and Soni, Section (5.3).

Recurrence Relations

Many recurrence relations can be derived for $C_N{}^M(x)$ and $dC_N{}^M(x)/dx$. The only one used in the present work comes from the identity

$$\frac{h}{2M}\frac{\partial}{\partial h}\frac{1}{(1-2hx+h^2)^M} = \frac{1-hx}{(1-2hx+h^2)^{M+1}} - \frac{1}{(1-2hx+h^2)^M} \tag{A-5}$$

Equating like powers of h after using the generating function Eq. (A-1) leads directly to

$$C_N{}^{M+1}(x) - xC_{N-1}^{M+1}(x) = \left(1+\frac{N}{2M}\right)C_N{}^M(x) \tag{A-6}$$

Sum Relation

We shall find it useful to evaluate the following sum.

$$U = \sum_{\text{all } r} C_{N-r}^j(x)C_{r-M}^k(x), \qquad 0 < M < N. \tag{A-7}$$

where only terms $M \leq r \leq N$ will contribute. Using Eq. (A-2), one can write

$$U = \frac{1}{(2\pi i)^2}\oint\frac{du}{u^{N+1}}\frac{1}{(1-2ux+u^2)^j}\oint\frac{dv}{v^{1-M}}\frac{1}{(1-2vx+v^2)^k}\sum_{r=0}^{\infty}\left(\frac{u}{v}\right)^r \tag{A-8}$$

where we have extended the sum over r from 0 to ∞ for convenience. The sum over r leads to

$$\sum_{r=0}^{\infty}\left(\frac{u}{v}\right)^r = \frac{v}{v-u} \tag{A-9}$$

provided $|v| > |u|$. We then find

$$U = \frac{1}{(2\pi i)^2}\oint\frac{du}{u^{N+1}(1-2ux+u^2)^j}\oint\frac{dv\, v^M}{(v-u)}\frac{1}{(1-2vx+v^2)^k} \tag{A-10}$$

The integral over v no longer has a singularity at $v = 0$, nor does it enclose the poles on the unit circle. But it does enclose the first-order pole at $v = u$, since we have required that $|u| < |v|$. The residue is readily evaluated to give

$$U = \frac{1}{2\pi i}\oint\frac{du}{u^{N-M+1}}\frac{1}{(1-2ux+u^2)^{j+k}} = C_{N-M}^{j+k}(x). \tag{A-11}$$

where the last expression follows directly from Eq. (A-2). Thus we have the sum relation

$$\sum_r C_{N-r}^j(x)C_{r-M}^k(x) = C_{N-M}^{j+k}(x) \tag{A-12}$$

Special Values ($M = 1$)

Let us rewrite Eq. (A-2) in the form

$$C^1_{N-1}(\cos\theta) = \frac{1}{2\pi i}\oint \frac{dh}{h^N}\frac{1}{(1 - 2h\cos\theta + h^2)} \tag{A-13}$$

The integrand has poles at $h = 0$, and $h = e^{\pm i\theta}$, but the contour only encloses the pole at $h = 0$. However, since there is no singularity at $h = \infty$, the boundary can be distorted so that the result is the negative of the sum of the residues at $h = e^{\pm i\theta}$. This leads to

$$C^1_{N-1}(\cos\theta) = \frac{\sin N\theta}{\sin\theta}, \tag{A-14}$$

which will be used in Appendix C.

Special Values ($M = 0$)

We can see directly from Eq. (A-2) that

$$C_N{}^0(\cos\theta) = \begin{Bmatrix} 1, & N = 0 \\ 0, & N \geq 1 \end{Bmatrix} = \delta_{N0}. \tag{A-15}$$

This result also follows from the absence of all but the first term in the power series in h, when $m = 0$ in Eq. (A-1). An alternate definition of $C_N{}^0$ exists in the literature.

Special Values ($x = 1$)

If we put $x = 1$ in Eq. (A-1), the left side becomes $(1 - h)^{-2M}$. From its expansion in powers of h, one can show that

$$C_N{}^M(1) = \frac{(2M + N - 1)!}{(2M - 1)!N!}. \tag{A-16}$$

Asymptotic Form for Small θ, *Large* N

It can be shown that

$$y = \sin^M\theta\, C^M_{N-M}(\cos\theta) \tag{A-17}$$

satisfies the differential equation (see Ref. 6)

$$\frac{d^2y}{d\theta^2} + \left[N^2 - \frac{M(M-1)}{\sin^2\theta}\right]y = 0. \tag{A-18}$$

For small values of θ, we can replace $\sin^2\theta$ by θ^2, obtaining

$$\frac{d^2\bar{y}}{d\theta^2} + \left[N^2 - \frac{M(M-1)}{\theta^2}\right]\bar{y} = 0, \tag{A-19}$$

where we are neglecting terms of relative order M^2/N^2 which appear from the next term in the expansion of $\sin\theta$ in powers of θ. Equation (A-19) is closely related to the differential equation for the spherical Bessel function. Specifically

$$\bar{y} = A\, N\theta\, j_{M-1}(N\theta) \tag{A-20}$$

where A is to be determined by required agreement between Eqs. (A-17) and (A-20) for small θ. Using Eq. (A-16) and the known power series for $j_{M-1}(x)$ we can write

$$\sin^M\theta\, C^M_{N-M}(\cos\theta) \simeq \frac{(N+M-1)!}{(N-M)!(M-1)!N^M 2^{M-1}}(N\theta)j_{M-1}(N\theta) \tag{A-21}$$

APPENDIX B

Let us use the integral representation in Eq. (A-2), in the alternate forms (the second is obtained from the first by integrating by parts)

$$\begin{aligned} C^j_{N-j}(x) &= \frac{1}{2\pi i}\oint \frac{du}{u^{N+1-j}}(1-2ux+u^2)^{-j} \\ &= -\frac{1}{2\pi i}\cdot\frac{2j}{(N-j)}\oint \frac{du(u-x)}{u^{N-j}(1-2ux+u^2)^{j+1}}. \end{aligned} \tag{B-1}$$

Here the path of integral in the complex plane is a circle around $u = 0$ which passes inside the poles on the unit circle at $u = x \pm i\sqrt{1-x^2}$. We then can write for Eq. (45)

$$\xi(N, M) = \sum_{m=0}^{M} e^{-m(\omega\tau/2Q)}[\xi(0, M-m)G(x, y) + \sin\mu\,\eta(0, M-m)H(x, y)]. \tag{B-2}$$

Here

$$x = \cos\omega\tau, \tag{B-3}$$

$$y = \cos\mu, \tag{B-4}$$

$$G(x, y) = \frac{1}{2\pi i}\oint \frac{du}{u^{m+1}}\frac{1}{2\pi i}\oint \frac{dv}{v^{N+1}}(1-vy)K(u, v), \tag{B-5}$$

$$H(x, y) = \frac{1}{2\pi i}\oint \frac{du}{u^{m+1}}\frac{1}{2\pi i}\oint \frac{dv}{v^{N}}K(u, v), \tag{B-6}$$

where

$$K(u, v) = \frac{1}{1-2vy+v^2}\sum_{j=0}^{\infty}\left[\frac{4\Delta uv}{(1-2ux+u^2)(1-2vy+v^2)}\right]^j, \tag{B-7}$$

and

$$\Delta = \frac{M_{12} R \sin \omega\tau}{4\gamma}. \tag{B-8}$$

The sum over j can be performed (keeping $|u|$ and $|v|$ small compared with 1 at this point to ensure convergence), to obtain

$$K(u, v) = \frac{1}{1 - 2vy + v^2 - 2vJ(u)}, \tag{B-9}$$

where

$$J(u) = \frac{2\Delta u}{1 - 2ux + u^2}. \tag{B-10}$$

Clearly $K(u, v)$ has poles in the v plane located at

$$v_{\pm} = e^{\pm\sigma}, \tag{B-11}$$

where

$$\cosh \sigma(u) = y + J(u). \tag{B-12}$$

Since the contour in the v plane in Eqs. (B-5) and (B-6) only encloses the origin, and since there is no pole at ∞, the contour can be deformed to enclose the two poles in Eqs. (B-9) instead, for which the residues are readily calculable. This leads to

$$G(x, y) = \frac{1}{2\pi i} \oint \frac{du}{u^{m+1}} \left[\cosh \left(N\sigma(u)\right) + \frac{J(u)}{\sinh \sigma(u)} \sinh \left(N\sigma(u)\right) \right] \tag{B-13}$$

$$H(x, y) = -\frac{1}{2\pi} \oint \frac{du}{u^{m+1}} \sinh \left(N\sigma(u)\right). \tag{B-14}$$

If we now replace u by $e^{i\theta}$ we have

$$G(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-im\theta} \left[\cosh N\sigma + J \frac{\sinh N\sigma}{\sinh \sigma} \right] \tag{B-15}$$

$$H(x, y) = -\frac{i}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{-im\theta} \sinh N\sigma, \tag{B-16}$$

where

$$J(\theta) = \frac{\Delta}{\cos \theta - \cos \omega\tau} \tag{B-17}$$

$$\cosh \sigma(\theta) = J(\theta) + \cos \mu, \tag{B-18}$$

and where the contour in the complex θ plane must pass above the singularities at

$$\theta = \pm\omega\tau, \tag{B-19}$$

in order that $|u|$ remain inside the unit circle. Thus we have reduced our final expression for the displacement to a sum over initial pulse displacements and angles and to a single integral over θ.

APPENDIX C

The evaluation of S^j can proceed as follows. The definition of s_k in Eq. (4) permits us to write

$$s_k = e^{-k(\omega\tau/2Q)} \sin k\omega\tau = e^{-k(\omega\tau/2Q)} \sin \omega\tau \, C^1_{k-1}(\cos \omega\tau), \tag{C-1}$$

where we have used the explicit form for $C^1_{k-1}(\cos \omega\tau)$ in Eq. (A-14). From the matrix form in Eq. (14) it is clear that

$$S_{pq} = \begin{Bmatrix} s_{p-q}, & p > q \\ 0, & p \leq q \end{Bmatrix}. \tag{C-2}$$

Since

$$(S^j)_{MM'} = \sum_{\alpha,\beta,\gamma,\ldots} s_{M-\alpha} s_{\alpha-\beta} s_{\beta-\gamma} \cdots s_{\delta-M'}, \tag{C-3}$$

we can use Eq. (C-1) to write

$$(S^j)_{MM'} = e^{-(M-M')\omega\tau/2Q} \sin^j \omega\tau \sum_{\alpha,\beta,\gamma} C^1_{M-\alpha-1} C^1_{\alpha-\beta-1} C^1_{\beta-\gamma-1} \cdots C^1_{\delta-M'-1}. \tag{C-4}$$

Repeated application of Eq. (A-12) then gives

$$(S^j)_{MM'} = e^{-(M-M')\omega\tau/2Q} \sin^j \omega\tau \, C^j_{M-M'-j}(\cos \omega\tau). \tag{C-5}$$

APPENDIX D

In order to proceed from Eq. (68) we shall evaluate $\xi(N, M)$ for large M, N by using a saddle-point calculation. It is clear that only $e^{N\sqrt{2J}}$ has the potential to give a large contribution to the integral over θ. For the exponent

$$f(\theta) = -iM\theta + N\sqrt{2J(\theta)} \tag{D-1}$$

the three saddle points occur where

$$\frac{df}{d\theta} = 0 = -iM + \frac{N}{\sqrt{2J}} \frac{dJ}{d\theta} = -iM + \frac{N \sin \theta (2J)^{3/2}}{4\Delta} \tag{D-2}$$

$$\sqrt{2J_s} = \left(\frac{4M\Delta}{N \sin \theta}\right)^{1/3} (e^{+\pi i/6}, e^{+5\pi i/6}, e^{+3\pi i/2}) \tag{D-3}$$

Clearly, the only saddle point of importance among the three identified in Eq. (D-3) is that for which Re $f(\theta_s)$ has the largest value. For the parameters of interest, $\Delta \ll |J_s| \ll 1$, enabling us to write for Eq. (54)

$$\cos\theta_s - \cos\omega\tau = \frac{\Delta}{J_s}, \qquad \theta_s \simeq \omega\tau - \frac{\Delta}{J_s \sin\omega\tau} \tag{D-4}$$

and for Eq. (D-3)

$$\sqrt{2J_s} \simeq \left(\frac{4M\Delta}{N\sin\omega\tau}\right)^{1/3} (e^{+\pi i/6}, e^{+5\pi i/6}, e^{+3\pi i/2}) \tag{D-5}$$

Using Eq. (D-4), we find

$$f(\theta_s) = -iM\omega\tau + \frac{3}{2}\left(\frac{4\Delta N^2 M}{\sin\omega\tau}\right)^{1/3} (e^{\pi i/6}, e^{5\pi i/6}, e^{3\pi i/2}). \tag{D-6}$$

Thus we obtain a contribution in Eq. (68) for the first of the three values of $f(\theta_s)$ in Eq. (D-6), which is the one having the largest real part. Specifically, we find

$$\frac{\xi(N,M)}{\xi_0} \simeq \frac{\sqrt{E}}{2M\sqrt{6\pi}} e^{-M\omega\tau/2Q - iM\omega\tau + 3E/2} + \text{complex conjugate}, \tag{D-7}$$

where

$$E = \left(\frac{4\Delta N^2 M}{\sin\omega\tau}\right)^{1/3} e^{\pi i/6} = \left(\frac{M_{12}R}{\gamma} N^2 M\right)^{1/3} e^{\pi i/6} \tag{D-8}$$

The second term in Eq. (D-7) comes from the additional saddle points resulting from changing θ_s to $-\theta_s$ in Eq. (D-4).

APPENDIX E

The generalization of the single-pulse exponent in Eq. (69) is somewhat more complicated. Starting with Eqs. (16) and (17), and allowing M_{ij} and R to have N dependence, one can eliminate $\theta_M{}^N$ to obtain

$$\begin{aligned} &\xi_M{}^{N+1} - \left(M_{11}^N + \frac{M_{12}^N \gamma_N}{M_{12}^{N-1}\gamma_{N+1}} M_{22}^{N-1}\right)\xi_M{}^N + \frac{M_{12}^N\gamma_N}{M_{12}^{N-1}\gamma_{N+1}}\xi_M{}^{N-1} \\ &= \frac{M_{12}^N R_N}{\gamma_{N+1}} \sum_{l=0}^{M-1} s_{M-l}\xi_l{}^N. \end{aligned} \tag{E-1}$$

For small μ, $M_{11} = M_{22} = 1$, and the left side of Eq. (82) can be written as

$$\frac{M_{12}^N}{\gamma_{N+1}}\left[\frac{\gamma_{N+1}}{M_{12}^N}(\xi_M{}^{N+1} - \xi_M{}^N) - \frac{\gamma_N}{M_{12}^{N-1}}(\xi_M{}^N - \xi_M{}^{N-1})\right], \tag{E-2}$$

which is equivalent to the differential operator

$$\frac{M_{12}}{\gamma}\frac{\partial}{\partial N}\left(\frac{\gamma}{M_{12}}\frac{\partial \xi}{\partial N}\right). \tag{E-3}$$

Thus Eq. (E-1) is approximately equivalent to the differential equation

$$\frac{\partial}{\partial N}\left(\frac{\gamma}{M_{12}}\frac{\partial \xi}{\partial N}\right) \simeq R_N \sum_{l=0}^{M-1} s_{M-l}\xi_l^{\,N}. \tag{E-4}$$

If we now replace ξ by the complex function w such that $\xi = (w + w^*)/2$ and write s_k in Eq. (11) as $i(r_k - r_k^*)/2$, where

$$r_k = e^{-k(i\omega\tau + \omega\tau/2Q)}, \tag{E-5}$$

we obtain

$$\frac{\partial}{\partial N}\frac{\gamma}{M_{12}}\frac{\partial}{\partial N}(w_M^{\,N} + w_M^{\,N*}) = \frac{iR_N}{2}\sum_{l=0}^{M-1}(r_{M-l} - r^*_{M-l})(w_l^{\,N} + w_l^{\,N*}). \tag{E-6}$$

At this point we will assume that the primary dependence of $w_M^{\,N}$ on M is that of the first term of Eq. (69), namely

$$w_M^{\,N} \sim e^{-M(i\omega\tau + \omega\tau/2Q)} z_M^{\,N}. \tag{E-7}$$

Neglecting terms in Eq. (E-6) with rapidly oscillating phase permits us to write

$$\frac{\partial}{\partial N}\left(\frac{\gamma}{M_{12}}\frac{\partial w_M^{\,N}}{\partial N}\right) \simeq \frac{iR_N}{2}\sum_{l=0}^{M-1} r_{M-l}w_l^{\,N}, \tag{E-8}$$

which becomes, by virtue of Eq. (E-7)

$$\frac{\partial}{\partial N}\left(\frac{\gamma}{M_{12}}\frac{\partial z_M^{\,N}}{\partial N}\right) \simeq \frac{iR_N}{2}\sum_{l=0}^{M-1} z_l^{\,N}. \tag{E-9}$$

In analogy with Eq. (70), we shall now assume that $z_M^{\,N}$ is of the form

$$z_M^{\,N} \simeq e^{f(N)M^{1/3}} \tag{E-10}$$

for large M, and convert the sum in Eq. (E-9) into an integral over $x = 1 - (l/M)$. The dominant terms in Eq. (E-9) then become

$$\frac{\gamma}{M_{12}} M^{2/3}\left(\frac{\partial f}{\partial N}\right)^2 e^{f(N)M^{1/3}} \simeq \frac{iR_N M}{2}\int_0^1 dx e^{f(N)M^{1/3}(1-x)^{1/3}}. \tag{E-11}$$

Assuming that the primary contribution to the integral for large M comes from small x, we can expand the right side of Eq. (E-11) in powers of x, to obtain

$$\left(\frac{\partial f}{\partial N}\right)^2 \simeq \frac{3iM_{12}^N R_N}{2\gamma_N f}, \tag{E-12}$$

where the M dependence has dropped out as expected. Equation (E-12) can clearly be integrated to obtain

$$f(N) \simeq \frac{3}{2} e^{\pi i/6}\left[\int\left(\frac{M_{12}R}{\gamma}\right)^{1/2} dN\right]^{2/3}. \tag{E-13}$$

Our final result for the exponent in the first term of Eq. (69), for a single pulse, is therefore

$$e_{s.p.} \simeq -\frac{M\omega\tau}{2Q} - iM\omega\tau + \frac{3}{2} e^{\pi i/6} M^{1/3}\left[\int\left(\frac{M_{12}R}{\gamma}\right)^{1/2} dN\right]^{2/3}. \tag{E-14}$$

Note that Eq. (E-14) agrees with Eq. (69) for $M_{12}R/\gamma$ independent of N. Equation (E-14) therefore suggests the replacement

$$\left(\frac{M_{12}R}{\gamma} N^2 M\right)^{1/3} \to M^{1/3}\left[\int\left(\frac{M_{12}R}{\gamma}\right)^{1/2} dN\right]^{2/3} \tag{E-15}$$

in Eq. (70) when γ, M_{12}, R have N dependence. For the case of constant M_{12} and R, and γ depending linearly on N, one can show that the replacement in Eq. (E-15) is equivalent to

$$\left(\frac{M_{12}RN^2M}{\gamma}\right)^{1/3} \to (M_{12}RN^2M)^{1/3}\left(\frac{\sqrt{\gamma_f} + \sqrt{\gamma_i}}{2}\right)^{-2/3}. \tag{E-16}$$

Thus γ in Eq. (70) is to be replaced by the square of the average value of $\gamma^{1/2}$.

APPENDIX F

Steady-State Solution for Modulated Displacement

It is also possible to obtain an analytic steady-state solution for a beam with modulated displacement and angle. Specifically, if we start with Eq. (49) and set

$$\xi(0, l) = \mathrm{Re}(\bar{\xi}_0 e^{il\bar{\omega}\tau}),$$

$$\eta(0, l) = \mathrm{Re}(\bar{\eta}_0 e^{il\bar{\omega}\tau}), \tag{F-1}$$

we can rewrite Eq. (49) in the form

$$\bar{\xi}(N, M) = \mathrm{Re}\left\{e^{iM\bar{\omega}\tau} \sum_{m=0}^{M} e^{-m(\omega\tau/2Q + i\bar{\omega}\tau)}[\bar{\xi}_0 G(x, y) + \bar{\eta}_0 \sin\mu\, H(x, y)]\right\}. \tag{F-2}$$

Here $\bar{\xi}_0$ and $\bar{\eta}_0$ are complex, allowing for the phase of the modulations which are at frequency $\bar{\omega}/2\pi$.

Equation (F-2) represents a solution where $\bar{\xi}(N, \infty)$ is modulated at frequency $\bar{\omega}/2\pi$ with a steady-state amplitude given by

$$\Xi = \left| \sum_{m=0}^{\infty} e^{-m(\omega\tau/2Q + i\bar{\omega}\tau)} [\bar{\xi}_0 G(x, y) + \sin\mu\, \bar{\eta}_0 H(x, y)] \right|. \tag{F-3}$$

Clearly Ξ can be obtained from the steady-state solution in Eq. (57) by replacing $\omega/2Q$ by $\omega/2Q + i\bar{\omega}$ and taking the absolute value. Thus

$$\Xi = \left| \bar{\xi}_0 \cosh N\bar{\sigma} + (\bar{J}\bar{\xi}_0 + \bar{\eta}_0 \sin\mu) \frac{\sinh N\bar{\sigma}}{\sinh \bar{\sigma}} \right|, \tag{F-4}$$

where

$$\left.\begin{aligned} \cosh\bar{\sigma} &= \cos\mu + \bar{J} \\ \bar{J} &= \frac{M_{12}R}{8\gamma}\,\bar{p}(\omega\tau, \bar{\omega}\tau) \end{aligned}\right\} \tag{F-5}$$

and

$$\bar{p}(\omega\tau, \bar{\omega}\tau) = \frac{\sin\omega\tau}{\sin\left[\left(\frac{\omega - \bar{\omega}}{2}\right)\tau + \frac{i\omega\tau}{4Q}\right] \sin\left[\left(\frac{\omega + \bar{\omega}}{2}\right)\tau - \frac{i\omega\tau}{4Q}\right]}. \tag{F-6}$$

If $\bar{\sigma}$ has a positive real part, we obtain exponential growth with the exponent

$$\bar{e}_0 = N\mathrm{Re}\bar{\sigma} \simeq N\mathrm{Re}(2\bar{J})^{1/2}, \tag{F-7}$$

where the last form is valid for $\mu = 0$ and $|\bar{J}| \ll 1$. In this limit equation (F-6) exhibits resonant behavior for

$$\bar{\omega}\tau = (2l + 1)\pi \pm \left(\omega\tau - \frac{\omega\tau}{2\sqrt{3}Q} - \pi\right), \quad l = 0, 1, 2, \ldots . \tag{F-8}$$

with resonant value

$$\bar{p}_r(\omega\tau, \bar{\omega}\tau) \simeq \frac{4\sqrt{3}Q}{\omega\tau} e^{\mp i\pi/3} \tag{F-9}$$

This is to be compared with Eq. (63b) and is clearly the same order of magnitude unless $\omega\tau$ is small compared with 2π. For $|\bar{J}| \ll 1$, one finally obtains

$$(\bar{e}_0)_{\max} \simeq N\left(\frac{3\sqrt{3}}{8}\frac{M_{12}RQ}{\gamma\omega\tau}\right)^{1/2}. \tag{F-10}$$