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SECOND-ORDER PATH LENGTH TERMS FOR A GENERAL BENDING MAGNET

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Second-order terms are derived for the path length through a non-uniform-field bending magnet. The field is specified by its value on the reference trajectory, and its first and second derivatives with respect to the radial coordinate. This extends the catalogue of second-order path length terms, as previously derived by Davies and Helm. The results have been incorporated into the computer programs TRANSPORT, TURTLE, and DIMAT.

I. INTRODUCTION

In high-energy particle storage rings, attention must be given to preserving the longitudinal bunch structure, so as to ensure adequate luminosity. Bunch structure is also important in transferring the beam from an accelerator to a storage ring, or between storage devices, to maintain synchronism with the accelerating field. In nuclear and particle physics analyzing spectrometers, precise timing information is useful for determining particle identity. For these reasons, careful attention must be paid to the path length the particles take through the various transport systems. In particular, it is useful to be able to calculate the second-order terms contributing to path length. Those for a quadrupole and for a uniform-field bending magnet have been calculated by Davies,[1] while those for the fringe field of a general bending magnet have been calculated by Helm.[2] Here, results are presented for a non-uniform-field bending magnet. We assume that the magnet preserves midplane symmetry and that the field can have both linear and quadratic dependences on the horizontal transverse coordinate. A similar second-order calculation has been performed independently by Matsuda, Matsuo, Ioanoviciu, Wollnik, and Rabbel.[3] However, we find our results to be not entirely consistent with theirs, and deem it worthwhile to publish ours separately.

Our exposition begins with a brief review of the meaning and derivation of the transverse second-order matrix elements. From there, we develop the formalism necessary to derive the matrix elements for the path-length difference. Having done so, we give explicitly the form of the required transverse terms. We then list our results, which are the the second-order matrix elements for the path-length difference. Finally, we relate the concepts of path-length difference and longitudinal separation.

II. TRANSVERSE COORDINATES

The coordinate system used to specify the position and direction of an arbitrary particle trajectory is based on the reference trajectory. The reference trajectory is a particular trajectory of a particle having the central design momentum of the beam line. It passes through quadrupoles on their magnetic axes and experiences uniform field in the central part of bending magnets.

At any point on the central trajectory, we may construct a plane perpendicular to the central trajectory at that point. As the point moves along the central trajectory, the plane will sweep out the adjacent region of space. Any point not on the central trajectory may be identified with a point on the central trajectory if it lies in the plane perpendicular to the central trajectory at that point. Moveover, in any practical situation the distance of a particle from the central trajectory is small enough so that the identification will be unambiguous.

Three coordinates, x, y, and t, are used to specify the position of a particle on any trajectory. The arc length along the central trajectory from the beginning of the beam line to the corresponding point is given by the coordinate t. The two transverse coordinates in the plane perpendicular to the central trajectory are denoted by x and y. The coordinate x is in the plane of bend of the central trajectory. If the bend plane is horizontal, then, facing along the reference trajectory in the direction of increasing t, the x axis points to the left and the y axis points up. The three coordinates x, y, and t form a right-handed system.

Two additional coordinates can be introduced to specify the direction of the trajectory. They are the two direction parameters x' and y', which are the derivatives of x and y with respect to t.

The differential line element is given in terms of the coordinates as

$$ds^2 = dx^2 + dy^2 + (1 + hx)^2\, dt^2. \tag{1}$$

where h is the curvature (or reciprocal of the radius of curvature) of the reference trajectory. The path length of a portion of the trajectory is simply the integral of this quantity over the part of the beam line under consideration. The path-length difference l is obtained by subtracting the path length along the reference trajectory. The sixth coordinate δ is taken as the fractional deviation of the particle momentum p from the reference trajectory momentum p_0. This is written as

$$p = p_0(1 + \delta). \tag{2}$$

Thus, a complete specification of a trajectory at any value of t is given by a six-component vector X, where

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix}. \tag{3}$$

In first order, the value of this vector at any point t_1 can be obtained from the initial value at t_0 by multiplication by a six-by-six transfer matrix.

$$X_1 = RX_0 \tag{4}$$

In a static magnetic system with midplane symmetry, i.e., where all scalar magnetic potentials are odd in y, δ is influenced by no other coordinate and l affects no other coordinate. Further, the two transverse planes are independent, so that R is block diagonalized with two by two submatrices along the diagonal.

The coordinates may alternatively be written as

$$\begin{aligned} x_1 &= (x|x_0)x_0 + (x|x_0')x_0' + (x|\delta)\delta \\ x_1' &= (x'|x_0)x_0 + (x'|x_0')x_0' + (x'|\delta)\delta \\ y_1 &= (y|y_0)y_0 + (y'|y_0')y_0' \\ y_1' &= (y'|y_0)y_0 + (y'|y_0')y_0' \\ l &= (l|x_0)x_0 + (l|x_0')x_0' + (l|\delta)\delta \end{aligned} \tag{5}$$

The transverse elements are often called sinelike, cosinelike, or dispersion rays and are denoted by the letters s, c, and d respectively, so that

$$\begin{aligned} &(x|x_0) = c_x(t) \qquad &(x|x_0') = s_x(t) \qquad &(x|\delta) = d_x(t) \\ &(x'|x_0) = c_x'(t) \qquad &(x'|x_0') = s_x'(t) \qquad &(x'|\delta) = d_x'(t) \\ &(y|y_0) = c_y(t) \qquad &(y|y_0') = s_y(t) & \\ &(y'|y_0) = c_y'(t) \qquad &(y'|y_0') = s_y'(t) & \end{aligned} \tag{6}$$

This formalism can be extended to include quadratic terms, so that for x we have

$$\begin{aligned} x_1 = {} & x_0 c_x(t) + x_0' s_x(t) + \delta d_x(t) \\ & + (x|x_0^2)x_0^2 + (x|x_0 x_0')x_0 x_0' + (x|x_0'^2)x_0'^2 \\ & + (x|y_0^2)y_0^2 + (x|y_0 y_0')y_0 y_0' + (x|y_0'^2)y_0'^2 \\ & + (x|x_0\delta)x_0\delta + (x|x_0'\delta)x_0'\delta + (x|\delta^2)\delta^2 \end{aligned} \tag{7}$$

Only terms which survive midplane symmetry are shown. The expansion, using the above notation, was first formulated by Streib[4] and the actual second-order coefficients for the transverse terms were calculated by Brown.[5]

III. PATH LENGTH DIFFERENCES

From Eq. (1), we see that the path-length difference is given by

$$l = \int_0^t \{[x'^2 + y'^2 + (1 + hx)^2]^{1/2} - 1\}\, d\tau. \tag{8}$$

Expanding by the binomial theorem to second order, we derive

$$l = \int_0^t \left\{hx(\tau) + \frac{1}{2}[x'(\tau)^2 + y'(\tau)^2]\right\} d\tau. \tag{9}$$

In first order, only the first term inside the braces survives. The first-order matrix elements for the longitudinal coordinate l are then the integrals of the first-order matrix elements for the transverse coordinate x. Considering only the first term in the brackets, we similarly see that one contribution to the second-order matrix elements for l is the integral of the second-order matrix elements for x. The remaining contributions come from the terms in the brackets. Since the expression for l is already quadratic in the two derivatives x' and y', we need consider only first-order contributions to these terms. We can expand $x(\tau)$ and the two derivatives x' and y' in terms of the initial coordinates and the appropriate matrix elements to derive explicit expressions for each of the longitudinal second-order matrix elements.

Because of midplane symmetry, there are only nine such matrix elements. They are the nine which correspond to the nine second-order matrix elements for the transverse coordinate x. The expressions for the second-order matrix elements for l are

$$(l|x_0^2) = \int_0^t \left[h(x|x_0^2) + \frac{1}{2}c_x'^2\right] d\tau \tag{10}$$

$$(l|x_0x_0') = \int_0^t [h(x|x_0x_0') + c_x's_x']\, d\tau$$

$$(l|x_0'^2) = \int_0^t \left[h(x|x_0'^2) + \frac{1}{2}s_x'^2\right] d\tau$$

$$(l|y_0^2) = \int_0^t \left[h(x|y_0^2) + \frac{1}{2}c_y'^2\right] d\tau$$

$$(l|y_0y_0') = \int_0^t [h(x|y_0y_0') + c_y's_y']\, d\tau$$

$$(l|y_0'^2) = \int_0^t \left[h(x|y_0'^2) + \frac{1}{2}s_y'^2\right] d\tau$$

$$(l|x_0\delta) = \int_0^t [h(x|x_0\delta) + c_x'd_x']\, d\tau$$

$$(l|x_0'\delta) = \int_0^t [h(x|x_0'\delta) + s_x'd_x']\, d\tau$$

$$(l|\delta^2) = \int_0^t \left[h(x|\delta^2) + \frac{1}{2}d_x'^2\right] d\tau$$

IV. SECOND-ORDER TRANSVERSE COORDINATES

The magnetic field on the midplane of a general bending magnet has no x component. The y component may be expanded to second order as

$$B_y(x, 0, t) = B_y(0, 0, t)[1 - nhx + \beta h^2 x^2]. \tag{11}$$

The characteristic rays or first-order transfer matrix elements are given by

$$c_x(t) = \cos k_x t \qquad s_x(t) = \frac{1}{k_x}\sin k_x t \qquad d_x(t) = \frac{h}{k_x^2}(1 - c_x(t)) \tag{12}$$

$$c_y(t) = \cos k_y t \qquad s_y(t) = \frac{1}{k_y}\sin k_y t,$$

where

$$k_x^2 = (1 - n)h^2$$
$$k_y^2 = nh^2.$$

If the value of k^2 in either plane becomes negative, the functions cos and sin are replaced cosh and sinh respectively.

If the second-order matrix elements are expressed in terms of the characteristic functions, then their form does not depend on the sign of k^2 in either plane. Taking the work of Brown, and simplifying a bit, the second-order terms for x are

$$(x|x_0^2) = \frac{h^2}{6} d_x(t)\,[2(n - \beta) + (5n - 2\beta - 3)(1 + c_x(t))] \tag{13}$$

$$(x|x_0 x_0') = \frac{h}{3(1 - n)} s_x(t)\,[2(n - \beta) - (5n - 2\beta - 3)c_x(t)]$$

$$(x|x_0'^2) = \frac{1}{6(1 - n)} d_x(t)\,[2(n - \beta) - (5n - 2\beta - 3)c_x(t)]$$

$$(x|y_0^2) = \frac{h^2}{1 - 5n}\left[\frac{1}{2}(5n^2 - 6n\beta + 2\beta - n)d_x(t) - \beta n h s_y^2(t)\right]$$

$$(x|y_0 y_0') = \frac{2h\beta}{1 - 5n}(c_y(t)s_y(t) - s_x(t))$$

$$(x|y_0'^2) = \frac{1}{1 - 5n}\left[h\beta s_y^2(t) + \frac{1}{2}(5n - 4\beta - 1)d_x(t)\right]$$

$$(x|x_0\delta) = \frac{h}{6(1 - n)}[-4(n - \beta)d_x(t) - 2h(5n - 2\beta - 3)s_x^2(t)$$
$$+ 3(n(n + 1) - 2\beta)hts_x(t)]$$

$$(x|x_0'\delta) = \frac{1}{6(1-n)^2}[2(5n - 2\beta - 3)s_x(t)(c_x(t) - 1) + 3(n(n+1) - 2\beta)(s_x(t) - tc_x(t))]$$

$$(x|\delta^2) = \frac{1}{6(1-n)^2}[8(n - \beta)d_x(t) + (5n - 2\beta - 3)hs_x{}^2(t) - 3(n(n+1) - 2\beta)hts_x(t)].$$

V. SECOND-ORDER PATH LENGTH DIFFERENCES

Having obtained both first- and second-order expressions for the transverse coordinates, we may now substitute the expressions from Eqs. (13) into Eqs. (10). The results are the second-order expressions for the path-length difference.

$$(l|x_0{}^2) = -\frac{h^2}{12(1-n)}[(n(3n-1) - 2\beta)(t - s_x(t)c_x(t)) + 4(n - \beta)(t - s_x(t))$$

$$(l|x_0x_0') = -\frac{h}{6(1-n)}[4(n-\beta)d_x(t) - (n(1-3n) + 2\beta)hs_x{}^2(t)]$$

$$(l|x_0'^2) = -\frac{1}{12(1-n)^2}[2(7n - 4\beta - 3)(t - s_x(t)) - (5n - 2\beta - 3)(t - s_x(t)c_x(t))] - \frac{1}{4}(t + s_x(t)c_x(t))$$

$$(l|y_0{}^2) = -\frac{1}{2(1-n)}\left[2\beta\frac{1-3n}{1-5n} - n\right]h^2(t - s_x(t)) + \frac{1}{4}\left(\frac{2\beta}{1-5n} - n\right)h^2(t - s_y(t)c_y(t))$$

$$(l|y_0y_0') = \left(\frac{\beta}{1-5n} - \frac{1}{2}n\right)h^2s_y{}^2(t) + 2h\frac{\beta}{1-5n}d_x(t)$$

$$(l|y_0'^2) = -\frac{\beta}{2n(1-5n)}(t - s_y(t)c_y(t)) + \frac{1}{2(1-n)}\left(\frac{4\beta}{1-5n} + 1\right)(t - s_x(t)) - \frac{1}{4}(t + s_y(t)c_y(t))$$

$$(l|x_0\delta) = \frac{1}{(1-n)^2}\left[-\frac{1}{2}(n(n+1) - 2\beta)h(s_x(t) - tc_x(t)) + \frac{2}{3}(n-\beta)h(t - s_x(t)) + \frac{1}{6}(n(3n-1) - 2\beta)h(t - s_x(t)c_x(t))\right]$$

$$(l|x_0'\delta) = \frac{1}{(1-n)^2}\left[-\frac{1}{3}(3n^2 - 2n - 4\beta + 3)d_x(t) + \frac{1}{2}(n^2 + n - 2\beta)hts_x(t) - \frac{1}{6}(3n^2 - n - 2\beta)hs_x^2(t)\right]$$

$$(l|\delta^2) = -\frac{1}{(1-n)^3}\left[\frac{4}{3}(n - \beta)(t - s_x(t)) - \frac{1}{2}(n(n+1) - 2\beta)(s_x(t) - tc_x(t)) + \frac{1}{12}(3n^2 - n - 2\beta)(t - c_x(t)s_x(t))\right]$$

The expressions have been checked in a variety of ways. First, they reduce to the expressions given by Davies for quadrupoles and uniform-field bending magnets. Second, they give identical results independent of whether a bending magnet is calculated as a single entity or has been segmented into several smaller magnets. This segmentation test has been done on bending magnets which are zero-gradient, weak focusing, and strongly focusing or defocusing, and with both zero and non-zero values of β. Finally, all but the $(l|\delta^2)$ term vanish for the Brown achromat,[6] which was analysed by Carey and Servranckx.[7]

Second-order longitudinal matrix elements have now been incorporated into several computer programs. These include the matrix calculation and fitting program TRANSPORT,[8] the Monte-Carlo beam simulation program TURTLE,[9] and the circular machine analysis program DIMAT.[10]

VI. LONGITUDINAL SEPARATION

The difference in path length corresponds to the longitudinal separation of the particles only for a highly relativistic particle beam, where the velocities of the particles are essentially that of light. For a non-relativistic beam, we must consider the fact that particles of different momentum will have different velocities. The longitudinal separation will receive contributions both from the path-length difference and from the velocity difference. To calculate the combined effect, we multiply the central velocity by the time difference.

$$l = v_0\left(\frac{L}{v} - \frac{L_0}{v_0}\right) \tag{15}$$

If we expand this expression in terms of the differences $L - L_0$ and $\Delta v = v - v_0$, and retain only terms which are at most second order in those differences, then we derive

$$l = L - L_0 - L_0\frac{\Delta v}{v_0} - (L - L_0)\frac{\Delta v}{v_0} - L_0\left(\frac{\Delta v}{v_0}\right)^2. \tag{16}$$

It remains to determine an expression for the fractional velocity deviation $\Delta v/v_0$, which is valid to second order in the fractional momentum deviation $\delta = \Delta p/p_0$. The particle velocity is given by β times c, the velocity of light, where β can be calculated in terms of the particle energy E and momentum p. The energy, in turn, can be expressed in terms of the momentum so that we can derive the following sequence of expressions.

$$v = \beta c = \frac{pc^2}{E} \tag{17}$$

$$= \frac{pc^2}{\sqrt{m^2c^4 + p^2c^2}}$$

$$= \frac{p_0c^2(1 + \delta)}{\sqrt{m^2c^4 + p_0{}^2c^2(1 + \delta)^2}}$$

If we now take the quotient $\Delta v/v_0$ and eliminate p_0 and m using the β and γ of the reference trajectory, we derive

$$\frac{\Delta v}{v_0} = \frac{1 + \delta}{\sqrt{1 + 2\beta\delta^2 + \beta^2\delta^2}} - 1. \tag{18}$$

Expanding the denominator using the binomial theorem finally yields

$$\frac{\Delta v}{v_0} = \frac{\delta}{\gamma^2} - \frac{3}{2}\frac{\beta^2}{\gamma^2}\delta^2. \tag{19}$$

The complete expression for longitudinal separation of two particles now becomes

$$l = (L - L_0)\left(1 - \frac{\delta}{\gamma^2}\right) - L_0\frac{\delta}{\gamma^2} + L_0\left[\frac{1}{\gamma^4} + \frac{3}{2}\left(\frac{\beta}{\gamma}\right)^2\right]\delta^2. \tag{20}$$

For the non-relativistic case, only those second-order terms involving the fractional momentum deviation δ need be modified. We now label with a subscript R, for relativistic, the second-order matrix elements for path-length difference. In terms of the relativistic matrix elements, the matrix elements for the longitudinal separation are given by

$$(l|x_0\delta) = (l|x_0\delta)_R - (l|x_0)\frac{1}{\gamma^2} \tag{21}$$

$$(l|x_0{}'\delta) = (l|x_0{}'\delta)_R - (l|x_0{}')\frac{1}{\gamma^2}$$

$$(l|\delta^2) = (l|\delta^2)_R - (l|\delta)\frac{1}{\gamma^2} + L_0\left[\frac{1}{\gamma^4} + \frac{3}{2}\left(\frac{\beta}{\gamma}\right)^2\right]$$

There is also a single additional longitudinal second-order matrix element

$$(l|l_0\delta) = -\frac{1}{\gamma^2} \tag{22}$$

REFERENCES

1. W. G. Davies, Nucl. Instrum. Methods **169**, 337 (1980).
2. R. H. Helm, SLAC Report No. 24, 1963.

3. H. Matsuda, T. Matsuo, D. Ioanoviciu, H. Wollnik, and V. Rabbel, private communication'.
4. J. F. Streib, HEPL Report No. 104, Stanford University, Stanford, California, 1960.
5. Karl L. Brown, SLAC Report No. 75, 1982.
6. Karl L. Brown, SLAC-PUB-2257, 1979.
7. David C. Carey, Nucl. Instrum. Methods **189,** 365 (1981). R. V. Servranckx, PEP Technical Memo, PTM-249, 1982.
8. K. L. Brown, D. C. Carey, Ch. Iselin, and F. Rothacker, SLAC Report No. 91, NAL Report No. 91, CERN 80–40.
9. D. C. Carey, NAL Report No. 64, 1978.
10. R. V. Servranckx, DIMAT.