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PRUD-CODE FOR CALCULATION OF THE NONSYMMETRIC MODES IN AXIAL SYMMETRIC CAVITIES

A. G. DAIKOVSKY, YU. I. PORTUGALOV, and A. D. RYABOV

Institute for High Energy Physics, Serpukhov , USSR

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This paper offers the method and brief description of the program package for calculating eigenfrequencies and electromagnetic fields with azimuthal variations in axial-symmetric cavities of an arbitrary sahpe. The method is based on the representation of the equations of electrodynamics in variables ρh_φ, ρe_φ. Apart from frequencies and fields, the accumulated energy, distribution of losses in the metal, other characteristics important for application are also computed. The package offers wide possibilities for graphic representation of the field topology, facilitating the analysis, and optimization of complicated accelerating structures.

Axially symmetric cavities are utilized in many radio-frequency devices. In particular, they form the basis of the accelerating structures in linear and cyclic particle accelerators.

Accelerating structures commonly utilize symmetric waves (having no variation in φ) as "working" ones. To compute such fields, the well-known programs SUPERFISH,[1,2] LANS,[3] and others are used.

Waves having no axial symmetry are used less commonly and in many cases are treated as "stray" ones. Nevertheless, as far as rf separators and *H*-wave structures are concerned calculations of such waves would be useful.

In conventional structures, azimuthally varying waves may be present for many reasons: technology-affected violation of axial symmetry, presence of coupling or tuning elements, asymmetric beam excitation, etc. Therefore to complete the analysis of accelerating structures, one must provide for the analysis of azimuthally varying fields. The problems associated with such modes are related to the presence of all six components of the electromagnetic field.

This paper presents a brief description of the PRUD program package designed to compute resonance frequencies and azimuthally varying fields in an axially symmetric cavity of essentially arbitrary cross section. The description of the method, package facilities, typical accuracy, examples and applications are given.

1. BASIC EQUATIONS. THE CHOICE OF INDEPENDENT VARIABLES

1.1. We solve the problem of finding eigenfrequencies and their corresponding electromagnetic fields in an axially symmetric cavity with symmetry axis z and an arbitrary boundary $\rho(z)$; only those modes that vary in the cyclic coordinate φ are considered.

For time-harmonic fields let us with no loss of generality put $\mathbf{H} = \mathbf{h} \operatorname{Cos} \omega t$ and $\mathbf{E} = \mathbf{e} \operatorname{Sin} \omega t$. With $\kappa = \omega/c$ the Maxwell equations in free space become

$$\begin{aligned} \operatorname{rot} \mathbf{h} &= \kappa \mathbf{e} \quad \operatorname{div} \mathbf{h} = 0 \\ \operatorname{rot} \mathbf{e} &= \kappa \mathbf{h} \quad \operatorname{div} \mathbf{e} = 0. \end{aligned} \tag{1}$$

The system of eqs. (1) can be written in the form of pair of second order equations for $\mathbf{h}$ and $\mathbf{e}$

$$\begin{aligned} \operatorname{rot} \operatorname{rot} \mathbf{h} &= \kappa^2 \mathbf{h} \\ \operatorname{rot} \operatorname{rot} \mathbf{e} &= \kappa^2 \mathbf{e}, \end{aligned} \tag{2}$$

or in components using cylindrical coordinates (ρ, φ, z),

$$\begin{aligned} \Delta h_\rho + \kappa^2 h_\rho &= \frac{2}{\rho^2} \frac{\partial h_\varphi}{\partial \varphi} + \frac{1}{\rho^2} h_\rho \\ \Delta h_\varphi + \kappa^2 h_\varphi &= -\frac{2}{\rho^2} \frac{\partial h_\rho}{\partial \varphi} + \frac{1}{\rho^2} h_\varphi \\ \Delta h_z + \kappa^2 h_z &= 0. \end{aligned} \tag{3}$$

The equations for (e_ρ, e_φ, e_z) are similar. Here Δ is the Laplace operator in cylindrical coordinates.

Application of the method of separation of variables to the system in Eq. (3) together with div $\mathbf{h} = 0$ yields

$$
\begin{aligned}
h_\rho &= \mathcal{H}_\rho(\rho, z)\,\Phi(n\varphi) \\
h_\varphi &= \mathcal{H}_\varphi(\rho, z)\,\Phi(n\varphi) \\
h_z &= \mathcal{H}_z(\rho, z)\,\Phi(n\varphi),
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
&\Phi(n\varphi) = \operatorname{Sin} n\varphi \qquad \Phi(n\varphi) = \operatorname{Cos} n\varphi \\
&\qquad\qquad\qquad \text{or} \\
&\Phi'(n\varphi) = \operatorname{Cos} n\varphi \qquad \Phi'(n\varphi) = -\operatorname{Sin} n\varphi.
\end{aligned}
\tag{5}
$$

For the electric field one obtains

$$
\begin{aligned}
e_\rho &= \mathcal{E}_\rho(\rho, z)\,\Phi'(n\varphi) \\
e_\varphi &= \mathcal{E}_\varphi(\rho, z)\,\Phi(n\varphi) \\
e_z &= \mathcal{E}_z(\rho, z)\,\Phi'(n\varphi).
\end{aligned}
\tag{6}
$$

1.2 It is common knowledge[4] that any free-space electromagnetic field can be expressed as a sum of solutions derived from two scalar functions. If the system possesses an axis, then the fields derived from the two solutions are generally referred to as the TE and TM modes with respect to that axis. In general each mode will have five components of **E**, **H**. If the boundaries possess rotational symmetry about the axis, the number of field components is not reduced. But if the field is rotationally symmetric, the number of field components is reduced to three for each mode.

We consider the case of a structure possessing rotational symmetry. Write out Eq. (1) in cylindrical coordinates and focus attention on the two pairs of arrowed equations

$$
\begin{aligned}
&\frac{1}{\rho}\left(\frac{\partial h_z}{\partial\varphi} - \frac{\partial \rho h_\varphi}{\partial z}\right) = \kappa e_\rho \uparrow &&\uparrow \frac{1}{\rho}\left(\frac{\partial e_z}{\partial\varphi} - \frac{\partial \rho e_\varphi}{\partial z}\right) = \kappa h_\rho \\
&\left(\frac{\partial h_\rho}{\partial z} - \frac{\partial h_z}{\partial\rho}\right) = \kappa e_\varphi &&\left(\frac{\partial e_\rho}{\partial z} - \frac{\partial e_z}{\partial\rho}\right) = \kappa h_\varphi \\
&\frac{1}{\rho}\left(\frac{\partial \rho h_\varphi}{\partial\rho} - \frac{\partial h_\rho}{\partial\varphi}\right) = \kappa e_z \downarrow &&\downarrow \frac{1}{\rho}\left(\frac{\partial \rho e_\varphi}{\partial\rho} - \frac{\partial e_\rho}{\partial\varphi}\right) = \kappa h_z.
\end{aligned}
\tag{7}
$$

Differentiate one of the equations of the pair with respect to φ and combine it with the other member of the pair. This can be done in four ways to yield h_z, h_φ, e_z and e_ρ in terms of ρh_φ and ρe_φ

$$
\begin{aligned}
\left(\kappa^2 - \frac{n^2}{\rho^2}\right) h_z &= \frac{1}{\rho^2}\frac{\partial^2 \rho h_\varphi}{\partial_\varphi \partial_z} + \frac{\kappa}{\rho}\frac{\partial \rho e_\varphi}{\partial_\rho} \\
\left(\kappa^2 - \frac{n^2}{\rho^2}\right) h_\rho &= \frac{1}{\rho^2}\frac{\partial^2 \rho h_\varphi}{\partial_\varphi \partial_\rho} - \frac{\kappa}{\rho}\frac{\partial \rho e_\varphi}{\partial_z} \\
\left(\kappa^2 - \frac{n^2}{\rho^2}\right) e_z &= \frac{1}{\rho^2}\frac{\partial^2 \rho e_\varphi}{\partial_\varphi \partial_z} + \frac{\kappa}{\rho}\frac{\partial \rho h_\varphi}{\partial_\rho} \\
\left(\kappa^2 - \frac{n^2}{\rho^2}\right) e_\rho &= \frac{1}{\rho^2}\frac{\partial^2 \rho e_\varphi}{\partial_\varphi \partial_\rho} - \frac{\kappa}{\rho}\frac{\partial \rho h_\varphi}{\partial_z}.
\end{aligned}
\tag{8}
$$

Here we have taken into account the fact that double differentiation of any component with respect to φ is equivalent to multiplication by $-n^2$. Equation (8) can be written in a vector form

$$
\begin{aligned}
\mathbf{h}_\perp &= \frac{1}{x}\left(\nabla_\perp \frac{\partial \rho h_\varphi}{\partial\varphi} + \kappa\rho\nabla_\perp \times \rho e_\varphi\right) \\
\mathbf{e}_\perp &= \frac{1}{x}\left(\nabla_\perp \frac{\partial \rho e_\varphi}{\partial_\varphi} + \kappa\rho\nabla_\perp \times \rho h_\varphi\right),
\end{aligned}
\tag{9}
$$

where $x = \kappa^2\rho^2 - n^2$ and $\nabla_\perp \equiv \left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial z}\right)$.

Thus in this case the two scalar-functions are ρh_φ and ρe_φ. For $n \neq 0$, each function generates five field components of **E**, **H**. However, if $n = 0$, ρh_φ by itself generates only (e_ρ, h_φ, e_z) with (h_ρ, e_φ, h_z) equal to zero and ρe_φ by itself generates only (h_ρ, e_φ, h_z) with (e_ρ, h_φ, e_z) equal to zero.

From Eq. (9) it follows that the general problem of eigenfrequencies and fields of an axially symmetric cavity can be reduced to a two-dimensional problem, where the two functions are

$$
\begin{aligned}
\hat{H}(\rho, z) &= \rho\mathcal{H}_\varphi(\rho, z) \\
\hat{E}(\rho, z) &= \rho\mathcal{E}_\varphi(\rho, z).
\end{aligned}
\tag{10}
$$

We shall show below that the choice of these variables is the most natural also from the viewpoint of satisfying the boundary conditions.

Let us write another system which follows from Eq. (3), (4), (9) and relates $\mathcal{H}_\varphi$ and $\mathcal{E}_\varphi$; it will be helpful in the following.

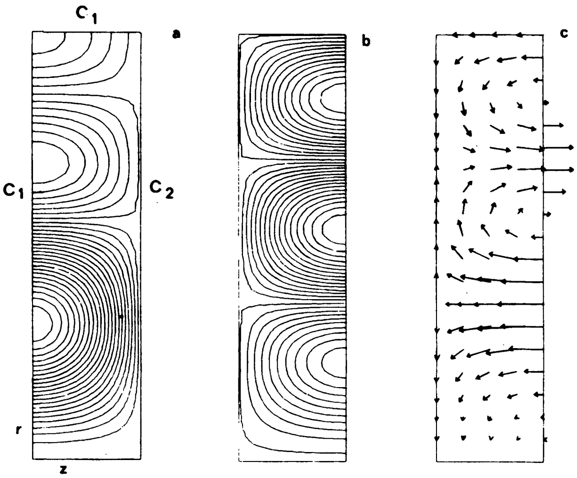

FIGURE 1 Results in a cylindrical cavity. The surface C_1 is a conducting boundary and C_2 is a plane of symmetry. (a) Isolines of H_ϕ. (b) Isolines of E_ϕ. (c) Vector field h in the $\phi = 0$ plane.

$$
\begin{aligned}
&\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \mathcal{H}_\varphi}{\partial \rho}\right) + \rho^2 \frac{\partial^2 \mathcal{H}_\varphi}{\partial z^2} + (x - 1)\mathcal{H}_\varphi \\
&\quad = \frac{2}{x}\left(n^2 \frac{\partial \rho \mathcal{H}_\varphi}{\partial_\rho} + n\kappa\rho \frac{\partial \rho \mathcal{E}_\varphi}{\partial z}\right) \\
&\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \mathcal{E}_\varphi}{\partial \rho}\right) + \rho^2 \frac{\partial^2 \mathcal{E}_\varphi}{\partial z^2} + (x - 1)\mathcal{E}_\varphi \\
&\quad = \frac{2}{x}\left(n^2 \frac{\partial \rho \mathcal{E}_\varphi}{\partial_\rho} - n\kappa\rho \frac{\partial \rho \mathcal{H}_\varphi}{\partial_z}\right)
\end{aligned} \tag{11}
$$

1.3 The fields should also satisfy some boundary conditions besides Eq. (1). Figure 1 shows the boundaries C_1 and C_2 in the plane (z, ρ): C_1 is the boundary of an ideally conductive metal, C_2 is the symmetry plane. For free oscillations to exist, the boundary values H_τ or E_τ should be equal to zero. That is,

$$
\begin{aligned}
&e_\tau = 0|_{C_1} \\
&e_\tau = 0|_{C_2} \quad \text{or} \quad h_\tau = 0/C_2,
\end{aligned} \tag{12}
$$

the latter depending on the symmetry of the modes sought.

To estimate the order of smallness of the field components in the axis neighbourhood, let us put $\mathcal{H}_\varphi$ and $\mathcal{E}_\varphi$ in the form

$$
\begin{aligned}
\mathcal{H}_\varphi &= \alpha_m{}^\varphi \cdot \rho^m + \alpha_{m+1}^\varphi \cdot \rho^{m+1} + \cdots \\
\mathcal{E}_\varphi &= \beta_m{}^\varphi \cdot \rho^m + \beta_{m+1}^\varphi \cdot \rho^{m+1} + \cdots.
\end{aligned} \tag{13}
$$

Putting (13) into Eq. (11), we obtain a recursion relation from which follow series containing terms of the same parity. The series begins with $m = n - 1$. That is,

$$
\begin{aligned}
\mathcal{H}_\varphi &= \alpha_{n-1}^\varphi \cdot \rho^{n-1} + \alpha_{n+1}^\varphi \cdot \rho^{n+1} + \cdots \\
\mathcal{E}_\varphi &= \beta_{n-1}^\varphi \cdot \rho^{n-1} + \beta_{n+1}^\varphi \cdot \rho^{n+1} + \cdots.
\end{aligned} \tag{14}
$$

Using Eq. (8) and (14), we obtain immediately

$$
\begin{aligned}
\mathcal{H}_\rho &= \alpha_{n-1}^\rho \cdot \rho^{n-1} + \alpha_{n+1}^\rho \cdot \rho^{n+1} + \cdots \\
\mathcal{H}_z &= \alpha_n{}^z \cdot \rho^n + \alpha_{n+2}^z \cdot \rho^{n+2} + \cdots \\
\mathcal{E}_\rho &= \beta_{n-1}^\rho \cdot \rho^{n-1} + \beta_{n+1}^\rho \cdot \rho^{n+1} + \cdots \\
\mathcal{E}_z &= \beta_n{}^z \cdot \rho^n + \beta_{n+2}^z \cdot \rho^{n+2} + \cdots,
\end{aligned} \tag{15}
$$

with $\alpha_{n-1}^\rho = \alpha_{n-1}^\varphi$, $\beta_{n-1}^\rho = -\beta_{n-1}^\varphi$.

Thus with $n > 1$ all the field components are equal to zero on the axis. When $n = 1$ with account of (4), (5) $\mathcal{H}_\varphi$, $\mathcal{H}_\rho$, $\mathcal{E}_\varphi$, $\mathcal{E}_\rho$ yield in the (ρ, φ)-plane a pair of nonzero mutually perpendicular vectors $\mathbf{h}$ and $\mathbf{e}$. For the variables $\hat{H}$, $\hat{E}$, we have zero boundary conditions on the axis for any $n \geq 1$.

2. OBTAINING A SYSTEM OF DISCRETE EQUATIONS

2.1. To obtain a system of discrete equations with respect to the variables $\hat{H}(\rho, z) = \rho\mathcal{H}_\varphi(\rho, z)$ and $\hat{E}(\rho, z) = \rho\mathcal{E}_\varphi(\rho, z)$, let us do the following. Let there be set a triangulation[5] of the cavity cross section in the plane (z, ρ) and basis $\{\psi_i(z, \rho)\}$; here i is the node number of the triangulation. Multiply Eq. (1) by a basis function ψ_i and by a φ-oriented area element ds and integrate the result over the cross-sectional area. Then

$$
\begin{aligned}
\int \psi_i \operatorname{rot} \mathbf{e} \cdot ds &= \kappa \int \psi_i \mathbf{h} \cdot ds \\
\int \psi_i \operatorname{rot} \mathbf{h} \cdot ds &= \kappa \int \psi_i \mathbf{e} ds.
\end{aligned} \tag{16}
$$

Applying Stokes' theorem, write Eq. (16) in the form

$$
\begin{aligned}
\int [\nabla\psi_i \times \mathbf{e}]_\varphi ds + \kappa \int \psi_i h_\varphi ds &= \oint \psi_i e_\tau dC \\
\int [\nabla\psi_i \times \mathbf{h}]_\varphi ds + \kappa \int \psi_i e_\varphi ds &= \oint \psi_i h_\tau dC.
\end{aligned} \tag{17}
$$

On performing simple transformations and using relations (4)–(6), (9) and the expansion

$$\hat{H} = \sum_i^N \hat{H}_i \psi_i$$
$$\hat{E} = \sum_i^N \hat{E}_i \psi_i, \tag{18}$$

we obtain a system of $2N$ equations

$$\sum_j^N A_{ij}\hat{H}_j + B_{ij}\hat{E}_j = \oint \psi_i E_\tau dC$$
$$\sum_j^N -B_{ij}\hat{H}_j + A_{ij}\hat{E}_j = \oint \psi_i \mathcal{H}_\tau dC, \tag{19}$$

where

$$A_{ij} = \int \left[\kappa \frac{\psi_i \psi_j}{\rho} - \frac{\kappa\rho}{x} (\nabla\psi_i \cdot \nabla\psi_j) \right] ds$$
$$B_{ij} = \int \frac{n}{x} [\nabla\psi_i \times \nabla\psi_j]_\varphi ds. \tag{20}$$

Note that the matrix A_{ij} is symmetric, the matrix B_{ij} is antisymmetric, and the complete matrix (19) is also symmetric. The final system of equations will be derived from Eq. (19) with the boundary conditions taken into account.

1. For the interior nodes of a triangle mesh, the contour integrals are equal to zero because $\psi_i = 0|_C$
2. For the nodes on the z axis, Eqs. (19) are replaced by $\hat{H}_i = 0$, $\hat{E}_i = 0$.
3. For the nodes on the metal C_1, the contour integral of the first equation is equal to zero because $\mathcal{E}_\tau = 0$, and the second equation is replaced by $\hat{E}_i = 0$.

We treat the nodes on the C_2 boundary in similar fashion: depending on the type of the symmetry of the modes sought, we put either $\hat{E}_i = 0$, $\mathcal{E}_\tau = 0$ or $\hat{H}_i = 0$, $\mathcal{H}_\tau = 0$. The derivation performed corresponds completely to the standard procedure of the finite-element method and for the case $n = 0$ leads to the equations coinciding with the flux scheme used in SUPERFISH.

2.2 Equations (19) have been obtained formally without consideration for χ taking the value zero on the $\rho^* = n/\kappa$ line, dividing the problem region into two parts. It would be natural to put down the equations for each subregion separately and impose the related continuity conditions on this line. In this case, both the form of Eq. (19) and uniformity in computing the coefficients are maintained.

Let us bring formulas for calculation of some integrals over a triangle which are present in (20)

$$J_i = \int_{S\Delta} \frac{\kappa\rho}{n^2 - \kappa^2\rho^2} ds \quad J_2 = \int_{S\Delta} \frac{n}{n^2 - \kappa^2\rho^2} ds.$$

If

$$G_1 = \int \frac{1}{n + \kappa\rho} ds \quad G_2 = \int \frac{1}{n - \kappa\rho} ds$$

are introduced, then $J_1 = (G_2 - G_1)/2$, $J_2 = (G_2 + G_1)/2$. To compute the integrals G_1 and G_2 it is convenient to apply the Stokes formula

$$\int_{S\Delta} \frac{\partial}{\partial z} p(z, \rho) ds = \oint_C p(z, \rho) d\rho$$

and reduce integration over the square to integration over the contour. The contribution to the integrals G_1 and G_2 from the side of the triangle $(z_1, \rho_1) - (z_2, \rho_2)$ is

$$JG_1 = \frac{1}{\kappa}(z_2 - z_1) + \frac{\kappa\beta - n\alpha}{\kappa^2} \times \ln \left| \frac{n + \kappa\rho_2}{n + \kappa\rho_1} \right|$$

$$TG_2 = -\frac{1}{\kappa}(z_2 - z_1) - \frac{\kappa\beta + n\alpha}{\kappa^2} \times \ln \left| \frac{n - \kappa\rho_2}{n - \kappa\rho_1} \right|,$$

where α and β are the coefficients of the equation for the given side

$$z = \alpha\rho + \beta.$$

3. EVALUATION OF THE EIGENFREQUENCY

The system of finite equations (19) has singularities which do not make it possible to consider it as a standard eigenvalue problem. One of the singularities lies in the nonlinear dependence of the coefficients (20) of the matrix on κ. Another singularity is the implicit dependence of the whole matrix on κ and reflects the fact that the triangle mesh necessary to calculate the coefficients should be consistent with κ in terms of Section 2.2. To determine the resonance values, the method of exciting current[1] should be generalized. This method applied in such a fashion looks as follows.

For some value of κ a triangle mesh consistent with it is constructed. In one of its nodes, say i, the so-called driving point, we put $\hat{H}_i = 1$ (or $\hat{E}_i = 1$). Then Eqs. (19) are constructed; in this case $\hat{H}_i = 1$ ($\hat{E}_i = 1$) are treated as a boundary condition. The system of the equations obtained is solved with respect to the remaining node unknown values. It is always solvable, including the resonance values of κ. Then one can calculate $\hat{H}_i^*$ ($\hat{E}_i^*$) and evaluate the residual function $I(\kappa) = 1 - \hat{H}_i^*$ ($I(\kappa) = 1 - \hat{E}_i^*$) from the first (and correspondingly the second provided $\hat{E}_i = 1$) equation of (19) written in the i-th node. It is easily seen that $I(\kappa) = 0$ for resonance and only for resonance values of κ. Thus the problem is reduced to finding the zeroes of $I(\kappa)$. The residual $I(\kappa)$ being equal to zero is interpreted simply in a physics way: resonance excitation of the field by vanishingly small intensity current (electric or magnetic current depending on the type of oscillation).

4. THE PACKAGE FACILITIES

The package facilities a uniform computation on any type of oscillations with azimuthal variations in the specified frequency band in axially symmetric cavities of arbitrary shape. Below we shall present package applications to cavities of various shapes. One can compute resonance frequencies, all six components of the fields **e**, **h**, the derived quantities, i.e., the cavity characteristics: the stored energy, distribution of losses in the metal walls, etc. If required by the user, the list of derived quantities can easily be made longer. The package is furnished with extensive facilities of graphic representation of the field topology, which are helpful in mode analysis and optimization of accelerating structures. Data preparation for the package programs takes up 5–20 minutes. The solution to the problem on the mesh containing about 1000 nodes takes up 3–5 minutes of CPU time of a computer with the speed of about 1 million operations per second.

The general features of the computing procedure are as follows. A user specifies the cavity geometry in the (z, ρ)-plane, the form of boundary

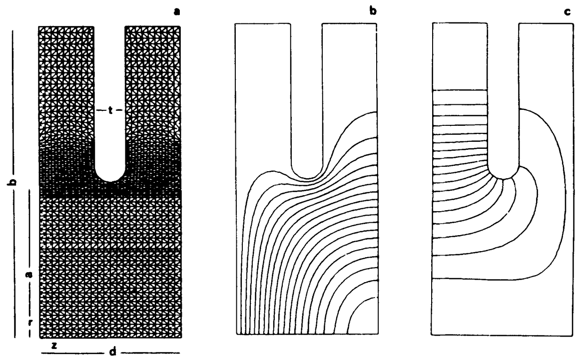

FIGURE 2 Results in a diaphragmed waveguide. (a) Triangular mesh. (c) Isolines of E_Φ. (c) Isolines of H_Φ.

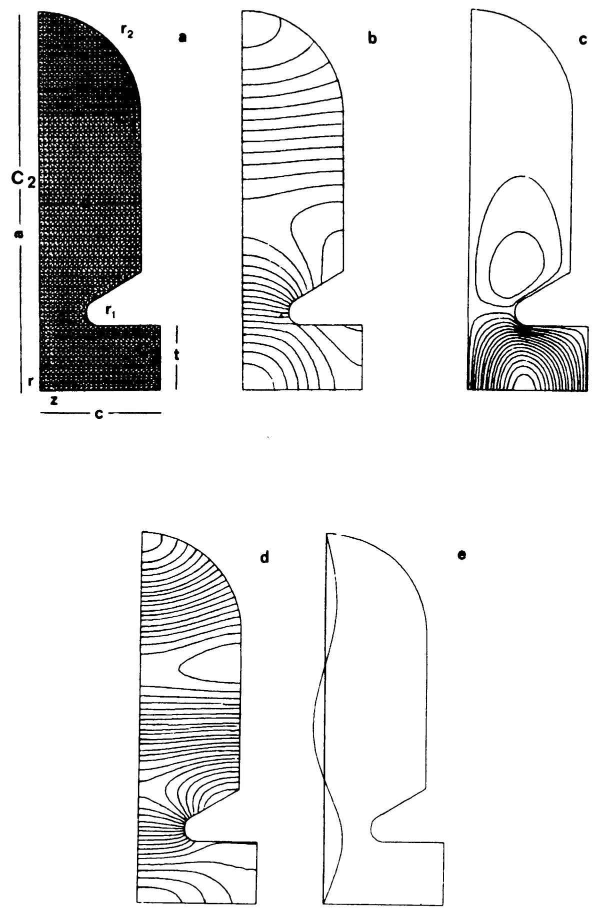

FIGURE 3 Results in a side-coupled cavity cell. (a) Triangular mesh. (b) Isolines for H_Φ for the E_{120} mode. (c) Isolines of E_Φ for the E_{120} mode. (d) Isolines of H_Φ for the E_{130} mode. (e) Electric-field component e_z along $z = 0$ for the E_{130} mode.

conditions on separate parts of the boundary: $e_\tau = 0$ on the surface of ideally conductive metal, $e_\tau = 0$ or $h_\tau = 0$ on the symmetry planes depending on the type of modes sought, the number of azimuthal variations, the mode of operation with the program.

The program works in two modes, a mode of estimating the frequency spectrum in the specified interval and a mode of accurate computation of a specific frequency and related fields and derived quantities.

In the first mode, the program computes the residual function $I(\kappa)$ whose zeroes specify resonance frequencies. The distribution of these zeroes are initial information for the user and serve as the initial approximations in precise computation in the second mode. The frequency band and resonance-tracing step are set up by a user.

In the second mode, the initial approximation of frequency is specified more precisely by iteration and if required by the user, the derived quantities are computed and, if necessary, the information is dumped for subsequent operation, for example, for graphic representation of the data.

All the processes such as construction of triangle meshes, optimal enumeration of the nodes with view to minimize the matrix bandwidth, derivation and solution of the systems of equations, the choice of the optimal position of the driving point, testing the mesh against the current value of the wave number, data control are carried out automatically. However, the operational efficiency of the package is dependent on user's experience and knowledge.

5. EXAMPLES

The examples presented below are aimed to show the package facilities for typical solutions and the form of their representation. The computational accuracy was estimated by testing the numerical solution against the analytic one for a cylindrical cavity and against the experimental data for a diaphragmed waveguide.

Example 1. A Cylindrical Cavity.

Each of Figs. 1a,b,c shows a quarter of the cavity dimension in the z-direction with the structure of H_{332}-type oscillations. The frequency obtained from the analytic solution is $f = 6188.2$ MHz, and the one computed with the help of the PRUD package is $f = 6191.9$ MHz. Thus the relative inaccuracy is $\Delta f/f = 6 \cdot 10^{-4}$. Figures 1a,b show the isolines of $\mathcal{H}_\varphi$, $\mathcal{E}_\varphi$, Fig. 1c shows the vector field $\mathbf{h}$ in the $\varphi = 0$ plane; C_1 is the metal surface, C_2 is the $h_\tau = 0$ symmetry plane.

Example 2. A Cell of a Diaphragmed Waveguide.

The cell parameters have been borrowed from Ref. 6 and are as follows: $d = 2.5$ cm, $a = 2.759$ cm, $a/b = 0.5$, $t = 0.56$ cm. The oscillation frequency EH_{11} of the $\pi/2$ type, according to the data from Ref. 6, is $f = 2997.93$ MHz in this cell and the one computed by the technique described on a mesh containing about 1500 nodes is $f = 2999.1$ MHz. Thus the discrepancy is $\Delta f/f = 4 \cdot 10^{-4}$. Figure 2a shows the typical triangle mesh of the cross section, Fig. 2b shows the isolines of $\mathcal{E}_\varphi$ and Fig. 2c those of $\mathcal{H}_\varphi$.

Example 3. An Accelerating Structure Cavity with Side-Coupled Cells.

The parameters are $a = 10$ cm, $c = 3.261$ cm, $d = 2.761$ cm, $t = 1.7$ cm. The boundary condition is $C_2 - e_\tau = 0$. Figure 3 shows two possible modes in this structure. The former is similar to that of the E_{120} type with the frequency $f = 3212$ MHz and the latter to that of the E_{130} type with the frequency $f = 4606.4$ MHz. Figure 3a presents the typical triangle mesh of the cavity region, Figs. 3b,c show the isolines of $\mathcal{H}_\varphi$ and $\mathcal{E}_\varphi$ for the E_{120} mode, Fig. 3d shows the isolines for $\mathcal{H}_\varphi$ for the E_{130} mode, Fig. 3e shows the component e_z along the $z = 0$ line for the E_{130} mode.

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