Particle Accelerators
1980, Vol. 11, pp. 71-79
0031-2460/80/1102-0071 \$06.50/0

TRANSVERSE RESISTIVE WALL INSTABILITY OF A RELATIVISTIC ELECTRON BEAM*

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(Received June 19, 1980)

The transverse stability of a relativistic electron beam propagating in an evacuated, smooth pipe of finite conductivity is examined. Exact and asymptotic solutions are obtained in the limit of continuous external focusing. An exact solution for the transport through a single focusing period with discrete focusing is extended to provide an asymptotic solution for a multiperiod transport system. It is found in both cases that focusing can only reduce the growth rate; it cannot suppress the instability. Applications to pulsed electric-power transmission are discussed.

I. INTRODUCTION

The displacement of the centroid of a relativistic electron beam from the axis of an evacuated conducting pipe produces charges and currents in the pipe walls which will affect the subsequent motion of the beam. The behavior of the beam in the presence of these wall charges and currents, in addition to various focusing forces, is examined.

The surface charges which are induced on the inner pipe wall by the nonaxisymmetric component of the beam's electric field generate an electric field which reacts on the beam to pull it further along in the direction of its initial displacement. The magnetic field of the displaced beam generates surface currents in the pipe which set up their own magnetic fields that act to push the beam in a direction opposite to that of its original displacement. At the front, or head of the beam, these forces cancel to within a factor of $1/\gamma^2$ resulting in a net destabilizing force. Here γ is the particle's energy in units of the rest energy.

Gradually, the magnetic field of the beam diffuses through the conducting pipe wall. As this occurs, the induced current decreases, reducing the magnetic restoring force on the beam. The electric force, however, does not weaken. Thus, the net destabilizing force grows with distance back from the head of the beam.

In Section II we formulate the beam dynamics problem for continuous focusing. An exact solution is then presented in Section III. An asymptotic solution is then given in Section IV and the growth of the beam displacement for parameters of interest for pulsed electric-power transmission is discussed. In Section V the problem is reformulated for the case of discrete focusing elements. An exact and an asymptotic solution are obtained. In Section VI the relationship between the continuous focusing and discrete focusing solutions is exhibited. Finally, the conclusions are discussed in Section VII. The use of various focusing schemes will be shown to reduce the growth rate. However, focusing cannot suppress the instability.

II. FORMULATION OF THE PROBLEM FOR CONTINUOUS FOCUSING

Consider the geometry shown in Fig. 1. The beam is slightly displaced in the positive x-direction. The resulting beam current and charge densities may be thought of as those arising from the undisplaced beam plus surface contributions of the form $\mathbf{J} = (I\xi/\pi a^2)\ \delta(r - a) \cos\phi\ \mathbf{e}_z$ and $\rho = (I\xi/\pi a^2 v)\ \delta(r - a)\cos\phi$. Here a is the beam radius, I the beam current and ξ is the displacement of the beam centroid from the pipe axis (z-axis). More precisely $\boldsymbol{\xi}(z,t) = \xi(z,t)\ \mathbf{e}_x$, v is the z-component of the beam velocity and ϕ is the azi-

* Work performed under the auspices of the U.S. Department of Energy by the Lawrence Livermore National Laboratory under contract number W-7405-ENG-48.

This work is performed by LLNL for the Department of Defense under DARPA (DOD) ARPA Order 3718, Amendment #12, monitored by NSWC under Contract No. N60921-80-PO-WO188.

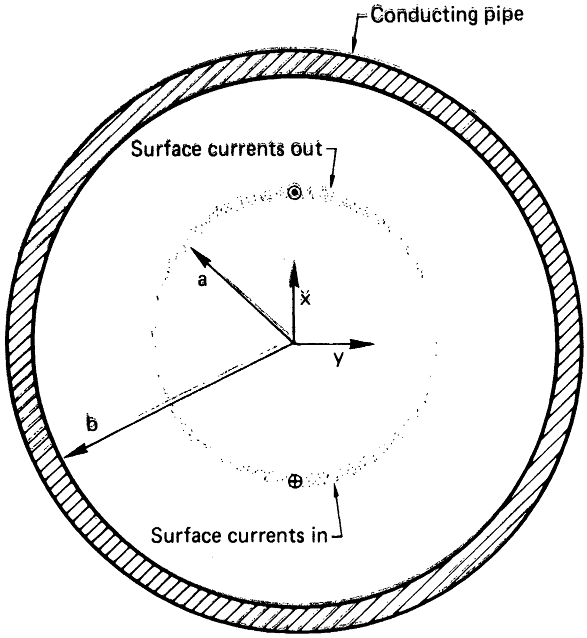

FIGURE 1 Equivalent surface charge and current density layers of beam displaced from the axis of a conducting pipe.

muthal angle measured from the positive x-axis. The quantity $\delta(x)$ is the Dirac delta function.

Only the fields due to the displacement of the beam centroid from the pipe axis need be considered. The fields produced by these beam charge and current densities induce corresponding charge and current densities in the walls. The fields produced by these induced charge and current densities were computed† to be[1]

$$\mathbf{B} = \frac{2I}{cb^2}\,\xi(z,t)\,\mathbf{e}_y - \frac{4I}{\pi b^3\sigma^{1/2}}\,\mathbf{e}_y \times \int_{-\infty}^{t} \frac{\partial\xi}{\partial t'}\sqrt{t - t'}\,dt'$$

$$\mathbf{E} = \frac{2I}{vb^2}\,\xi(z,t)\,\mathbf{e}_x.$$

Here b and σ are the pipe radius and conductivity, respectively, and c is the speed of light. Gaussian units are used in this paper.

† It is assumed in the derivation that the fields which precede the beam can be neglected either because the speed of the beam is so close to that of light or by noting that, in any case, the fields leading the pulse cannot extend more than a few beam radii in front of the head. If $a/L \ll 1$, where L is the beam length, then these fields cannot have an appreciable effect on the beam dynamics.

The integral term in the expression for **B** represents the weakening of the magnetic field with distance back from the head of the beam caused by the induced current diffusing into the pipe wall. The resulting Lorentz force acting upon the beam is

$$\mathbf{F}(z,t) = \mathbf{e}_x\left[\frac{2Ie}{vb^2\gamma^2}\,\xi(z,t) + \frac{4Iev}{\pi cb^3\sigma^{1/2}}\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\sqrt{t - t'}\,dt'\right].$$

The equation of motion for the beam displacement now becomes

$$\frac{d^2\xi}{dt^2} + k_\beta{}^2v^2\xi = \frac{2Ie}{vb^2\gamma^3 m}\,\xi + \frac{4Iev}{\gamma\pi cb^3\sigma^{1/2}m}\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\sqrt{t - t'}\,dt' \quad (1)$$

in which m and e are the electron rest mass and charge, respectively, and $k_\beta{}^2v^2\xi$ is a focusing term similar to the restoring force in a simple harmonic oscillator. This term represents the effects of allowing the number of discrete "lenses" to increase without limit in such a way that the product of the inter-lens distance and the focal length remains constant. We consider this limiting case in Section VI.

Noting that $4Iev/(\gamma\pi cb^3\sigma^{1/2}m)$ has the dimensions of $\sec^{-5/2}$ we define a frequency

$$\Omega^{5/2} \equiv \frac{4evI}{\gamma m\pi cb^3\sigma^{1/2}}.$$

The equation for ξ then can be written as

$$\frac{d^2\xi}{dt^2} + \omega_0{}^2\xi = \Omega^{5/2}\int_{-\infty}^{t}\frac{\partial\xi}{\partial t'}\sqrt{t - t'}\,dt' \quad (2)$$

where $\omega_0{}^2 \equiv k_\beta{}^2v^2 - 2Ie/(mvb^2\gamma^3)$. The solutions to Eq. (2) fall into three classes, depending upon the value of $\omega_0{}^2$. We have:

positive focusing: $\omega_0{}^2 > 0$,

critical focusing: $\omega_0{}^2 = 0$, and

negative focusing: $\omega_0{}^2 < 0$.

Only the case of positive focusing is of practical interest.

Some information on the growth rate of the instability may be obtained from dimensional arguments. For simplicity consider the critical focusing case:

$$\frac{d^2\xi}{dt^2} = \Omega^{5/2} \int_{-\infty}^{t} \frac{\partial \xi}{\partial t'} \sqrt{t - t'}\, dt'.$$

The right-hand side of this expression represents the effects of the diffusion of the B-field of the beam into the pipe wall. If the pulse is τ seconds in length then the maximum range of t' over which the integrand is nonzero is τ. Thus, dimensionally the equation becomes

$$\xi/t_g^{\,2} \approx \Omega^{5/2}\xi\, \tau^{1/2}$$

where t_g is a characteristic growth time. Hence $t_g \sim \Omega^{-5/4}\,\tau^{-1/4}$ and the corresponding growth length l_g is $l_g \sim vt_g \sim v\Omega^{-5/4}\,\tau^{-1/4}$. Inserting numerical values, we obtain

$$l_g \approx \frac{3.8\,\gamma^{1/2}}{I_{\mathrm{kA}}^{1/2}\,\tau_{(\mu\mathrm{sec})}^{1/4}} \left(\frac{\sigma}{10^{17}\,(\mathrm{sec}^{-1})}\right)^{1/4} \times \left(\frac{b_{\mathrm{cm}}}{10}\right)^{3/2} \text{(meters)}.$$

We will compare this length to the growth length with continuous focusing in Section IV and show that focusing substantially increases the growth length.

III. EXACT SOLUTION FOR CONTINUOUS FOCUSING

In the equation of motion (2), we have $d/dt = \partial/\partial t + v\partial/\partial z$. For the initial conditions $\partial\xi/\partial z\,(0,t) = 0$; $\xi(0,t) = d_0$, the solution is of the form

$$\xi(z,t) = u_{-1}(t - z/v) \sum_{n=0}^{\infty} (t - z/v)^{n/2} g_n(z), \tag{3}$$

where $u_{-1}(x)$ is the unit step function and the functions $g_n(z)$ must be determined. Substituting Eq. (3) into Eq. (2) yields

$$\sum_{n=0}^{\infty} (t - z/v)^{n/2} \frac{d^2 g_n}{dz^2}(z) v^2 + \omega_0^{\,2} \sum_{n=0}^{\infty} (t - z/v)^{n/2} g_n(z) = \Omega^{5/2} \sum_{n=0}^{\infty} g_n(z) \tag{4}$$

$$\times \int_{z/v}^{t} \frac{n}{2} (t' - z/v)^{n/2 - 1} (t - t')^{1/2}\, dt,'$$

where we have used the fact that

$$\frac{\partial}{\partial t} u_{-1}(t - z/v) = \delta(t - z/v).$$

After rewriting Eq. (4) in terms of the retarded time $\tau = t - z/v$, we can evaluate the integral on the right-hand side by use of the convolution theorem for Laplace transforms.

The resulting differential equation for the functions $g_n(z)$ is

$$v^2 \sum_{n=0}^{\infty} \tau^{n/2} \frac{d^2 g_n}{dz^2}(z) + \omega_0^{\,2} \sum_{n=0}^{\infty} \tau^{n/2} g_n(z) = \frac{\sqrt{\pi}}{2}\Omega^{5/2} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)} \tau^{(n+1)/2} g_n(z). \tag{5}$$

Equating coefficients of equal powers of τ yields

$$\text{for } n = 0: \quad \frac{d^2 g_0}{dz^2}(z) + k_0^{\,2} g_0(z) = 0, \tag{6}$$

$$\text{for } n > 0: \quad \frac{d^2 g_{n+1}}{dz^2} + k_0^{\,2} g_{n+1} = \frac{\sqrt{\pi}}{2}\frac{\Omega^{5/2}}{v^2} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)} g_n, \tag{7}$$

where $k_0 \equiv \omega_0/v$. The boundary conditions now become $g_0(0) = d_0$, $g_n(0) = 0$ for $n > 0$, and $g_n'(0) = 0$ for all n, where a prime denotes differentiation with respect to z.

With these boundary conditions, the solutions of Eqs. (6) and (7) are

$$g_0(z) = d_0 \cos k_0 z,$$

$$g_{n+1}(z) = \frac{\Omega^{5/2}\sqrt{\pi}}{2k_0 v^2} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{2}\right)} \times \int_0^z g_n(z') \sin k_0(z - z')\, dz'.$$

Using the convolution theorem for Laplace transforms, we obtain

$$\tilde{g}_{n+1}(s) = \frac{\sqrt{\pi}\Omega^{5/2}}{2k_0 v^2} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)} \tilde{g}_n(s) \frac{k_0}{k_0^2+s^2},$$

where the tilde denotes the Laplace transform and s is the Laplace transform variable.

This recursion relation may be solved to yield $\tilde{g}_{n+1}(s)$ in terms of $\tilde{g}_0(s)$. Thus

$$\tilde{g}_{n+1}(s) = d_0 \left(\frac{\sqrt{\pi}\,\Omega^{5/2}}{2v^2}\right)^{n+1} \times \frac{1}{\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)} \frac{s}{(k_0^2+s^2)^{n+2}},$$

which may be inverted to yield

$$g_{n+1}(z) = d_0 \left(\frac{\sqrt{\pi}\,\Omega^{5/2}}{4v^2}\frac{z}{k_0}\right)^{n+1} \times \frac{k_0 z\, j_n(k_0 z)}{\Gamma(n+2)\,\Gamma\left(\frac{n}{2}+\frac{3}{2}\right)},$$

where j_n is the spherical Bessel function of order n. Thus, the solution is

$$\xi(z,t) = d_0 u_{-1}(t - z/v) \times \sum_{n=0}^{\infty} \left(\frac{\sqrt{\pi(t-z/v)}\,\Omega^{5/2} z}{4v^2 k_0}\right)^n \times \frac{k_0 z\, j_{n-1}(k_0 z)}{\Gamma(n+1)\,\Gamma\left(\frac{n}{2}+1\right)}. \quad (8)$$

If we define the transformed variables

$$\bar{\tau} \equiv \left(\frac{\Omega^{5/2}}{k_0^2 v^2}\right)^2 \tau = \left(\frac{\Omega^{5/2}}{k_0^2 v^2}\right)^2 (t - z/v), \quad (8a)$$

$$\bar{z} \equiv k_0 z, \quad (8b)$$

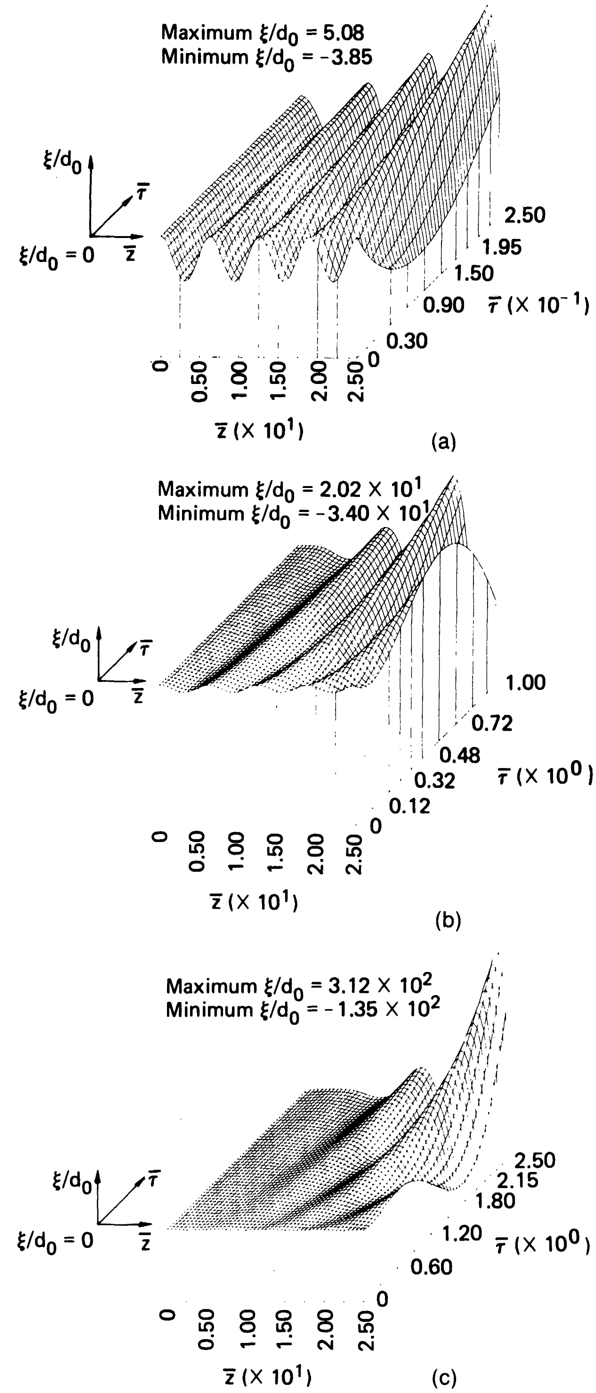

FIGURE 2 Normalized beam displacement for continuous focusing solution Eq. (8c) as a function of $\bar{z}$ and $\bar{\tau}$ for (a) $\bar{z}$ = [0 to 25], $\bar{\tau}$ = [0 to 0.25], (b) $\bar{z}$ = [0 to 25], $\bar{\tau}$ = [0 to 1] and (c) $\bar{z}$ = [0 to 25], $\bar{\tau}$ = [0 to 2.5].

the solution (8) can be written as

$$\xi(\bar{z},\bar{\tau}) = d_0 u_{-1}(\bar{\tau}) \sum_{n=0}^{\infty} \left(\frac{\sqrt{\pi\bar{\tau}}\,\bar{z}}{4}\right)^n \times \frac{\bar{z} j_{n-1}(\bar{z})}{\Gamma(n+1)\,\Gamma\left(\frac{n}{2}+1\right)}. \tag{8c}$$

The value of $\xi(\bar{z},\bar{\tau})/d_0$ determined from Eq. (8c) is plotted vs $\bar{z}$ and $\bar{\tau}$ in Fig. 2. The displacement as a function of z and t may be found by applying the inverses of the transformations (8a) and (8b) to the $\bar{z}$ and $\bar{\tau}$ axes of the plots. In this manner the solution for many values of the parameters k_0 and $\Omega^{5/2}$ may be determined from one series of plots.

IV. ASYMPTOTIC SOLUTION FOR CONTINUOUS FOCUSING

The solution for positive focusing (8) may be written as

$$\xi(z,t) = d_0 u_{-1}(t - z/v) \times \sum_{n=0}^{\infty} \frac{A^n k_0 z\, j_{n-1}(k_0 z)}{\Gamma(n+1)\,\Gamma\left(\frac{n}{2}+1\right)}, \tag{9}$$

where

$$A = \sqrt{\pi(t - z/v)}\,\Omega^{5/2} z/(4k_0 v^2).$$

Our aim is to derive an integral representation of this solution so that an asymptotic formula may be obtained.

Laplace transforming the sum in the variable A^2 gives

$$\xi(z,t) = d_0 u_{-1}(t - z/v) \sqrt{\frac{\pi k_0 z}{2}}\, \mathscr{L}^{-1} \times \left\{\frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{s}}\right)^n \frac{1}{n!} J_{n-1/2}(k_0 z)\right\},$$

where the inverse Laplace transform is denoted by $\mathscr{L}^{-1}$ and where J_ν is the cylindrical Bessel function of order ν. Use of the relation[2]

$$\left(\frac{2}{\pi z}\right)^{1/2} \cos\sqrt{z^2 - 2zt} = \sum_{m=0}^{\infty} \frac{t^m}{m!} J_{m-1/2}(z),$$

gives

$$\xi(z,t) = d_0 u_{-1}(t - z/v)\, \mathscr{L}^{-1} \times \left\{\frac{1}{s} \cos\sqrt{(k_0 z)^2 - \frac{2k_0 z}{\sqrt{s}}}\right\}.$$

Using the Bromwich integral to invert the Laplace transform gives

$$\begin{aligned}\xi(z,t) = {} & d_0 u_{-1}(t - z/v) \frac{1}{4\pi i} \times \int_{c-i\infty}^{c+i\infty} \exp(A^2 s) \\ & \times \left[\exp\left\{i\left[(k_0 z)^2 - \frac{2k_0 z}{\sqrt{s}}\right]^{1/2}\right\} + \exp\left\{-i\left[(k_0 z)^2 - \frac{2k_0 z}{\sqrt{s}}\right]^{1/2}\right\}\right] \frac{ds}{s}.\end{aligned} \tag{10}$$

If $k_0 z \gg A^{2/3} \gg 1$, the saddle points of the integrand are found to be at $s \simeq (\mp i)^{2/3}/(2A^2)^{2/3}$ where the upper (lower) sign corresponds to the saddle points of the first (second) term in Eq. (10). For each sign one of the saddle points lies on the branch cut of the integrand and is inaccessible. The integration path is shown in Fig. 3. The result of the steepest-descent calculation is

$$\xi(z,t) \sim \frac{2d_0}{\sqrt{3\pi}} u_{-1}(t - z/v) \left(\frac{2}{A}\right)^{1/3} \times \exp\left[\frac{3}{2}\left(\frac{A}{2}\right)^{2/3}\right] \cos k_0 z \times \sin\left[\frac{2\pi}{3} - \frac{3\sqrt{3}}{2}\left(\frac{A}{2}\right)^{2/3}\right]. \tag{11}$$

We may now define a characteristic length for the growth of the instability from the relation

$$\frac{z}{z_g} \equiv \frac{A}{2} = \frac{\sqrt{\pi\tau}\,\Omega^{5/2}}{8k_0 v^2} z$$

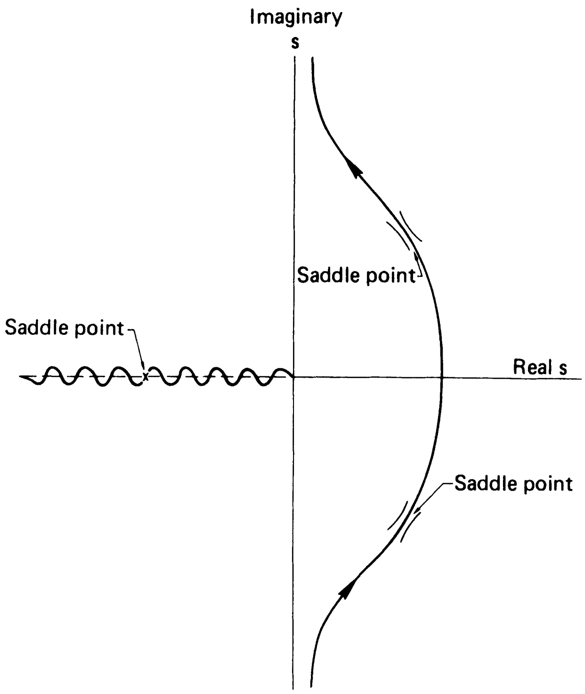

FIGURE 3 Contour for Eq. (10).

or

$$z_g = \frac{8k_0 v^2}{\sqrt{\pi\tau}\,\Omega^{5/2}}.$$

Inserting numerical values, we have

$$z_g = \frac{4.0\,\gamma}{I_{(\mathrm{kA})}\,\tau^{1/2}_{(\mu\mathrm{sec})}} \left(\frac{\sigma}{10^{17}\,(\mathrm{sec}^{-1})}\right)^{1/2} \times \left(\frac{b_{(\mathrm{cm})}}{10}\right)^3 \left(\frac{100(\mathrm{m})}{\lambda_0}\right) \text{ meters,} \tag{12}$$

where $\lambda_0 = 2\pi/k_0$, the net coherent betatron wavelength due to the external continuous focusing. This expression should be compared with that in Section II for the critical focusing case. The growth length with focusing grows more rapidly with pipe radius, conductivity and γ, but less rapidly with increasing beam current than the growth length in the absence of focusing. In addition, the growth length for the focusing case varies inversely with the coherent betatron wavelength. The inverse dependence of z_g with beam current and the square root of the pulse length illustrates the difficulty of attempting to propagate long high-current pulses over great distances. The most sensitive parameter in the growth length expression is the pipe radius, while one of the least is the pipe conductivity. A value of 10^{17} sec^{-1} corresponds to the σ of aluminum, which would be difficult to exceed in practical situations.

Consider a pulsed electric-power transmission example.[3] For a 5 MeV, 2 kA beam with a pulse length of 100 nsec, the critical focusing case gives a growth length of $z_g \simeq 15$ m, while the positive focusing expression gives $z_g \simeq 63\ (100\ \mathrm{m}/\lambda_0)$ m, where σ and b have been taken as 10^{17} sec^{-1} and 10 cm, respectively. These growth lengths are disturbingly short even though the latter growth length can be extended by increased focusing (decreased λ_0).

V. FORMULATION OF THE PROBLEM FOR DISCRETE FOCUSING

Consider the single transport cell shown in Fig. 4 consisting of two half-lenses each of focal length $2f$ and a drift length L. The transport matrix for each half lens is

$$M_l = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{2f} & 1 \end{pmatrix}, \tag{13}$$

in the sense that if a prime and a double prime

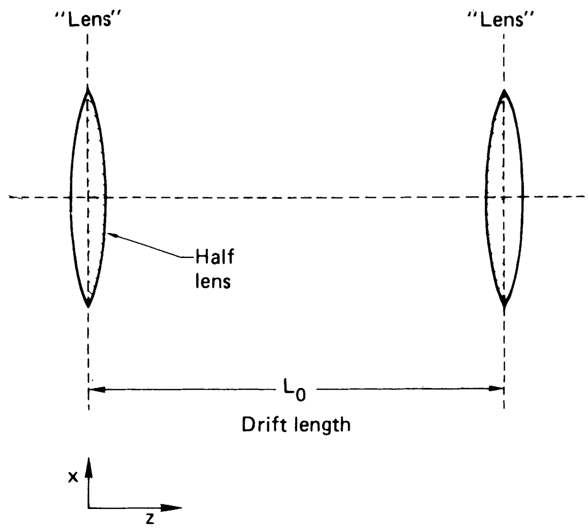

FIGURE 4 A single half-lens drift length half-lens combination.

denote quantities immediately before and after passing through the half-lens respectively, then

$$\begin{pmatrix} \xi \\ \frac{\partial \xi}{\partial z} \end{pmatrix}'' = M_l \begin{pmatrix} \xi \\ \frac{\partial \xi}{\partial z} \end{pmatrix}' \tag{14}$$

In the drift-length region, the beam displacement varies according to the equation of motion

$$\frac{d^2\xi}{dt^2} = \Omega^{5/2} \int_{-\infty}^{t} \frac{\partial \xi}{\partial t'} \sqrt{t - t'}\, dt', \tag{15}$$

where we have assumed that $\gamma \gg 1$. If we change variables, from z, t to z, $\tau = t - z/v$, Eq. (15) can be written as

$$\frac{\partial^2 \xi}{\partial z^2} = \frac{\Omega^{5/2}}{v^2} \int_{-\infty}^{\tau} \frac{\partial \xi}{\partial \tau'} \sqrt{\tau - \tau'}\, dt'. \tag{16}$$

Laplace transforming in τ gives

$$\frac{\partial^2 \tilde{\xi}}{\partial z^2} - \frac{\sqrt{\pi}\Omega^{5/2}}{2v^2\sqrt{s}} \tilde{\xi} = 0 \tag{17}$$

with solution $\tilde{\xi} = D \cosh \alpha z + E \sinh \alpha z$ and $\partial\tilde{\xi}/\partial z = D\alpha \sinh \alpha z + E\alpha \cosh \alpha z$, where we have defined

$$\alpha \equiv \pi^{1/4}\Omega^{5/4}/[\sqrt{2}v(s)^{1/4}].$$

Now at $z = 0$, $\tilde{\xi} = \tilde{\xi}^{(0)} = D$ and $\partial\tilde{\xi}/\partial z = E\alpha$. Therefore

$$\tilde{\xi}(z) = \tilde{\xi}^{(0)} \cosh \alpha z + \frac{1}{\alpha} \frac{\partial \tilde{\xi}^{(0)}}{\partial z} \sinh \alpha z,$$

$$\frac{\partial \tilde{\xi}}{\partial z}(z) = \alpha\tilde{\xi}^{(0)} \sinh \alpha z + \frac{\partial \tilde{\xi}^{(0)}}{\partial z} \cosh \alpha z.$$

Then the drift-length transfer matrix is

$$M_d = \begin{pmatrix} \cosh \alpha L & \frac{\sinh \alpha L}{\alpha} \\ \alpha \sinh \alpha L & \cosh \alpha L \end{pmatrix},$$

such that after traversing one complete cell,

$$\begin{pmatrix} \tilde{\xi} \\ \frac{\partial \tilde{\xi}}{\partial z} \end{pmatrix}^{(1)} = M_l M_d M_l \begin{pmatrix} \tilde{\xi} \\ \frac{\partial \tilde{\xi}}{\partial z} \end{pmatrix}^{(0)}.$$

We can thus write the unit-cell transport matrix $M_l M_d M_l$ as

$$M_l M_d M_l = \begin{pmatrix} \cosh \alpha L - \frac{\sinh \alpha L}{2\alpha f} & \frac{\sinh \alpha L}{\alpha} \\ \alpha \sinh \alpha L - \frac{\cosh \alpha L}{f} + \frac{\sinh \alpha L}{4\alpha f^2} & \cosh \beta L - \frac{\sinh \alpha L}{2\alpha f} \end{pmatrix}.$$

This unimodular matrix can be written as[4]

$$M_l M_d M_l = \cos \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin \mu \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & -m_{11} \end{bmatrix}.$$

Here

$$\cos \mu = \cosh \alpha L - \frac{\sinh \alpha L}{2\alpha f} = \frac{1}{2} \mathrm{Tr}(M_l M_d M_l)$$

and $m_{11} = 0$. After traversing N-cells,

$$\begin{pmatrix} \tilde{\xi} \\ \frac{\partial \tilde{\xi}}{\partial z} \end{pmatrix}^{(N)} = (M_l M_d M_l)^N \begin{pmatrix} \tilde{\xi} \\ \frac{\partial \tilde{\xi}}{\partial z} \end{pmatrix}^{(0)}.$$

Now

$$(M_l M_d M_l)^N = \cos N\mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin N\mu \begin{bmatrix} 0 & m_{12} \\ m_{21} & 0 \end{bmatrix}.$$

In the problem under consideration we choose $\partial\tilde{\xi}^{(0)}/\partial z = 0$; therefore $\xi^{(N)} = \tilde{\xi}^{(0)} \cos N\mu$ or

$$\tilde{\xi}^{(N)} = \frac{\tilde{\xi}^{(0)}}{2} \left\{ \exp\left[iN \cos^{-1} \times \left(\cosh \alpha L - \frac{\sinh \alpha L}{2\alpha f} \right) \right] + \exp\left[-iN \cos^{-1} \times \left(\cosh \alpha L - \frac{\sinh \alpha L}{2\alpha f} \right) \right] \right\}.$$

Now $i \cos^{-1} x = \log [x \pm \sqrt{x^2 - 1}]$, so that

$$\xi^{(N)} = \frac{d_0}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{ds}{s} \exp(\tau s) \times \{\exp[N \log(x \pm \sqrt{x^2 - 1})] + \exp[-N \log(x \pm \sqrt{x^2 - 1})]\},$$

where

$$x \equiv \cosh \alpha L - \frac{\sinh \alpha L}{2\alpha f}.$$

For large N the saddle points of the integrand occur at

$$s \simeq (\mp i)^{2/3} \left[\frac{Nh^2L^2}{4\tau} \frac{\left(1 - \frac{L}{6f}\right)}{\sqrt{1 - \left(1 - \frac{L}{2f}\right)^2}} \right]^{2/3},$$

where $h = \alpha s^{1/4} = \pi^{1/4}\Omega^{5/4}/(\sqrt{2}v)$ and the upper (lower) sign corresponds to the saddle points for the first (second) term. For each term one of the saddle points lies on a branch cut and cannot be traversed. The integration contour is basically the same as that for Eq. (10) and is shown in Fig. 3. A steepest-descent evaluation of the integral yields

$$\xi^{(N)} \sim \frac{2d_0}{\sqrt{3\pi}} \left[\frac{8v^2\sqrt{1 - \left(1 - \frac{L}{2f}\right)^2}}{NL^2\Omega^{5/2}\sqrt{\pi\tau}\left(1 - \frac{L}{6f}\right)} \right]^{1/3} \times \exp\left\{ \frac{3}{2} \left[\frac{NL^2\sqrt{\pi\tau}\Omega^{5/2}\left(1 - \frac{L}{6f}\right)}{8v^2\sqrt{1 - \left(1 - \frac{L}{2f}\right)^2}} \right]^{2/3} \right\}$$
$$\times \cos\left\{ N \tan^{-1} \left[\frac{\sqrt{1 - \left(1 - \frac{L}{2f}\right)^2}}{1 - \frac{L}{2f}} \right] \right\} \times \sin\left\{ \frac{2\pi}{3} - \frac{3\sqrt{3}}{2} \left[\frac{NL^2\sqrt{\pi\tau}\Omega^{5/2}\left(1 - \frac{L}{6f}\right)}{8v^2\sqrt{1 - \left(1 - \frac{L}{2f}\right)^2}} \right]^{2/3} \right\},$$

where we have put $\xi^{(0)} = d_0$, the initial displacement of the beam.

VI. RELATION OF THE DISCRETE FOCUSING SOLUTION TO THE CONTINUOUS FOCUSING SOLUTION

Consider the cell shown in Fig. 4. Let the number of lenses increase such that the interlens distance $L = L_0/N$, while at the same time extending the focal length so that $f = f_0 N$. Then the displacement at the Nth cell is given by

$$\begin{pmatrix} \tilde{\xi} \\ \frac{\partial \tilde{\xi}}{\partial z} \end{pmatrix}^{(N)} = \left[\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_0N} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_0/N \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_0N} & 1 \end{pmatrix} \right]^{(N)} \begin{pmatrix} \xi \\ \frac{\partial \xi}{\partial z} \end{pmatrix}^{(0)}$$
$$= \begin{pmatrix} 1 - \frac{L_0}{2f_0N^2} & \frac{L_0}{N} \\ \frac{L_0}{4f_0{}^2N^2} - \frac{1}{f_0N} & 1 - \frac{L_0}{2f_0N^2} \end{pmatrix}^{(N)} \times \begin{pmatrix} \xi^{(0)} \\ 0 \end{pmatrix}$$

where we have chosen the beam to have an initial displacement but no initial slope; i.e., $\partial\xi/\partial x = 0$ at $z = 0$. As in Section V, this may be written as

$$\begin{pmatrix} \xi \\ \dfrac{\partial \xi}{\partial z} \end{pmatrix}^{(N)} = \left\{ \cos N\mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin N\mu \begin{bmatrix} 0 & m_{12} \\ m_{21} & 0 \end{bmatrix} \right\} \begin{pmatrix} \xi^{(0)} \\ 0 \end{pmatrix},$$

where

$$\mu = \cos^{-1}\left(1 - \frac{L_0}{2f_0N^2}\right).$$

Thus

$$\xi^{(N)} = \xi^{(0)} \cos\{N \cos^{-1}[1 - L_0/(2f_0N^2)]\}.$$

As $N \rightarrow \infty$ this equation becomes

$$\xi^{(N)} = \xi^{(0)} \cos\left(N\sqrt{\frac{L_0}{f_0N^2}}\right) = \xi^{(0)} \cos\sqrt{\frac{L_0}{f_0}}. \tag{19}$$

Now in the continuous focusing limit we have

$$\frac{d^2\xi}{dt^2} + \omega_\beta{}^2\xi = 0.$$

Using the variables z, $\tau = t - z/v$ this becomes

$$\frac{\partial^2\xi}{\partial z^2} + k_\beta{}^2\xi = 0.$$

For the initial conditions $\xi^{(0)} \neq 0$; $\partial\xi(0)/\partial z = 0$ this gives

$$\xi(z) = \xi^{(0)} \cos k_\beta z. \tag{20}$$

In particular for $z = L_0$, comparison of Eq. (19) with Eq. (20) gives

$$k_\beta = \frac{1}{\sqrt{L_0 f_0}}. \tag{21}$$

If we now put $L = L_0/N$, $f = f_0N$ in Eq. (18) and let $N \rightarrow \infty$, we obtain the continuous focusing asymptotic solution (11), as we should.

VII. CONCLUSION

In summary, an exact solution for the growth of the transverse displacement of a beam traveling in an evacuated pipe of finite conductivity with continuous focusing has been presented. In addition, asymptotic growth rates for both continuous and discrete element focusing were obtained. The instability was shown to be significant for the parameters of interest for pulsed electric-power transmission.

For both continuous and discrete element focusing, the instability grows. Focusing can only act to slow the growth: it cannot suppress the instability.

The authors wish to thank R. Melendez for performing the numerical work and E. P. Lee for helpful discussions.

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