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STABILITY OF A SELF-CONSISTENT LONGITUDINAL PHASE-SPACE DISTRIBUTION UNDER SPACE CHARGE PERTURBATIONS

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The equations of longitudinal motion for particles in a beam bunch with a linear external bunching field and with space charge forces are presented. A self-consistent phase-space distribution with an envelope equation, which is a solution to the Vlasov equation with these equations of motion, is described. The stability of the stationary distribution, the phase-space distribution with bunching and debunching balanced, is analyzed under normal mode perturbations. Eigenfrequencies and eigenmodes of these perturbations are descrrbed; they are found to be stable. Equations for numerical analysis of normal mode oscillations of the particle distribution in a system in which the external bunching force is periodic are derived. Results of the numerical analysis indicate that these oscillations can be unstable when the period of the bunching force and the period of the normal mode oscillation are in resonance. However, the numerical results also indicate that these instabilities are quite small in the case in which the bunching force period is much smaller then the individual particle motion period.

NOTE: The lower case italic letter "ν" should not be misconstrued with the lower case Greek letter nu "ν".

INTRODUCTION

Many applications of particle acceleration, such as heavy ion fusion, require longitudinal bunching of a high intensity particle beam to extremely high particle currents with correspondingly large space charge forces. Proper accelerator design requires a precise analysis of longitudinal and transverse motion with a stability analysis.

In previous work by L. Smith et al.,[1] an analysis of transverse stability was presented. Those calculations were based upon the Kapchinskij-Vladimirskij (K-V) distribution,[2] which is a self-consistent transverse phase-space distribution with coupled envelope equations to describe the motion of that distribution. Perturbations of the K-V distribution were analyzed to determine conditions of instability, and that analysis places important constraints on accelerator design at high currents.

In a previous paper[3] we derived a self-consistent distribution function in longitudinal phase space with an envelope equation to describe the transport of that distribution through an arbitrary linear bunching system. In this paper we analyze the stability of that distribution under small perturbations.

In Section I we describe the longitudinal equations of motion and present the self-consistent phase space distribution which is the solution of the Vlasov equation with these equations of motion. In Section II we analyze the stability of the stationary distribution; this is the case in which the external bunching is constant and is balanced by the space charge and phase space debunching. Eigenmodes and eigenfrequencies of normal mode perturbations are described and are searched for instability.

In Sections III and IV, the stability analysis is extended to the case in which the external bunching is periodic and the unperturbed distribution is extracted from the periodic solution of the envelope equation of Section I. In Section III, coupled linear differential equations are derived which can be integrated numerically to find the eigenfrequencies and eigenmodes of perturbations of the phase space distribution. In Section IV-A, scaled variables for longitudinal motion are defined, so that the stability of periodic systems can be described in terms of three dimensionless parameters (ψ_0, ψ and l/L, defined below). In Section IV-B we present results of the numerical analysis of the stability of a beam bunch in a periodic system, and describe conditions of instability in terms of the parameters of Section IV-A.

I. The Longitudinal Equation of Motion and the Self-Consistent Phase Space Distribution

In this paper we assume that the transverse (x-y) and longitudinal (z) motions of particles in the beam

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bunch are completely decoupled with the beam length much greater than the beam radius. We choose the longitudinal distance from the center of the bunch z and the position of the center of the bunch s as the dependent and independent variables.

Under these assumptions, the ions in the bunch experience a space charge force given, from Maxwell's equations,[3] by

$$F_z = -\frac{gq^2e^2}{\gamma^2}\frac{d\lambda}{dz}, \tag{1}$$

where $e = 4.8 \times 10^{-10}$ esu, q is the ion charge state, γ is the usual relativistic kinematic factor, λ is the number of ions per unit length, and g is a geometrical factor of order unity. For the particular case of an ion at the center of a constant transverse density round beam of radius a inside a round conducting beam pipe of radius b, $g = 1 + 2\ln(b/a)$. We assume that transverse variations of particle density and motion simply produce some average value of g which we treat as constant.

We simplify the discussion by assuming the particles are nonrelativistic ($\gamma = 1$) and by assuming that the center of the bunch is not accelerating but moves with constant speed βc and rewrite (1) as

$$\frac{d^2z}{ds^2} \equiv z'' = -\frac{q^2e^2g}{\beta^2c^2M}\frac{d\lambda}{dz} \equiv -A\frac{d\lambda}{dz}, \tag{2}$$

where M is the ion mass and the symbol (′) denotes differentiation with respect to s. This is a debunching force and tends to extend the bunch. We add a linear bunching force F_B by applying a linearly - ramped external electric field

$$E_z = \left(\frac{dE}{dz}\right) z \quad \text{so that} \quad F_B = qe\left(\frac{dE}{dz}\right) z.$$

We define a bunching parameter K by the equation

$$K = -\frac{qe}{M\beta^2c^2}\left(\frac{dE}{dz}\right)$$

and obtain the equation of motion

$$z'' = -A\frac{d\lambda}{dz} - Kz. \tag{3}$$

The Vlasov equation for the z–z' phase space distribution function $f(z,z',s)$ is:

$$\frac{\partial f}{\partial s} + z'\frac{\partial f}{\partial z} + \left(-Kz - A\frac{d\lambda}{dz}\right)\frac{\partial f}{\partial z'} = 0. \tag{4}$$

The solution of Eq. (4) must be self-consistent; that is, the charge density function $\lambda(z,s)$ must be given by

$$\lambda(z,s) = \int f(z,z',s)\, dz'. \tag{5}$$

In Ref. 2 we have derived a solution to (4) defined by

$$f(z,z',s) = \frac{3M}{2\pi\varepsilon_L}\sqrt{-\frac{z^2}{z_0^2} - \frac{z_0^2}{\varepsilon_L^2}\left(z' - \frac{z_0'z}{z_0}\right)^2} \tag{6}$$

wherever the argument of the square root is real ($f = 0$ otherwise). N is the total number of ions in the beam, and ε_L is a constant (the longitudinal emittance). z_0 and $z'_0 = dz_0/ds$ are found by integration of the envelope equation:

$$\frac{d^2z_0}{ds^2} = \frac{\varepsilon_L^2}{z_0^3} + \tfrac{3}{2}\frac{AN}{z_0^2} - K(s)z_0. \tag{7}$$

The initial conditions $z_0(s=0)$ and $z'_0(s=0)$ may be chosen arbitrarily and the bunching parameter $K(s)$ may be an arbitrary function of s.

The line charge density $\lambda(z,s)$ is found using Eq. (5) to be a parabolic function

$$\begin{aligned} \lambda(z,s) &= \tfrac{3}{4}\frac{N}{z_0}\left(1 - \frac{z^2}{z_0^2}\right) \qquad z < z_0 \\ &\equiv \lambda_0\left(1 - \frac{z^2}{z_0^2}\right) \\ \lambda(z,s) &= 0 \qquad z > z_0 \end{aligned} \tag{8}$$

The equation of motion of an individual particle in the distribution, from Ref. 3, is

$$z'' = \left(\frac{2A\lambda_0}{z_0^2} - K(s)\right) z \tag{9}$$

and is a linear force for all s.

This distribution function has an elliptical outer boundary in phase space where the argument of the square root is zero. The area of the ellipse is $\pi\varepsilon_L$, the longitudinal emittance, and ε_L remains constant. This distribution function [Eq. (6)] is similar to the K-V distribution in transverse space in that it is a self-consistent distribution with a linear space charge force, and it has a second order linear differential envelope equation to describe the motion of the distribution. Perturbations of this distribution are analyzed below to determine longitudinal stability.

For steady-state transport ($K(s)$ constant), a phase space distribution which is also constant can be used. This is found by choosing $z_0(s=0)$ such that $z'_0 = 0$ and setting $z'_0(s=0) = 0$. From Eq. (7) we require

$$\varepsilon_L^2 + \frac{3}{2} AN z_0 - Kz_0^4 = 0. \qquad (10)$$

Then $z''_0 = 0$ for all s. If we introduce parameters v and v_0, defined by $v = z'$ and $z_0 v_0 = \varepsilon_L$, we can rewrite the distribution function in a symmetric form

$$f(z,v) = \frac{3N}{2\pi v_0 z_0}\sqrt{1 - \frac{z^2}{z_0^2} - \frac{v^2}{v_0^2}}, \quad \frac{z^2}{z_0^2} + \frac{v^2}{v_0^2} < 1 \qquad (11)$$

$$f(z,v) = 0, \quad \frac{z^2}{z_0^2} + \frac{v^2}{v_0^2} > 1$$

In the next section we analyze the stability of perturbations of this stationary distribution.

II. Normal Modes of the Stationary Longitudinal Distribution

Our investigation of the longitudinal stability of the transport of high current particle beams begins with an analysis of perturbations of the standard stationary solution. The eigenmodes and eigenfrequencies of these perturbations are described in this section, and these characteristic eigenmodes are searched for instability.

The standard stationary distribution in longitudinal phase space is rewritten as

$$f_0(z,v) = \frac{3N}{2\pi v_0 z_0}\sqrt{1 - \frac{z^2}{z_0^2} - \frac{v^2}{v_0^2}}. \qquad (12)$$

The theta-function is used to indicate that the distribution f_0 is confined to the ellipitical region of phase space where the argument of the square root in f_0 is positive. We have also introduced an unnormalized function f_0^1 which contains the principal variation of f_0 for future calculation convenience.

The single particle equation of motion, (3), can be rewritten as

$$z'' = -Kz - A\frac{\partial\lambda}{\partial z} = -\left(K - \frac{3AN}{2z_0^3}\right) z$$
$$\equiv -(\nu_0^2 - \omega_p^2)z \equiv -\nu^2 z. \qquad (13)$$

This includes the external bunching force $Kz = \nu_0^2 z$, and we have defined a plasma frequency ω_p^2 associated with the space charge force $[3AN/2z_0^3]z$.

Oscillations of the standard distribution are described by adding a perturbation

$$f(z,v,s) = f_0(z,v) + f_p(z,v,s)$$

which in turn leads to a perturbed space charge field

$$A\frac{\partial\lambda}{\partial z} = A\frac{\partial\lambda_0}{\partial z} + A\frac{\partial\lambda_p}{\partial z}$$

In the analysis below we will use techniques previously developed by Sacherer[4] for analysis of a one-dimensional transverse distribution. The Vlasov equation is linearized about $f_0(z,v)$ to obtain

$$\frac{\partial f_p}{\partial s} + v\frac{\partial f_p}{\partial z} - \nu^2 z\frac{\partial f_p}{\partial v} = \frac{z_0\omega_p^2}{\pi v}\frac{\partial\lambda_p}{\partial z}\frac{\partial f_0^1}{\partial v}. \qquad (14)$$

We will use coordinates (r,ϕ) interchangeably with (z,v). These coordinates are defined by $z/z_0 = r\cos\phi$ and $v/v_0 = r\sin\phi$. We will search for normal mode solutions which are of the form

$$f_p(z,v,s) = e^{-i\omega s} f(z,v) = e^{-i\omega s} f(r,\phi),$$

where ω is an eigenfrequency to be determined. In these new coordinates, Eq. (14) becomes

$$e^{-i(\omega/v)\phi} \frac{d}{d\phi}\left[e^{i(\omega/v)\phi} f(r,\phi)\right] = \frac{\omega_p^2}{\pi v^3}\frac{\partial \lambda_p}{\partial z}\sin\phi\left(-\frac{\partial f_0^1}{\partial r}\right). \tag{15}$$

The unique solution of Eq. (15) which is periodic in ϕ is

$$f(r,\phi) = \frac{\omega_p^2}{\pi v^3}\frac{e^{-i(\omega/v)\phi}}{1-e^{-2\pi i(\omega/v)}} \times \int_{\phi-2\pi}^{\phi} e^{i(\omega/v)\phi}\frac{\partial \lambda_p}{\partial z_1}\sin\phi_1\, d\phi_1 \cdot \left(-\frac{\partial f_0^1}{\partial r}\right). \tag{16}$$

This may be rewritten in terms of a new integration parameter u as

$$f(z,v) = -\frac{1}{v_0 r}\frac{\partial f_0^1}{\partial r}\frac{\omega_p^2}{\pi v^2(e^{2\pi i(\omega/v)}-1)} \times \int_0^{2\pi} e^{i(\omega/v)u}\frac{\partial \lambda_p}{\partial z_1}\frac{v_1}{v}\, du,$$

where

$$u = \phi_1 - \phi + 2\pi$$

$$z_1 = z\cos u - v/v \sin u$$

$$v_1 = vz\sin u + v\cos u.$$

Our integro-differential equation for $\lambda_p(z)$ is

$$\lambda_p(z) = \int_{-\infty}^{\infty} -\frac{1}{v_0 r}\frac{\partial f_0^1}{\partial r}\frac{\omega_p^2}{\pi v^2(e^{2\pi i(\omega/v)}-1)} \left(\int_0^{2\pi} e^{i(\omega/v)u}\frac{\partial \lambda_p(z_1)}{\partial z_1}\frac{v_1}{v}\, du\right) dv \tag{17}$$

The solutions of Eq. (17) provide the eigenmodes and eigenfrequencies of the perturbations. Equation (17) is reducible to an integral equation by a series of manipulations somewhat similar to those used by Sacherer. The derivative of the unperturbed function is replaced by

$$\frac{\partial f_0^1}{\partial r} \cong -\frac{r}{\sqrt{1-r^2}}\,\theta(1-r^2). \tag{18}$$

We have eliminated the term in (18) proportional to the derivative of the theta function, since the multiplying factor $(1-r^2)^{1/2}$ is zero where this derivative is non-zero and the term makes no contribution to the integrations in (17). Another change of variables

$$v = v_0\left[\sqrt{1-\left(\frac{z}{z_0}\right)^2}\cos\eta\right],$$

$$\text{whence } dv = -v\sqrt{1-r^2}\, d\eta,$$

conveniently simplifies Eq. (17) and we can write

$$\lambda_p(z) = \int_0^{2\pi}\int_0^{2\pi}\frac{\omega_p^2}{2\pi v^2}\frac{e^{i(\omega/v)u}}{e^{2\pi i(\omega/v)}-1} \times \frac{\partial \lambda_p(z_1)}{\partial z_1}\frac{v'}{v}\, du\, d\eta$$

$$= \frac{1}{v^2+\omega_p^2}\frac{i\omega}{v}\int_0^{2\pi}\int_0^{2\pi}\frac{\varepsilon^{i(\omega/v)u}}{2\pi(e^{2\pi i(\omega/v)}-1)} \lambda_p(z_1)\, d\eta\, du, \tag{19}$$

where in the last step we have integrated by parts over u. In another change of variables we can write

$$z = z_0\cos\xi$$

and

$$z_1 = z_0\cos\psi = z_0(\cos\xi\cos u + \sin\xi\sin u\cos\eta).$$

We now assert that the solutions of Eq. (19) are Legendre polynomials

$$\lambda_p(z) = \lambda_n(z/z_0) = aP_n(\cos\xi). \tag{20}$$

This ansatz is demonstrated by inserting

$$\lambda_n(\cos\psi) = aP_n(\cos\psi) = \frac{4\pi}{(2n+1)} \cdot a\sum_m Y_{nm}(\xi,\eta)Y_{nm}(u,0) \tag{21}$$

into Eq. (19). The second substitution in Eq. (21) is an application of the addition theorem for spherical harmonics. Because of the integration over η, only the $m = 0$ term of the expansion in Eq. (21) contributes and we find that the solutions $\lambda_n(z/z_0)$ are indeed Legendre polynomials of order n greater than 0, provided that ω satisfies the eigenvalue condition

$$\nu^2 + \omega_p^2 = \frac{\omega_p^2}{(e^{2\pi i(\omega/\nu)} - 1)} \frac{i\omega}{\nu} \times \int_0^{2\pi} e^{i(\omega/\nu)u} P_n(\cos u)\, du. \tag{22}$$

Equation (22) can be reduced by use of the following two identities

$$P_n(\cos u) = \sum_{m=0}^{n} \frac{1}{4^n} \binom{2m}{m} \binom{2n-m}{n-m} \cdot \cos(n-2m)u \tag{23a}$$

and

$$\int_0^{2\pi} e^{i(\omega/\nu)} \cos mu\, du = -u \frac{\omega}{\nu}\left(e^{2\pi i(\omega/\nu)} - 1\right) \frac{\nu^2}{\omega^2 - m^2\nu^2} \tag{23b}$$

and we find the eigenvalue equation

$$\frac{\nu^2 + \omega_p^2}{\omega^2} = \omega_p^2 \sum_{m=0}^{n} \frac{1}{4^n} \binom{2m}{m} \binom{2n-m}{n-m} \times \frac{1}{\omega^2 - ((n-2m)\nu)^2}. \tag{24}$$

By grouping similar terms, (24) can be rewritten as

$$K_n(\omega) \equiv \left[\frac{(1\cdot 3\ldots 2n-1)}{(2\cdot 4\ldots 2n)} \left\{ \frac{2n^2\omega_p^2}{\omega^2 - n^2\nu^2} + \tfrac{1}{2} \frac{2n}{(2n-1)} \frac{2(n-2)^2\omega_p^2}{(\omega^2 - (n-2)^2\nu^2} + \frac{1\cdot 3\cdot(2n)\cdot(2n-2)}{2\cdot 4\cdot(2n-1)\cdot(2n-3)} \frac{2(n-4)^2\omega_p^2}{(\omega^2 - (n-4)^2\nu^2)} + \ldots \right\} \right] = 1. \tag{25}$$

The sum inside the parentheses is terminated at $n - 2m = 2$ if n is even, and at $n - 2m = 1$ if n is odd. Some of the lowest order eigenvalue equations are tabulated in Table 1 for convenience. We find that for even n there are $n/2$ eigenvalues of ω^2, and for odd n there are $(n + 1)/2$ eigenvalues of ω^2. The eigenfunctions $f_{ni}(z,v)$ can be found by substituting ω_{ni} and $\lambda_n(z/z_0)$ into Eq. (4) or (5) and integrating. We do not display these functions explicitly in this paper.

It can be seen by inspecting Eq. (23) or (24) that the eigenvalues ω_{ni}^2 are real and positive for all n and i when ω_p^2 has a value within the range $0 < \omega_p^2 < \nu_0^2$; that is, a normal, non-negative value. This indicates that perturbations are bounded. This is unlike the corresponding problem in two-dimensional (x-y) transverse space, where the K-V distribution shows instabilities when ω_p^2 is sufficiently close to ν_0^2.[6]

We state below the asymptotic values of ω_{ni}^2 in the limits of interest; that is (1) $\omega_p^2 \to 0$, the limit of zero space charge, and (2) $\omega_p^2 \to \nu_0^2$, the limit of maximum space charge

$$\begin{aligned} &(1) \qquad \omega_{ni}^2 \to ((n-2i)\nu)^2 \text{ as } \omega_p^2 \to 0 \\ &(2) \qquad \omega_{no}^2 \to (1 + 2 + \ldots + n)\omega_p^2 \\ &\qquad\qquad \omega_{ni}^2 \to 0 \qquad (i \neq 0) \\ &\qquad\qquad \text{as} \quad \omega_p^2 \to \nu_0^2 \end{aligned} \tag{26}$$

We conclude that the normal modes of oscillation of the standard stationary longitudinal distribution are stable, independent of current.

III. Stability of the Standard Longitudinal Distribution in a Periodic Transport System

In the previous section we derived the eigenfrequencies and eigenmodes of oscillation of the stationary standard longitudinal distribution. These

TABLE 1

Eigenvalue equations for $\lambda_n(z/z_0)$; $n \leq 6$.

n	Eigenvalue equation: $K_n(\omega) = 1$
1	$\dfrac{\omega^2}{\omega^2 - \nu^2} = 1$
2	$\dfrac{3\omega_p^2}{\omega^2 - (2\nu)^2} = 1$
3	$\dfrac{3\omega_p^2}{8(\omega^2 - \nu^2)} + \dfrac{45\omega_p^2}{8(\omega^2 - 9\nu^2)} = 1$
4	$\dfrac{5\omega_p^2}{4(\omega^2 - 4\nu^2)} + \dfrac{35\omega_p^2}{4(\omega^2 - 16\nu^2)} = 1$
5	$\dfrac{30}{128}\dfrac{\omega_p^2}{(\omega^2 - \nu^2)} + \dfrac{35 \cdot 9}{128}\dfrac{\omega_p^2}{(\omega^2 - 9\nu^2)} \quad \dfrac{63 \cdot 25}{128}\dfrac{\omega_p^2}{(\omega^2 - 25\nu^2)} = 1$
6	$\dfrac{105 \cdot 4}{512}\dfrac{\omega_p^2}{(\omega^2 - 4\nu^2)} + \dfrac{126 \cdot 16}{512}\dfrac{\omega_p^2}{(\omega^2 - 16\nu^2)} + \dfrac{213 \cdot 36}{512}\dfrac{\omega_p^2}{(\omega^2 - 36\nu^2)} = 1$

modes are found to undergo stable oscillations. In that analysis we required the external longitudinal bunching force to be constant throughout the accelerator transport system. A more accurate approximation would require the bunching force to be periodic in the accelerator, as will occur when bunching elements are alternated with other transport elements in the transport structure. The envelope equation (7) can be solved to find a periodic solution for the unperturbed longitudinal phase space distribution. In this paper we consider perturbations of this solution, and we derive coupled linear differential equations which can be used to find the eigenfrequencies of these perturbations. These equations can be integrated numerically and searched for regions of stability.

A. Solution of the Equations of Motion for the Periodic Solution. The constant bunching force used in Section II

$$\frac{d^2z}{ds^2} = -K_0 z = \left(\frac{qe}{M\beta^2c^2}\frac{dE}{dz}\right) z \tag{27}$$

is replaced by a periodic bunching force of period length L of the following form, as shown in Fig. 1A

$$K(s)z = K_0 z \qquad 0 < s < l$$

$$K(s) = 0 \qquad l < s < L$$

This corresponds to an accelerator composed of bunching sections of length l alternating with longitudinal drift sections of length L-l, which may contain transverse focusing elements.

In order to exhibit similarities between this work and corresponding analyses of transverse motion, we will use longitudinal beta functions β_z, α_z which correspond to the transverse Courant-Snyder functions. These are defined in terms of the beam envelope parameters z_0 and ε_L of Section I by

$$\beta_z = \frac{z_0^2}{\varepsilon_L} \qquad \text{and} \qquad \alpha_z = -\frac{\beta_z'}{2}. \tag{28}$$

The unperturbed self-consistent longitudinal phase

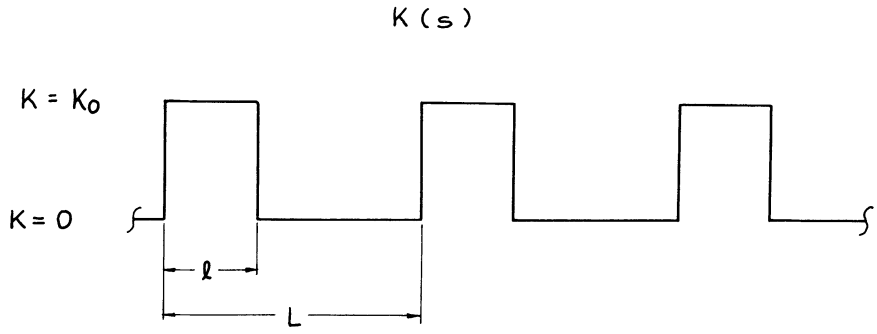

FIGURE 1A Bunching force $K(s)$ in a periodic longitudinal transport system.

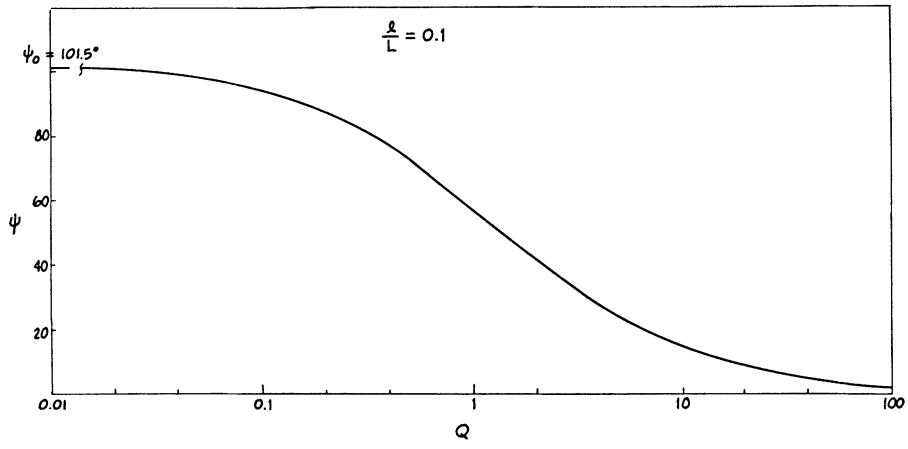

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FIGURE 1B The tune ψ as a function of the dimensionless space change parameter Q with $\psi_0 = 101.5°$ and $l/L = 0.1$.

space distribution

$$f_0(z,z',s) = \frac{3N}{2\pi\varepsilon_L} \cdot \sqrt{1 - \frac{z^2}{z_0^2} - \frac{z_0^2}{\varepsilon_L^2}\left(z' - \frac{z_0'}{z_0} z\right)^2} \tag{29}$$

is rewritten as

$$f_0(z,z',s) = \frac{3N}{2\pi\varepsilon_L^{3/2}} \cdot \sqrt{\varepsilon_L - \frac{z^2}{\beta_z} - \beta_z\left(z' + \frac{\alpha_z}{\beta_z} z\right)^2} \tag{30}$$

where $z_0 = (\varepsilon_L\beta_z)^{1/2}$ is the periodic solution of the evelope equation

$$z_0'' = \frac{\varepsilon_L}{z_0^3} + {}^3\!/_2 \frac{AN}{z_0^2} - K(s)z_0. \tag{31}$$

The equation of motion of an individual particle in the distribution is

$$z'' = -\left(K(s) - {}^3\!/_2 \frac{AN}{z_0^3}\right) z$$

$$\equiv -\left(K(s) - \frac{Q_0}{z_0^3}\right) z. \tag{32}$$

The Hamiltonian associated with this equation of motion is

$$H = \tfrac{1}{2}z'^2 + \tfrac{1}{2}\left(K(s) - \frac{Q_0}{z_0^3}\right)z^2. \tag{33}$$

Following L. Smith's analysis for transverse motion,[1] we change variables from z, z' to z, p using the generating function

$$\phi(z,p) = z\left(p - \frac{\alpha_z}{2\beta_z}z\right) \tag{34}$$

from which $\quad p = z' + \dfrac{\alpha_z}{\beta_z}z$

and

$$\bar{H} = H + \frac{\partial\phi}{\partial s} = \tfrac{1}{2}p^2 + \tfrac{1}{2}\frac{z^2}{\beta_z^2} - \frac{\alpha_z}{\beta_z}zp \tag{36}$$

and, precisely as in the transverse analysis, the solution of the equations of particle motion is

$$\begin{aligned} \frac{z(s)}{\sqrt{\beta_z(s)}} &= \frac{z(s_0)}{\sqrt{\beta_0}}\cos(\psi - \psi_0) \\ &\quad + p(s_0)\sqrt{\beta_0}\sin(\psi - \psi_0), \\ \sqrt{\beta_z(s)}\,p(s)\sqrt{\beta_0}\cos(\psi - \psi_0) &- \frac{z(s_0)}{\sqrt{\beta_0}}\cdot \\ &\sin(\psi - W_0), \end{aligned} \tag{37}$$

where $z(s_0)$, $p(s_0)$, $\beta_0 = \beta_z(s_0)$, $\psi_0 = \psi_z(s_0)$ are the initial particle coordinates and beta function values. The quantity ψ is the longitudinal phase advance defined in the same manner as for transverse space

$$\psi = \psi_z(s) = \int \frac{1}{\beta_z}\,ds = \int \frac{\varepsilon_l}{z_0^2}\,ds. \tag{38}$$

From Eq. (37), it follows that

$$R^2 = \frac{1}{\varepsilon_l}\left(\frac{z^2}{\beta_z} + \beta_z p^2\right) \tag{39}$$

is a constant of motion. Our unperturbed distribution function can be written in terms of this constant of motion as

$$\begin{aligned} f_0(z,z') = f_0(z,p) &= \frac{3N}{2\pi\varepsilon_L}\sqrt{1 - R^2} \\ &\cdot \theta(1 - R^2) \equiv \frac{3N}{2\pi\varepsilon_L}\bar{f}_0(R^2). \end{aligned} \tag{40}$$

B. Perturbations of the Periodic Solution. As in our analysis of the stationary solution we add a perturbed distribution $f_p(z,p,s)$ to $f_0(z,p)$ and search for self-consistent solutions of the Vlasov equation linearized about the unperturbed solution. The perturbed distribution adds a perturbed line charge density

$$\lambda_p(z,s) = \int f_p(z,p,s)\,dp \tag{41}$$

with a space charge potential

$$V_p = A\,\lambda_p(z,s) \tag{42}$$

The linearized Vlasov equation is

$$\begin{aligned} \frac{df_p}{ds} &= \nabla V\cdot\nabla_p f_0(z,p) = -\frac{3NA}{2\pi\varepsilon_l^2}\frac{\partial\lambda_p}{\partial z} \\ &\beta_z p_z \frac{1}{\sqrt{1 - R^2}}\cdot\theta(1 - R^2). \end{aligned} \tag{43}$$

In Eq. (43) we have ignored a term proportional to the derivative of the theta function (as in Section II) since it makes no contribution to the analysis below.

Our method of solution for eigenfrequencies is similar to that of L. Smith et al[1] for transverse oscillations. We choose eigenfunctions of the form

$$\begin{aligned} f_n(z,p,s) = \frac{\theta(1 - R^2)}{\sqrt{1 - R^2}}\cdot\Bigg\{&\sum_{i=0}^{n} C_i(s)\left(\frac{z}{\sqrt{\beta_z}}\right)^{n-i} \\ &(\sqrt{\beta_z}\,p_z)^i + \sum_{i=0}^{nh} D_n(s)\left(\frac{z}{\sqrt{\beta_z}}\right)^{n-2i} \\ &(\sqrt{\beta_z}\,p_z)^i + \ldots\Bigg\}. \end{aligned} \tag{44}$$

These eigenfunctions are characterized by an order n, which is the highest power of z appearing in $\lambda_n(z)$. In the above expansion the progression $C, D, E, \ldots$ in $n, n-2, \ldots, n-2m, \ldots$ terminates at $n-2m=1$ (n odd) or $n-2m=0$ (n even). Also in this progression $n, n-2, \ldots$, we include only odd or only even values of $n-2m$, since odd and even powers decouple completely.

To find the eigenfrequencies of $f_n(z,p,s)$ using Eq. (43) we need only retain the terms proportional to $z^{n-i}p^i$, that is, the highest order terms. The lower order terms will only repeat eignefrequencies obtained by equations of lower order in n.

Thus we write for our charge density

$$\lambda_{pn}(z) = \pi\left(\frac{z}{\sqrt{\beta_z}}\right)^n \frac{\varepsilon_L}{\sqrt{\beta_z}}\left\{C_0 - \tfrac{1}{2}C_2 + \tfrac{3}{8}C_4 + \ldots + (-1)^j \frac{(2j-1)!!}{(2j)!!} C_{2j} + \ldots\right\} + 0(z)^{n-2}. \tag{45}$$

The sum in j terminates at $2j = n$ (n even) or $2j = n - 1$ (n odd). We have used the relation

$$\int_{-|a|}^{+|a|} \frac{u^{2n}}{\sqrt{a^2-u^2}}\, du = \frac{\pi(2n-1)!!}{(2n)!!}\, a^{2n}. \tag{46}$$

Substitution of (44) into (43), using (45), gives us $(n+1)$ coupled linear differential equations for the $(n+1)$ functions $C_i(s)$. These linear differential equations are of the form

$$\frac{dC_0}{ds} = \frac{C_1}{\beta_z}$$

$$\frac{dC_1}{ds} = -\frac{n}{\beta_z}C_0 + \frac{2C_2}{\beta_z} - \frac{n}{\sqrt{\beta_z}}\, Q'\{C_0 - \tfrac{1}{2}C_2 + \tfrac{3}{8}C_4 + \ldots\}$$

$$\frac{dC_i}{ds} = -\frac{(n-i+1)}{\beta_z}C_{i-1} + \frac{(i+1)}{\beta_z}C_{i+1}$$

$$\vdots$$

$$\frac{dC_n}{ds} = -\frac{1}{\beta_z}C_{n-1}$$

and we define

$$Q' \equiv \tfrac{3}{2}\frac{AN}{\varepsilon_L^{3/2}} = \beta_z^{3/2}\frac{Q_0}{z_0^3}.$$

The differential equations for the lowest order perturbations ($n \leq 4$) are tabulated explicitly in Appendix I.

The equations for $C_0, C_1, \ldots, C_n$ are coupled linear equations with periodic coefficients. Their solutions are

$$C_i = e^{i\omega s}\, c_i(s), \qquad i = 0, \ldots, n$$

where c_i is a periodic function of s with the focusing period, and $e^{i\omega s}$ is a growth function. There are $(n+1)$ eigenvalues of ω and each eigenvalue is associated with a set of periodic functions $c_i(s)$ and the eigenfunctions $e^{i\omega s}$ occur in complex conjugate reciprocal pairs. If n is even, $\omega = 0$ is one of the eigenvalues.

The eigenvalues can be found by integrating the coupled linear equations over a period of the accelerator structure, using the periodic solution of the equation for β_z. A matrix can be formed by choosing an $(n+1)$ orthogonal set of initial values of $(C_0, C_1, \ldots, C_n)$ and integrating the differential equation over one period to find final values of $(C_0, C_1, \ldots C_n)$. An $(n+1)\times(n+1)$ transfer matrix A can be constructed so that

$$\underset{\text{final}}{\begin{pmatrix} C_0 \\ C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{pmatrix}} = A \times \underset{\text{initial}}{\begin{pmatrix} C_0 \\ C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{pmatrix}}$$

The eigenvalues of A provide the $(n+1)$ eigenfunctions $e^{i\omega s}$.

If ω is not real the perturbation grows in amplitude and the oscillation is unstable. In the next section we will follow the procedure outlined above to identify periodic systems with longitudinal instabilities.

The procedure outlined above can be used to find the eigenfrequencies and eigenmodes of oscillation of the stationary distribution by setting the focusing force $K(s)$ constant and the beta function β_z also constant. The resulting eigenvalues ω, as we derive above by a different method, are all real.

IV. Stability of the Standard Longitudinal Distribution in a Periodic Transport System—Numerical Analysis

In the previous section we derived coupled linear differential equations for perturbations of the standard distribution[3] which can be solved numerically in a particular transport system to determine the longitudinal stability of that system. In this section we define scaled variables so that the stability of a periodic transport system can be described in terms of three dimensionless variables: ψ_0, ψ, and l/L, and we present a summary of the numerical analysis of the stability of longitudinal transport in terms of these variables.

A. Scaled Variables. The envelope equation (31) can be set in a dimensionless form by the following change of variables, similar but not identical to the scaled transverse variables used by Lambertson et al.[8] We replace s by θ, with θ given by

$$\theta \equiv \sqrt{K_0}\, s.$$

We replace z_0 with u_z, where

$$u_z = K_0^{+1/4} \varepsilon_L^{-1/2}\, z_0$$

and we replace the constants in the space charge term by a scaled parameter Q with

$$Q \equiv {}^3\!/_2 \, \frac{AN}{(\varepsilon_L)^{3/2}(K_0)^{1/4}} \,. \tag{47}$$

In these coordinates the envelope equation becomes

$$\frac{d^2 u_z(\theta)}{d\theta^2} = -S(\theta) u_z + \frac{1}{u_z^3} + \frac{Q}{u_z^2}\,, \tag{48}$$

where

$$S(\theta) = \frac{K(s)}{K_0} = 1, \qquad 0 < s < l$$
$$S(\theta) = 0, \qquad l < s < L.$$

The variables l, L, and Q, completely determine the scaled motion. We can express these variables in terms of the longitudinal phase advance per period ψ, where

$$\psi = \int_0^L \frac{\varepsilon_L}{z_0^2}\, ds = \int_0^{\sqrt{K_0}L} \frac{d\theta}{u_z^2} \quad \text{in radians.} \tag{49}$$

In Eq. (49) we use the periodic solution of the envelope equation (31, or 48) to determine z_0 or u_z. Through the envelope equation ψ is a function of the space charge parameter Q. We define a phase advance at zero current ψ_0 by setting $Q = 0$. The variation of ψ with Q for a typical case is shown in Fig. 1B. The transport system can be described by these three variables: ψ_0, ψ, and l/L.

The special case where $K(s)$ is constant (or $l = L$) is the continuous focusing system analyzed in Section II. In this limit ($l \to L$) ψ_0 and ψ have the values

$$\psi_0 = \nu_0 L \quad and \quad \psi = \nu L,$$

where ν_0 and ν are the bunching frequencies with and without space charge defined in Section II. The scaled variables may be used to parameterize an equation for the total number of ions N, which is similar to the Maschke formula for the current I determined from the transverse stability requirement.[9] We replace dE/dz in K_0 with

$$\frac{dE}{dz} = \frac{E_{\max}}{z_{\max}} = \frac{E_{\max} K_0^{1/4}}{\varepsilon_L^{1/2} u_{\max}}\,,$$

where $E_{\max}$ is the maximum bunching field. We solve for N in Eq (47), obtaining

$$N = \frac{2M^{2/3}(\beta c)^{4/3}}{3(q^2 e^2 g)}\, \varepsilon_L^{4/3} (qe\, E_{\max})^{1/3}\, \frac{Q}{u_{\max}^{1/3}} \tag{50}$$

In this expression we have cgs units. This can be written as

$$N = 4.4 \times 10^{14} A^{2/3} q^{-5/3} \beta^{4/3} (E_{\max})^{1/3} (\varepsilon_L)^{4/3} Q(u_{\max})^{-1/3} g^{-1}, \qquad (51)$$

with A the ion atomic weight, q the ion charge, $E_{\max}$ in volts/m, ε_L in meter-radians, and Q and $u_{\max}$ are dimensionless. We have used non-relativistic kinematics ($\gamma = 1$) and the emittance ε_L is not normalized. However, unlike the transverse case, stability does not limit Q and $u_{\max}$ to values of order unity.

B. Eigenvalues of Perturbation Oscillations. In the previous section we derived $(n + 1)$ coupled linear differential equations for the $(n + 1)$ coefficients $C_n(z,p,s)$ of a particle distribution perturbation of order n, where n is simply the highest power of z appearing in the perturbed density $\lambda_p(z,s)$. These questions can be integrated over one period to form a transport matrix A and the eigenvalues of the matrix A provide the eigenfrequencies of the perturbations. There are $(n + 1)$ complex eigenvalues E_m of the form

$$E_m = \lambda_m e^{i\phi_m} \qquad m = \pm n, \pm n - 2, \ldots$$

and these eigenvalues occur in complex conjugate reciprocal pairs.

In the case where $K(s)$ is constant, λ_m is 1 for all m and the phase θ_m is simply $\theta_m = \nu_m L$ with asymptotic values given in Section II

$$\phi_m \to \pm n\nu_0 L, \pm(n-2)\nu_0 L, \ldots \qquad Q \to 0$$

$$\phi_m \to \pm\sqrt{1 + 2 + \ldots + n}\, \nu_0 L \qquad m = \pm n, \quad Q \to \infty$$

$$\to 0 \qquad \text{for all other } \phi_m,\ Q \to \infty.$$

For the periodic focusing case the eigenvalue phases have similar behavior, with the important difference that the phase must lie between $-180°$ and $+180°$ per period. Thus the eigenvalue phases ϕ_m have asymptotic values

$$\phi_m \to \pm n\psi_0, \pm(n-2)\psi_0, \ldots \qquad \text{modulo } 180°$$

as $\psi \to \psi_0$ or $Q \to 0$, and

$$\phi_m \to \pm\sqrt{1 + 2 + \ldots n}\, \psi_0 \qquad \text{modulo } 180°$$

$$\phi_m \to 0 \qquad (m = \pm n)$$

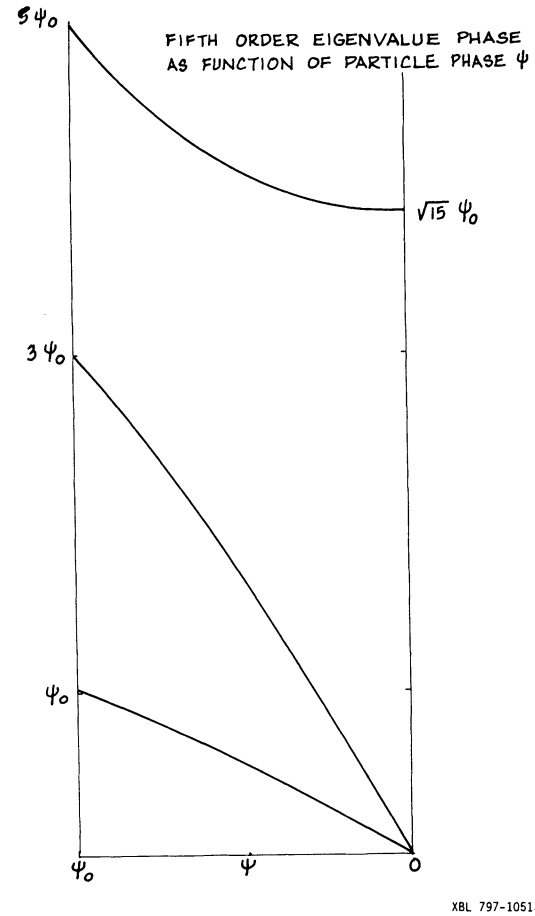

FIGURE 2A Positive phases ϕ of fifth order perturbation eigenvalues ($E_m = \lambda_m^{\pm i\phi_m}$) with $5\psi_0 \ll 180°$.

as $\psi \to 0$ or $Q \to \infty$. The difference between these two types of behavior is illustrated in Figs. 2A and 2B for a particular fifth-order set of phases (only the three positive phases are shown).

This difference is important in determining regions of possible instability, since the eigenvalues $\lambda_m e^{i\phi_m}$ occur in complex conjugate reciprocal pairs. Instability occurs in regions where $\lambda_m \neq 1$, and this can occur only where two different phases are equal or where a phase is 0° or 180°. In the fifth order case shown in Fig. 2B there are three possible regions with instabilities (which are circled) and the phase crossings (which in Fig. 2B are shown as point crossings) may stretch over finite regions in ψ (or Q). The fifth order case shown in Fig. 2B has been calculated in detail for $l/L = 0.5$ and all three areas are found to contain instabilities. The values of

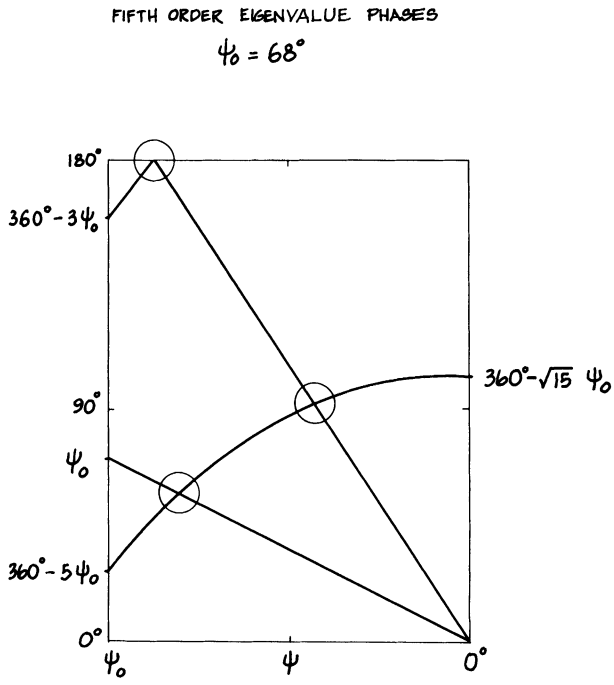

FIGURE 2B Positive phases ϕ of fifth order perturbation eigenvalues with $\psi_0 = 68°$.

$|\lambda_m|$ are found to lie in the range $0.995 < |\lambda_m| < 1.005$ and the instabilities stretch over ranges of $\Delta\psi/\psi_0 \lesssim 0.005$.

In Fig. 3A we show the fourth order phase diagram for $\psi_0 = 101.6°$, $l/L = 0.1$. The instabilities are somewhat larger; $|\lambda_m|_{\max} = 1.032$ and $\Delta\psi/\psi_0 = 0.03$ for the 0° instability at $\psi = 70°$. The sixth order phase diagram for the same case is shown in Fig. 3B and shows seven possible instabilities, all but the one labeled S are unstable; the values lie within: $0.983 < |\lambda_m| < 1.017$, $\Delta\psi/\psi_0 \lesssim 0.1$. This case ($\psi_0 = 101.6°$, $l/L = 0.1$) has three small fifth order instabilities with $|\lambda_m| < 1.01$ and $\Delta\psi/\psi_0 < 0.01$. However, there is a large second order instability also shown in Fig. 3A. For $\psi < 55°$, the second order perturbations (which are perturbations of the envelope boundary) are unstable with $|\lambda_m|_{\max} \cong 1.11$. In fact, for ψ_0 within the range $90° < \psi_0 < 104°$ large regions of phase advance ψ are unstable, indicating a serious space charge instability.

An instability similar to the above second order instability occurs in each order n in the region $180°/n < \psi_0 < 180°/(1 + 2 + \ldots + n)^{1/2}$. The instability is greatest for ψ_0 close to the upper boundary $(180°/(1 + 2 + \ldots + n)^{1/2}$ and for Q large. The corresponding fourth order instability occurs for $45° < \psi_0 < 56.9°$, and we find a growth $\lambda_{\max}$ of ~ 1.035 near 56°. The sixth order instability occurs for $30° < \psi_0 < 39.3°$. It has a maximum growth near 39° of 1.015. We find that as the order increases, the sizes of the growth function and of the regions of ψ and ψ_0 with instability rapidly decrease, so that we expect that only the lowest order instabilities of this type may be important.

We summarize some general conclusions on stability obtained by analyzing a number of cases with different values of l/L, ψ_0, and for $n \leq 8$.

1. Regions of instability occur whenever phase crossings occur for $n > 3$ (third order crossings are stable), unless $d\phi_i/d\psi$ and $d\phi_j/d\phi$ (where ϕ_i, ϕ_j are the crossing phases) have the same sign (as in Fig.

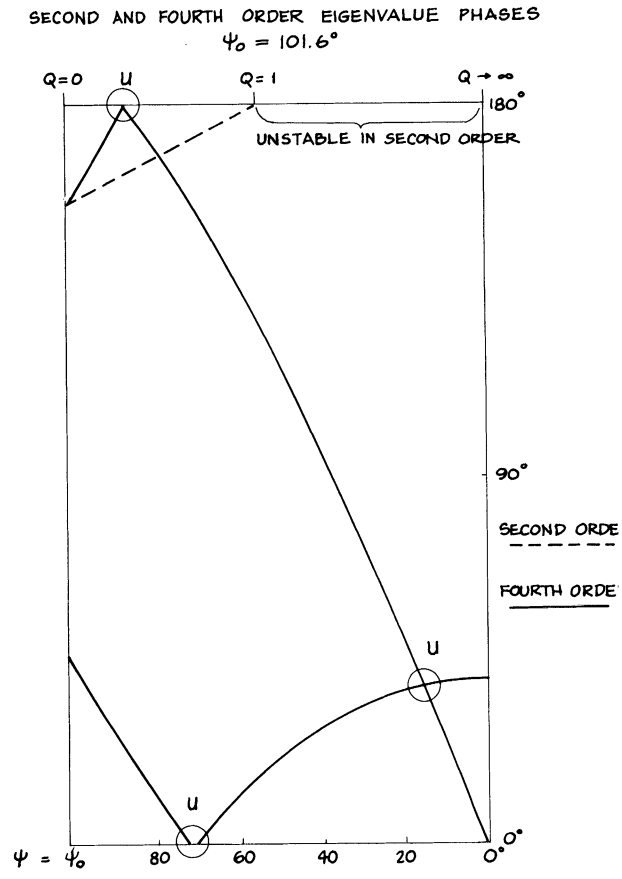

FIGURE 3A Positive phases of ϕ of second and fourth order perturbation eigenvalues with $\psi_0 = 101.6°$ and $l/L = 0.1$. Instability ($\lambda_m \neq 1$) occurs in all regions of phase crossings and where ϕ_m crosses 0° or 180°.

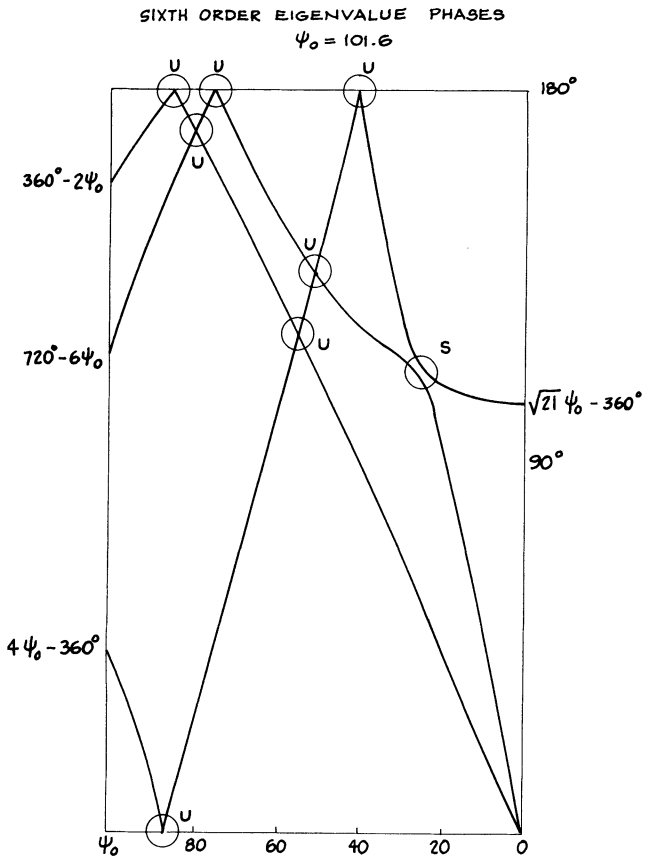

FIGURE 3B Positive phases ϕ of sixth order perturbation eigenvalues with $\psi_0 = 101.6°$ and $l/L = 0.1$. Instability occurs at all phase crossings labeled U, but does not occur at the phase encounter region labeled S.

3B, the case of stability). All 0° and 180° crossings are unstable if $n \neq 3$.

2. The size $(\Delta\psi/\psi_0)$ and amplitude $|\lambda_m|_{max}$ of these instability regions decrease with increasing order n. The size and amplitude also increase somewhat as l/L decreases at constant ψ_0; however, the instability is not a sensitive function of this parameter.

3. We do not expect instabilities of small size $(\Delta\psi/\psi_0 \lesssim 0.05)$ and small amplitude $(|\lambda|_{max} < 1.05)$ to be of significant importance in accelerator design. Our analysis indicates that only the second order instability which occurs for $90° < \psi_0 < 104°$ is large in these terms.

4. Currently existing or proposed accelerators have a rather small longitudinal phase advance per period ψ_0. The lowest order n at which an instability may occur is $n = 180°/\psi_0$, which would be large. Numerical analysis shows that as n increases the degree of instability becomes vanishingly small. However, superperiods of accelerator design should avoid resonances with the largest low order instabilities.

Our conclusion is that as long as ψ_0 is relatively small, space charge perturbations of the standard longitudinal distribution show no large instabilities.

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APPENDIX I

COUPLED DIFFERENTIAL EQUATIONS FOR LOWEST ORDERS (n = 1,2,3,4)

A: $n = 1$ (dipole mode)

$$\frac{dC_0}{ds} = \frac{C_1}{\beta_z}$$

$$\frac{dC_1}{ds} = -\frac{C_0}{\beta_z} - \frac{Q'}{\sqrt{\beta_z}}\, C_0$$

B: $n = 2$ (oscillations of envelope boundary)

$$\frac{dC_0}{ds} = \frac{C_1}{\beta_z}$$

$$\frac{dC_1}{ds} = -\frac{2C_0}{\beta_z} + \frac{2C_2}{\beta_z} - \frac{Q'}{\sqrt{\beta_z}}\,(2C_0 - C_2)$$

$$\frac{dC_2}{ds} = -\frac{C_1}{\beta_z}$$

C: $n = 3$

$$\frac{dC_0}{ds} = \frac{C_1}{\beta_z}$$

$$\frac{dC_1}{ds} = -\frac{3C_0}{\beta_z} + \frac{2C_2}{\beta_z} - \frac{Q'}{\sqrt{\beta_z}}\,(3C_0 - {}^3\!/_2\, C_2)$$

$$\frac{dC_2}{ds} \quad \frac{2C_1}{\beta_z} \quad \frac{3C_3}{\beta_z}$$

$$\frac{dC_0}{ds} = -\frac{C_1}{\beta_z}$$

D: $n = 4$

$$\frac{dC_1}{ds} = \frac{C_0}{\beta_z}$$

$$\frac{dC_2}{ds} = -\frac{4C_0}{\beta_z} + \frac{2C_2}{\beta_z} - \frac{Q'}{\sqrt{\beta_z}}\,\cdot$$

$$\cdot\ (4C_0 - 2C_2 + {}^3\!/_2\, C_4)$$

$$\frac{dC_2}{ds} = -\frac{3C_1}{\beta_z} + \frac{3C_3}{\beta_z}$$

$$\frac{dC_3}{ds} = -\frac{2C_2}{\beta_z} + \frac{4C_4}{\beta_z}$$

$$\frac{dC_4}{ds} = -\frac{C_3}{\beta_z}$$

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