

MOMENT METHOD IN CHARGED-PARTICLE BEAM DYNAMICS

N. YU. KAZARINOV, E. A. PERELSTEIN, and V. F. SHEVTSOV

Department of New Acceleration Methods, Joint Institute for Nuclear Research, Dubna, USSR

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I. INTRODUCTION

In the investigation of particle beams, the averaged characteristics, such as mean velocity, temperature, mean-square dimensions, etc., are often of practical interest.

The problem of the time dependence of the charged-particle beam mean-square dimensions (the envelopes) in electromagnetic fields was investigated by several authors.¹

In Refs. 2 and 3, the self-consistent envelope equations were obtained for a beam with elliptical cross section and a uniform charge density (Vladimirsky-Kapchinsky equations). It is characteristic for this model that the beam boundary does not rotate in configuration space. Later, these equations have been generalized for a beam with a charge density having elliptic symmetry.^{4,5}

In Ref. 6 a nonstationary self-consistent beam model was proposed which eliminated the restrictions of the Vladimirsky-Kapchinsky model and included it as a special case. The envelope equations obtained in that study represent a system of nonlinear differential matrix equations which makes their use more difficult.

Recently an equation was obtained for the variation with time of the mean-square radius of a beam in a longitudinal magnetic field for the case of azimuthal symmetry.⁷

By dealing with a distribution function which is non-zero in a finite region of the phase space or is integrated with arbitrary powers of phase variables, it is possible to introduce full moments of the distribution function, i.e., moments for all phase variables. In Ref. 8, in the case of relativistic motion in arbitrary electromagnetic fields of bunched charged particle beams of a low density, the hierarchy of equations for the moments of

different order was obtained. If the electromagnetic fields are linear, this hierarchy degenerates into independent subsystems for moments of one order.

In this paper, on the basis of the method proposed in Ref. 8, a system of equations for mean-square dimensions of charged particle beams in the linear electromagnetic fields is derived. The influence of space charge of the beam is taken into account. Effects connected with nonlinearity of the forces acting on the particles are also considered.

II. EQUATIONS OF MOTION

As in Ref. 8, we will obtain equations for the mean-square beam dimensions in linear external electromagnetic fields. For nonrelativistic particle motion, the equations of motion can be expressed in the following form:

$$\begin{aligned} \frac{dx_i}{dt} &= v_i \\ \frac{dv_i}{dt} &= \sum_{k=1}^3 [b_{ik}(t)x_k + a_{ik}(t)v_k] \\ &+ b_i^0(t) + F_i^s(\mathbf{x}, t), \quad i = 1, 2, 3 \end{aligned} \quad (1)$$

where x_i , v_i are the particle coordinates and velocities in a rectangular coordinate system, b_{ik} , a_{ik} are square matrices; and $F^s(\mathbf{x}, t)$ is the Lorentz force of the beam electromagnetic field.

For simplification we will express the system (1) in matrix form. Let us define a vector Y in 6-

dimensional phase space as

$$Y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \equiv \begin{pmatrix} X \\ V \end{pmatrix}, \quad (2)$$

where X , V , are vectors of particle coordinates and velocities.

According to the system (1), the vector Y satisfies the matrix equation

$$\frac{dY}{dt} = A(t)Y + B_0(t) + F_s, \quad (3)$$

$$A(t) = \begin{pmatrix} 0 & I \\ b(t) & a(t) \end{pmatrix}; \quad B_0 = \begin{pmatrix} 0 \\ b_0 \end{pmatrix}.$$

Here $A(t)$ is a block matrix constructed from the matrices a and b , I is the unit matrix, the notation of B_0 is evident, and the vector F_s is constructed from the components of the Lorentz force.

$$F_s = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F_1^s \\ F_2^s \\ F_3^s \end{pmatrix} \equiv \begin{pmatrix} 0 \\ F \end{pmatrix}. \quad (4)$$

If collisions are neglected, the distribution function f satisfies the Liouville equation

$$\frac{df}{dt} = 0. \quad (5)$$

Let us define the moments of the distribution function over the total set of phase coordinates as

$$\begin{aligned} M^0 &= \int f dy; \\ M^I &= \bar{Y} = \frac{1}{M^0} \int Yf dy, \\ M^{II} &= \overline{(Y - \bar{Y})(Y - \bar{Y})^*} \\ &= \frac{1}{M^0} \int (Y - \bar{Y})(Y - \bar{Y})^* f dy, \end{aligned} \quad (6)$$

where the symbol “*” denotes the transposed

matrix. Integration in (6) is performed relative to the phase volume occupied by particles ($dy = dx dv$).

Let us form a block matrix from the second order moments:

$$M^{II} = \begin{pmatrix} M_{xx} & M_{xv} \\ M_{xv}^* & M_{vv} \end{pmatrix}. \quad (7)$$

The elements of this block matrix are square matrices. Thus, for example, M_{xx} is the symmetric matrix of the mean-square dimensions of the beam:

$$M_{xx} = \overline{(X - \bar{X})(X - \bar{X})^*}. \quad (8)$$

From Eqs. (3), (5) we get the system of equations for the moments (6). That is,

$$\begin{aligned} \frac{dM^0}{dt} &= 0; \\ \frac{dM^I}{dt} &= AM^I + B_0 \\ \frac{dM^{II}}{dt} &= AM^{II} + M^{II}A^* \\ &\quad + \overline{F_s(Y - \bar{Y})^*} + \overline{(Y - \bar{Y})F_s^*}. \end{aligned} \quad (9)$$

To derive Eq. (9) we take into account the consequence of Newton's third law, i.e.,

$$\bar{F}_s = 0. \quad (10)$$

By using the definition of the matrix A and Eq. (9), it is easy to obtain equations for the elements of the matrix M^{II} .

$$\frac{dM_{xx}}{dt} = M_{xv} + M_{xv}^* \quad (11a)$$

$$\begin{aligned} \frac{dM_{vv}}{dt} &= bM_{xv} + M_{xv}^*b^* + aM_{vv} \\ &\quad + M_{vv}a^* + F_{sv} + F_{sv}^* \end{aligned} \quad (11b)$$

$$\frac{dM_{xv}}{dt} = M_{vv} + M_{xx}b^* + F_{sx}^* + M_{xv}a^*, \quad (11c)$$

where the matrices F_{sx} , F_{sv} are defined,

according to Eqs. (2), (4) as

$$\begin{aligned} \overline{(Y - \bar{Y})F_s^*} &\equiv \begin{pmatrix} 0 & F_{sx}^* \\ 0 & F_{sv}^* \end{pmatrix} \\ &= \begin{pmatrix} 0 & \overline{(X - \bar{X})F^*} \\ 0 & \overline{(V - \bar{V})F^*} \end{pmatrix}. \end{aligned} \quad (12)$$

The system (11) will be closed if the elements of the matrices F_{sx} , F_{sv} are expressed in terms of second-order moments. If the self-fields are linear, these expressions are found easily⁶ and give the dependence of the matrix b elements on the mean-square dimensions. The problem of self-fields will be discussed in detail in Section II.

Let us consider for simplicity the case of a beam with negligibly small self-fields. We introduce a matrix J defined as

$$J = M_{xv} - M_{xv}^*. \quad (13)$$

The elements of this matrix are connected with the mean angular momentum

$$\begin{aligned} \mathbf{j} &= \frac{1}{M^0} \int [(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{v} - \bar{\mathbf{v}})] f dy \\ J &= \begin{pmatrix} 0 & j_3 & -j_2 \\ -j_3 & 0 & j_1 \\ j_2 & -j_1 & 0 \end{pmatrix}. \end{aligned} \quad (14)$$

To eliminate the matrix M_{xv} from the system (11), we differentiate Eq. (11a) with respect to time.

$$\begin{aligned} \frac{d^2 M_{xx}}{dt^2} &= 2M_{vv} + bM_{xx} + M_{xx}b^* \\ &+ \frac{1}{2} \left(a \frac{dM_{xx}}{dt} + \frac{dM_{xx}}{dt} a^* \right) \\ &+ \frac{1}{2}(aJ^* + Ja^*). \end{aligned} \quad (15)$$

By differentiating Eq. (15) and using Eq. (11b), we get

$$\begin{aligned} \frac{d^3 M_{xx}}{dt^3} &= \left(2b - \frac{1}{4}a^2 + \frac{1}{2} \frac{da}{dt} \right) \frac{dM_{xx}}{dt} \\ &+ \frac{dM_{xx}}{dt} \left(2b^* - \frac{1}{4}a^{*2} + \frac{1}{2} \frac{da^*}{dt} \right) \end{aligned}$$

$$\begin{aligned} & - \frac{3}{2} a \frac{dM_{xx}}{dt} a^* \\ & + \frac{3}{2} \left(a \frac{d^2 M_{xx}}{dt^2} + \frac{d^2 M_{xx}}{dt^2} a^* \right) \\ & + \left(\frac{db}{dt} - \frac{1}{2} ab \right) M_{xx} \\ & + M_{xx} \left(\frac{db^*}{dt} - \frac{1}{2} b^* a^* \right) \\ & - \frac{3}{2} a M_{xx} b^* - \frac{3}{2} b M_{xx} a^* \\ & + \left(b + \frac{1}{4} a^2 - \frac{1}{2} \frac{da}{dt} \right) J \\ & + J^* \left(b^* + \frac{1}{4} a^{*2} - \frac{1}{2} \frac{da^*}{dt} \right). \end{aligned} \quad (16)$$

As can be concluded from Eqs. (11c) and (14), the matrix J satisfies the equation:

$$\begin{aligned} \frac{dJ}{dt} &= M_{xx}b^* - bM_{xx} \\ &+ \frac{1}{2} \left(\frac{dM_{xx}}{dt} a^* - a \frac{dM_{xx}}{dt} \right) \\ &+ \frac{1}{2}(Ja^* - aJ^*) \end{aligned} \quad (17)$$

The systems (16), (17), in conjunction with initial conditions for the moments and Eqs. (8), (11), taken at the initial moment of the time, solve the problem.

Note that in the case of low particle density (when the self-fields of the bunch may be neglected), Eqs. (16), (17) are linear and do not depend on the distribution-function form. The same result was obtained in the one-dimensional case in Ref. 8. In the two-dimensional case (infinite in the beam direction) these equations are valid in the presence of the beam self-fields for a wide class of distribution functions f . For linear self-fields solutions of Eqs. (16), (17) give self-consistent oscillations of the beam.

III. TRANSVERSE MOTION

When the motions are separated, beam infinite in one direction (x_3 axis) can be considered, and the moments (6) for the transverse motion can

be constructed. The matrices M_{xx} and J entering the system (16, 17) will then be square matrices of the second order and the derivative with respect to time is understood as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_0 \frac{\partial}{\partial x_3}, \quad (18)$$

where v_0 is the longitudinal particle velocity. The transverse particle motion is assumed to be non-relativistic.

Let us discuss the problem involving oscillations of mean-square beam dimensions in a static potential

$$U(x_1, x_2) = \frac{b_1}{2} x_1^2 + b x_1 x_2 + \frac{b_2}{2} x_2^2 \quad (19)$$

and in a stationary longitudinal magnetic field

$$\mathbf{H} = (0, 0, H_3). \quad (20)$$

The matrices a and b are expressed in the form

$$a = \alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad b = \begin{pmatrix} b_1 & b \\ b & b_2 \end{pmatrix}. \quad (21)$$

The parameter α is equal to the cyclotron frequency

$$\alpha = 2\omega_L = \frac{eH_3}{mc\gamma_0}, \quad (22)$$

where e and m are the particle charge and mass, respectively, c is the speed of the light in vacuum, and γ_0 is the relativistic factor of longitudinal motion.

Equations (16, 17) are invariant with respect to a rotational transformation which diagonalizes the matrix b . The matrix a does not change under the transformation. For this reason, the matrix b may be considered to be diagonal without limiting the general nature of the problem. That is,

$$b = - \begin{pmatrix} v_1^2 & 0 \\ 0 & v_2^2 \end{pmatrix}. \quad (23)$$

Here the elements v_1 and v_2 are the normal oscillation frequencies in the potential (19).

In this case, the components of matrices M_{xx} and J satisfy the equations

$$\begin{aligned} \frac{d^3 M_{xx}^{11}}{dt^3} &= \ddot{g}_1 = - \left(4v_1^2 - \frac{\alpha^2}{2} \right) \dot{g}_1 \\ &- \frac{3}{2} \alpha^2 \dot{g}_2 + \alpha(v_2^2 + 3v_1^2)g + 3\alpha\ddot{g} \end{aligned}$$

$$\begin{aligned} \frac{d^3 M_{xx}^{22}}{dt^3} &= \ddot{g}_2 = - \left(4v_2^2 - \frac{\alpha^2}{2} \right) \dot{g}_2 \\ &- \frac{3}{2} \alpha^2 \dot{g}_1 - \alpha(v_1^2 + 3v_2^2)g - 3\alpha\ddot{g} \\ \frac{d^3 M_{xx}^{12}}{dt^3} &= \frac{d^3 M_{xx}^{21}}{dt^3} = \ddot{g} = - 2(v_1^2 + v_2^2 - \alpha^2) \dot{g} \\ &+ 2\alpha(v_2^2 g_2 - v_1^2 g_1) \\ &+ \frac{3}{2} \alpha(\ddot{g}_2 - \ddot{g}_1) - (v_1^2 - v_2^2)J \\ \frac{dJ_{12}}{dt} &= - \frac{dJ_{21}}{dt} = \dot{J} = (v_1^2 - v_2^2)g \\ &- \frac{1}{2} \alpha(\dot{g}_1 + \dot{g}_2). \end{aligned} \quad (24)$$

The oscillation frequencies ω of the system (24) are determined from the equation

$$\begin{aligned} \omega^2[\omega^4 - 4\omega^2(v_1^2 + v_2^2 + \alpha^2) + 16v_1^2 v_2^2] \\ [(\omega^2 - v_1^2 - v_2^2 - \alpha^2)^2 - 4v_1^2 v_2^2] = 0. \end{aligned} \quad (25)$$

The solutions of Eq. (25) are real for real values of the parameters v_1 , v_2 , α . Consequently, the system (24) determines the stable oscillations of the mean-square dimensions of the beam. As will be shown below, degeneration of the characteristic frequencies in the case $v_1 = v_2 = 0$ does not cause a linear increase of the oscillation amplitude. Note that the first of the factors in the square brackets generates double oscillation frequencies of a particle in the fields (19, 20).

The self-field is easily introduced into consideration in the case of beam with elliptic cross section and uniform charge density (self-fields are linear). Then the elements of matrices F_{sx} and F_{sv} entering Eqs. (11) are expressed in terms of second-order moments. These expressions are valid for a wider class of distribution functions f .

Let us define the distribution function f as

$$\begin{aligned} f &= \frac{N}{\pi^2 |M''|^{1/2}} \\ &\times \Phi'[(Y - \bar{Y})^*(M'')^{-1}(Y - \bar{Y})], \end{aligned} \quad (26)$$

where N is the number of particles per unit beam length, $(M'')^{-1}$ is the inverse of the matrix M'' , $|M''|$ is the determinant of the matrix M'' , a prime denotes differentiation with respect to the argument, and Φ is an arbitrary function

satisfying the normalization condition:

$$\int_0^\infty \Phi(x) dx = 1. \quad (27)$$

Obtaining densities of charge ρ and current i as in Ref. 6, we get according to Eq. (26)

$$\begin{aligned} \rho &= \int f dv_1 dv_2 \\ &= \frac{eN}{\pi |M_{xx}|^{1/2}} \Phi[(X - \bar{X})M_{xx}^{-1}(X - \bar{X})] \end{aligned} \quad (28a)$$

$$\begin{aligned} i &= \int (V - \bar{V})f dv_1 dv_2 \\ &= M_{xv}^* M_{xx}^{-1}(X - \bar{X})\rho. \end{aligned} \quad (28b)$$

In accordance with Eq. (28a), the charge density ρ at a fixed time is constant on ellipses with centers at the point (\bar{x}_1, \bar{x}_2) and rotated about the axis x_1 by some angle φ . Let us choose a new system of coordinates (x_1', x_2')

$$x_1 = \bar{x}_1 + x_1' \cos \varphi - x_2' \sin \varphi \quad (29)$$

$$x_2 = \bar{x}_2 + x_1' \sin \varphi + x_2' \cos \varphi.$$

In the primed system of coordinates the charge-density ρ is expressed in the form

$$\begin{aligned} \rho &= \frac{eN}{\pi |M_{xx}|^{1/2}} \Phi \left(\frac{x_1'^2}{d_1^2} + \frac{x_2'^2}{d_2^2} \right) \\ &= \rho \left(\frac{x_1'^2}{d_1^2} + \frac{x_2'^2}{d_2^2}, t \right), \end{aligned} \quad (30)$$

where $d_{1,2}^2$ are the eigenvalues of the matrix M_{xx} .

The case of beams with charge density (30) has been considered in Ref. 5. It can be shown^{4,5} that for the self electrical field E' of beams with such density a valid relation which does not depend on the functional form of ρ is

$$\overline{x_1' E_1'} = eN \frac{d_1}{d_1 + d_2}; \quad (31)$$

$$\overline{x_2' E_2'} = eN \frac{d_2}{d_1 + d_2}.$$

In accordance with the symmetry of the system

$$\overline{x_1' E_2'} = \overline{x_2' E_1'} = 0. \quad (32)$$

Turning back to the initial system of coordinates, by using formulae (31, 32), we get an expression for the matrix F_{sx}

$$\begin{aligned} F_{sx} &= F_{sx}^* = \frac{\nu c^2}{\gamma_0^3} \frac{M_{xx}^{1/2}}{Sp M_{xx}^{1/2}} \\ &= b^s M_{xx} = M_{xx} b^{s*} \quad (33) \\ b^s &= \frac{\nu c^2}{\gamma_0^3} \frac{M_{xx}^{-1/2}}{Sp M_{xx}^{1/2}}. \end{aligned}$$

In formula (33) the following designations have been used: $\nu = Nr_0$ is the Budker parameter, $r_0 = (e^2/mc^2)$, $Sp M_{xx}^{1/2}$ is the trace of the matrix $M_{xx}^{1/2}$, and the matrix $M_{xx}^{1/2}$ is

$$M_{xx}^{1/2} \cdot M_{xx}^{1/2} = M_{xx}. \quad (34)$$

The expression for the matrix b^s agrees with that obtained in Ref. 6 in the case of linear self-fields.

By using formulas (28b) and (33) we find the dependence of matrix F_{sv} elements on the second-order moments

$$F_{sv}^* = M_{xv}^* M_{xx}^{-1} \overline{(X - \bar{X}) F^*} = M_{xv}^* b^{s*}. \quad (35)$$

This result may also be obtained by using the linear approximation of the Lorentz force proposed in Ref. 5

$$F = \epsilon(X - \bar{X}), \quad (36)$$

where the matrix ϵ is a solution to the problem of minimizing the following functionals at a fixed time:

$$\begin{aligned} D_1 &= \frac{1}{N} \int [\epsilon_{11}(x_1 - \bar{x}_1) + \epsilon_{12}(x_2 - \bar{x}_2) \\ &\quad - F_1^s(\mathbf{x}, t)]^2 f d\mathbf{x} d\mathbf{v} = \min \end{aligned} \quad (37)$$

$$\begin{aligned} D_2 &= \frac{1}{N} \int [\epsilon_{21}(x_1 - \bar{x}_1) + \epsilon_{22}(x_2 - \bar{x}_2) \\ &\quad - F_2^s(\mathbf{x}, t)]^2 f d\mathbf{x} d\mathbf{v} = \min. \end{aligned}$$

It is straightforward to show that

$$\epsilon = F_{sx} M_{xx}^{-1} = b^s. \quad (38)$$

By using formulae (36) and (38), an expression for matrix F_{sv} is easily found and coincides with formula (35).

It follows from (33), (35) the influence of the self-fields leads to the dependence of the matrix b elements on the second-order moments:

$$b = b^s + b^{ext}, \quad (39)$$

where the matrix b^{ext} is defined by the external electric field (19).

Thus the system of equations of the second-order moments (11) with definition of the matrix F_{sx} and F_{sv} in accordance with formulae (33), (35) does not depend on the distribution function form in the class of functions (26). Consequently, the equations for mean-square dimensions (16, 17) with the formula (39) yield *self-consistent* oscillations of the beam dimensions for a wide class of distribution functions.

The important characteristic of the system (11) is the determinant D of the second-order moment matrix M'' . The quantity defines the effective phase volume of the beam. It can be shown⁶ that determinant D changes in time as

$$D = D_0 \exp 2 \int_0^t Spa dt', \quad (40)$$

where D_0 is the initial phase volume. In the absence of energy dissipation ($Spa = 0$) the effective phase volume is an integral of the motion. Thus the restriction of the distribution function class in the form (26) leads to conservation of the beam phase volume. An analogous result in the case of azimuthally symmetric beams has been obtained in Ref. 7.

IV. EXTERNAL FIELDS

In this section we will consider some consequences of Eqs. (16, 17) in the special case of external electromagnetic fields.

In the absence of an external longitudinal magnetic field ($\alpha = 0$), Eqs. (24) are simplified and have solutions

$$g_1 = g_1^0 + \frac{\ddot{g}_1^0}{4\nu_1^2} - \frac{\ddot{g}_1^0}{4\nu_1^2} \cos 2\nu_1 t + \frac{\dot{g}_1^0}{2\nu_1} \sin 2\nu_1 t$$

$$g_2 = g_2^0 + \frac{\ddot{g}_2^0}{4\nu_2^2} - \frac{\ddot{g}_2^0}{4\nu_2^2} \cos 2\nu_2 t + \frac{\dot{g}_2^0}{2\nu_2} \sin 2\nu_2 t$$

$$\begin{aligned} g = & \frac{1}{4\nu_1\nu_2} \{ -[\ddot{g}^0 + (\nu_1 - \nu_2)^2 g^0] \cos(\nu_1 + \nu_2)t \\ & + [(\nu_1 + \nu_2)\dot{g}^0 + (\nu_1 - \nu_2)J^0] \\ & \times \sin(\nu_1 + \nu_2)t + [\ddot{g}^0 + (\nu_1 + \nu_2)^2 g^0] \\ & \times \cos(\nu_1 - \nu_2)t - [(\nu_1 - \nu_2)\dot{g}^0 \\ & + (\nu_1 + \nu_2)J^0] \sin(\nu_1 - \nu_2)t \} \end{aligned} \quad (41)$$

$$\begin{aligned} J = & -\frac{\nu_1 - \nu_2}{4\nu_1\nu_2} \{ [(\nu_1 + \nu_2)\dot{g}^0 + (\nu_1 - \nu_2)J^0] \\ & \times \cos(\nu_1 + \nu_2)t + [(\nu_1 - \nu_2)^2 g^0 + \ddot{g}^0] \\ & \times \sin(\nu_1 + \nu_2)t + \frac{\nu_1 + \nu_2}{4\nu_1\nu_2} \{ [(\nu_1 - \nu_2)\dot{g}^0 \\ & + (\nu_1 + \nu_2)J^0] \cos(\nu_1 - \nu_2)t \\ & + [\ddot{g}^0 + (\nu_1 + \nu_2)^2 g^0] \sin(\nu_1 + \nu_2)t \}. \end{aligned}$$

In formulae (41) and below, the index "0" denotes the initial value of the moments and its derivatives.

The diagonal matrix elements g_1, g_2 vary with time according to a harmonic law with frequencies $\omega_{1,2} = 2\nu_{1,2}$ and solutions coincide in form with the general solution of the Vladimirsky-Kapchinsky equation.³

The variation of element g and mean moment J with time depends on the rotations of the elliptic beam cross-section with respect to the coordinate axes. These rotations represent oscillations with combined frequencies $\omega_{3,4} = \nu_1 \pm \nu_2$. Indeed, the matrix M_{xx} has the form in the case of a beam with uniform charge density and elliptic cross section

$$M_{xx} = \frac{1}{4} \begin{pmatrix} r_1^2 \cos^2 \varphi + r_2^2 \sin^2 \varphi \times \\ \times \frac{1}{2} (r_1^2 - r_2^2) \sin 2\varphi \\ (r_1^2 - r_2^2) \sin 2\varphi \times \\ \times r_1^2 \sin^2 \varphi + r_2^2 \cos^2 \varphi \end{pmatrix}, \quad (42)$$

where $r_{1,2}$ are the half-axes of the ellipse, and φ is the angle between the half-axes r_1 and axis x_1 . The absence of rotation ($\varphi = 0$) means $g \equiv 0$.

In contrast to Ref. 3, the general solution of Eqs. (24) in the case of $\alpha = 0$ yields not only the

oscillations of the half-axes of the elliptic beam boundary but also its rotation in the plane (x_1, x_2) .

Let us assume that the ellipse does not rotate ($g \equiv 0$). It is straightforward to show that Eqs. (16, 17) are equivalent to the Vladimirsky-Kapchinsky equations. Let's consider for example the element g_1 . According to (16), it obeys the equation

$$\frac{d^3 g_1}{dt^3} + 4v_1^2 \frac{dg_1}{dt} + 2g_1 \frac{dv_1^2}{dt} = 0. \quad (43)$$

Eq. (43) coincides with the equation obtained in Ref. 8 in the one-dimensional case.

Let us transform the dependent variable

$$g_1 = \frac{1}{4} r_1^2. \quad (44)$$

Then

$$\frac{d}{dt} \left[r_1^3 \left(\frac{d^2 r_1}{dt^2} + v_1^2 r_1 \right) \right] = 0, \quad (45)$$

or

$$\frac{d^2 r_1}{dt^2} + v_1^2 r_1 = \frac{E_1^2}{r_1^3}. \quad (46)$$

The constant E_1 is determined from the initial conditions for the moments and Eqs. (8), (11), taken at the initial time.

$$E_1^2 = 16 \left\{ \overline{(v_1 - \bar{v}_1)^2 (x_1 - \bar{x}_1)^2} - \overline{[(x_1 - \bar{x}_1)(v_1 - \bar{v}_1)]^2} \right\} \quad (46.1)$$

According to Schwarz's inequality, the quantity in brackets is positive. It is straightforward to show that

$$\frac{dE_1^2}{dt} = 0. \quad (47)$$

The constant E_1 for an arbitrary distribution function is the analog of the phase volume contained in the Vladimirsky-Kapchinsky equations. An expression for the square of the phase volume which is analogous to formula (46.1) was obtained in Refs. 4 and 5.

Considering the effect of the self-fields of the beam (39), we get the Vladimirsky-Kapchinsky equations

$$\begin{aligned} \frac{d^2 r_1}{dt^2} + v_1^2 r_1 - \frac{E_1^2}{r_1^3} - \frac{4vc^2}{\gamma_0^3} \frac{1}{r_1 + r_2} &= 0 \\ \frac{d^2 r_2}{dt^2} + v_2^2 r_2 - \frac{E_2^2}{r_2^3} - \frac{4vc^2}{\gamma_0^3} \frac{1}{r_1 + r_2} &= 0. \end{aligned} \quad (48)$$

Now let us discuss oscillations of the beam mean-square dimensions in a longitudinal magnetic field (matrix $b = 0$). In this case, in accordance with Eq. (25), the oscillation frequencies are degenerate, but this does not cause any increase in oscillation amplitude. The general solution of the system (24) is

$$\begin{aligned} g_1 &= \left[\dot{g}_1^0 + \frac{\ddot{g}_1^0}{4\alpha^2} + \frac{3}{4} \left(\frac{\ddot{g}_2^0}{\alpha^2} + \frac{2}{\alpha} \dot{g}^0 \right) \right] \\ &+ \left[\frac{3}{2\alpha} \dot{g}_1^0 - \frac{1}{2\alpha} \dot{g}_2^0 + \frac{\dot{g}^0}{\alpha^2} \right] \sin \alpha t \\ &- \left[\frac{2}{\alpha} \dot{g}^0 + \frac{\ddot{g}_2^0}{\alpha^2} \right] \cos \alpha t \\ &+ \left[\frac{1}{4\alpha} \dot{g}_2^0 - \frac{1}{4\alpha} \dot{g}_1^0 - \frac{\dot{g}^0}{2\alpha^2} \right] \sin \alpha t \\ &+ \left[\frac{\ddot{g}_2^0}{4\alpha^2} - \frac{\dot{g}_1^0}{4\alpha^2} + \frac{\dot{g}^0}{2\alpha} \right] \cos 2\alpha t \\ g_2 &= \left[g_2^0 + \frac{\ddot{g}_2^0}{4\alpha^2} + \frac{3}{4} \left(\frac{\dot{g}_1^0}{\alpha^2} + \frac{2}{\alpha} \dot{g}^0 \right) \right] \\ &+ \left[\frac{3}{2\alpha} \dot{g}_2^0 - \frac{1}{2\alpha} \dot{g}_1^0 + \frac{\dot{g}^0}{\alpha^2} \right] \sin \alpha t \\ &- \left[\frac{2}{\alpha} \dot{g}^0 + \frac{\dot{g}_1^0}{\alpha^2} \right] \cos \alpha t \\ &+ \left[\frac{1}{4\alpha} \dot{g}_1^0 - \frac{1}{4\alpha} \dot{g}_2^0 + \frac{\dot{g}^0}{2\alpha^2} \right] \sin \alpha t \\ &+ \left[\frac{\dot{g}_1^0}{4\alpha^2} - \frac{\ddot{g}_2^0}{4\alpha^2} + \frac{\dot{g}^0}{2\alpha} \right] \cos 2\alpha t \end{aligned} \quad (49)$$

$$\begin{aligned}
g = & - \left[\frac{3}{4\alpha} (\dot{g}_1^0 - \dot{g}_2^0) + g^0 - \frac{\dot{g}^0}{2\alpha^2} \right] \\
& + \left[\frac{\ddot{g}_1^0}{2\alpha^2} - \frac{\ddot{g}_2^0}{2\alpha^2} - \frac{2}{\alpha} \dot{g}^0 \right] \sin \alpha t \\
& + \left[\frac{\dot{g}^0}{\alpha^2} + \frac{\dot{g}_1^0}{\alpha} - \frac{\dot{g}_2^0}{\alpha} \right] \cos \alpha t \\
& + \left[\frac{\ddot{g}_1^0}{4\alpha^2} - \frac{\ddot{g}_2^0}{4\alpha^2} - \frac{\dot{g}^0}{2\alpha} \right] \sin 2\alpha t \\
& + \left[\frac{\ddot{g}_1^0}{4\alpha^2} - \frac{\ddot{g}_2^0}{4\alpha^2} + \frac{\dot{g}^0}{2\alpha} \right] \cos 2\alpha t
\end{aligned}$$

$$\begin{aligned}
J = J^0 - \frac{1}{2\alpha} (\ddot{g}_1^0 + \ddot{g}_2^0) \\
- \frac{1}{2} (\dot{g}_1^0 + \dot{g}_2^0) \sin \alpha t \\
+ \frac{1}{2\alpha} (\ddot{g}_1^0 + \ddot{g}_2^0) \cos \alpha t.
\end{aligned}$$

Thus the mean-square dimensions of the beam in the longitudinal magnetic field oscillate with frequencies $\omega_1 = \alpha$, $\omega_2 = 2\alpha$.

In the case of azimuthal symmetry (beam with a circular cross section) from the system of Eqs. (16, 17) an equation analogous to the Vladimirsky-Kapchinsky equation may be obtained.⁷ For a circular beam, we have

$$g_1 = g_2; \quad g = 0, \quad (50)$$

and, as a consequence of (50), $v_1 = v_2$. In a varying magnetic field it is necessary to take into account the induction electric field. This field makes an additional contribution to the matrix b :

$$b^{ind} = \frac{1}{2} \frac{da}{dt}. \quad (51)$$

By introducing formulae (50, 51) into the system (16, 17), we get two independent equations

$$\begin{aligned}
\frac{d^3 g_1}{dt^3} + 4 \left(v_1^2 + \frac{\alpha^2}{4} \right) \frac{dg_1}{dt} \\
+ 2g_1 \frac{d}{dt} \left(v_1^2 + \frac{\alpha^2}{4} \right) = 0 \quad (52) \\
\frac{d}{dt} (J + \alpha g_1) = 0.
\end{aligned}$$

It is not difficult to see that the first of Eqs. (52) coincides with Eq. (43) with a change of v_1^2 to $v_1^2 + \alpha^2/4$. For this reason, we have an equation for the radius of the envelope $R^2 = 4g_1$ (for a uniformly charged beam)

$$\frac{d^2 R}{dt^2} + \left(v_1^2 + \frac{\alpha^2}{4} \right) R - \frac{E_3^2}{R^3} - \frac{2v_1 c^2}{\gamma_0^3 R} = 0. \quad (53)$$

The second of Eqs. (52) is the expression of the conservation law of the canonical angular momentum (Busch's theorem):

$$\begin{aligned}
M_3 = \overline{(x_1 - \bar{x}_1)(P_2 - \bar{P}_2)} \\
- \overline{(x_2 - \bar{x}_2)(P_1 - \bar{P}_1)}. \quad (54)
\end{aligned}$$

Indeed, in the case discussed the canonical momenta $P_{1,2}$ are

$$P_1 = v_1 - \omega_L x_2; \quad P_2 = v_1 + \omega_L x_1 \quad (55)$$

By introducing Eq. (55) into Eq. (54) and averaging, we find

$$M_3 = J + \alpha g_1 = \text{const.} \quad (56)$$

For determination of the constant E_3 in Eq. (53) we use Eqs. (11), (15) and find

$$\begin{aligned}
E_3^2 = E_1^2 - 4J^2 + 4M_3^2 \\
= 4M_3^2 + 16\{ \overline{(x_1 - \bar{x}_1)^2 (v_1 - \bar{v}_1)^2} \\
- \overline{[(x_1 - \bar{x}_1)(v_1 - \bar{v}_1)]^2} \\
- \overline{[(x_1 - \bar{x}_1)(v_2 - \bar{v}_2)]^2} \}. \quad (57)
\end{aligned}$$

It can be shown that the expression in brackets is invariant with respect to rotation of the coordinate system

$$X = \bar{X} + QX'; \quad V = \bar{V} + QX' + QV', \quad (58)$$

where the matrix Q is

$$Q = \begin{pmatrix} \cos \psi(t) & \sin \psi(t) \\ -\sin \psi(t) & \cos \psi(t) \end{pmatrix}. \quad (59)$$

By representing formula (57) in the coordinate system where the condition

$$\overline{x_1' v_2'} = 0 \quad (60)$$

is valid, it can be easily seen that the resulting expression coincides with (46.1) and consequently is positive. By calculating the moments in the coordinate systems (X, V) and (X', V') , we find

$$\begin{aligned} \overline{(x_1 - \bar{x}_1)(v_2 - \bar{v}_2)} &= \overline{x_1' v_2'} + \overline{\psi x_1'^2} \\ \overline{(x_1 - \bar{x}_1)^2} &= \overline{x_1'^2}. \end{aligned} \quad (61)$$

To satisfy Eq. (60), we have

$$\overline{\psi} = \frac{(x_1 - \bar{x}_1)(v_2 - \bar{v}_2)}{(x_1 - \bar{x}_1)^2} = \frac{J}{2g_1} \quad (62)$$

Equation (53) in conjunction with formula (57) yields stable nonlinear oscillations of the mean-square radius of the azimuthally symmetric charged beam in a longitudinal magnetic field in the vicinity of a stationary point which is

$$R_c = \frac{\frac{vc^2}{\gamma_0^3} + \left\{ \left(\frac{vc^2}{\gamma_0^3} \right)^2 + E_3^2 \left(v_1^2 + \frac{\alpha^2}{4} \right) \right\}^{1/2}}{\left(v_1^2 + \frac{\alpha^2}{4} \right)}. \quad (63)$$

IV. NONLINEAR FIELDS

In this section effects connected with the nonlinearity of the external electromagnetic field will be considered in the case of a one-dimensional problem.

Let us assume that in Lorentz force of an external electromagnetic field, there is a nonlinearity x^n , where n is an odd integer. The equations of a motion for particles in this case are expressed in the form

$$\frac{dx}{dt} = v \quad (64)$$

$$\frac{dv}{dt} = -\omega_0^2 x - \beta x^n$$

and for the second-order moments we have the system

$$\begin{aligned} \frac{d\overline{x^2}}{dt} &= 2\overline{xv} \\ \frac{d\overline{xv}}{dt} &= \overline{v^2} - \omega_0^2 \overline{x^2} - \beta \overline{x^{n+1}} \end{aligned} \quad (65)$$

$$\frac{d\overline{v^2}}{dt} = -2\omega_0^2 \overline{xv} - 2\beta \overline{vx^n}$$

As in the previous sections, the bar in Eqs. (65) designates averaging with distribution function f . It follows from Eqs. (65) that the second order moment is connected with the higher-order moments $\overline{x^{n+1}}$, $\overline{vx^n}$.

As in Section III, an equation for change with time of the mean-square dimension $r = \sqrt{\overline{x^2}}$ may be derived

$$\frac{d^2 r}{dt^2} + \omega_0^2 r - \frac{E^2}{r^3} + \beta \frac{\overline{x^{n+1}}}{r} = 0, \quad (66)$$

where the emittance E is defined in accordance with formula (46.1)

$$E^2 = \overline{x^2 v^2} - (\overline{xv})^2. \quad (67)$$

However, in this case E is a function of time. By using the definition (67) we get from Eqs. (65)

$$\frac{dE^2}{dt} = \beta \overline{x^{n+1}} \frac{dr^2}{dt} - r^2 \frac{2\beta}{n+1} \frac{d\overline{x^{n+1}}}{dt}. \quad (68)$$

Accounting for the variation of emittance permits one to close the system (65) for the second-order moments, i.e., to obtain a closed system of equations for the mean-square quantities. Indeed, let us solve Eq. (68) with respect to $\overline{x^{n+1}}$ and introduce the resulting expression into Eq. (66). We find

$$\begin{aligned} \frac{d^2 r}{dt^2} + \omega_0^2 r - \frac{E^2}{r^3} - \frac{\beta}{2} (n+1) r^{n+1} \\ \times \int_0^t \frac{1}{\beta r^{n+3}} \frac{dE^2}{dt'} dt' + \tilde{\beta} r^n = 0, \end{aligned} \quad (69)$$

where $\tilde{\beta} = \beta \overline{x_0^{n+1}} / r_0^{n+1}$, and $\overline{x_0^{n+1}}$, $r_0^2 = \overline{x_0^2}$ are moments of the initial distribution function.

If the variation of emittance E can be neglected, then, according to Eq. (69), the mean-square dimension r satisfies the equation:

$$\frac{d^2 r}{dt^2} + \omega_0^2 r - \frac{E^2}{r^3} + \tilde{\beta} r^n = 0. \quad (70)$$

In this case, a small static ($d\tilde{\beta}/dt = 0$) nonlinearity results in a nonlinear shift of the oscillation frequency. In the absence of the nonlinear term ($\tilde{\beta} = 0$), the general solution of Eq. (70) can be

expressed in the form

$$\left(\frac{r}{r_m}\right)^2 = \sqrt{1 + A^2} + A \sin(2\omega_0 t + \varphi), \quad (71)$$

where r_m – radius of matched beam. In the first order of smallness of β , the amplitude A and phase φ in Eq. (71) should be treated as slowly varying functions which, according to the Krylov-Bogolyubov method,⁹ satisfy the equations

$$\begin{aligned} \frac{dA}{dt} &= 0 \\ \frac{d\varphi}{dt} &= \frac{\tilde{\beta} r_m^{n-1}}{2\pi\omega_0} \int_0^{2\pi} \left(1 + \frac{\sqrt{1 + A^2}}{A} \sin \Psi\right) \\ &\quad \times (\sqrt{1 + A^2} + A \sin \Psi)^{n-1/2} d\Psi \end{aligned} \quad (72)$$

For a cubic nonlinearity ($n = 3$), calculation of the integral in Eq. (72) gives the result

$$\frac{d\varphi}{dt} = \frac{3}{2} \tilde{\beta} \frac{r_m^2}{\omega_0} \sqrt{1 + A^2} \approx \frac{3}{2} \frac{\tilde{\beta} r_m^2}{\omega_0} \left(1 + \frac{A^2}{2}\right) \quad (73)$$

Therefore, in addition to the nonlinear frequency shift for one-particle oscillations, proportional to the square of the amplitude,⁹ there is a correction for the frequency independent of the oscillation amplitude.

In the case of a small static nonlinearity, it is also possible to obtain an expression for the variation with time of the mean-square beam dimension, taking into account the change in emittance of Eq. (67). In the first order of smallness in respect to β , according to Ref. 9, the equations of motion (64) are equivalent to the system

$$\begin{aligned} \frac{d^2x}{dt^2} + \omega^2(\epsilon_0)x &= 0 \\ \frac{d\epsilon_0}{dt} &= \frac{d}{dt} \left(\frac{v^2}{2} + \frac{\omega_0^2 x^2}{2} \right) = 0. \end{aligned} \quad (74)$$

For cubic nonlinearity, the dependence of the frequency on energy is

$$\omega(\epsilon_0) = \omega_0 + \frac{3}{4} \frac{\beta \epsilon_0}{\omega_0^3}. \quad (75)$$

Let us introduce the moments of the distribution function

$$M_n'' = \overline{\omega^{2n}(\epsilon_0)x^2}; \quad \epsilon_n = \overline{\omega^{2n}(\epsilon_0)\epsilon}. \quad (76)$$

Here ϵ is the integral of the system (74)

$$\epsilon = \frac{v^2}{2} + \frac{\omega^2(\epsilon_0)x^2}{2} = \text{const} \quad (77)$$

The moments M_n'' and ϵ_n satisfy the series of equations

$$\frac{d^2 M_n''}{dt^2} + 4M_{n+1}'' = 4\epsilon_n; \quad n = 0, 1, 2, \dots \quad (78)$$

and the solution for the mean-square dimension $\overline{x^2}$ has the following form (see Appendix):

$$\begin{aligned} \overline{x^2} &= M_0'' = \iint_{-\infty}^{\infty} f(x, v, t = 0) \\ &\quad \times \left[\frac{x^2}{2} (1 + \cos 2\omega(\epsilon_0)t) \right. \\ &\quad \left. + \frac{xv}{\omega(\epsilon_0)} \sin 2\omega(\epsilon_0)t \right. \\ &\quad \left. + \frac{v^2}{2\omega^2(\epsilon_0)} (1 - \cos 2\omega(\epsilon_0)t) \right] dx dv, \end{aligned} \quad (79)$$

where $f(x, v, t = 0)$ is the initial distribution function.

This result may be obtained from the more general formula. In the case of a particle oscillation in an arbitrary static potential, the equations of motion in angle-action variables (ψ, I) are

$$\frac{dI}{dt} = 0 \quad (80)$$

$$\frac{d\psi}{dt} = \omega(I)$$

For the Liouville equation we get

$$\frac{\partial f}{\partial t} + \omega(I) \frac{\partial f}{\partial \psi} = 0 \quad (81)$$

The solution of this equation is easily found to

be

$$f(t) = f(I, \psi - \omega(I)t), \quad (82)$$

where $f(I, \psi)$ is the initial distribution function.

The average value of an arbitrary function $A(I, \psi)$ is given by

$$\bar{A} = \int A(I, \psi + \omega(I)t) f(I, \psi) dI d\psi. \quad (83)$$

Thus, we obtain the formula (79) by considering elementary beams with transverse oscillation energy ϵ_0 and averaging the resulting expression with initial distribution function f .

In analogy to (79), it is possible to obtain an expression for the change with time of the position of the beam center of mass. For the moments of the distribution function

$$M_n^I = \overline{\omega^{2n}(\epsilon_0)x}; \quad n = 0, 1, 2, \dots, \quad (84)$$

the series of equation

$$\frac{d^2 M_n^I}{dt^2} + M_{n+1}^I = 0 \quad (85)$$

is valid. The solution of the system (85) is found in a manner such as that for Eqs. (78)

$$\begin{aligned} \bar{x} = M_0^I &= \iint_{-\infty}^{\infty} f(x, v, t = 0) \\ &\times \left[x \cos \omega(\epsilon_0)t \right. \\ &\quad \left. + \frac{v}{\omega(\epsilon_0)} \sin \omega(\epsilon_0)t \right] dx dv. \quad (86) \end{aligned}$$

Similar expressions for variation with time of \bar{x} and \bar{x}^2 may be obtained in the case of a circular charged-particle beam in a linear external field, with a distribution in angular frequency ω_0 . For example, for the mean-square dimension \bar{x}^2 we will have

$$\begin{aligned} \bar{x}^2 &= \iiint_{-\infty}^{\infty} f(x, v, \omega_0, t = 0) \\ &\times \left[\frac{x^2}{2} (1 + \cos 2\omega_0 t) + \frac{xv}{\omega_0} \sin 2\omega_0 t \right. \\ &\quad \left. + \frac{v^2}{2\omega_0^2} (1 - \cos 2\omega_0 t) \right] dx dv d\omega_0. \quad (87) \end{aligned}$$

Let us consider an initial distribution function of the form

$$\begin{aligned} f(x, v, t = 0) \\ = \frac{N}{2\pi E_0} \exp\left(-\frac{1}{2E_0^2} (\overline{v_0^2} x^2 - 2xv_0xv + \overline{x_0^2} v^2)\right) \end{aligned} \quad (88)$$

where the index "0" denotes the initial values of the second-order moment, E_0 is the initial value of the effective phase volume (67). Let us substitute expression (88) into (79) and perform the integration, neglecting the inessential dependence of the frequency on energy in denominators in (79). We find the result

$$\begin{aligned} \bar{x}^2 &= \frac{\overline{x_0^2}}{2} + \frac{\overline{v_0^2}}{2\omega_0^2} + \frac{1}{2} \left\{ \left[1 + \left(\frac{3}{2} \frac{\beta}{\omega_0} \lambda_{1t} \right)^2 \right] \right. \\ &\quad \times \left. \left[1 + \left(\frac{3}{2} \frac{\beta}{\omega_0} \lambda_{2t} \right)^2 \right] \right\}^{-1/4} \\ &\times \left[\frac{\lambda_2 \cos(2\omega_0 t + \alpha_2 - 2\Psi)}{\sqrt{1 + \left(\frac{3}{2} \frac{\beta}{\omega_0} \lambda_{2t} \right)^2}} \right. \\ &\quad \left. - \frac{\lambda_1 \cos(2\omega_0 t + \alpha_1 - 2\Psi)}{\sqrt{1 + \left(\frac{3}{2} \frac{\beta}{\omega_0} \lambda_{1t} \right)^2}} \right]. \quad (89) \end{aligned}$$

Here the notations are

$$\begin{aligned} \lambda_{1,2} &= \frac{\overline{x_0^2}}{2} + \frac{\overline{v_0^2}}{2\omega_0^2} \\ &= \frac{1}{2} \sqrt{\left(\frac{\overline{x_0^2}}{2} - \frac{\overline{v_0^2}}{2\omega_0^2}\right)^2 + 4\left(\frac{xv_0}{\omega_0}\right)^2} \\ \alpha_{1,2} &= \frac{3}{2} \arctan \frac{3}{2} \frac{\beta}{\omega_0} \lambda_{1,2t} \\ &\quad + \frac{1}{2} \arctan \frac{3}{2} \frac{\beta}{\omega_0} \lambda_{2,1t} \\ \cos 2\Psi &= \frac{\overline{x_0^2} - \frac{\overline{v_0^2}}{\omega_0^2}}{\lambda_2 - \lambda_1}; \quad (90) \end{aligned}$$

$$\sin 2\Psi = \frac{2 \frac{\overline{xv_0}}{\omega_0}}{\lambda_2 - \lambda_1}.$$

In the absence of nonlinearity ($\beta = 0$), the expression for $\overline{x^2}$ coincides with the general solution of the Vladimirovsky-Kapchinsky equation (41). The constants $\lambda_{1,2}$ are connected with the maximum and minimum mean-square dimensions

$$\begin{aligned} \lambda_1 + \lambda_2 &= \overline{x_{max}^2} + \overline{x_{min}^2} \\ |\lambda_1 - \lambda_2| &= \overline{x_{max}^2} - \overline{x_{min}^2}. \end{aligned} \quad (91)$$

At times $t \ll [(\beta/\omega_0)\lambda_{1,2}]^{-1}$, formula (89) describes the oscillation of the mean-square dimension with constant amplitude and frequency dependent on oscillation amplitude. For the almost matched beam ($\lambda_1 \approx \lambda_2 \approx r_m^2$), the nonlinear frequency shift $\alpha_{1,2}$ does not depend on oscillation amplitude and differs from formula (73) only by the factor 3/2. Therefore, the Eq. (70) deriving in the case of small emittance variation is valid for time interval less than the inverse nonlinear frequency shift.

In the case of a circular beam with a spread in angular frequency, for an initial distribution function

$$\begin{aligned} f(x, v, \omega_0, t = 0) \\ = \frac{N}{\sqrt{2\pi\Delta\omega_0}} \Phi\left(x, \frac{v}{\omega_0}\right) e^{-(\omega_0 - \overline{\omega_0})^2/2\Delta^2}, \end{aligned} \quad (92)$$

the mean-square dimension varies with the time as

$$\begin{aligned} \overline{x^2} &= \frac{\overline{x_0^2}}{2} + \frac{\overline{v_0^2}}{2\omega_0^2} + e^{-2\Delta^2 t^2} \left[\frac{\overline{xv_0}}{\omega_0} \sin 2\omega_0 t \right. \\ &\quad \left. + \left(\frac{\overline{x_0^2}}{2} - \frac{\overline{v_0^2}}{2\omega_0^2} \right) \cos 2\omega_0 t \right]. \end{aligned} \quad (93)$$

As can be shown from formulae (89), (93), in both cases the mean-square dimension $\overline{x^2}$ oscillates with the double frequency $2\omega_0$ near the equilibrium value $\frac{1}{2}(\overline{x_0^2} + \overline{v_0^2}/\omega_0^2)$ and with decreased oscillation amplitude. At times greater than the inverse nonlinear frequency shift or inverse fre-

quency spread

$$t \gg \left(\frac{\beta}{\omega_0} \lambda_{1,2} \right)^{-1}; \quad t \gg \frac{1}{\Delta}, \quad (94)$$

these oscillations are damped and the mean-square dimension tends toward the limiting value

$$\overline{x^2} = \frac{1}{2} \left(\overline{x_0^2} + \frac{\overline{v_0^2}}{\omega_0^2} \right) = \frac{1}{2} (\overline{x_{max}^2} + \overline{x_{min}^2}). \quad (95)$$

This result is valid for every initial distribution function with finite second-order moments. Indeed, when $t > 0$, the terms dependent on time in (79) are the sign-changed functions of the coordinates and velocities and the distance between two zeros is $\Delta x \sim (\omega_0/\beta t)^{1/2}$. If d is the characteristic length of the variation of the function $f(x, v, t = 0)$, then for $\Delta x \ll d$ corresponding integrals in (79) are equal to zero and the damping time is $t \gg [(\beta/\omega_0)d^2]^{-1}$.

If the position of the system center of mass is not equal to zero at initial moment of the time, then, as follows from (86), \bar{x} oscillates with frequency ω_0 . For longer time period, if the inequalities (94) are satisfied, these oscillations are damped. In this case the variation of the beam dimension with time is of particular interest

$$r^2 = \overline{x^2} - \bar{x}^2 \quad (96)$$

Let us assume that at the initial time the condition

$$\overline{x_0^2} = \frac{\overline{v_0^2}}{\omega_0^2}; \quad \overline{xv_0} = 0 \quad (97)$$

is satisfied. Then for longer time periods, the dimension of the beam tends to the value

$$r^2 = r_0^2 + \bar{x}_0^2, \quad (98)$$

where r_0 is the initial beam dimension. The increase in beam dimension is connected with transition of the energy of coherent oscillations into the energy of incoherent motion because of the external fields nonlinearity (or of the frequency spread).

As mentioned earlier, the oscillation nonlinearity (frequency spread) leads to the infinite series of equations (65) for the distribution-function moments. Nevertheless, this infinite system may be reduced to one equation (69) for mean-square

dimension if the beam emittance (67) is a known function of the time. In the case of a small static nonlinearity, the emittance variation may be easily obtained. The moments entering the definition (67) are found in accordance with the formulae

$$\begin{aligned} \overline{xv} &= \iint_{-\infty}^{\infty} f(x, v, t = 0) \left[xv \cos \omega(\epsilon_0)t \right. \\ &\quad - \frac{x^2 \omega(\epsilon_0)}{2} \sin 2\omega(\epsilon_0)t \\ &\quad \left. + \frac{v^2}{2\omega(\epsilon_0)} \sin 2\omega(\epsilon_0)t \right] dx dv \\ \overline{v^2} &= \iint_{-\infty}^{\infty} f(x, v, t = 0) \quad (99) \\ &\quad \times \left[\frac{x^2 \omega^2(\epsilon_0)}{2} (1 - \cos 2\omega(\epsilon_0)t) \right. \\ &\quad - xv \omega(\epsilon_0) \sin 2\omega(\epsilon_0)t \\ &\quad \left. + \frac{v^2}{2} (1 + \cos 2\omega(\epsilon_0)t) \right] dx dv. \end{aligned}$$

Calculation of the integrals in (99) with the initial distribution function (88) gives the emittance variation with time

$$\begin{aligned} E^2 &\approx \frac{\omega_0^2}{4} \left\{ (\lambda_1 + \lambda_2)^2 \right. \\ &\quad - \frac{1}{\sqrt{\left[1 + \left(\frac{3\beta}{2\omega_0} \lambda_1 t \right)^2 \right] \left[1 + \left(\frac{3\beta}{2\omega_0} \lambda_2 t \right)^2 \right]}} \\ &\quad \times \left[\frac{\lambda_1^2}{1 + \left(\frac{3\beta}{2\omega_0} \lambda_1 t \right)^2} \right. \\ &\quad \left. + \frac{\lambda_2^2}{1 + \left(\frac{3\beta}{2\omega_0} \lambda_2 t \right)^2} \right. \\ &\quad \left. - \frac{2\lambda_1 \lambda_2 \cos(\alpha_2 - \alpha_1)}{\sqrt{\left[1 + \left(\frac{3\beta}{2\omega_0} \lambda_1 t \right)^2 \right] \left[1 + \left(\frac{3\beta}{2\omega_0} \lambda_2 t \right)^2 \right]}} \right] \quad (100) \end{aligned}$$

From Eq. (100), the emittance E slowly increases from its initial value $E_0^2 = x_0^2 v_0^2 - (xv_0)^2 = \omega_0^2 \lambda_1 \lambda_2$ and for a time period longer than the inverse nonlinear frequency shift tends toward the value

$$E_k^2 = \frac{\omega_0^2}{4} (\lambda_1 + \lambda_2)^2 = \frac{\omega_0^2}{4} (\overline{x_{max}^2} + \overline{x_{min}^2})^2. \quad (101)$$

The particles occupy a region of effective area E_k on the phase plane. The full emittance change is given by

$$\begin{aligned} \Delta E^2 &= E_k^2 - E_0^2 = \frac{\omega_0^2}{4} (\lambda_1 - \lambda_2)^2 \\ &= \frac{\omega_0^2}{4} \overline{x_{max}^2} \left(1 - \frac{\overline{x_{min}^2}}{\overline{x_{max}^2}} \right)^2. \quad (102) \end{aligned}$$

This expression for the full emittance change is valid also for a circular beam with frequency spread.

Therefore the external-field nonlinearity (frequency spread) leads to emittance growth. This growth increases with increase of the initial beam mismatching $\overline{x_{max}^2}/\overline{x_{min}^2}$. This fact had been noted in Refs. 4 and 11, where emittance growth caused by beam self-fields nonlinearity had been obtained by numerical simulation of the beam transported in a smooth focusing channel.

In conclusion of this section, we consider the problem of the emittance growth caused by the instantaneous change of the oscillation frequency. Let us assume that at some time a change

$$\omega^2 = \omega_0^2 + \delta \quad (103)$$

of the oscillation frequency occurs and at the initial time the matched beam conditions (97) were satisfied. This situation takes place in the use of an electron ring for ion-acceleration several times in electron ring accelerators.¹⁰ When the ions are lost ω^2 is represented by Eq. (103). If the time period between two successive acceleration cycles is longer than inverse nonlinear oscillation frequency shift (or inverse electron frequency spread) then for one acceleration cycle, the emittance E growth is given by

$$\Delta E^2 = \frac{E_0^2}{4} \frac{\delta^2}{\omega_0^4}. \quad (104)$$

This result follows from (102) when $\delta \ll \omega_0^2$. Therefore for k acceleration cycles the emittance increases as

$$E^2 \approx E_0^2 e^{k\delta^2/4\omega_0^4}. \quad (105)$$

In this case δ is proportional to the neutralization factor $f_n = ZN_i/N_e$, where $N_{e,i}$ are the numbers of electrons and ions and Z is the ion charge. For $f_n \lesssim 1$, the emittance growth may be significant and put a limit on the number of acceleration cycles.

V. CONCLUSION

In conclusion, we summarize the basic results. The system of equations for the mean-square dimensions of the charged-particle bunched beams was obtained by using the equations for the full second-order moments of the distribution function. If self-fields of the beam can be neglected, the equations will be linear and will not depend on the form of the distribution function of particles in the beam. The effect of space charge will be easily taken into account if the self-fields are linear.

In the two-dimensional case (infinite in the beam direction) the equations obtained are valid not only for a beam with elliptical cross section and uniform charge density (linear self-fields), but for a wider class of distribution functions. In this case the equations do not depend on distribution-function form in the presence of self-fields.

The equations derived permit a consideration of the mean-square dimension oscillations of a linear beam in a focusing potential and a longitudinal magnetic field by a unified approach.

In the absence of a longitudinal magnetic field, the equations are equivalent to the Vladimírsky-Kapchinsky equations, provided that the elliptic beam boundary does not rotate.

For the mean-square radius of a circular beam in a longitudinal magnetic field, the equation coincides with that obtained by Lee and Cooper (particle collisions not taken into account).

The restriction of the distribution function class does not allow us to account for the emittance changing caused by oscillation nonlinearity. The effects connected with a variation of emittance were investigated with simple one-dimensional problems with nonlinearity of the Lorentz force of the external electromagnetic field.

Accounting for the change of emittance permits one to close the equations for the second order moments, i.e., to obtain closed equations for the mean-square quantities. The law of variation of emittance may be applied from outside, particularly on the basis of the solution of model problems.

From the results obtained, it is concluded that, during the time of the inverse nonlinear oscillation frequency shift (inverse frequency spread in the circular beam) the emittance may be considered constant. For time periods much longer than those considered, the effective emittance increases to a limiting value and this emittance growth increases with increase of the initial beam mismatching. During the same time period, the oscillations of the beam center of mass decay and, because of the energy transition from coherent into incoherent oscillations, the mean square dimension increases. The full emittance growth caused by instantaneous frequency changing is also obtained.

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APPENDIX

For the Laplace transform of moments M_{np}'' we get according to Eq. (78):

$$p^2 M_{np}'' + 4M_{n+1}'' p = \frac{4\epsilon_n}{p} \quad (A.1)$$

$$+ pM_n''(t=0) + \left. \frac{dM_n''}{dt} \right|_{t=0} = F_n$$

Let us define the sequence of determinants of $(N+1)$ -order:

$$D_N = \begin{vmatrix} p^2 & 4 & 0 & \cdots & 0 \\ 0 & p^2 & 4 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & p^2 \end{vmatrix}; \quad (A.2)$$

$$D_N^0 = \begin{vmatrix} F_0 & 4 & \cdots & \cdots & 0 \\ F_1 & p^2 & 4 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_N & 0 & \cdots & \cdots & p^2 \end{vmatrix}$$

Then M_{op}'' may be determined according to for-

mula:

$$M_{op}'' = \lim_{N \rightarrow \infty} \frac{D_N^0}{D_N} \quad (\text{A.3})$$

Calculation of the determinant D_N is easily performed

$$D_N = -p^{2(N+1)} \quad (\text{A.4})$$

Let us expand determinant D_N^0 according to the elements of the first column. We obtain in result:

$$D_N^0 = p^{2N} \sum_{n=0}^N (-1)^n \left(\frac{2}{p}\right)^{2n} F_n \quad (\text{A.5})$$

By introducing (A.4) and (A.5) into (A.3) and by converging to the limit, we get:

$$M_{op}'' = \frac{1}{p^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{p}\right)^{2n} F_n \quad (\text{A.6})$$

By using the definition of moments (76), expression (A.6) can be written in the following form:

$$\begin{aligned} M_{op}'' &= \iint_{-\infty}^{\infty} f(x, v, t = 0) \\ &\times \left[\left(\frac{4}{p} \epsilon + px^2 + 2xv \right) \frac{1}{p^2} \right. \\ &\times \left. \sum_{n=0}^{\infty} (-1)^n \left(\frac{2\omega(\epsilon_0)}{p} \right)^{2n} \right] dx dv \end{aligned} \quad (\text{A.7})$$

By summarizing the power series in subintegral

expression and by performing an inverse Laplace transformation, we obtain the final result:

$$\begin{aligned} M_0'' &= \iint_{-\infty}^{\infty} f(x, v, t = 0) \\ &\times \left[\frac{\epsilon}{\omega^2(\epsilon_0)} (1 - \cos 2\omega(\epsilon_0)t) \right. \\ &+ x^2 \cos 2\omega(\epsilon_0)t \\ &\left. + \frac{xv}{\omega(\epsilon_0)} \sin 2\omega(\epsilon_0)t \right] dx dv \end{aligned} \quad (\text{A.8})$$

REFERENCES

1. J. D. Lawson, *Proceedings of the Symposium on Ion Sources and Formation of Ion Beams*, Brookhaven National Laboratory Report BNL-50310, 1971.
2. I. M. Kapchinsky and V. V. Vladimirsky, *Proceedings of the International Conference on High Energy Accelerators and Instruments*, CERN, Geneva, 1959, p. 274.
3. I. M. Kapchinsky, *Particle Dynamics in Linear Resonance Accelerators*, Atomizdat, Moscow, 1966.
4. P. M. Lapostolle, *IEEE Trans. Nucl. Sci.* **18**, 1101 (1971).
5. F. J. Sacherer, *IEEE Trans. Nucl. Sci.* **18**, 1101 (1971).
6. O. I. Yarkovoy, *Zh. Tekh. Fiz. (USSR)* **36**, 986 (1966).
7. E. P. Lee and R. K. Cooper, *Particle Accelerators* **7**, 83 (1976).
8. A. D. Dymnikov and E. A. Perelstein, *Nucl. Instrum. Methods* **148**, 567 (1978).
9. N. N. Bogolyubov, and Yu A. Mitropol'sky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, Nauka, Moscow, 1974.
10. M. L. Iovnovich et al., JINR Preprint P9-11686, Dubna, 1978.
11. V. S. Kuznetsov et al., *Proceedings of the 8th International Conference on High-Energy Accelerators*, CERN, Geneva, 1971.