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RADIATIVE POLARIZATION AT ULTRA-HIGH ENERGIES†

Ya. S. DERBENEV, A. M. KONDRATENKO, and A. N. SKRINSKY

Institute of Nuclear Physics, Siberian Division, USSR Academy of Sciences, Novosibirsk 90, USSR

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Possibilities of obtaining polarized electron and positron beams at arbitrarily high energies are studied in cases when spin-diffusion processes in inhomogeneous fields due to quantum fluctuations of radiation become essential. The requirements for precision of the magnetic systems are determined for providing a high degree of radiative polarization. Specially considered is the situation, characteristic of storage rings with a large number of elements, when perturbations of separate elements are statistically independent. The depolarizing effects are investigated at extremely high energies, when the spread of spin-precession frequencies exceeds the revolution frequency of particles in a storage ring. It is shown that the depolarizing influence of the colliding beam in the storage ring decreases as the maximum energy grows.

In a special section, the main formulae are given and the energy dependence of the criteria for existence of the radiative polarization in the conventional-type storage rings is discussed. In conclusion, additional measures are considered that may be useful for obtaining radiative polarization of high-energy electrons and positrons.

I INTRODUCTION

Design and construction of the electron and positron storage rings with energy 10–100 GeV has stimulated investigation of the question of the possibility of obtaining radiative polarization at such high energies. As is known, quantum fluctuations of the radiation result in spin diffusion caused by stochastic mixing of particle trajectories in fields with constant direction, with the diffusion rate quickly growing with energy.

Estimates of depolarizing effects in some cases have led to the conclusion that radiative polarization could not be achieved in practice at high energies. However, in our view, based on the theoretical papers,[1–4] the situation is not so hopeless. The decrease of our pessimism was partially due to discussions between the Novosibirsk group and other laboratories having interest in the problem (1976–1977). An analysis of the high-energy situation made in this work revealed good hopes for obtaining polarized beams (including colliding beams) at arbitrarily high energies. Our consideration uses as a basis the theory of spin diffusion, developed in detail in Refs. 2 and 3 for a single beam and in Ref. 4 for colliding beams. The importance of the problem has made it necessary to reanalyze the main theoretical prerequisites of these papers, but no essential changes or additions to the theory have appeared.

Depolarizing effects would be negligibly small in an ideal storage ring with plane closed orbits where the magnetic field is strictly vertical. When the situation differs from this ideal one, spin diffusion can exceed the polarizing influence of radiation, thereby depolarizing a beam. Radial magnetic field and its radial gradient, as well as the longitudinal field on the orbit, appear dangerous.‡ The field of colliding bunches in colliding beam machines provides a special type of perturbation.[4]

Radiative kinetics of polarization at arbitrary conditions of the motion in storage rings has been studied in detail in Refs. 2–4 and 7. Based on them, a general analysis of the high-energy situation has been performed in this work; also formulated are the requirements for the accuracy of realizing the magnetic system necessary to obtain a high degree of radiative polarization.

The presentation is the following. In Section II the kinetic equation is written describing spin

† The work was performed in 1977 (Preprint INP 77-60).

‡ Here we mean the usual situation with almost vertical magnetic field. In principle, a high degree of radiative polarization can be achieved in storage rings with the field changing its direction along the orbit and complicated equilibrium (periodical) spin motion.[5] Dynamical stability of spin motion in such storage rings was proved in Ref. 6.

relaxation due to radiation processes during cyclic motion of the particles with arbitrary energy. The equation includes two possible diffusion mechanisms, a non-resonant one, when in the process of stochastic walks of the spin and orbital motion frequencies, crossing of spin resonances does not occur and the resonant one, when crossing does occur.

In Sections III and IV the main effects connected with the non-ideal character of the magnetic system are considered at energies at which the spread of precession frequency is small and the non-resonant mechanism of diffusion predominates. In this treatment we have taken into account the circumstance, important for storage rings with a large number of elements, that there is no correlation of the perturbative fields of separate elements.

In Section V spin diffusion is considered at "ultra-high" energies, at which the spread of precession frequency is of the order or larger than the distance between the neighboring Fourier harmonics of the perturbative fields, i.e., the revolution frequency. One must distinguish here the situations when either subsequent passings of separate resonances at synchrotron energy oscillations are correlated, or when such a correlation disappears due to the quantum diffusion of energy.[2–8] In the latter case, the rate of spin diffusion grows more slowly with energy than does the rate of polarizing process, weakening the criteria for existence of polarization.

Section VI treats the stability polarization of the colliding beams. Because of the strong non-linearity of its field, a bunch of colliding particles results in a series of resonances whose strengths decrease slightly with increase of the resonance order. Under these conditions, it is stochastic passing of the resonances due to quantum energy fluctuations that provides the depolarization mechanism.

Section VII summarizes the main results. In conclusion some methods of increasing the radiative polarization stability are discussed.

II MAIN FORMULAE

In an inhomogeneous field, variation of the polarization state occurs not only due to the direct action of radiation during emission of quanta, but also due to perturbation of the orbital motion as well, when the variation is accumulated "integrally" in the following moments because of the deflection of a particle trajectory.

The importance of radiation influence upon the polarization through the orbital motion is connected with the fact that relaxation times of the orbital motion are many orders of magnitude less than the polarization time. Therefore even small variations of the field direction on the particle trajectory can strongly influence the degree of radiative polarization.

Spin precession in the storage ring field is described by the well-known equation

$$\dot{\mathbf{s}} = [\mathbf{W}_l \mathbf{s}], \tag{2.1}$$

where

$$\mathbf{W}_l = \left(1 + \gamma \frac{q_a}{q_0}\right) \frac{[\mathbf{v}\dot{\mathbf{v}}]}{v^2} - \frac{q}{\gamma} \frac{(\mathbf{B}\mathbf{v})\mathbf{v}}{v^2} - \frac{q}{\gamma^2 v^2} [\mathbf{v}\mathbf{E}],$$

$q = q_0 + q_a = (e/m) + q_a$ is the gyromagnetic ratio, q_a its anomalous part, $\gamma = (1 - v^2)^{-1/2}$, $(c = 1)$, and $\mathbf{v}$ and $\dot{\mathbf{v}}$ are particle velocity and acceleration in the electromagnetic field $\mathbf{E}$, $\mathbf{B}$ (for electrons and positrons $q_a/q_0 \approx \alpha/2\pi \approx 10^{-3}$).

We begin with the description of spin relaxation far from spin resonances. In this case, a general method is applied, well-known in the theory of radiative effects of orbital motion. First integrals of motion are determined, action (or amplitude) and phase. Then by perturbation theory, one finds rates of variation of action variables averaged over phases under radiation influence. Due to phase mixing this is sufficient for finding relaxation times and the equilibrium state.

In a homogeneous magnetic field, spin projections on the field direction and phases of precession around the field serve as action-phase spin variables. As (2.1) describes rotation, the action variable in the general case of motion with an angular velocity of varying direction has the form

$$s_{\mathbf{n}} = \mathbf{sn}(\mathbf{p}, \mathbf{r}),$$

where $\mathbf{n}$ is a unit vector giving the direction of a precession axis and depending on the momentum $\mathbf{p}$ and coordinates $\mathbf{r}$ of a particle.[6] The phase ψ of precession around $\mathbf{n}$ is a variable conjugate to $s_{\mathbf{n}}$.

The vector $\mathbf{n}$ is a solution of Eq. (1.1) not depending explicitly on time and is completely determined by the particle trajectory. Correspondingly, its spectrum contains the frequencies of orbital motion only. The axis $\mathbf{n}$ has maximum sensitivity to trajectory parameters in the neighborhood of a spin resonance, when the average precession frequency $\langle\dot{\psi}\rangle$ is close to some combination of orbital-motion frequencies. The axis $\mathbf{n}$

plays the same role with respect to the spin motion as do energy-dependent closed particle orbits with respect to betatron oscillations. Due to the spread of precession frequencies $\langle\dot{\psi}\rangle$ after mixing over phases, the average spin for a group of the particles moving near the equilibrium orbit is directed along the precession axis on the equilibrium trajectory $\mathbf{n}_s(\theta) \equiv \mathbf{n}(\mathbf{p}_s, \mathbf{r}_s)$: $\langle\mathbf{s}\rangle = \langle s_\mathbf{n}\rangle\mathbf{n}_s(\theta)$. The vector $\mathbf{n}_s$ is evidently periodic with respect to a generalized azimuth θ of a particle: $\mathbf{n}_s(\theta) = \mathbf{n}_s(\theta + 2\pi)$. Thus, a stable polarization direction, repeating from turn to turn, is realized at arbitrary (stationary) fields.[6]

In this work, a situation is considered when the $\mathbf{W}_l$ direction differs slightly from the vertical one. Small deviations of $\mathbf{n}$ from the vertical, necessary for quantitative description of the depolarizing influence of the quantum fluctuations of radiation, can be found by a perturbation theory.[2]

As usual, the radius-vector $\mathbf{r}$ of a particle is presented in the form:

$$\mathbf{r} = \mathbf{r}_0(\theta) + x\mathbf{e}_x(\theta) + z\mathbf{e}_z$$

where x is the radial and z the vertical deviation of a particle from some ideal plane orbit $\mathbf{r}_0(\theta)$, $\mathbf{e}_x$ and $\mathbf{e}_z$ are corresponding unit vectors, and θ is a generalized azimuth of the particle. It is clear that

$$\mathbf{r}_0' \equiv d\mathbf{r}_0/d\theta = R\mathbf{e}_y = R[\mathbf{e}_z\mathbf{e}_x],$$
$$\mathbf{e}_x' = K\mathbf{e}_y,\ \mathbf{e}_y' = -K\mathbf{e}_x,\ \mathbf{e}_z' = 0,$$

where $K(\theta) = B_z/\langle B_z\rangle$ is the dimensionless curvature of the ideal trajectory ($\langle B_z\rangle = \int_0^{2\pi} B_z d\theta/2\pi$), and R is the orbit perimeter divided by 2π.

Spin motion is conveniently described in a system of unit vectors:

$$\mathbf{e}_1 = \frac{[\mathbf{v}\mathbf{e}_z]}{|[\mathbf{v}\mathbf{e}_z]|},\quad \mathbf{e}_2 = \frac{\mathbf{v}}{v},\quad \mathbf{e}_3 = [\mathbf{e}_1\mathbf{e}_2], \tag{2.2}$$

whose difference from $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ is small. In the system (2.2), components of the angular velocity of the precession in the $\mathbf{e}_1$ and $\mathbf{e}_3$ directions are directly proportional to an anomalous moment, while the component along $\mathbf{e}_2$ slightly depends on it ($q_a \ll q$). The calculations are therefore simplified. Subtracting the angular velocity of the basic motion from $\mathbf{W}_l$ one obtains the following expression for the precession angular velocity $\mathbf{W}$ in the system (2.2):

$$\begin{aligned} W_1 &= \mathbf{W}\mathbf{e}_1 \approx q_a(B_y\dot{x} - B_x), \\ W_2 &\approx (q_0/\gamma)(B_y + B_x\dot{x}), \\ W_3 &\approx -q_aB_z. \end{aligned} \tag{2.3}$$

The components W_1 and W_2 are small compared with W_3 and in a linear approximation one has

$$\mathbf{n} = \mathbf{e}_3 + \operatorname{Im}\left\{(\mathbf{e}_1 - i\mathbf{e}_2)\exp\left(i\int_0^t W_3\,dt\right)\right.$$
$$\left.\times\int_{-\infty}^t (W_1 + iW_2)\exp\left(-i\int_0^t W_3\,dt\right)dt\right\}. \tag{2.4}$$

The integration in (2.4) is performed by adding an imaginary part $+i\epsilon$ to W_3. The spectrum of the solution thus constructed contains only the frequencies of orbital motion, in accordance with the definition of a precession axis.

Proceeding to integration over θ and expanding the integrand in a Fourier series, one can write the solution (2.4) for $\mathbf{n}$ in the form

$$\mathbf{n} = \mathbf{e}_3 + \operatorname{Im}\left\{(\mathbf{e}_1 - i\mathbf{e}_2)\exp\left[i\int_0^\theta\left(\frac{W_3}{\dot{\theta}} - \nu\right)d\theta\right]\right.$$
$$\left.\times\sum_k \frac{w_k}{\nu - \nu_k}\exp(i\psi_k)\right\}, \tag{2.5}$$

where $\nu = \langle W_3/\dot{\theta}\rangle \approx \gamma q_a/q_0$ is the average frequency of spin precession around $\mathbf{n}$ (in units of the revolution frequency), the w_k are amplitudes of the resonance perturbation harmonics, and ψ_k and $\nu_k = \psi_k'$ are combinations of phases and frequencies of orbital motion with integer coefficients. That is

$$(W_1 + iW_2)\exp\left[-i\int_0^\theta\left(\frac{W_3}{\dot{\theta}} - \nu\right)d\theta\right] = \sum_k w_k e^{i\psi_k}$$

The quantity $|w_k|$ plays the role of a resonance strength at $\nu = \nu_k$.[8] Equations (2.4) and (2.5) are valid, if $|\nu - \nu_k| \gg |w_k|$.

The spin projection on $\mathbf{n}$ varies slowly due to radiation influence, approaching some value determined by the equilibrium of polarizing processes. The average velocity $s_\mathbf{n}$ of the variation can be written as

$$\bar{\dot{s}}_\mathbf{n} = \overline{\frac{d}{dt}\delta s_\mathbf{n}} = \overline{\mathbf{n}\frac{d}{dt}\delta\mathbf{s}} + \overline{\mathbf{s}\frac{d}{dt}\delta\mathbf{n}} + \overline{2\frac{d}{dt}\delta\mathbf{s}\cdot\delta\mathbf{n}},$$

where $\delta\mathbf{s}$ and $\delta\mathbf{n}$ are increments of $\mathbf{s}$ and $\mathbf{n}$ due to radiation. The first term $\overline{\mathbf{n}(d/dt)\delta\mathbf{s}}$ describes the direct influence of radiation upon the spin, and the second one the radiation influence on the precession axis by perturbation of the orbital motion. The correlation term $\overline{2(d/dt)\delta\mathbf{s}\cdot\delta\mathbf{n}}$ is small for ultrarelativistic electrons and can be neglected.

In the case when a $\mathbf{W}$ direction is close to the vertical one, the term $\overline{\mathbf{s}(d/dt)\delta\mathbf{n}}$ describes the depolarizing action of the energy quantum fluctuations†

$$\overline{\mathbf{s}\frac{d}{dt}\delta\mathbf{n}} = \overline{s_{\mathbf{n}}\mathbf{n}\frac{d}{dt}\delta\mathbf{n}} = -\frac{s_{\mathbf{n}}}{2}\overline{\frac{d}{dt}(\delta\mathbf{n})^2}$$
$$= -\frac{s_{\mathbf{n}}}{2}\left\langle\left(\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2\frac{d}{dt}(\delta\gamma)^2\right\rangle.$$

For a polarization degree $\zeta = \langle s_{\mathbf{n}}\rangle/s$, an equation is obtained (time is measured in units of the period of particle revolution in the storage ring divided by 2π)

$$\dot{\zeta} = -(\lambda + \lambda_d^0)\zeta - \frac{\lambda r_e}{R^2}\gamma^5\langle K^3\rangle, \qquad (2.6)$$

where

$$\lambda = \frac{5\sqrt{3}}{8}\frac{\lambdabar r_e}{R^2}\gamma^5\langle|K|^3\rangle,$$
$$\lambda_d^0 = \frac{55}{48\sqrt{3}}\frac{\lambdabar r_e}{R^2}\gamma^7\left\langle\left(\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2|K|^3\right\rangle,$$

$\lambdabar = \hbar/m$ and $r_e = e^2/m$ are the Compton wavelength and classical radius of an electron.

Thus the particles get polarized along the direction of the average angular velocity of their rotation ($\zeta > 0$) or opposite to it ($\zeta < 0$), the polarization degree being

$$\zeta_{t\to\infty} = -\frac{8}{5\sqrt{3}}\frac{\langle K^3\rangle}{\left\langle|K|^3\left[1 + \frac{11}{18}\left(\gamma\frac{\partial n}{\partial\gamma}\right)^2\right]\right\rangle}. \qquad (2.7)$$

The characteristic time of the polarization is $T = (\lambda + \lambda_d^0)^{-1}$.

In an ideal storage ring, when closed orbits are plane and there is no longitudinal magnetic field on them, the precession axis is determined by vertical deviations of the momentum and cordinate, being independent of the energy ($\partial\mathbf{n}/\partial\gamma = 0$). The depolarizing influence of quantum fluctuations of the radiation is here due only to the vertical diffusion of the momentum, which is by a factor γ less than the longitudinal one. This diffusion can be essential only near spin resonances with a betatron frequency of vertical oscillations, and one is always able to operate far from them. Thus in the ideal storage ring (with a constant sign of field), the degree of equilibrium polarization and its characteristic time are

$$\zeta_{t\to\infty} = -\frac{5\sqrt{3}}{8} = -92\%,$$
$$T = \lambda^{-1} = \left[\frac{5\sqrt{3}}{8}\frac{\lambdabar r_e}{R^2}\gamma^5\langle|K|^3\rangle\right]^{-1}.$$

As in the ideal storage ring, only effects of direct spin-radiation interaction are essential. This result can be obtained using probabilities of spin-flips during radiation, first found for the homogeneous field in Ref. 9 and for the inhomogeneous one in Refs. 10 and 11.

For a magnetic field deviating from the ideal one, the precession axis becomes energy dependent. Quantum fluctuations of the energy lead to "trembling" of the precession axis, i.e., to additional spin diffusion. The depolarizing influence of quantum fluctuations of the radiation increases considerably when spin resonances become closer. Usually one must choose the particle energy and the magnetic-system parameters to avoid the resonances with the highest strength.

The situation is different for a sufficiently high energy or in the case of colliding beams, when resonances are unavoidable. The perturbation theory developed in Ref. 7 allows quantitative description of the depolarization in the resonace region. The rate of these processes is given by

$$\lambda_d^r = \pi\sum_k\langle|w_k|^2\delta(\nu - \nu_k)\rangle, \qquad (2.8)$$

where $\delta(\nu - \nu_k)$ is the Dirac delta function and brackets $\langle\cdots\rangle$ designate averaging over the equilibrium distribution in the beam. The formula (2.8) describes the depolarizing influence of multiple uncorrelated resonance passages during random walks of the tune shift $\nu - \nu_k$, arising from the radiative mixing of particle trajectories. For applicability of (2.8), the root mean square tune shift $\sigma_0 = \langle(\nu - \nu_k)^2\rangle^{1/2}$ must exceed the inverse time of the radiative mixing of a w_k harmonic, as well as that of the tune shift itself (Λ_k and Λ correspondingly), i.e.,

$$\sigma_0 \gg \Lambda_k, \Lambda.$$

The diffusive passing of the resonance is also required to be rapid,

$$\sigma_0\Lambda \gg |w_k|^2.$$

† In storage rings with a magnetic field strongly varying its direction, an additional polarizing effect arises from the spin dependence of the radiative friction force.

This condition is satisfied in the practically interesting region of possible radiative polarization where

$$\lambda_d^r \approx \lambda \ll \Lambda.$$

The complete equation for the polarization degree ζ has the form (2.6), in which λ_d^0 must be replaced by $\lambda_d = \lambda_d^0 + \lambda_d^r$.

III INFLUENCE OF DISTORTION OF THE EQUILIBRIUM MOTION

As already noted, different non-idealities of the magnetic system play essential roles in the processes of spin diffusion. At ultra-high energies $\gamma q_a/q_0 \gg 1$, the most dangerous are the influence of the radial magnetic field resulting in the vertical distortion of closed orbits and that of its radial gradient connecting vertical betatron oscillations with the radial motion as seen from (2.3). Depolarizing effects due to the longitudinal field are less than those of the radial field with the same magnitude by a factor $(\gamma q_a/q_0)^2$.

In this and the following chapter, the spin diffusion is considered in the energy region where the spread in $\nu = \gamma q_a/q_0$ arising from the spread in beam energy is much less than unity (the distance between spin resonances).

At the optimum energy choice, the frequency distance from the dangerous resonances is considerably larger than the spread of $\nu(\gamma)$. One can neglect here slow synchrotron oscillations and use the formula (2.7) describing a non-resonant mechanism of diffusion.

Consider now the influence of the radial magnetic field $B_x(\mathbf{r}_s)$ on the equilibrium orbit which can be caused, for example, by tilts of the magnets in bending regions or by vertical shifts of the the focusing elements. Study of these effects was started in Ref. 1 and continued in Ref. 2. Energy dependence of $\mathbf{n}$ in this case is mainly determined by the energy dependence of the precession frequency W_3, proportional to a large value of ν.

In the case of interest,

$$(W_1 + iW_2)dt = -q_a B_x dt = \nu \frac{B_x}{\langle B_z\rangle} d\theta,$$

$$\int_0^t W_3\, dt = \nu\left(\tilde{K} - \frac{x'}{R}\right),$$

where $\tilde{K} = \int_0^\theta K d\theta$.

The radial deviation x satisfies the equation

$$x'' + g_x x = \frac{\Delta\gamma}{\gamma} KR, \tag{3.1}$$

where $g_x = K^2 + R\,\partial B_z/\langle B_z\rangle\,\partial x$. Thus using (2.4) one obtains:

$$\begin{aligned}\left(\gamma\frac{\partial \mathbf{n}}{\partial\gamma}\right)^2 &= \frac{\nu^2}{\langle B_z\rangle^2}\left|\gamma\frac{\partial}{\partial\gamma}\left[e^{i\nu\tilde{K}}\int_{-\infty}^{\theta}\right.\right.\\ &\quad\left.\left.\times\left(1 + i\nu\frac{x'}{R}\right)B_x e^{-i\nu\tilde{K}}d\theta\right]\right|^2\\ &= \frac{\nu^4}{\langle B_z\rangle^2}\left|\int_{-\infty}^{\theta} K\left(\int_{-\infty}^{\theta} B_x e^{-i\nu\tilde{K}}d\theta\right)d\theta\right.\\ &\quad + \int_{-\infty}^{\theta} K(\theta)\int_{-\infty}^{\theta} B_x(\theta_1)\\ &\quad\left.\times\,\mathrm{Im}[f_x(\theta)f_x^{*\prime}(\theta_1)]e^{-i\nu\tilde{K}}d\theta d\theta_1\right|^2.\end{aligned}$$

Here $f_x(\theta) = e^{2\pi i\nu_x}f_x(\theta - 2\pi)$ is the solution of the Floquet equation (3.1) with the normalization $\mathrm{Im}\, f'_x f_x^* = 1$ (ν_x is frequency of radial betatron oscillations in units of revolution frequency). In the calculation of the energy derivative we have taken into account that x and x' are continuous at the moment of an energy jump [according to (3.1) only x'' shows a discontinuity].

The first term of (3.2) is due to energy dependence of the average frequency of spin precession. For harmonics with a number k close to ν, this term increases resonantly proportional to $(\nu - k)^{-2}$. The second term, connected with the betatron modulation of the precession frequency, is universely proportional to the first power of tune shift near the resonances $k \approx \nu$ and $k \approx \nu \pm \nu_x$. For high-energy storage rings, a large number of magnetic-system elements, as well as an alternating gradient, are characteristic (ν_x, $\nu_z \gg 1$). Therefore, for these machines the most dangerous effects are due to an energy dependence of the average precession frequency [the second term in (3.2) is about ν_x times less than the first one]. The strength of the resonances $\nu = k \pm \nu_x$ is about ν_x times less than that of the resonances $\nu = k$. Thus the radial field influence can be described by the following formula,†

† All the final formulae of this work (with the exception of those in Section VIII) concern the case of a usual storage ring with a constant sign of the vertical field ($K > 0$) and a large number of periods per particle revolution.

excluding small regions near the points $\nu = \pm\nu_x + k$

$$\frac{\langle K^3[\gamma(\partial\mathbf{n}/\partial\gamma)]^2\rangle}{\langle K^3\rangle} = \frac{\nu^4}{\langle K^3\rangle\langle B_z\rangle^2}\left\langle K^3 \left| \int_{-\infty}^{\theta} K \times \left(\int_{-\infty}^{\theta} B_x e^{-i\nu\tilde{K}} d\theta\right) d\theta \right|^2 \right\rangle$$

$$\approx \nu^2 \sum_k \frac{|w_k|^2}{(\nu - k)^2}, \tag{3.3}$$

where the w_k are Fourier-series harmonics of the following value:

$$\nu \frac{B_x}{\langle B_z\rangle} \exp\left[-i\nu \int_0^{\theta} (K - 1) d\theta\right] = \sum_k w_k e^{ik\theta}.$$

The quantity $B_x(\mathbf{r}_s)$ can be expressed through the vertical deviations z_s of the equilibrium orbit from the ideal plane;

$$B_x = \langle B_z\rangle \frac{z_s''}{R}.$$

The deviations z_s satisfy the equation

$$z_s'' + g_z z_s = RH, \tag{3.5}$$

where $g_z = -R\,\partial B_z/\langle B_z\rangle\,\partial x$ and the quantity H can be expressed through the radial fields on the ideal trajectory as

$$H = B_x(\mathbf{r}_0)/\langle B_z\rangle.$$

If the angle of spin rotation around the field during a transit of one periodicity element is small ($\nu \ll N$), then one can put $K = 1$, $w_k = \nu k^2 z_s$ and the formula (3.3) corresponds to that first obtained for this case in Ref. 1.

To preserve a degree of radiative polarization, it is necessary, as seen from (2.7) and (3.3), that the harmonics B_x nearest to ν should be sufficiently small, i.e.,

$$|B_x^k| \lesssim \langle B_z\rangle \frac{(\nu - k)^2}{\nu^2}. \tag{3.6}$$

To understand whether or not this requirement can be really fulfilled, one must take into account the following. The perturbative fields can be due to systematic inaccuracies of the elements of the storage-ring periodic system or those of a random nature (for example, random errors in the construction of the system elements, inaccuracy in the alignment of the elements, etc.). Distances between the lines in a frequency spectrum of systematic perturbations are of the order of the number of periodicity elements (in units of the revolution frequency). The storage-ring parameters must be chosen so that the frequency of spin precession is far enough from the frequencies of these perturbations.

The distance between spectral lines of random perturbations is of the order of the revolution frequency. However, owing to the fact that there is no correlation between perturbations of separate elements, the resulting amplitude of the perturbative field harmonics is approximately $\sqrt{Q}$ times less than that for systematic perturbations, where Q is the number of independent elements in the system.

Let us calculate now the effect of random perturbations. Assume that on the orbit there is a large number Q of regions where perturbative radial fields are completely uncorrelated (the occupation of the orbit by these regions must not be complete).

Usually the condition is fulfilled that in each element with the radial field, the angle of spin rotation around the vertical direction, as well as solutions of the Floquet equation of betatron oscillations, do not significantly change. We restrict ourselves to writing down formulae in this approximation when the perturbative field of one element can be represented by a delta function.

Strengths of the resonances $|w_k^{(n)}|$ due to the radial field having a magnitude $H_n = H(\theta_n)$ in dimensionless units in one nth region occupying a fraction of the orbit $\eta_n = l_n/2\pi R$ can be calculated by the following formula obtained from (3.4) and (3.5)

$$|w_k^{(n)}| = \frac{\nu}{2\pi R}\left|\int_0^{2\pi} z_s'' \exp[-i\nu(\tilde{K} - \theta) - ik\theta] d\theta\right|$$

$$\approx \frac{\nu^2}{2\pi R}\left|\int_0^{2\pi} K z_s' \exp(-ik\tilde{K}) d\theta\right|$$

$$= \frac{\nu}{2\pi}\left|\int_0^{2\pi} H F^{\nu=k} \exp(-ik\theta) d\theta\right|$$

$$\approx \nu\eta_n |H_n F_n^{\nu=k}|,$$

where F^ν is the characteristic function of a storage ring determined by the Floquet solution f_z of the equation for vertical betatron oscillations:

$$F^{\nu} = \frac{\nu}{2}\left[f_z \int_{-\infty}^{\theta} K f_z^{*} e^{-i\nu\tilde{K}} d\theta - f_z^{*} \int_{-\infty}^{\theta} K f'_z e^{-i\nu\tilde{K}} d\theta\right] e^{i\nu\theta}$$

$$= \frac{\nu}{2}\left\{\frac{f_z \int_{\theta-2\pi/p}^{\theta} K f_z^{*\prime} e^{-i\nu\tilde{K}}\, d\theta}{1 - \exp[i(2\pi/p)(\nu + \nu_z)]} - \frac{f_z^{*} \int_{\theta-2\pi/p}^{\theta} K f'_z e^{-i\nu\tilde{K}} d\theta}{1 - \exp[i(2\pi/p)(\nu - \nu_z)]}\right\}, \quad (3.7)$$

where p is the number of the magnetic-system superperiods per one particle revolution. The function F^{ν} grows near the resonances $\nu = mp \pm \nu_z$; between them its absolute value is unity or less, with the exception of comparatively small regions of ν where it can become of the order of ν/p. The integrand (3.7) contains the Floquet function in the bending sections of the orbit only. The value of $|F^{\nu}|$ is proportional to $|f_z|$, i.e., the main contribution to the strength $|w_k|$ is provided by regions with large β functions.

For a round storage ring ($K = 1$) and uniform focusing, when the absolute value of the Floquet function $|f_z|$ is slightly dependent on azimuth, F^{ν} is approximately equal to

$$F^{\nu} = \frac{\nu^2}{\nu^2 - \nu_z^2}.$$

The resulting influence of all the regions due to the absence of their correlation is given by a sum

$$\overline{|w_k|^2} = \sum_{n=1}^{Q} \overline{|w_k^{(n)}|^2} = \nu^2 \sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{\nu=k}|^2. \quad (3.8)$$

To obtain the conditions for radiative polarization existence, one must sum over k in (3.3). Harmonics of number k close to ν give the main contribution. As seen from (3.7), the values of $|F^{\nu=k}|$ and hence of $|w_k|$ depend on k very slightly, excluding the resonance cases when $[\nu]$, the integer part of ν, is close to $\pm\nu_z + mp$ and $|w_{k=[\nu]}|$ grows considerably. Thus, by using the formula

$$\sum_{k=-\infty}^{\infty} (\nu - k)^{-4} = \frac{\pi^4}{3} \frac{(1 + 2\cos^2 \pi\nu)}{\sin^4 \pi\nu},$$

we find that for the radiative polarization with average degree higher than 50% to exist, the condition

$$\sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{[\nu]}|^2 \le \frac{54}{11\pi^4} \frac{\sin^4 \pi\nu}{\nu^4(1 + 2\cos^2 \pi\nu)} \quad (3.9)$$

must be satisfied. This condition has a probability meaning and determines the accuracy of construction and mounting of the magnetic-system elements, allowing also an estimate of technical difficulties arising in the construction of storage rings used for obtaining polarized electrons and positrons.

For example, at vertical shifts of the lenses by Δz_L, a radial field with magnitude $H = g_z \Delta z_L/R$ appears in the region of the lenses. Assuming $\eta_n |g_z|_n |F_n^{[\nu]}|$ to be equal for all Q_L of the lenses, one can obtain from (3.9) the condition for allowable vertical displacements of the lenses ($\nu = [\nu] + 1/2$):

$$\sqrt{\overline{(\Delta z_l)^2}} \le \frac{\sqrt{54/11}}{\pi^2} \frac{R\sqrt{Q_L}}{|g_z|_n |F_n^{[\nu]}| \varepsilon \nu^2}, \quad (3.10)$$

where $\varepsilon = \sum_{n=1}^{Q_L} \eta_n$ is the fraction of the orbit occupied by all the lenses. At random inclinations of the magnets with the vertical field by an angle α_M, the radial field $H = \alpha_M K$ appears. If all Q_M regions are the same, a formula can be obtained for the required accuracy of magnet alignment†

$$\sqrt{\overline{\alpha_M^2}} \le \frac{\sqrt{54/11}}{\pi^2} \frac{\sqrt{Q_M}}{\nu^2 |F^{[\nu]}|}. \quad (3.11)$$

For random perturbations, all the harmonics of the field H up to order of $1/n$ are approximately identical. In this case, it is possible to rewrite (3.9) as the condition for maximum possible vertical deviations z_s of the equilibrium orbit, which are sometimes interesting. The formula for the mean probable deviation $\overline{z_s^2}$ at azimuth θ with random radial fields in Q regions has the form[12]

$$\overline{z_s^2} = \frac{\pi^2}{2} \frac{R^2 |f_z(\theta)|^2}{\sin^2 \pi\nu_z} \sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |f_z|_n^2. \quad (3.12)$$

For simplicity, we assume that $|f_z|_n$ and $|F_n^{[\nu]}|$ are equal in all Q regions. Then, substituting (3.12) in (3.9), one obtains

$$|\sin \pi\nu^z| \frac{\sqrt{\overline{z_s^2}}}{|f_z(\theta)|} \le \frac{\sqrt{27/11}}{\pi} \frac{R|f_z|_n}{|F_n^{[\nu]}|} \times \frac{\sin^2 \pi\nu}{\nu^2\sqrt{1 + 2\cos^2 \pi\nu}}. \quad (3.13)$$

† Strictly speaking, in obtaining the condition for α_M it is not correct to approximate the bending region by a delta function. If one takes into account the finite length of the magnet, the formula (3.9) is obtained where $|F_n^{[\nu]}|$ in the magnet center must be replaced by $|F_n^{[\nu]}| \to \int_n H F^{[\nu]} \exp(-i\nu\theta) d\theta/(2\pi\eta_n)$. This correction is usually numerically small.

This condition determines the acceptable deviation of the equilibrium orbit.†

Note that when ν_z tends to an integer k the condition for radiative polarization remains the same despite the increase $\sqrt{\overline{z_s^2}} \sim |\nu_z - k|^{-1}$ (with the exception of the resonances $\pm\nu_z = mp + [\nu]$).

Conditions (3.9) and (3.13) have been written for random perturbations. As harmonics of the radial field essentially influencing vertical deviations and spin diffusion are different, it is possible to have for special perturbations arbitrary deviations of the equilibrium orbit from the plane and independently a high degree of beam polarization.

IV INFLUENCE OF THE RADIAL FIELD GRADIENT‡

Consider now the effects due to the part of the radial field $\Delta B_x = x\,\partial B_x/\partial x + \Delta z\,\partial B_x/\partial z$, inhomogeneous over the beam section. Here x and Δz satisfy the equations

$$\begin{aligned} x'' + g_x x &= RK\frac{\Delta\gamma}{\gamma} - \kappa\Delta z, \\ (\Delta z)'' &= \frac{R}{\langle B_z\rangle}\Delta B_x = -g_z\Delta z + \kappa x. \end{aligned} \tag{4.1}$$

The parameter $\kappa(\theta) = R\,\partial B_x/(\langle B_z\rangle\,\partial x)$ as a function of azimuth θ can contain harmonics of any number k. A solution for small coupling κ can be written as

$$\begin{aligned} x &= c_x f_x + c_x^* f_x^* + \frac{\Delta\gamma}{\gamma}\psi_x, \\ \Delta z &= c_z f_z + c_z^* f_z^* + c_x f_{xz} + c_x^* f_{xz}^* + \frac{\Delta\gamma}{\gamma}\psi_z, \end{aligned} \tag{4.2}$$

where c_x, c_z are complex amplitudes of free x and z osillcations with frequencies $\pm\nu_x + mp$, $\pm\nu_z + mp$. The terms $c_x f_{xz} + c_x^* f_{xz}^*$ and $(\Delta\gamma/\gamma)\psi_z$ describe constrained vertical motion with frequencies $\pm\nu_x + k$ and k respectively. The transverse x and z motions are excited by energy quantum fluctuations. At the moment when a quantum is radiated, x, x' and z, z' remain continuous, while the amplitudes c_x and c_z undergo a jump together with $\Delta\gamma$. Therefore the constrained and free parts of z-oscillations at the excitation of a vertical size by energy fluctuations are of the same order of magnitude. Thus, a gradient $\partial B_x/\partial x$ leads to appearance in the spin motion of the resonances with the frequencies $\pm\nu_z + mp$, $\pm\nu_x + mp$, k, near which the spin diffusion increases considerably.

Formulae for the rate of the diffusion arising due to a gradient $\partial B_x/\partial x$ on the orbit have been obtained earlier in Ref. 2. We discuss this question in more detail because $\partial B_x/\partial x$ can be used to increase vertical size in order to obtain a higher luminosity of colliding beams. Here a question arises about the relation between an established beam size and the spin-diffusion rate, i.e., a question about an optimum luminosity of the polarized beams.

Now we present briefly how a general formula can be derived. The direct calculation of the quantity

$$\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2 = \nu^2\left|\gamma\frac{\partial}{\partial\gamma}\int_{-\infty}^{\theta}(\Delta z)''e^{-i\nu\tilde{K}}d\theta\right|, \tag{4.3}$$

with a substitution (4.12) is rather tedious. The calculation can be simplified by using integration by parts and the condition of continuity of transverse motion at the moment of quanta emission ($\partial z/\partial\gamma = \partial z'/\partial\gamma = \partial x/\partial\gamma = \partial x'/\partial\gamma = 0$). The equation (4.1) and its solution can be conveniently written in the equivalent form, introducing a time-dependent amplitude $A_z(\theta)$,

$$\begin{aligned} z &= A_z f_z + A_z^* f_z^*, \\ A_z' &= \frac{1}{2i}\kappa x f_z^*. \end{aligned} \tag{4.4}$$

Then (4.3) takes the form

$$\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2 = \nu^2\left|\gamma\frac{\partial}{\partial\gamma}\int_{-\infty}^{\theta}\kappa x F^{\nu}e^{-i\nu\theta}d\theta\right|^2, \tag{4.5}$$

† Formulae (3.9) and (3.13) are obtained if one assumes that perturbations of separate elements with respect to the plane orbit are completely uncorrelated. In the general case, if correlations are present, (3.9) and (3.13) determine acceptable deviations of the radial field and the vertical coordinate from the values on the average orbit obtained by smoothing over a betatron period $2\pi R/\nu_z$. In a length considerably exceeding $2\pi R/\nu_z$ the deviation of such an average orbit from the plane can exceed the mean square deviation given by (3.12).

‡ Depolarization effects due to the longitudinal field can be easily calculated in a way similar to that for B_x and $\partial B_x/\partial x$. However, they can be essential only if special longitudinal fields of large magnitude are introduced.[2]

where F^ν is determined by (3.7). Representing the x motion in a form similar to (4.4) one finally obtains a formula, completely describing spin diffusion in presence of the field gradient κ.

$$\left\langle K^3\left(\gamma\frac{\partial\mathbf{n}}{\partial\gamma}\right)^2\right\rangle/\langle K^3\rangle$$

$$= \frac{\nu^2}{4\langle K^3\rangle}\left\langle K^3\left|\int_{-\infty}^{\theta} K\left[f_x^*\int_{-\infty}^{\theta}\kappa f_x F^\nu e^{-i\nu\theta}d\theta - f_x\int_{-\infty}^{\theta}\kappa f_x^* F^\nu e^{-i\nu\theta}d\theta\right]d\theta\right|^2\right\rangle. \quad (4.6)$$

It can be seen that a main influence is due to harmonics of number k closest to $\nu + mp$ and $\nu + mp \pm \nu_x$, whereas the vertical size is evidently very sensitive to the resonances $\nu_z - \nu_x + mp \approx k$. For example, when the frequencies ν_x and ν_z get closer, the size grows proportionally to $|\nu_x - \nu_z|^{-1}$, the diffusion rate being practically unchanged. This phenomenon can easily be understood from general considerations. For spin diffusion, z-motion deviations are essential, arising after quantum emission during a time of approximately $|\nu - \nu_k|^{-1}$. These deviations do not depend evidently (at $|\nu - \nu_k| \gg |\nu_z - \nu_x|$) on the closeness of ν_z and ν_x and are determined by the value κ of the coupling. The jumps of the free part of z oscillations, as well as those of the constrained one, increase proportionally to $|\nu_z - \nu_x|^{-1}$, but their sum remains the same during a time $|\nu - \nu_k|^{-1}$ due to the continuity of z, z', x, x' at the moment of energy fluctuation. The independence of the spin-diffusion rate on the resonances in the orbital motion is a general property of the diffusion mechanism considered. We come to the important conclusion that beam size can be increased without changing the radiative polarization stability. To this end "natural" perturbations connected with inaccuracy in construction of a focusing system can be used, as well as those specially introduced and not containing harmonics close to $\nu + mp$ and $\nu + mp \pm \nu_x$.

By means of (4.6), one can formulate the requirements necessary to provide radiative polarization on the accuracy with which the focusing elements are made and mounted. If, for example, $\kappa(\theta)$ is a random function given by Q uncorrelated regions, then to provide an average radiative polarization higher than 50% the following condition should be fulfilled

$$\sum_{n=1}^{Q}\eta_n^2\overline{\kappa_n^2}|F_n^\nu|^2\left\{[\sin^{-2}\pi(\nu+\nu_x)+\sin^{-2}\pi(\nu-\nu_x)]\times\frac{|f_x|_n^2}{\langle K^3\rangle}\left\langle K^3\left|\int_{-\infty}^{\theta}Kf_x d\theta\right|^2\right\rangle + 4(\psi_x)_n^2\sin^{-2}\pi\nu\right\} \leq \frac{72}{11\pi^2\nu^2}, \quad (4.7)$$

where $\psi_x(\theta) = \mathrm{Im}(f_x\int_{-\infty}^{\theta}Kf_x^* d\theta)$ is the ψ function describing radial deviations of the closed orbits. It can be seen that the main contribution is due to those regions where $\kappa^2|f_x f_z|^2$ is the largest.

A radial gradient of the radial field can arise from random turns of the magnetic lenses around the orbit direction. If the lenses are turned at a small angle α_L, a gradient $\kappa = \alpha_L g_z$ arises. Assuming for simplicity that all Q_L regions are identical, one obtains from (4.5) a formula for the required accuracy of a lens orientation ($\nu = [\nu] + 1/2$, $\nu_x = [\nu_x] + 1/4$).

$$\sqrt{\overline{\alpha_L^2}} \leq \frac{\sqrt{9/11}}{\pi}\frac{\nu_x^2\sqrt{Q_L}}{\nu\varepsilon|g_z|_n|F_n^\nu|}. \quad (4.8)$$

For perturbations κ uncorrelated along the orbit a relation can be obtained between the spin-diffusion rate and the vertical size excited by energy fluctuations due to the coupling of z and x motion. The formula for the mean-probable value $\overline{(\Delta z)^2}$ has the form ($|\nu_z - \nu_x| \ll 1$)

$$\overline{(\Delta z)^2} = \frac{11}{576\cdot\pi^2}\frac{R^2|f_z(\theta)|^2}{|\nu_z-\nu_x|^2}\frac{\langle K^3|\int_{-\infty}^{\theta}Kf_x d\theta|^2\rangle}{\langle K^3\rangle}\cdot\left(\frac{\lambda}{\Lambda_x}+\frac{\lambda}{\Lambda_z}\right)\sum_{n=1}^{Q}\eta_n^2\kappa_n^2|f_x f_z|_n^2 \quad (4.9)$$

where Λ_x and Λ_z are decrements of radiative damping and λ is the polarization decrement [see (2.6)]. We have used here that $(d/dt)(\delta\gamma)^2/\gamma^2 = \frac{11}{9}\lambda$. Assuming $|f_x|_n$, $|f_z|_n$ and $|F_n^\nu|$ equal at all Q regions, one obtains from (4.7) and (4.9) at $\nu = [\nu] + \frac{1}{2}$, $\nu_x = [\nu_x] + \frac{1}{4}$ a restriction on the maximum size $\overline{(\Delta z)^2}$ at which the radiative polarization is preserved

$$|\nu_z - \nu_x|\sqrt{\overline{(\Delta z)^2}}/|f_z(\theta)| \leq \frac{R}{4\pi}\frac{|f_z|_n}{|F_n^\nu|}\left(\frac{\lambda}{\Lambda_x}+\frac{\lambda}{\Lambda_z}\right)^{1/2}. \quad (4.10)$$

This formula gives the relation between the vertical size and the spin-diffusion rate under the conditions when vertical oscillations are excited by quantum fluctuations of the energy because of the coupling of x and z motion due to uncorrelated perturbations.

Another possible mechanism of spin diffusion can be provided by random hits in the vertical direction exciting vertical betatron oscillations. Such a diffusion mechanism has been considered in the old paper.[2] It has been shown for electron and positron storage rings that fluctuations of angle of the quantum emission give a small effect. Other real factors (for example, scattering of the particles inside a bunch, collisions with the residual gas) also give negligible contribution to the vertical size and the spin–diffusion rate.†

In papers[13,14] a phenomenological model has been used in consideration of the depolarization effects, in which the vertical motion was described by free oscillations with frequencies $\pm\nu_z + mp$. In such a model, there are no resonances $\nu = k \pm \nu_x$, $\nu = k$; on the other hand, the diffusion rate appeared to be overestimated in a practically interesting case when betatron frequencies are close to each other ($|\nu_z - \nu_x| \ll 1$).

V RESONANT SPIN DIFFUSION IN THE STORAGE RING MAGNETIC FIELD

When the energy increases the spread of precession frequencies ν grows because of the growth of the absolute energy spread of the beam. The frequency ν as a function of energy undergoes diffusion variation by quantum fluctuations of the radiation and synchrotron oscillations with a frequency ν_γ

$$\nu = \bar{\nu} + \Delta \cos \psi_\gamma$$

where $\Delta = \nu(\Delta\gamma)_0/\gamma$ is the amplitude and ψ_γ the phase of synchrotron oscillations of the precession frequency ($\psi'_\gamma = \nu_\gamma$).

† In Ref. 15 the transverse hits by a field of the colliding beam were suggested to be uncorrelated from one revolution to another. However, such a situation is not practically real, because it results in destruction of the natural beam state with a sharp decrease of luminosity. Under optimum conditions depolarization effects due to the colliding beam (as well as beam-beam effects in orbital motion) are caused by nonlinear resonances in betatron oscillations, the stochasticity of whose influence is again connected with the quantum diffusion of energy (see Section VI).

The mean-square precession-frequency spread is equal to

$$\sigma_\nu^2 = \langle \Delta^2 \rangle = 2\nu^2 \left\langle \left(\frac{\Delta\gamma}{\gamma} \right)^2 \right\rangle = \frac{11\nu^2\lambda}{18\Lambda_\gamma}$$

where λ is the polarization rate and Λ_γ the decrement of the energy radiative damping.

In the case $\sigma_\nu \ll 1$, one can be far from spin resonances by choosing appropriately frequencies of precession and orbital motion. Here one can neglect the synchrotron energy modulation. In the case considered here, when $\sigma_\nu \gtrsim 1$, the resonances are periodically passed during synchrotron oscillations. In the region of possible radiative polarization, the condition of rapidity of each passing must be fulfilled beyond any doubt ($|w_k|^2 \ll \nu_\gamma \Delta$).

The problem of periodic passings of resonances has been solved in Ref. 8, where it has been shown that at completely correlated successive rapid passings, depolarization is possible only in the narrow regions of spin resonances with a synchrotron frequency of oscillations. Stochastic variation of the precession frequency due to quantum fluctuations of the energy violates the phase correlation of the passings resulting in the spin diffusion.[2]

First we consider a case when the shift of the spin-precession phase during one period of synchrotron oscillations caused by diffusion ν is much less than one. Successive passings are correlated if the following condition is satisfied‡[2]

$$\frac{d}{dt} \overline{(\delta\nu)^2/\nu_\gamma^2} \approx \frac{\nu^2\lambda}{\nu_\gamma^2} \ll \nu_\gamma \qquad (5.1)$$

In Ref. 16 it has been assumed that there is already no correlation between successive passings at the energies of the order of 30 to 100 GeV. This led to a too pessimistic estimate of the possibility of providing radiative polarization in storage rings at these energies. According to (5.1), at these energies the correlation is conserved during a large number of synchrotron oscillations and the diffusion rate is considerably smaller.

Provided the condition (5.1) is satisfied, one can use formulae (2.4) for quantitative description of the depolarization process, describing the nonresonant diffusion mechanism, in which modulation resonances with a synchrotron oscillation

‡ We assume that the synchrotron frequency does not exceed the revolution frequency of the particles ($\nu_\gamma \lesssim 1$).

frequency ν_γ must be taken into account. The condition of obtaining the radiative polarization higher than 50% has the form†

$$\frac{11}{18}\nu^2\sum_{k,m}\frac{|w_k|^2\langle T_m^2(\Delta/\nu_\gamma)\rangle}{[(k-\bar{\nu}-m\nu_\gamma)^2-\nu_\gamma^2]^2}\leq 1 \quad (5.2)$$

Here T_m is a Bessel function and the brackets $\langle\cdots\rangle$ designate averaging over the equilibrium distribution of amplitudes Δ in the beam. For example, for a Gaussian distribution over squared amplitudes Δ in the beam one obtains

$$\langle T_m^2\rangle = I_m\left(\frac{\sigma_\nu^2}{2\nu_\gamma^2}\right)\exp\left(-\frac{\sigma_\nu^2}{2\nu_\gamma^2}\right), \quad (5.3)$$

where I_m is the Bessel function of imaginary argument. In (5.2), the influence of resonances $\bar{\nu}=k$ has been taken into account only for those whose strengths $|w_k|$ considerably exceed those of the resonances with betatron frequencies.

If $\sigma_\nu^2 \ll \nu_\gamma$, the contribution of modulation resonances is very small. If in this case one takes into account synchrotron oscillations of the precession frequency, it results only in the appearance of sharp lines in regions of modulation resonances in which radiative polarization can be absent.

In the inverse situation $\sigma_\nu^2 \gg \nu_\gamma$, the condition for obtaining polarized beams can change essentially. At $\nu_\gamma \ll 1$ it becomes more severe. One must note here that with an increase of the synchrotron-oscillation frequency the influence of modulation resonances substantially decreases. Moreover, the condition of obtaining polarization becomes even less severe than at $\Delta = 0$ if the synchrotron oscillation frequency is chosen to be much greater than 1 ($\nu_\gamma \gg 1$). This is possible in the storage rings at very high energies.

At very high energies, the conditions (5.1) can be violated and $\sigma_\nu \gg \nu_\gamma$. In this case, synchrotron passings of a resonance are completely uncorrelated and the diffusion rate is independent of the value of the synchrotron frequency ν_γ. Spin diffusion is described by (2.8). The same formula can be applied in the case when successive passings are correlated, but modulation resonances $\bar{\nu} = k + m\nu_\gamma$ overlap because of the spread of the synchrotron frequency, i.e. if the condition is held that

$$(\Delta\nu_\gamma) \gg \nu_\gamma^2. \quad (5.4)$$

† We assume that the spread of the frequency shift $(\bar{\nu} - k + m\nu_\gamma)$ due to the spread of frequencies $\bar{\nu}$ and ν_γ in the beam is much less than the distance between the resonances ν_γ.

For a Gaussian distribution over the energy deviations from equilibrium, one has ($\nu = \bar{\nu} + \Delta\nu$)

$$\langle\delta(\nu-k)\rangle = \int_{-\infty}^{\infty}\delta(\bar{\nu}-k+\Delta\nu) \times \frac{\exp[-(\Delta\nu)^2/\sigma_\nu^2]}{\sqrt{\pi}\sigma_\nu}\,d\Delta\nu = \frac{\exp[-(\bar{\nu}-k)^2/\sigma_\nu^2]}{\sqrt{\pi}\sigma_\nu}.$$

We shall assume that resonance strengths in the interval $|\bar{\nu}-k| \lesssim \sigma_\nu$ are approximately equal and their value $|w_k|$ in (1.3) can be taken out the summation sign.

At $\sigma_\nu \ll 1$, the spin diffusion due to uncorrelated passings is exponentially small and can be comparable to the usual one caused by "trembling" of the precession axis n. Therefore the condition for obtaining radiative polarization $\lambda_d^0 + \lambda_d^r \leq \lambda$ at $\sigma_\nu \ll 1$ will be written as (at the point $\sin^2\pi\bar{\nu} = 1$)

$$\frac{11\pi^4}{54}\nu^2|w_{[\nu]}|^2\left\{1+\frac{108\exp(-2\sigma_\nu)^{-2}}{11\pi^3\sqrt{\pi}\nu^2\lambda}\right\}\leq 1. \quad (5.5)$$

At $\sigma_\nu \gg 1$, the nonresonant diffusion is always negligibly small and the condition for obtaining polarized electrons and positrons is

$$\pi|w_k|^2 \leq \lambda. \quad (5.6)$$

One should also note that at $\sigma_\nu \gg 1$, there is no resonance dependence of spin diffusion on energy. With energy increase, the conditions for obtaining radiative polarization can be satisfied more easily due to the growth of the polarization rate λ (in units of the revolution frequency).

VI STABILITY OF THE POLARIZATION OF COLLIDING BEAMS

The problem of stability of radiative polarization is very important in practice for colliding beams. Because of the strong nonlinearity of the collective field of the colliding particles, strengths of the spin resonances connected with them decrease slightly with an increase of their numbers. Frequencies of the orbital motion become dependent on the amplitudes of particle oscillations around the equilibrium orbit. At high energies, the average precession frequency is characteristic of synchrotron oscillations with large amplitudes. Under these conditions, there are always resonating harmonics determining the depolarizing influence of the colliding beam.

The investigation of spin diffusion during collisions presented here is based on the results of Ref. 4. We have additionally considered the case when synchrotron modulation of the precession frequency is predominant. Also discussed are the possibilities of conserving radiative polarization under the conditions of maximum luminosity.

Under these conditions, particles of the colliding beam do not essentially influence the beam size. The size is mainly due to radiative effects caused by the coupling of the vertical and radial motion. The colliding-beam influence is connected with the betatron oscillation nonlinear resonances

$$\nu = \nu_k \equiv k_\theta + k_z \nu_z + k_x \nu_x, \qquad (6.1)$$

with $|k_x| + |k_z| \geq 2$. Effects of resonances with $|k_x| + |k_z| \geq 2$ are connected with the non-ideal character of a focusing system, because the colliding beam only slightly changes the orbital motion. It is evident that in situations when resonances with the main perturbation harmonics with $|k_x| + |k_z| \leq 1$ are inevitable (for example, in the case when the amplitude of synchrotron oscillations of the precession frequency is of the order or larger than the distance between linear resonances, which is about unity for random perturbations), the depolarization effects of the colliding beam cannot be the most important.

We shall write down the condition for conserving radiative polarization for head-on collisions, when the equilibrium orbits of the bunches in the interaction region coincide. The formula (6.1) for resonance strengths has the form

$$|w_k| = \frac{\nu}{\langle B_z \rangle} \langle 2B_x \exp(-i\nu\tilde{K} + i\nu\theta - i\psi_k)\rangle$$

$$= \frac{\nu}{R} \langle z'' \exp(-i\nu\tilde{K} + i\nu\theta - i\psi_k)\rangle, \qquad (6.2)$$

where $\psi'_k = \nu_k$, B_x is the magnetic field of the colliding bunches, and averaging is performed at a fixed energy. The vertical motion obeys the equation

$$z'' + g_z z = \kappa x + 2(R/\langle B_z \rangle) B_x(z, x, \theta). \qquad (6.3)$$

Assuming that the coupling of $z - x$ motion is small (in this case one can neglect the inverse influence of the vertical motion upon the radial one), during the solution of (6.3) by perturbation theory one must substitute into the function $B_x(z, x, \theta)$ the solutions of (4.2) for z and those of (3.1) for x.

Calculations of the strengths for the higher-order resonances caused by the colliding beam can be conveniently performed by using the formula following from (6.2) and (6.3)

$$w_k = 2\nu \langle B_x F^{\nu_k} e^{-i\nu_k\theta}\rangle / \langle B_z \rangle, \qquad (6.4)$$

where $F^{\nu_k} \equiv F^{\nu=\nu_k}$ is a periodic function of azimuth (3.7).

Calculation of $\langle |w_k|^2 \rangle$ is rather tedious in the general case and to this end digital computation proves convenient. For illustration, consider as electron-positron colliding beam in a one-path storage ring. We restrict ourselves to a study of spin resonances with vertical oscillations ($k_x = 0$, $|k_z| \geq 2$) and assume that $B_x = B_x(z, 0, \theta)$. The azimuthal distribution of the particle density is assumed to be proportional to delta functions, whose number per turn is half the number p of storage-ring intersection regions, which for simplicity is equal to the number of magnetic-system superperiods. In the intersection regions, only free vertical oscillations $z_{\theta=0} = a_z \cos\psi_z$ will be taken into account, the constrained oscillations being neglected,† where the amplitude a_z is determined by the β-function value in the intersection region ($a_z = 2|c_z||f_z|_0$). Such approximations reveal the peculiarities of the influence of colliding particles on polarization.

In this approximation, at even $|k_z|$ the strength w_k equals zero. At odd $|k_z|$, for a Gaussian distribution in the transverse section of the beam, the quantity $\langle |w_k|^2 \rangle$ takes the following value ($|k_z| \gg 1$)

i) at $\langle a_z^2 \rangle \ll k_z^2 z_0^2$

$$\langle |w_k|^2 \rangle = \frac{8}{\pi^2} \sqrt{\frac{2}{\pi |k_z|^3}} \frac{N_e^2 r_e^2 \nu^2 |F^\nu|^2}{\gamma^2 x_0^2} \times \frac{(1 + 2\langle a_z^2 \rangle / z_0^2)^{1/4}}{(1 + \langle a_z^2 \rangle / z_0^2)^{1/2}} \times \left[\frac{\langle a_z^2 \rangle}{\langle a_z^2 \rangle + z_0^2 + z_0\sqrt{z_0^2 + 2\langle a_z^2 \rangle}}\right]^{|K_z|},$$

ii) at $k_z^2 x_0^2 \gg \langle a_z^2 \rangle \gg k_z^2 z_0^2$ (6.5)

$$\langle |w_k|^2 \rangle = \frac{16}{\pi^3 |k_z|^2} \frac{N_e^2 r_e^2 \nu^2 |F^\nu|^2}{\gamma^2 x_0^2},$$

iii) at $\langle a_z^2 \rangle \gg k_z^2 x_0^2$

$$\langle |w_k|^2 \rangle = \frac{4}{\pi^2} \frac{N_e^2 r_e^2 \nu^2 |F^\nu|^2}{\gamma^2 \langle a_z^2 \rangle} \ln \frac{\langle a_z^2 \rangle}{k_z^2 x_0^2}.$$

† Taking account of constrained oscillations in the resonant diffusion mechanism (by contrast to the nonresonant one) cannot lead to a substantial variation of the effect.

Here N_e is the number of colliding particles in the storage ring, $2x_0$ and $2z_0$ are the radial and vertical size of the colliding beam in the intersection regions, and $\langle a_z^2\rangle$ is the mean square amplitude of vertical betatron oscillations in the intersection regions.

The spread of betatron frequencies ν_z and ν_x introduced by the colliding particles is of the order of the frequency shift $(\Delta\nu_z)_s$ and $(\Delta\nu_x)_s$ of the equilibrium particle. For a Gaussian distribution of the particle density over the transverse section of the colliding beam one obtains

$$(\Delta\nu_z)_s = \frac{2N_e r_e R|f_z|_0^2}{\pi\gamma z_0(z_0 + x_0)}, \quad (\Delta\nu_x)_s = \frac{2N_e r_e R|f_x|_0^2}{\pi\gamma x_0(z_0 + x_0)}, \tag{6.6}$$

where $|f_z|_0$ and $|f_x|_0$ are the values of the Floquet functions in the intersection regions.

The σ_0 value equal to the spread of the frequency shift $|\nu - kp - k_z\nu_z|$ averaged over the phase oscillations is determined by the frequency shift of vertical oscillations (due practically to the small vertical size)

$$\sigma_0 \simeq |k_z(\Delta\nu_z)_s|.$$

Under the conditions when† $\sigma_\nu \ll \sigma_0$, synchrotron oscillations of ν can be neglected and the diffusion rate from (2.8) is

$$\lambda_d = \pi\sum_k \frac{|w_k|^2 \exp(-a_z^2/\langle a_z^2\rangle)}{|k_z|\langle a_z^2\rangle|\partial\nu_z/\partial a_z^2|}, \tag{6.7}$$

where all the quantities under the summation sign are taken at the points a_z^2, for which $\nu = \nu_k(a_z^2)$. λ_d in (6.7) is approximately

$$\lambda_d \approx \pi\frac{\langle|w_k|^2\rangle}{\sigma_0}.$$

Here the $\langle|w_k|^2\rangle$ are the harmonics with maximum value for which the values $\nu - \nu_k(a_z^2)$ belong to the interval 0 to $k_z(\Delta\nu_z)_s$. The width of the region of most effective influence of each harmonic is usually about σ_0.

The vertical sizes of the colliding beams are of the same order at maximum luminosity. Spin diffusion in this case occurs with a maximum rate if the resonance $\nu = kp + k_z\nu_z(a_z^2)$ (for the numbers k, k_z providing maximum $|w_k|^2$ at a width σ_0) is realized for an amplitude equal to ($|k_z| \gg 1$)

$$a_z^2 = |k_z|\langle a_z^2\rangle\frac{z_0}{\sqrt{z_0^2 + 2\langle a_z^2\rangle}}.$$

The radiative polarization is conserved if the condition is fulfilled that

$$\lambda_d^{\max} = \frac{4}{\pi}\frac{N_e r_e \nu^2}{\gamma R}\frac{|F^{\nu_k}|_0}{|f_z|_0} \times A\left[\frac{\langle a_z^2\rangle}{\langle a_z^2\rangle + z_0^2 + z_0\sqrt{z_0^2 + 2\langle a_z^2\rangle}}\right]^{|k_z|} \le \lambda, \tag{6.8}$$

where A is a number depending on the value of $a_z^2 = |k_z|(\langle a_z^2\rangle z_0/\sqrt{z_0^2 + 2\langle a_z^2\rangle})$

$$A = \frac{\langle a_z^2\rangle}{|k_z|x_0^2}\frac{z_0^2}{z_0^2 + \langle a_z^2\rangle} \qquad \text{at } a_z^2 \gg x_0^2,$$

$$A = \left[\frac{\pi\langle a_z^2\rangle}{|k_z|^3 x_0^2}\right]^{1/2}\frac{[1 + 2\langle a_z^2\rangle/z_0^2]^{1/4}}{[1 + \langle a_z^2\rangle/z_0^2]} \qquad \text{at } x_0^2 \gg a_z^2 \gg z_0^2,$$

$$A = \frac{4z_0^3}{|k_z|^3 x_0\langle a_z^2\rangle} \qquad \text{at } z_0^2 \gg a_z^2.$$

The equality in (6.8) determines the maximum number $|k_z|_{\max}$ of the spin resonances "at work". Sufficiently close to the resonances with $|k_z| \le |k_z|_{\max}$, beam depolarization occurs. The "dangerous" interval in which it happens is approximately equal to

$$\Delta\nu \approx \sigma_0 \approx |k_z(\Delta\nu_z)_s|.$$

At large $|k_z|$, the ν distribution of the resonances is almost equally probable. If the sum of dangerous intervals ν does not exceed the number of storage-ring superperiods

$$\sum_{|k_z| \le |k_z|_{\max}} |\Delta\nu| \lesssim |k_z|_{\max}^2(\Delta\nu_z)_s \lesssim p, \tag{6.9}$$

then energy intervals always exist in which radiative polarization is conserved.‡

† Note that if constrained x and z oscillations are taken into account, the frequency shift $(\Delta\nu_z)_s$ undergoes slow variations with the frequency ν_γ of phase oscillations. However, in the first approximation these oscillations can be neglected.

‡ If ν_z is chosen close to a rational number m/n with low n ($n < |k_z|_{\max}$), spin resonances superimpose on each other and the possible frequency shift $(\Delta\nu_z)_s$ can be additionally increased compared to (6.9). However, at low n the stability of collisions can become worse.

This condition together with (6.8) determining $|k_z|_{\max}$, allows us to find the maximum luminosity that will not destroy the radiative polarization.

At high energies, the cases $\sigma_\nu \gg \sigma_0$ must be considered as well. The effective width of spin resonances becomes then of the order σ_ν and the probability to enter this interval of large strength harmonics increases. One should note that at such high energies the polarization rate λ also increases considerably (in units of the revolution frequency). If successive passings of the (6.1) resonance at the synchrotron precession modulation are completely uncorrelated, this is valid at $\bar{\sigma}_0 = \max(\sigma_0, \nu^2\lambda/\nu_\gamma^2) \gg \nu_\gamma$. Then the resonances with the synchrotron frequency are not revealed. The condition of polarization conservation for a Gaussian distribution over $\Delta\nu = \nu - \bar{\nu}$ can be written using (2.8) as

$$\lambda_d = \sqrt{\pi} \sum_k \frac{\langle |w_k|^2 \rangle}{\sigma_\nu} \exp[-(\bar{\nu} - \nu_k)^2/\sigma_\nu^2] \leq \lambda, \tag{6.10}$$

where the $\langle |w_k|^2 \rangle$ are determined from (6.5). The condition for maximum luminosity, when there is a large number of working resonances and their ν distribution can be assumed to be uniform, can be found from

$$|k_z|_{\max} \sigma_\nu \lesssim p, \tag{6.11}$$

in which the maximum number of working resonances is obtained from

$$\sqrt{\pi} \langle |w_k|^2 \rangle = \lambda \sigma_\nu. \tag{6.12}$$

At $\bar{\sigma}_0 \ll \nu_\gamma$, spin resonances with synchrotron oscillations occur. The diffusion rate λ_d is considerably smaller than that in (6.10) if $(\bar{\nu} - \nu_k)$ at a width $\approx \sigma_\nu$ does not coincide with $m\nu_\gamma$. In this case nonresonant diffusion occurs, whose rate can be found using the formulae of section 5.

For practical applications, the appearance of resonances with the frequencies of radial oscillations proves essential. In the first approximation, this results in an increase of the number of working resonances. The restriction on maximum luminosity can be found at $\sigma_\nu \ll \sigma_0$ from the condition

$$|k_z|_{\max}^2 |k_x|_{\max} (\Delta\nu_z)_s \lesssim p, \tag{6.13}$$

replacing (6.9), while at $\sigma_\nu \gg \nu_0$ from the condition

$$|k_z|_{\max} |k_x|_{\max} \sigma_\nu \lesssim p, \tag{6.14}$$

written instead of (6.11). The maximum number of working resonances with the frequencies of radial oscillations $|k_x|_{\max}$ is usually of the order of $|k_z|_{\max}$, determined from (6.8) and (6.12) respectively.

Let us discuss the results obtained. As follows from (6.8) and (6.10), the diffusion rate $\lambda_d^{\max}$ depends on the beam size and functions as

$$\lambda_d^{\max} \sim (\Delta\nu_z)_s z_0^3/(|f_z|_0^2 x_0) \quad \text{at } |K_z|_{\max} z_0^2 > x_0^2,$$

$$\lambda_d^{\max} \sim (\Delta\nu_z)_s z_0^2/|f_z|_0^2 \quad \text{at } |K_z|_{\max} z_0^2 < x_0^2,$$

$$\lambda_d^{\max} \sim [(\Delta\nu_z)_s^2/\sigma_\nu](z_0^2/|f_z|_0^2) \quad \text{at } \sigma_\nu \gtrsim |k_z|_{\max}(\Delta\nu_z)_s.$$

The luminosity is proportional to the factor $(\Delta\nu_z)_s^2 z_0 x_0/|f_z|_0^4$. Hence, by decreasing the β functions ($|f_z|_0 \sim |f_x|_0$) at a fixed frequency shift $(\Delta\nu_z)_s$, one can increase the luminosity, the polarization remaining stable. An additional gain can be obtained by increasing $|f_x|_0$. However, in this method to obtain the maximum luminosity one must increase the beam current.

It is interesting to trace how the polarization stability conditions change when the frequencies of betatron oscillations ν_z and ν_x get closer (the conditions of obtaining the radiative polarization without collisions are the same). As seen from (6.5), at fixed $(\Delta\nu_z)_s$ the resonance strengths $|w_k|$ increase proportionally to $|\nu_z - \nu_x|^{-1}$. This leads to an increase of the dangerous intervals of each resonance due to the colliding beam as well as to a small (logarithmic) increase of $|k_z|_{\max}$, $|k_x|_{\max}$. However, more essential is the coincidence of nonlinear resonances with the frequencies of vertical and radial oscillations. Instead of a two-dimensional resonance pattern, one finds a one-dimensional one. This allows us [as seen from comparison of (6.13) and (6.14) with (6.9) and (6.11)] at the optimum choice of frequency (energy) to increase $(\Delta\nu_z)_s$ by a factor of $|k_x|_{\max}$, therefore increasing the luminosity of the polarized beams.

Note also that to increase the luminosity of polarized beams it is useful to choose the frequency of betatron oscillations slightly above integer resonance (a similar method is employed to increase the luminosity of colliding beams). The effective spread of ν_z over the amplitudes of oscillations near the equilibrium orbit is thus descreased compared to (6.6) and hampers entering the nonlinear resonances.

The situation when the frequencies ν_z and ν_x are close not only to each other, but to an integer number as well, is probably optimum both for conservation of radiative polarization and for providing collision stability.

For storage rings with two independent paths a multi-bunch regime is useful, as it allows one to increase considerably the total luminosity of polarized beams at the same collision efficiency (if the number of the particles in the storage rings is not restricted for some other reason).

VII SUMMARY OF THE RESULTS

In this chapter we present the main formulae of the work and discuss them, using a simple example of a focusing system.

To obtain polarized electrons and positrons, it is necessary that the inaccuracies in the construction of the magnetic system do not result in the appearance of a radial field and its radial gradient of too large a magnitude. At energies when the spread of precession frequencies is very small ($\sigma_v = v\sqrt{11\lambda/(18\Lambda_y)} \ll 1$), for an average polarization degree higher than 50% the conditions should be fulfilled (Sections III and IV) that

$$\sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{[v]}|^2 \leq \frac{54}{11\pi^4} \frac{\sin^4 \pi v}{v^4(1 + 2\cos^2 \pi v)}, \quad (7.1)$$

$$\sum_{n=1}^{Q} \eta_n^2 \overline{\kappa_n^2} |F_n^v|^2 \Big\{ [\sin^{-2} \pi(v + v_x) + \sin^{-2} \pi(v - v_x)] \times \frac{|f_x|_n^2}{\langle K^3 \rangle} \left\langle K^3 \left| \int_{-\infty}^{\theta} K f_x d\theta \right|^2 \right\rangle + 4(\psi_x)_n^2 \sin^{-2} \pi v \Big\} \leq 72/(11\pi^2 v^2). \quad (7.2)$$

When the energy increases, the restrictions on the radial-field magnitude are the most severe. Therefore at $\sigma_v \gtrsim 1$ we write down only the requirements on the possible value of H_n^2 at $v_y \ll 1$ (Section V): at $v^2\lambda \ll v_y^3$, $(\Delta v_y) \ll v_y^2/\sigma_v$.

$$\left(\sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F^{[v]}|^2 \right) \sum_{k=-\infty}^{\infty} \frac{\exp[-(k - \bar{v})^2/\sigma_v^2]}{\sigma_v \sin^2 \pi[(k - \bar{v})/v_y]} \leq \frac{36}{11\pi\sqrt{\pi}} \frac{v_y^3}{v^4}, \quad (7.3)$$

at $v^2\lambda \gg v_y^3$ or $(\Delta v_y) \gg v_y^2/\sigma_v$:

$$\sqrt{\pi} \left(\sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{[v]}|^2 \right) \sum_{k=-\infty}^{\infty} \frac{1}{\sigma_v} \times \exp\left[-\frac{(k - \bar{v})^2}{\sigma_v^2} \right] \leq \frac{\lambda}{v^2}, \quad (7.4)$$

where (Δv_y) is the spread of the synchrotron oscillation frequency.

In particular, at $\sigma_v \gg 1$ one obtains {in (7.3) $\bar{v}$ and v_y are chosen so that for all essential resonances $\sin^2 \pi[(k - \bar{v})/v_y] = 1$}

$$\sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{[v]}|^2 \leq \frac{36}{11\pi^2} \frac{v_y^3}{v^4}, \quad (7.5)$$

$$\pi \sum_{n=1}^{Q} \eta_n^2 \overline{H_n^2} |F_n^{[v]}|^2 \leq \lambda/v^2 \quad (7.6)$$

Conditions for polarization conservation in collisions will be given here for the cases close to optimum, when the betatron oscillation frequencies v_z and v_x are close. Here the existence criteria are approximately the same as in the model taking into account the resonances with the vertical motion only. If the spread of the precession frequency does not exceed considerably the shift of the frequency of betatron oscillations [$\sigma_v \lesssim (\Delta v_z)_s$], the requirement of polarization conservation takes the form (at equal sizes of the colliding beams $z_0^2 \approx 2\langle a_z^2 \rangle$)

$$2A(\Delta v_z)_s \frac{v^2 |F^v|_0^2}{|f_z|_0^4} \frac{z_0 x_0}{R^2} (3 + 2\sqrt{2})^{-|k_z|} \lesssim \lambda, \quad (7.7)$$

where $|k_z|$ is the minimum harmonics number for which a resonance condition $v = kp + k_z v_z$ holds in the interval of variation of the betatron oscillation frequency $(\Delta v_z)_s$. The number A is equal to

$$A = \frac{z_0^2}{3|k_z|x_0^2} \quad \text{at } 2x_0^2 \ll |k_z| z_0^2$$

$$A = \frac{z_0}{|k_z|^{3/2} x_0} \quad \text{at } 2x_0^2 \gg |k_z| z_0^2.$$

In the region $\sigma_v \gg (\Delta v_z)_s$, radiative polarization exists at

$$3 \frac{(\Delta v_z)_s^2}{\sigma_v} \frac{v^2 |F^v|_0^2}{|f_z|_0^4} \frac{z_0^2}{R^2} \times \sum_{k_z} \frac{\exp[-(\bar{v} - kp - k_z v_z)^2/\sigma_v^2]}{(3 + 2\sqrt{2})^{|k_z|}} \lesssim \lambda. \quad (7.8)$$

Eqs. (7.7) and (7.8) allow estimation of the depolarizing influence of the colliding beam. In critical situations when $\sigma_v \sim (\Delta v_z)_s$, a more accurate calculation can be performed using the general formulae (2.8), (6.4). One should note that usually the criteria (7.7) and (7.8) can be fulfilled in the region of collision stability.

The quantity $|F^{\nu}|$ entering the criteria above is determined by the vertical focusing and arrangement of the bending magnets [see (3.7)]. Consider, for example, a strong-focusing system consisting of $2N$ identical thin focusing and defocusing magnetic lenses. Between the lenses, $2N$ bending magnets are symmetrically installed. For such a system, $\eta_n|g_z| = (N/\pi^2)\sin(\pi\nu_z/N)$. The average values of $\sqrt{|F_L^{\nu}|^2}$ in the lenses and F_M^{ν} in the magnets are respectively equal to†

$$|F_L^{\nu}| = \frac{[\sin^4(\pi\nu/N) + 4\sin^2(\pi\nu_z/N)\sin^4(\pi\nu/2N)]^{1/2}}{|\sin^2(\pi\nu/N) - \sin^2(\pi\nu_z/N)|},$$

$$|F_M^{\nu}| = \left\{\left[\frac{\cos(\pi\nu/2N)\sin^2(\pi\nu_z/N)}{\sin^2(\pi\nu/N) - \sin^2(\pi\nu_z/N)} + \frac{\sin(\pi\nu/2N)}{\pi\nu/2N}\right]^2 + \frac{4\sin^2(\pi\nu/N)\sin^6(\pi\nu/2N)}{\sin^2(\pi\nu/N) - \sin^2(\pi\nu_z/N)}\right\}^{1/2} \tag{7.9}$$

It can be seen that values of $|F^{\nu}|$ are usually unity or less with the exception of the region near $\nu = \pm\nu_z + kN$. $|F^{\nu}|$ decreases sharply when the number of system periods per a particle revolution increases.

$$|F_L^{\nu}| \approx \frac{\sqrt{9/2}\,\pi^2\nu^2}{N^2}, \qquad |F_M^{\nu}| \approx 2\frac{\pi^2\nu^2}{N^2}. \qquad \left(N \gg \nu,\ \nu_z = \frac{N}{4}\right) \tag{7.10}$$

Let us present, for example, restrictions on the accuracy of alignment of the magnetic-system elements, assuming that the number of independent elements is equal to the number of lenses or magnets respectively ($Q_L = Q_M = 2N$). We remind ourselves that restrictions on Δz_L, α_M (or H_n) concern in fact only the short-wavelength part of the perturbations with correlation lengths $\lesssim R/\nu$ and do not prohibit slow variations of Δz_L, α_M with large amplitudes, resulting in large deviations of the orbit from a plane at the lengths $\gg R/\nu$. In the region $\sigma_v \ll 1$ from (7.1) one obtains restrictions on the possible vertical shifts of the lenses and angles of inclination for magnets.

$$\sqrt{\overline{(\Delta z_L)^2}} \leq \frac{\sqrt{27/11}}{N^{3/2}}\frac{\sin^2\pi\nu}{\nu^2(1+2\cos^2\pi\nu)^{1/2}} \times \frac{R}{|F_L^{\nu}|\,|\sin\pi(\nu_z/N)|},$$
$$\sqrt{\overline{\alpha_M^2}} \leq \frac{2\sqrt{27/11}\sqrt{N}}{\pi^2\nu^2}\frac{\sin^2\pi\nu}{|F_M^{\nu}|} \times (1+2\cos^2\pi\nu)^{-1/2}, \tag{7.11}$$

where $|F_L^{\nu}|$ and $|F_M^{\nu}|$ are determined by (7.9), (7.10). The requirement on the rotation angle α_L of the lenses, obtained from (7.2), is usually easily satisfied and we estimate the acceptable value as [compare with (14.8)]

$$\sqrt{\overline{\alpha_L^2}} \lesssim 0.7\cdot 10^{-2}\frac{N^{5/2}}{\nu^3} \quad \left(\nu_z \approx \frac{N}{4},\ N \gg \nu\right). \tag{7.12}$$

From (7.5), (7.6) one obtains at $\sigma_v \gg 1$

$$\left.\begin{aligned} \sqrt{\overline{(\Delta z_L)^2}} &\leq \frac{\pi\sqrt{18/11}\,\nu_\gamma^3}{N^{3/2}\nu^2|F_L^{\nu}|}R \\ \sqrt{\overline{\alpha_M^2}} &\leq \frac{6\sqrt{2N}\,\nu_\gamma^3}{\pi\nu^2|F_M^{\nu}|} \end{aligned}\right\} \quad \text{at } \nu^2\lambda \ll \nu_\gamma^3,\ (\Delta\nu_\gamma) \ll \frac{\nu_\gamma^2}{\sigma_v}, \tag{7.13}$$

$$\left.\begin{aligned} \sqrt{\overline{(\Delta z_L)^2}} &\leq \frac{\pi\sqrt{\lambda}R}{\sqrt{2}\nu N^{3/2}|F_L^{\nu}|} \\ \sqrt{\overline{\alpha_M^2}} &\leq \frac{\sqrt{2N\lambda}}{\sqrt{\pi}\nu|F_M^{\nu}|} \end{aligned}\right\} \quad \text{at } \nu^2\lambda \gg \nu_\gamma^3 \text{ or } (\Delta\nu_\gamma) \gg \nu_\gamma^2/\sigma_v. \tag{7.14}$$

If each of the $2N$ elements of the magnetic system consists of Q independent elements, the criteria above become better by a factor about $\sqrt{Q/2N}$.

The conditions obtained show that to provide a high degree of radiative polarization, it is more profitable to have storage rings with a maximum possible number of magnetic-system periods. Criteria of polarization stability during collisions are usually satisfied easily at a natural vertical size (due to existing non-idealities of the magnetic system). If the vertical size is specially increased to provide large luminosity criteria, (7.7) and (7.8) can sometimes be violated. Similar to the situations without collisions, it is more profitable to have storage rings

† Here the effective value of $|F_M^{\nu}|$ in the magnet is given taking into account its finite length (see the note on p. 253).

with a large number of periodicity elements ($N \gg \nu$).

It is interesting to observe how the restrictions on the construction accuracy change with an increase of the maximum storage-ring energy. The main requirements concern (Δz_L) and α_M. We assume an energy dependence of the parameters for traditional storage rings

$$R \sim \gamma^2, N \sim \gamma, Q \sim R \sim \gamma^2, \nu_\gamma \sim \sqrt{\gamma},$$
$$\sigma_\nu \sim \gamma, \lambda \sim \gamma$$

where Q is the total number of independent elements, including a substructure of the system period. With energy increase $(\Delta z_L)_{\mathrm{adm}}$ and $(\alpha_M)_{\mathrm{adm}}$ vary as

i) at $\sigma_\nu \ll 1$,

$$(\Delta z_L)_{\mathrm{adm}} \sim (\alpha_M)_{\mathrm{adm}} \sim \gamma^{-1},$$

ii) at $\sigma_\nu \gtrsim 1$,

$$(\Delta z_L)_{\mathrm{adm}} \sim (\alpha_M)_{\mathrm{adm}} \sim \begin{cases} \gamma^{-1/4} & \nu^2\lambda \ll \nu_\gamma^3, (\Delta\nu_\gamma) \ll \nu_\gamma^2/\sigma_\nu, \\ \gamma^{1/2} & \nu^2\lambda > \nu_\gamma^3 \text{ or } (\Delta\nu_\gamma) > \nu_\gamma^2/\sigma_\nu. \end{cases}$$

Note that the criteria become stronger until the resonant diffusion mechanism appears dominant. After that, requirements on the construction accuracy begin to weaken with the growth of maximum energy. The most strict requirements are those for storage rings at the energy of several hundreds of GeV.

For colliding beams, the ratio λ_d/λ decreases with the growth of the maximum energy at fixed $(\Delta\nu_z)_s$.

$$\lambda d/\lambda \sim \gamma^{-1} \qquad \text{at } \sigma_\nu \lesssim (\Delta\nu_z)_s$$
$$\lambda d/\lambda \sim \gamma^{-2} \qquad \text{at } \sigma_\nu \gg (\Delta\nu_z)_s$$

Varying the energy in a given storage ring, one obtains, for the energies at which the spread σ_ν is small, requirements on d_M, Δz_L that reveal the energy dependence [see (7.12) and (7.13)]

$$\begin{aligned} (\Delta z_L)_{\mathrm{adm}} &\sim (\alpha_M)_{\mathrm{adm}} \sim \gamma^{-4} && \text{at } \nu \ll N, \\ (\Delta z_L)_{\mathrm{adm}} &\sim (\alpha_M)_{\mathrm{adm}} \sim \gamma^{-2} && \text{at } \nu \gtrsim N \end{aligned} \tag{7.15}$$

When σ_ν approaches 1, the criteria sharply become strict (at $\nu_\gamma \ll 1$) in a comparatively narrow region [see (7.5), (7.6)], and continue further to follow the former law (7.15) (at $\nu_\gamma =$ const.). For a sufficiently high energy, the resonant diffusion mechanism is switched on [see (7.14)]. Requirements on $(\alpha_M)_{\mathrm{adm}}$ and $(\Delta z_L)_{\mathrm{adm}}$ vary as $\gamma^{-1/2}$ at $\nu \ll N$ and as $\gamma^{3/2}$ at $\nu > N$.

For beam collisions in a storage ring, the condition of polarization conservation ($\lambda_d \lesssim \lambda$) at fixed $(\Delta\nu_z)_s$ varies [see (7.7) and (7.8)] as

i) at $\sigma_\nu \lesssim (\Delta\nu_z)_s$

$$\lambda_d/\lambda \sim \gamma^3 \qquad \text{if } \nu \ll N$$
$$\lambda_d/\lambda \sim \gamma^{-1} \qquad \text{if } \nu > N$$

ii) at $1 \gg \sigma_\nu \gg (\Delta\nu_z)_s$

$$\lambda_d/\lambda \sim \gamma \qquad \text{if } \nu \ll N,$$
$$\lambda_d/\lambda \sim \gamma^{-3} \qquad \text{if } \nu > N.$$

At $\sigma_\nu \gtrsim 1$ the influence of colliding beams can be neglected in comparison with the effects of the magnetic-system non-idealities.

VIII CONCLUSIONS

The investigation carried out in this work allows us to answer the question about the possibility of electron and positron self-polarization depending on the conditions of their motion. It may occur that in a given storage ring these conditions are not fulfilled. To prevent depolarization special measures are then needed. One evident way is to compensate dangerous perturbation harmonics. It is very important here to have the possibility of monitoring the beam-polarization degree without its destruction, in order to provide a continuous test of the compensation efficiency. However, with a large number of harmonics causing depolarization, the use of this method is not very simple.

To increase the role of polarizing processes, one can employ a "wiggler" installed in a straight section of the storage ring as proposed in Refs. 17–19. The wiggler is a set of regions with large alternating sign vertical magnetic field with zero average value and large $\langle B_z^3 \rangle$. The minimum number of field oscillations is determined by an acceptable amplitude of spatial beating of the orbit in the straight section. In the simplest variant, there are three regions with lengths θ_-, θ_+, θ_- and fields at them B_-, B_+, B_-. A zero average value $B_+\theta_+ + 2B_-\theta_- = 0$, as well as symmetry with respect to the center of the region, provide conservation of the particle trajectory outside the wiggler. To obtain a high polarization degree, the condition must be satisfied that $B_+^2 \gg B_-^2$.

As a consequence of the introduction of sufficiently high fields B_+ and B_-, both polarizing and depolarizing processes are determined by the radiation in the wiggler regions. At $\sigma_v \ll 1$, when $\lambda_d \sim \langle(\gamma\partial\mathbf{n}/\partial\gamma)^2|B|^3\rangle$, the depolarizing influence of the radiation is relatively small if in the wiggler region the parameter $|\gamma\partial\mathbf{n}/\partial\gamma|$ is much less than 1. As the fraction of the orbit occupied by the wiggler is small, the vector $\gamma\partial\mathbf{n}/\partial\gamma$ is determined primarily by perturbations in the other orbit regions and therefore is given by two parameters along the entire wiggler length. The energy dependence of $\mathbf{n}$ in this section can be compensated if, for example, radial magnetic fields are introduced at two points of the orbit (between which the angle of spin rotation around the vertical direction is not an integer multiple of π). Required values of these fields can be chosen so as to provide a minimum depolarization rate. One should remember that the wiggler increases the energy spread in the beam, hence σ_v as well.

At $\sigma_v \gtrsim 1$ or in colliding beams, when the depolarization is due to diffusive resonance passings, the wiggler does not increase the rate of spin diffusion, and at sufficient values of the wiggler fields, the polarizing influence of the radiation becomes predominant.

Under optimum conditions, in the presence of the wiggler the polarization degree tends to the value

$$\zeta_{t\to\infty} = \frac{8}{5\sqrt{3}} \qquad \frac{\langle B_z^3\rangle}{\langle|B_z|^3\rangle} = \frac{8}{5\sqrt{3}}\frac{B_+^2 - B_-^2}{B_+^2 + B_-^2},$$

while the polarization time tends to

$$T \equiv \lambda^{-1} = \frac{8}{5\sqrt{3}}\frac{R^2}{\lambda r_e}\gamma^{-5}\frac{2\pi|\langle B_0\rangle|^3}{|B_+^3|\theta_+ + 2|B_-^3|\theta_-}.$$

It is important that the wiggler also decreases considerably the polarization time. This method is especially convenient to increase the role of polarizing processes in large storage rings at limiting high energies with a small average field $\langle B_0\rangle$. At non-limiting (with respect to the radiative losses) energies, the polarization rate can be increased by a factor of about $B_+^3\theta_+/(2\pi\langle B_0^3\rangle)$; simultaneously the radiative losses gain a factor of $B_+^2\theta_+/2\pi\langle B_0^2\rangle$. If sufficiently high fields B_+ are available, this method can be also used in the cases when it is reasonable to increase values of λ proportional to $\langle|B_z|^3\rangle$ without substantial increase of the radiative losses per revolution proportional to $\langle B_z^2\rangle$. Provided this condition is held, the polarization rate can be increased by a factor of B_+/B_0. Situations are also possible when diffusion processes do not allow particle polarization at a required energy, but the depolarization time is sufficiently high to perform an experiment. Here it can be reasonable to polarize a beam using the wiggler at a smaller energy, when the parameter $\gamma\partial\mathbf{n}/\partial\gamma$ is small and need not be compensated. Depolarization caused by crossing of spin resonances when the beam energy is increased can be suppressed by the methods described in Refs. 20–22.

Polarization can be also accelerated (proportionally to the K^2 value in the magnets) by decreasing the length of bending sections (the ratio of the magnetic radius and the average one is simultaneously decreased). At $\sigma_v \gtrsim 1$ or for colliding beams, this causes an increase of the degree of the equilibrium polarization as well, because the rate of the depolarizing effects $\lambda_d \sim \lambda_d^r$ remains approximately the same with an increase of K. Radiative losses are proportional to the value of K in the magnets.

In this work, possibilities of providing a transverse polarization are considered for high-energy particles in storage rings with a vertical field. It is known that longitudinal polarization can be achieved by using radial fields in the section (or a combination of radial and vertical ones) with orbit and spin reconstruction at the section end (see Ref. 4 and review articles[18,19]). Investigation of the radiative stability of polarization in such a storage ring does not essentially differ from the usual case of the constant vertical field and there are reasons to believe that it is possible to provide a high degree of the polarization along the velocity.

Note also that spin reconstruction at the end of the section with bending fields is not an essential requirement. On the contrary, in special situations when the spin direction is not constant in the basic parts of the orbit, additional possibilities of the polarization control and suppression of depolarizing factors can appear.

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