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ON STOCHASTIC COOLING

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In the work presented here, we investigate the features and possibilities of a cooling technique of heavy particle beams with an electronic feedback system as suggested by S. Van der Meer. The consideration is based on the kinetic equation for systems with nonconservative interaction. Expressions are derived for damping decrements of transverse and longitudinal oscillations of particles in storage rings. Limits are obtained for the attainable value of the decrement, determined by the influence of particles on each other. Effects are considered that determine the equilibrium size of the beam and optimization of cooling methods is discussed. The main features of the method are also illustrated by a simple model.

The absence of radiation cooling for heavy particles requires some additional effort to damp particle beams in accelerators. At low energies, electron cooling[1] is quite effective, allowing one to achieve cooling times of the order of a fraction of a second. At high energies, however, the cooling rate goes down (as the 5th power of energy for a given angular spread of a beam) and a more effective damping may be that due to particle interaction with external dissipative device.

The stochastic cooling proposed by Van der Meer[2] by means of an electronic feedback system is such a damping. The simplest system of this kind consists of a wideband difference pickup and a kicker. A signal proportional to the particle mean transverse displacement is induced on the pickup and after amplification is transmitted to the kicker. The influence of this signal on the same particle at the kicker determines the damping effect (for example, reduces the particle transverse velocity at the point of the orbit where the velocity has its maximum). The delay time is selected to be equal to the time required for a particle to move from the pickup to the kicker. Currently there are experiments where this method is realized and oscillation dampling is observed to be of the order of several tens of per cent per hour.[3] In the work presented here an analysis of this stochastic cooling is given.

As is shown later, a useful effect will be determined by "selfaction" just as in electron cooling. The mutual influence of particles on each other via the external system decreases the damping decrement and becomes unavoidable as the beam density increases, resulting in a limit for the damping decrements attainable with this method. Note that since the characteristic distances associated with interaction via an external system are large compared with the distances between the particles in the beam, this effect becomes dominant much earlier than for electron cooling.

In Section 1, a simple model is considered which shows the observation of the main features of this method. In Section 2 the particle interaction form is obtained. Based on the results in Section 2, damping decrements are obtained in Section 3. In Section 4, the mutual influence of particles on each other is considered and limits are obtained for maximum decrements attainable with the method. Here we use the results of general considerations used by authors in Ref. 4 of particle interaction with a dissipative external device. In Section 5, effects that determine the beam dimensions are discussed. In Section 6 some ways of how to optimize this method are considered.

1 MODEL

Particle motion will be considered in terms of variables of action $\mathbf{I}$ and the phase $\boldsymbol{\psi}$; in the case of betatron oscillations I is proportional to the oscillation amplitude squared and for longitudinal motion $I_{\parallel}$ is proportional to the momentum deviation from its equilibrium value $I_{\|} \sim p - p_s = q$. Let us demonstrate the main features of the method by using it for the problem of betatron-oscillation damping.

Particle interaction can be considered to be dependent on the phase difference because the relative frequency spread $\Delta\omega/\omega$ is small and damping times are large. Taking the particle transverse displacements from an equilibrium orbit to be small compared with characteristic the apertures, one can write the equation of motion in the form

$$\dot{I}_a = -\sum_b \frac{\partial V(a, b)}{\partial \psi_a} \tag{1.1}$$

$$\dot{\psi}_a = \omega_\perp(a) + \sum_b \frac{\partial V(a, b)}{\partial I_a}. \tag{1.2}$$

Here

$$V(a, b) = \lambda\sqrt{I_a I_b}\, g(\theta_a - \theta_b)\sin(\psi_a - \psi_b) \tag{1.3}$$

describes the interaction between particles a and b. The factor $g(\theta)$ depends on the feedback-system structure and determines the azimuthal distance of the effective interaction between particles. Since there is no damping in a Hermitian interaction, as will be shown below, we shall consider the factor g to be symmetric: $g(\theta_a - \theta_b) = g(\theta_b - \theta_a)$ and its azimuthal harmonics

$$g_l = \int \frac{d\theta}{2\pi} e^{-il\theta} g(\theta)$$

to be real. The azimuthal interaction length will be taken to be equal to θ_0, so

$$g(\theta) \simeq 1, |\theta| < \theta_0/2; \qquad g(\theta) \simeq 0, |\theta| > \theta_0/2$$

Consequently, the harmonics g_l are of the same order of magnitude $g_l \simeq \theta_0$ up to $l \lesssim \theta_0^{-1}$; for larger l; their values decrease rapidly. For the sake of simplicity, let us also assume that at $\lambda = 0$, the unperturbed frequencies

$$\dot{\theta}_a = \omega_a^\circ(q), \qquad \dot{\psi}_a = \omega_{\perp a}(q)$$

are linear functions of momentum, neglecting their dependence on amplitudes. Since the number of characteristic harmonics is large ($l \gg 1$), it is sufficient to have this assumption satisfied only for the revolution frequency $\omega^\circ(q)$.

The term $b = a$ in (1.1) describes the useful "self-action" effect

$$\dot{I}_a = -\left(\frac{\partial V(a, b)}{\partial \psi_a}\right)_{b=a} = -\lambda I_a, \tag{1.4}$$

so that the parameter λ determines the single-particle damping decrement. A damping effect occurs in a nonconservative system (nonHermitian interaction). That is, $V(a, b) \neq V(b, a)$.

The remaining terms of the sum (1.1) describe the mutual influence of particles on each other. One can understand this interaction within the framework of perturbation theory. Let us find (from (1.2)) the first-order corrections in λ to the phase ψ due to the interaction in (1.2) and substitute them into (1.1). Let us average the expressions obtained over phase, taking initial phases for all particles as independent, random values. Then, within an accuracy of the order of $o(\lambda^4)$ we obtain

$$\lambda_{\text{ef}} = \lambda - \sum_l \frac{\pi N \lambda^2}{|\mathbf{n}\cdot\boldsymbol{\omega}'|} g_l^2 \int dq f^2(q), \tag{1.5}$$

where $\mathbf{n}$ is a vector whose longitudinal component is l and whose transverse components are either zero or unity and $\boldsymbol{\omega}' = \partial\boldsymbol{\omega}/\partial q$. The effect of the mutual influence of particles increases as λ_q^2. Therefore the maximum attainable decrement is limited by $\lambda < \lambda_{\max}$ given by

$$\lambda_{\max}^{-1} \simeq \sum_l \frac{\pi N g_l^2}{|\mathbf{n}\cdot\Delta\boldsymbol{\omega}|} \simeq \frac{N_{\text{ef}}}{\Delta\omega^\circ l_{\text{ef}}} \ln\frac{1}{\theta_0} \tag{1.6}$$

This value depends on the revolution frequency spread $\Delta\omega^\circ = |\partial\omega^\circ/\partial q|\Delta q$, where Δq is the momentum spread determined by the width of the distribution function. The dependence of $\lambda_{\max}$ on the frequency spread makes (1.6) different from the result obtained by Van der Meer. The difference appears as a consequence of conservation of the particle phase correlations over many turns because the interaction is weak and the relative spread of frequencies is small, whereas Van der Meer assumed splitting for correlations for one turn. $\lambda_{\max}$ is determined by the length of interaction region θ_0, $l_{\text{ef}} \simeq \theta_0^{-1}$ and by the number $N_{\text{ef}} \sim N\theta_0$ of particles located in this region. In this sense, the result (1.6) appears to be universal.

If both terms in (1.5) are of the same order of magnitude, perturbation theory over λ is inapplicable. In this case, the use of the variable X_a and X_a^* seems to be more appropriate, where

$$X_a = A_a e^{-i\psi_a}; \qquad A_a = \sqrt{2I_a}$$

With these variables, Eqs. (1.1) and (1.2) take the form

$$\dot{X}_a + i\omega_\perp(a) X_a = -\frac{\lambda}{2}\sum_l Z_l(t) g_l e^{il\theta_a(t)}. \tag{1.7}$$

Here a collective variable Z_l is introduced

$$Z_l(t) = \sum_b X_b(t)e^{-il\theta_b(t)}, \tag{1.8}$$

which may be interpreted as an azimuthal harmonic for the beam center of mass coordinates. Each term in the sum (1.8) oscillates with its own frequency $\mathbf{n}\cdot\boldsymbol{\omega}_b = l\omega^\circ(b) + \omega_\perp(b)$, so that $Z_l(t)$ is not zero because of dynamical (non-thermodynamic) fluctuations in a system with a finite number of particles N.

As is seen from (1.7), each particle, in addition to free oscillations of amplitude X_a^0, experiences constrained oscillations connected with fluctuations of the center-of-mass coordinate. Thus

$$X_a(t) = X_a^0 e^{-i\omega_\perp(a)t} - \frac{\lambda}{2}\sum_l g_l \int_0^t dt' Z_l(t')$$
$$\times \exp i[l\theta_a(t') - \omega_\perp(a)\cdot(t - t')]. \tag{1.10}$$

Because of this interaction, the motion of different particles turns out to be correlated, which finally leads to a restriction on the attainable decrement. Let us demonstrate this with a transverse-decrement calculation. The value

$$\langle \dot{I} \rangle = \frac{1}{2N}\frac{d}{dt}\sum_a |X_a(t)|^2$$

as follows from (1.7) is expressed through $Z_e(t)$ as

$$\langle \dot{I} \rangle = -\sum_l \frac{\lambda g_l}{2N}|Z_l(t)|^2. \tag{1.11}$$

If the initial distribution of particles is homogeneous over phases, then (1.10) gives the equation

$$Z_l(t) = Z_l^0(t) - \frac{\lambda g_l}{2}\int_0^t dt' Z_l(t') f_l(t - t'), \tag{1.12}$$

where

$$Z_l^0(t) = \sum_a X_a^0 e^{-i\omega_\perp(a)t - il\theta_a(t)}$$

$$f_l(t) = \sum_a e^{-i\mathbf{n}\boldsymbol{\omega}_a\cdot t}.$$

Assuming that the initial amplitudes X_a^0 are statistically independent and using the solution (1.12), we can rewrite (1.11) in the form

$$\langle \dot{I} \rangle = -\frac{1}{N}\sum_a |X_a^0|^2 \sum_l \frac{\lambda g_l}{2}$$
$$\times \left|\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\nu e^{-i\nu t}}{\varepsilon_l(\nu)\cdot(\nu - \mathbf{n}\boldsymbol{\omega}_a)}\right|^2, \tag{1.13}$$

where

$$\varepsilon_l(\nu) = 1 + i\frac{\lambda g_l}{2}\sum_b(\nu - \mathbf{n}\boldsymbol{\omega}_b)^{-1}.$$

The poles for the function under the integral are determined by the zeros of the functions $\varepsilon_l(\nu)$. If the coherent motion is stable, the significant zeros of $\varepsilon_l(\nu)$ lie near the single-particle frequencies

$$\nu = \mathbf{n}\boldsymbol{\omega}_c - i\zeta_c; \qquad \zeta_c = \frac{\lambda g_l}{2\varepsilon_l(\omega_c)},$$

where

$$\varepsilon_l(\omega_c) = 1 + \frac{i\lambda g_l}{2}\sum_{b \neq c}[\mathbf{n}(\boldsymbol{\omega}_c - \boldsymbol{\omega}_b) - \zeta_c]^{-1}. \tag{1.14}$$

Let us consider the case when $\lambda N_{\mathrm{ef}} g_l$ is less than the frequency spread $\Delta\omega$. In $\varepsilon_l(\omega)$ one can neglect ζ_l and the most significant contribution in (1.13) is given by the residue $c = a$. In this case, (1.13) is reduced to the expression

$$\langle \dot{I}(t) \rangle = -\frac{1}{2N}\sum_a |X_a^0|^2 \sum_l \gamma_l^{(a)} e^{-\gamma_l^a t}$$

This shows that $\langle I(t) \rangle$ damps from the value $\langle I(0) \rangle$ with the decrement

$$\gamma_l = \left\langle \frac{\lambda g_l}{|\varepsilon_l(q)|^2} \right\rangle. \tag{1.15}$$

The brackets here mean an average over momentum using the distribution function. Thus, the mutual influence of particles on each other result in a decrease of the partial decrements λg_l by a factor $|\varepsilon_l|^2$. The difference of $|\varepsilon_l|$ from unity is of the order of magnitude of the ratio of the coherent decrement $\gamma_l^{\mathrm{coh}} \equiv \frac{1}{2}\lambda g_l N$ to the beam frequency spread $\Delta\omega$.

At $|\varepsilon_l| \simeq 1$ (1.5) again follows from (1.14). With an increase in number of particles, the partial decrements vary inversely with N^2. The maximum attainable decrement is obtained if $|\varepsilon_l| \simeq 1$ at all l, which coincides with the condition (1.6).

In the converse limiting case $\lambda N g_l \gg \Delta\omega$,

$$\varepsilon_l(\nu)\cdot(\nu - \mathbf{n}\boldsymbol{\omega}_a) \simeq \nu - \mathbf{n}\boldsymbol{\omega}_a + \frac{i}{2}\lambda N g_l,$$

so (1.13) shows that damping $\langle I \rangle$ occurs with the coherent decrement γ_l^{coh}, but the equilibrium spread of amplitudes

$$\langle I(\infty) \rangle = \langle I(0) \rangle \left[1 - \frac{1}{N_{\mathrm{ef}}}\right]$$

does not differ from the initial spread, within statistical accuracy. In this sense, damping does not occur.

The result obtained (1.15) is in agreement with the more thorough considerations of the authors in Ref. 4 (see also Section 4).

2 FORM OF THE INTERACTION

In order to obtain certain formulae we shall consider pickup and kicker as a matched line. Such a choice is determined by the following considerations:

1) the kicker length $l_{\|}$ cannot be too short, otherwise due to revolution frequency spread the useful effect of self-action will be absent for most particles while they pass the kicker;

2) in order to get more effective interaction, the pickup fields should vary in a time on the order of that required for a particle to pass through the pickup;

3) the fields should damp sufficiently fast to decrease the influence of the signal on the other particles. Many-turn effects should not exist.

Therefore, the choice of pickup in the form of a matched line seems to be the most appropriate. In our description of particle interactions with matched lines we shall follow Ref. 5. On other methods see, for example, Refs. 6 and 7.

For low-frequency oscillations the field in the pickup has a form similar to that for an infinite waveguide; i.e., it can be represented as a superposition of travelling waves

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r}_\perp, \theta)\frac{c}{\sqrt{2\pi}}\int dk\varphi_k(t)e^{ikR\theta} \quad (2.1)$$

$\mathbf{A}_0$ here is an electrostatic field whose value differs from zero inside the pickup and falls exponentially outside the pickup in a distance of the order of the transverse dimention l

$$\nabla_\perp \cdot \mathbf{A}_0 = 0; \qquad \int d^2\mathbf{r}_\perp \cdot |\mathbf{A}_0|^2 = 1.$$

It can be considered as a potential field

$$\mathbf{A}_0 = -\nabla U(\mathbf{r}_\perp; \theta). \quad (2.2)$$

The amplitudes of the field $\varphi_k(t)$ are determined by all the particles which excites the field:

$$\ddot{\varphi}_k + c^2k^2\varphi_k = -\frac{4\pi e}{\sqrt{2\pi}}\sum_a \frac{dU(r_a(t), \theta)}{dt} e^{-ikR\theta_a(t)}.$$

A signal induced in the pickup is then amplified and transmitted to the kicker where the field $\tilde{\mathbf{A}}(\mathbf{r}; t)$ is excited. This has the same form (2.1) with amplitudes $\tilde{\varphi}_k$ determined by an amplification factor

$$\tilde{\varphi}_k(t) = K(k)\varphi_k(t - \tau),$$

where τ is the delay time for a signal.

The influence on particle 1 inside the kicker is described by the potential

$$V(1/t) = -\frac{e}{c}\mathbf{v}(1) \cdot \tilde{\mathbf{A}}(r_1, t),$$

where the dependence $V(1/t)$ on time reduces to the field dependence on the particle parameters exciting the field in the pickup.

As a result, the potential

$$V(1/t) = \sum_2 V(1, 2) \quad (2.3)$$

is expressed by the effective interaction $V(1, 2)$ of Eq. (1.3) between particles 2 that produce the field, and the affected particle 1.

We shall take into account only the interaction of particles via an amplifying system by considering a sufficiently large amplification factor and by neglecting the effects of particle interaction with a single pickup and kicker. These interactions are considered in Ref. 5, for instance, and have no relevance to stochastic cooling. Besides, we shall confine ourselves to the case of unbunched beam and the small corrections modulation of azimuthal motion caused by betatron oscillations will be neglected.

Let us introduce the interaction harmonics $V_{\mathbf{n}_1\mathbf{n}_2}$

$$V(1, 2) = \sum_{\mathbf{n}_1\mathbf{n}_2} V_{\mathbf{n}_1\mathbf{n}_2}(I_1, I_2)e^{i(\mathbf{n}_1\cdot\boldsymbol{\psi}_1 - \mathbf{n}_2\cdot\boldsymbol{\psi}_2)}, \quad (2.4)$$

where $\mathbf{n}$ means a set $(m_r; m_z; l)$. Following the procedure described above, the harmonics are expressed through the potential harmonics $U(\mathbf{r}_\perp; \theta)$,

$$U^l_{m_r m_z}(q) = \int \frac{d\theta}{2\pi} e^{il\theta}[\sqrt{v_r}|f_r(\theta)|]^{|m_r|}$$
$$\times [\sqrt{v_z}|f_z(\theta)|]^{|m_z|}.$$
$$e^{-i(m_r\eta_r + m_z\eta_z)}\left[\frac{\partial^{|m_r| + |m_z|}U(\mathbf{r}_\perp; \theta)}{\partial z^{|m_r|}\,\partial z^{|m_r|}}\right]_{z_b=0}$$

Here $f(\theta) = |f|e^{i\eta}$ is the Floquet function. The momentum dependence of U_m^l appears through the momentum dependence of the particle radial deviation

$$r_\perp = r_b + \frac{\psi(\theta)Rq}{p_s}$$

Let L be the distance between the centers of pickup and kicker. For the sake of simplicity let us neglect the difference between the value of Floquet functions in the region of the pickup and kicker, Assuming also that the potential $U(\mathbf{r}, \theta)$ is factorized, it can be rewritten in the form

$$U(\mathbf{r}; \theta) = U(\mathbf{r})g(\theta),$$

where the function $g(\theta)$ is equal to unity inside the pickup and falls rapidly to zero at distances $\Delta\theta \simeq l_\perp/R$ from the pickup edge.

$$g(\theta) \simeq 1; \qquad |\theta| < \frac{\theta_0}{2},$$

Here θ_0 is the azimuthal length of the pickup $\theta_0 = l_{\|}/R$. Its harmonics g_l are constant $g_l \simeq \theta_0/\pi$ for $|l| < \pi/2\theta_0$; falls as l^{-1} for $\pi/2\theta_0 < |l| < \Delta\theta^{-1}$ and are negligibly small for $|l| > \Delta\theta^{-1}$.

If $\sqrt{v_\perp}\, f_\perp \simeq 1$ in the region of the pickup, then

$$U_{m_r m_z}^l(q) = g(l + kR)U_{m_r m_z}(q) \qquad (2.5)$$

With minor simplifications, the interaction harmonics can be written in the form

$$V_{\mathbf{n}_1\mathbf{n}_2}(\mathbf{I}_1, \mathbf{I}_2) = \prod_{\alpha=r,z} I_{1\alpha}^{|m_{1\alpha}/2|} I_{2\alpha}^{|m_{2\alpha}/2|} v_{n_1 n_2}(q_1 q_2), \quad (2.6)$$

where

$$v_{n_1n_2}(q_1q_2) = U_{m_{1r}m_{1z}}(q_1)U_{m_{2r}m_{2z}}(q_2)T_{n_1n_2} \times \exp(i\chi_{n_1n_2}) \qquad (2.7)$$

and the constant $T_{n_1n_2}$ is equal to

$$T_{n_1n_2} = -\frac{2e^2}{R}\int \frac{dx \cdot K(x)C_{n_1}(x)C_{n_2}(x)}{\omega_0^2x^2 - (\mathbf{n}_2 \cdot \boldsymbol{\omega}^s + i\varepsilon)^2};$$

$$C_n(x) \equiv \frac{(\mathbf{n}\cdot\boldsymbol{\omega}^s + x\omega^\circ)\cdot g(l+x)}{|m_r|!|m_z|!}\left(\frac{R}{2v_r p_s}\right)^{|m_r/2|} \times \left(\frac{R}{2v_z p_s}\right)^{|m_z/2|} \qquad (2.8)$$

The frequencies here are taken as their equilibrium values.

The integral is defined in (2.8) in accordance with the causality principle. The phase $\chi_{n_1n_2}$ in (2.7) is equal to

$$\chi_{n_1n_2} = \mathbf{n}_2 \cdot \boldsymbol{\omega}_2\tau - l_1 \cdot \frac{L}{R}$$

If the delay time τ is approximately equal to the time required for a particle to pass the distance between the pickup and kicker,

$$\tau = \frac{L}{c} - \frac{\alpha}{\omega^\circ} \qquad (2.9)$$

where $\alpha \simeq \theta_0$ is a parameter whose value will be specified later, then

$$\chi_{n_1n_2} = \frac{L}{R}(l_1 - l_2) + \mathbf{m}_{2\perp}v_\perp\tau + \frac{L}{R\omega^\circ}\mathbf{n}\cdot\Delta\boldsymbol{\omega} - \alpha l_2. \qquad (2.10)$$

3 DAMPING DECREMENTS

After an averaging of the rapidly oscillating terms in the expansion (2.4), only the diagonal harmonics V_{nn} will be left and the interaction will become a function of the particle phase difference. The acceptability of this averaging will be discussed in Section 5.

When considering transverse motion with an accuracy $o(a^2/l_\perp^2)$, it is sufficient to confine ourselves to dipole oscillations ($|m_r| = 1$, $m_z = 0$ for radial, $m_r = 0$; $|m_z| = 1$ for vertical oscillations). If further, one takes the amplification factor to be constant $K(k) = K$, then (2.4) through (2.8) yield the dipole interaction in the form

$$V_1(1,2) = \frac{Ke^2}{v_\perp p_s}\sqrt{I_1I_2}\,[U_{10}^2(q)]^2 \times \left\{\left[g\left(\frac{\xi+\theta_0}{2}\right) - g\left(\frac{\xi-\theta_0}{2}\right)\right]\cos\Psi + v_\perp\left(\frac{\theta_0-\xi}{2}\right)\sin\Psi\right\} \qquad (3.1)$$

similar to the result (1.3) in the model interaction above. Here $\Psi = \psi_1 - \psi_2 + \tau\Delta\omega^\circ + \tau\omega_\perp + \frac{1}{2}v\xi$; $\xi = \alpha - \tau\Delta\omega^\circ - (\theta_1 - \theta_2)$ and $V_1(1,2)$ differs from zero for $\xi > 0$. Note that in the sum over azimuthal harmonics carried out for (3.1), harmonics up to $l \simeq \theta_0^{-1}$ are essential.

The transverse-oscillation decrement is defined "self-action" from (1.4). If one can neglect a small term ($\sim\theta_0$) in braces then the decrement

$$\lambda_\perp = -\frac{Ke^2}{v_\perp p_s}U_{10}^2(0) \times \left[g\left(\frac{\xi_0+\theta_0}{2}\right) - g\left(\frac{\xi_0-\theta_0}{2}\right)\right]\sin\Psi_0;$$

$$\xi_0 = \alpha - \tau\Delta\omega^\circ > 0; \qquad \Psi_0 = \tfrac{1}{2}v\xi + \tau\omega_\perp + \tau\Delta\omega^\circ$$

will have its maximum under the condition that α is taken to be equal to

$$\alpha = \theta_0 \tag{3.2}$$

and the distance between pickups L is chosen to meet the requirement

$$\left(\frac{v_\perp}{R}\right)\left(L - \frac{l_\parallel}{2}\right) = \frac{\pi}{2} + 2\pi k; \qquad k = 0, 1, \ldots \tag{3.3}$$

In this case, if the frequency spread $\Delta\omega$ is small, the decrement is equal to

$$\lambda_\perp = \frac{Ke^2}{v_\perp p_s} U_{10}^2(0); \qquad \lambda_\perp \simeq \frac{Ke^2}{v_\perp p_s l_\perp^2}. \tag{3.4}$$

Formula (3.4) is applicable as long as

$$\tau\Delta\omega^\circ < \theta_0 \quad \text{or} \quad l_\parallel > L\frac{\Delta\omega^\circ}{\omega^\circ}. \tag{3.5}$$

The conditions (3.2) to (3.5) have a simple meaning. The condition (3.3) corresponds to the ordinary requirement for the distance between pickups to be a multiple of a quarter wavelength of betatron oscillations. The minimum-distance limit (3.5) between pickups insures a kicker signal effect on a particle in spite of the time spread of motion of particles between pickups. Condition (3.2) shows that the optimum delay time (2.9) should be less than the distance between the pickup centers. This condition appears due to some structural features of the system under consideration when pickups are made as matched lines. Really, as follows from the structure of (2.8), the interaction is due to the effect on a particle of the wave travelling in the pickup in the direction opposite to the beam motion. For complete use of the pickup length, the signal should be picked up and transmitted to the opposite sides of the kicker and pickup. This leads to the condition (3.2). With the parameters chosen α, L an interaction with an accuracy to the small term of the order of $v\theta_0$ changes its sign at the transition $1 \leftrightarrow 2$. The back-coupling system under consideration is thus close to systems with purely non-Hermitian interaction, ensuring the best condition for oscillation damping.

For defining the damping rate for energy spread, it is sufficient to confine oneself to harmonics with $m_\perp = 0$,

$$V_0(1, 2) = -U_{00}(q_1)U_{00}(q_2)\frac{Ke^2}{R} \times \left[g\left(\frac{\xi - \theta_0}{2}\right) - g\left(\frac{\xi + \theta_0}{2}\right)\right], \tag{3.6}$$

with the condition that $\xi = \alpha - |\theta_1 - \theta_2| > 0$, whence

$$R\dot{q}_\perp = -\left(\frac{\partial V_0(1, 2)}{\partial \theta_1}\right)_{l=1} = -\frac{Ke^2}{R} U_{00}^2(q_1) \times \left[\left(\frac{\partial g}{\partial \theta}\right)_- - \left(\frac{\partial g}{\partial \theta}\right)_+\right], \tag{3.7}$$

where derivatives $(\partial g/\partial\theta)_\pm$ are defined as derivatives at the point $\theta = \frac{1}{2}(\alpha \pm \theta_0)$.

It is clear from (3.7) that $\dot{q} \neq 0$ due to edge effects where the derivative $\partial g/\partial\theta$ has its extremum.

The delay time (2.9) is optimum with

$$\alpha = 0; \qquad \alpha = 2\theta_0 \tag{3.8}$$

These values differ from those for the transverse motion (3.2). Note that the distance between the kicker and pickup is insignificant here.

$\alpha = 0$ means that at the moment of input (output) of the particle into (from) the kicker, there exists the same (but amplified) field that was induced by the particle at the input (output) of the pickup. Both of these effects have the same sign. The value $\alpha = 2\theta_0$ means that at the input of the kicker a particle experiences the effect of the amplified field induced by the particle at the output of the pickup.

The average value of the first part of (3.7) at $q = 0$ gives the average particle losses.

The energy-spread damping can be characterized by the decrement δ_p in the relation

$$\frac{d}{dt}\langle q^2\rangle = -\delta_p\langle q^2\rangle, \tag{3.9}$$

and its form follows from (3.7).

$$\delta_p = -\frac{Ke^2}{R^2}\left[\frac{\partial}{\partial q} U_{00}^2(q)\right]_{q=0}\left[\left(\frac{\partial g}{\partial \theta}\right)_- - \left(\frac{\partial g}{\partial \theta}\right)_+\right], \tag{3.10}$$

which gives

$$\delta_p = \frac{2Ke^2}{Rp_s} U_{00}(0)U_{10}(0)\frac{\psi_p}{\Delta\theta} \simeq \frac{2Ke^2}{p_s l_\perp^2}\psi_p. \tag{3.11}$$

Here ψ_p is the ψ-function value at the point of the pickup location for which the momentum dependence $U_{00}(q)$ is significant.

It can be seen from (3.10) that the decrement differs from zero due to excitment of the field at the ends of the pickup under the condition that in the pickup (or kicker) the harmonic $U_{00}(0) \neq 0$. In deriving the value (3.11), U_{00} was taken to be $U_{00} \simeq l_\perp U_{10}$; $|(\partial g/\partial\theta)_\pm| \simeq \Delta\theta^{-1} = R/l_\perp$.

Note that the interaction (3.6) is determined by the azimuthal harmonics up to $l \simeq \Delta\theta^{-1} = R/l_\perp$.

Earlier the amplification factor was considered to be constant. If the band width is limited, the effective length of the interaction changes. That is if harmonics with frequencies $n\omega^\circ$, $n > n_0$ are not amplified, it can be easily shown in all formulae that the interaction length along the azimuth θ_0 should be taken as the maximum of $\theta_0 = l_\parallel/R$ and n_0^{-1}.

4 ATTAINABLE DECREMENTS

In this section, we shall consider the effect of the particles' mutual influence. This effect, like the effect of self-action that has been already considered, can be described using a standard method of the kinetic equation. In the kinetic equation for the zeroth harmonics of the single-particle distribution function $f(\mathbf{I}_1) = f(1)$

$$\frac{\partial f(1)}{\partial t} - i \sum_{\mathbf{n}} \frac{\partial}{\partial \mathbf{I}_1} [\mathbf{n} V_{nn}(I_1; I_1) f(1)] = St \qquad (4.1)$$

$$\int d^3\Gamma_1 f(1) = 1; \qquad d^3\Gamma = d\mathbf{I} d\boldsymbol{\psi}.$$

The second term on the left-hand side describes the self-action effect, which differs from zero with a non-Hermitian interaction. As is shown in Ref. 4, the collision integral St for the non-Hermitian interaction has the form

$$\begin{aligned} St = \sum_{\mathbf{n}} \left(\mathbf{n}\frac{\partial}{\partial \mathbf{I}}\right)\Big\{&(1 - |\varepsilon_n|^{-2}) \cdot f(I_1) \cdot \operatorname{Im} V_{nn}(I_1; I_1) \\ &+ \pi N \int d\Gamma_2 \delta[\mathbf{n}\boldsymbol{\omega}_2 - \mathbf{n}\boldsymbol{\omega}_1] \cdot |\varepsilon_n(q)|^{-2} \\ &\times |V_{nn}(I_1 I_2)|^2 \\ &\times \left[\mathbf{n}\frac{\partial f(\mathbf{I}_1)}{\partial \mathbf{I}_1} f(I_2) - \mathbf{n}\frac{\partial f(\mathbf{I}_2)}{\partial \mathbf{I}_2} \cdot f(\mathbf{I}_1)\right]\Big\}. \end{aligned} \qquad (4.2)$$

Here we have taken into account only diagonal interaction harmonics $V_{nn}(I_1 I_2)$. We shall discuss the role of the neglected terms in Section 5. In addition to that, because the frequencies are included in combinations $\mathbf{n}\boldsymbol{\omega} = l\omega^\circ + \mathbf{m}_\perp \boldsymbol{\omega}_\perp$ with characteristic $m_\perp \simeq 1$; $l \gg 1$, the dependence of $\omega_\perp$ on the amplitude is insignificant, so the frequencies can be considered as linear functions of momentum.

The first term in (4.2), proportional to the imaginary part of the interaction, describes the dissipation of the process; this term is absent in Hermitian interactions. The second term corresponds to the ordinary collision integral, which gives only a redistribution of decrements.

The value $\varepsilon_n(q)$ in (4.2) is

$$\varepsilon_n(q) = 1 + i\pi N \int d\Gamma_2 \delta_2[\mathbf{n}\boldsymbol{\omega}_2 - \mathbf{n}\boldsymbol{\omega}_1] V^*_{nn}(I_2 I_2) \times \mathbf{n}\frac{\partial f(I_2)}{\partial \mathbf{I}_2} \qquad (4.3)$$

and plays a role analogous to the dielectric constant (or specific inductive capacitance) in plasmas. For the model considered in Section 1, (4.3) coincides with (1.13).

From (4.1) (4.2), one can obtain expressions for damping decrements.

If one can consider the momentum distribution and frequency $\omega(q)$ as constant in time (for instance, $\delta_p \ll \lambda_\perp$), the betatron-oscillation damping goes exponentially with decrement

$$\lambda_\perp = -\sum_{\mathbf{n}} m_\perp \frac{\operatorname{Im} V_{nn}(0)}{\langle I_\perp \rangle} \cdot \left\langle \frac{I_\perp^{|m_\perp|}}{|\varepsilon_n(q)|^2} \right\rangle, \qquad (4.4)$$

or

$$\lambda_\perp \simeq \frac{Ke^2}{v_\perp p_s} U_{10}^2(0) \frac{1}{l_m} \sum_{l=-l_n/2}^{l_m/2} \langle |\varepsilon_l(q)|^{-2} \rangle.$$

The decrement $\lambda_\perp$ differs from the single-particle decrement (3.4) (which is due to self-action) by the amount that $|\varepsilon_l|$ differs from unity. Thus, as in the model considered in Section 1, the particles' mutual effect leads to the result that the decrement decreases as N^{-2}, where N is the number of particles. The maximum attainable decrements (in order of magnitude are determined by the condition

$$|\varepsilon_l| \simeq 1 \quad \text{with} \quad l \simeq l_{\max} \qquad (4.5)$$

For transverse oscillations ($l_{\max} \simeq \pi/\theta_0$), this leads to the limitation

$$\lambda_\perp < \lambda_{\max} \simeq \frac{\Delta\omega}{\pi N \theta_0^2}, \qquad (4.6)$$

which corresponds to the estimate (1.6).

The decrement $\lambda_\perp$ (3.4) achieves its limiting value (4.6) with amplification factors of the order

$$K \simeq \left(\frac{p_s c}{e^2/R}\right) \cdot \left(\frac{l_\perp}{l_\parallel}\right)^2 \cdot \frac{\pi v_\perp \Delta\omega^\circ}{N\omega^\circ}$$

Similarly, damping of the momentum spread (3.9) is determined by the decrement

$$\delta_p \cdot \langle q^2 \rangle = -\sum_l \frac{2l}{R} \left\langle q \frac{\operatorname{Im} V_{ll}(II)}{|\varepsilon_l(q)|^2} \right\rangle. \quad (4.7)$$

At $|\varepsilon_l| \simeq 1$, this expression takes the form (3.11). In energy-spread damping, with the decrease in the frequency spread $\Delta\omega(q)$ the particles' mutual influence increases [that is, $\varepsilon_l(q)$ becomes higher]. Therefore, the damping does not have an exponential character.

If the interaction harmonics V_{00} is not equal to zero at $m_r = m_z = 0$, damping leads to a distribution with a non-zero equilibrium momentum spread

$$\left\langle \frac{a^2}{l_\perp^2} \right\rangle_{\text{eq}} = \left(\frac{\delta_p}{\lambda_0} \right)^{2/3}. \quad (4.8)$$

In this formula,

$$\lambda_0 \simeq \frac{4 p_s v_r^2}{\pi N \Delta\theta} \left(\frac{\partial \omega^\circ}{\partial q} \right) \ln^{-1} \left(\frac{\theta_0}{\Delta\theta} \right)$$

The spreads occuring are, generally speaking, large. In order to avoid an equilibrium momentum spread, it is desirable to choose the damping system in such a way to have the V_{00} harmonics small. For example, one can make a pickup in such a way as to get zero signal if a particle passes the pickup without deflection from its equilibrium orbit [field harmonics $U_{00}(0) = 0$]. In this case, the decrement will differ from zero [but half that in formula (3.11)], if in the kicker the field harmonic $U_{00}(0) \neq 0$. An equilibrium momentum spread under these conditions is determined by the accuracy of pickup adjustment.

The difference of $|\varepsilon_l|$ from unity in such a system is of order

$$\varepsilon_l(q) \simeq 1 + \frac{\delta_p}{\Delta} q \frac{\partial f}{\partial q}; \qquad \Delta = \frac{4 |\partial \omega^\circ / \partial q|}{\pi N (\Delta\theta)^2}. \quad (4.9)$$

As can be seen from (4.7) and (4.9), if the average energy losses are compensated, $\langle q \rangle = 0$. Then with a large initial spread $\Delta\omega(0) \gg N(\Delta\theta)^2 \delta p$, the damping of $\langle q^2 \rangle$ goes exponentially

$$\langle q^2(t) \rangle = \langle q^2(\theta) \rangle e^{-\delta_p t}$$

up to $\Delta\omega_c \simeq N\delta_p \cdot (\Delta\theta)^2$. Then the rms momentum spread decreases inversely proportionally to time as

$$\langle q^2(t) \rangle = \left(\frac{\delta_p}{\Delta} \right)^2 \cdot (1 + \delta_p t)^{-1} \quad (4.10)$$

If the initial spread $\langle q^2(0) \rangle \leq (\delta_p/\Delta)^2$ is small, the damping has the form

$$\langle q^2(t) \rangle = \langle q^2(0) \rangle \cdot \left[1 + \frac{\Delta^2 \langle q^2(0) \rangle}{\delta_p} t \right]^{-1}$$

In other words, the optimal value of δ_p for energy damping should be

$$(\delta_p)_{\text{opt}} = \frac{4 \Delta\omega(0)}{\pi N (\Delta\theta)^2}; \quad (4.11)$$

in this case

$$\langle q^2(t) \rangle = \langle q^2(0) \rangle \cdot [1 + (\delta_p)_{\text{opt}} t]^{-1} \quad (4.12)$$

5 EQUILIBRIUM SIZE OF A BEAM

Until now, we have taken into account only diagonal terms in the interaction (2.4), assuming harmonics with $\mathbf{n}_1 \neq \mathbf{n}_2$ to be rapidly varying in time. This is incorrect in the case when resonances $\mathbf{n}_1 \boldsymbol{\omega}(1) = \mathbf{n}_2 \boldsymbol{\omega}(2)$ with $\mathbf{n}_1 \neq \mathbf{n}_2$ are possible. Let us now take into account these resonances, assuming $|\varepsilon_l| \simeq 1$ when the interaction renormalization is negligible. To this end, to (4.2) one should add a collision integral

$$\pi N \sum_{\mathbf{n}_1} \left(\mathbf{n}_1 \frac{\partial}{\partial \mathbf{I}_1} \right) \sum_{\mathbf{n}_2 \neq \mathbf{n}_1} \int d\Gamma_2 \delta_+ [\mathbf{n}_2 \boldsymbol{\omega}(2) - \mathbf{n}_1 \boldsymbol{\omega}(1)]$$
$$\times V^*_{n_1 n_2}(I_1 I_2) \left\{ \mathbf{n}_1 \frac{\partial f(1)}{\partial \mathbf{I}_1} \cdot f(2) \cdot V_{n_1 n_2}(I_1 I_2) \right.$$
$$\left. - \mathbf{n}_2 \frac{\partial f(2)}{\partial \mathbf{I}_2} f(1) V^*_{n_2 n_1}(I_2 I_1) \right\} \quad (5.1)$$

As is shown in Ref. 4, the most dangerous resonances are those with $|m_{1\perp}| = 1$, $|m_{2\perp}| = 0$, leading to a final equilibrium size

$$\left\langle \frac{a^2}{l_\perp^2} \right\rangle \simeq \frac{\lambda_\perp}{(\lambda_\perp)_{\max}} \ln \left(\frac{\pi}{\theta_0} \cdot \frac{\Delta\omega}{\delta\omega} \right), \quad (5.3)$$

under the condition that the argument of the logarithm is larger than unity. Here $\delta\omega$ is the frequency detuning and $\Delta\omega$ is the frequency shift to the closest neighbouring resonance. It is assumed here that harmonics $U_{00} \simeq l_\perp U_{10}$.

The effect decrease with decrease in the frequency spread and it disappears at $\Delta\omega \leq \delta\omega(\theta_0/\pi)$. Another way to remove the effect is by appropriate choice of the pickup design to get no signal if a particle passes through it with no transverse deflection $[U_{00}(0) = 0]$. Then $\langle a^2 \rangle$ will be determined by the accuracy of alignment of the pickup.

Resonances $|m_{1\perp} = |m_{2\perp}| = 1$ decrease the damping decrement value and become lower with a decrease of $\Delta\omega$-spread.

Another factor that affects the final beam size is field fluctuations in pickup of the order of $\langle \varepsilon^2 \rangle \simeq (4\pi/l_\perp^3)T$ because of their further amplication. Here T is the pickup temperature (or that of matching element) given in energy units. The effect leads to the beam size of the order given in Ref. 4.

$$\langle a_r^2/R^2 \rangle \simeq \frac{KT\theta_0}{E_s v_r} \tag{5.4}$$

If the amplifier noise is substantial, it should also be taken into account.

6 FURTHER POSSIBILITIES OF THE METHOD

Let us consider the possibilities of this method in getting maximum decrements and minimum beam size. We discuss first the case when it is required to damp only the transverse dimension of the beam at a given frequency spread. In this case, transverse-oscillation damping is exponential. The maximum decrement attainable for transverse motion was already found in (4.6). It grows as θ_0^{-2} with decrease of interaction length, which corresponds to a decrease in mutual influence of particles on each other. However, θ_0 cannot be less than that required by the condition (3.5) with a final spread of the revolution frequencies. The optimum situation is

$$\theta_0 = \frac{l_\parallel}{R} \simeq \frac{L}{R}\frac{\Delta\omega^\circ}{\omega^\circ} \tag{6.1}$$

In this case, the decrement is inversely proportional to the frequency spread $\lambda_{\max} \simeq 1/\Delta\omega^\circ$. Since the pickup length is always larger than its transverse dimensions $l_\perp$, the choice (6.1) may require the pickup dimensions to be decreased. This, in its turn, gives a limit for the transverse dimensions of the beam. That is, the choice of a length $l_\parallel$ with (6.1) is possible if the pickup function ψ_p can be made sufficiently small

$$\psi_p < \frac{L}{R}\langle \psi \rangle.$$

At the same time, a corresponding decrease of the β function in the pickup is required.

The same formulae are applicable if the effective interaction length is determined by the band width. In the case when the resonance effect (5.3) is suppressed [field harmonic $U_{00}(0)$ is absent], the transverse beam size is determined by noise (5.4). The noise effect can be diminished by an increase in the number of the pickup-kicker systems per turn, decreasing correspondingly the amplification factor for each pair, maintaining the total effect amplification factor at the same level. It is assumed here that the signal from a pickup is transmitted only to its "own" kicker. Otherwise $\lambda_{\max}$ is decreased. Since the revolution frequency spread is small, phase correlations are maintained for many turns. Therefore, for the modification given, all the formula derived above are valid except for (5.4), where the interaction should be understood as a sum of interactions with all systems (i.e., the amplification factor is a summing factor nK). In particular, an increase in the number of systems does not alter the maximum attainable decrement. An exception is the effect of noise, which influences a beam in each system independently. As a result, in formula (5.4) one should mean as previously, an amplification factor of a single pickup-kicker pair. Thus, with the total amplification factor nK maintained, $\langle a^2 \rangle$ are decreasing as n^{-1} with an increases of the number of systems n and the beam is less affected by noise.

Recall that maintaining phase correlations for many turns led also to the difference in the expressions for the maximum decrement obtained (4.6) and the result obtained by Van der Meer. The difference is determined by a factor $\omega^\circ\theta_0/\Delta\omega^\circ$ and for θ_0 from (6.1), this difference is small (on the order of L/R).

Let us consider what will happen to a stored cooled beam with a number of particles N from injection of additional bunch of particles with number of particles $\tilde{N}$. Let the momentum spread for injected and cooled beams be the same. It is easy to show that at $\lambda \ll \lambda_{\max}$, the rms amplitude for the cooled beam satisfies the equation (here $m_z = 0; |m_r| = 1$):

$$\langle \dot{I} \rangle + \lambda\langle I \rangle = -\pi\tilde{N}\sum_l \frac{|V_n(\theta)|^2}{|\mathbf{n}\Delta\boldsymbol{\omega}|}[\langle I \rangle - \langle \tilde{I} \rangle], \tag{6.3}$$

where $\langle \tilde{I} \rangle$ is the rms amplitude of the injected beam.

As can be seen from (6.3), interaction of the beams leads to two effects; first, to a slight variation of the decrement and second, to excitation of the cooled beam by the injected beam (effect of temperature flattenning of beams). If the frequency spread is assumed to have a constant time dependence $\langle I(t) \rangle$, one obtains

$$\langle I(t) \rangle \sim \langle \tilde{I}(0) \rangle t e^{-\lambda t},$$

which gains its maximum at $t \simeq \lambda^{-1}$, $\langle I \rangle_{\max} \simeq (\lambda_\perp/\lambda_{\max})\langle \tilde{I}(0) \rangle$. Here $\tilde{\lambda}_{\max}$ is determined by formula (4.6) which $N = \tilde{N}$. In this case, an injected beam damps with decrement of the order $\lambda_\perp$.

Let us emphasize that if transverse oscillations are cooled with decrement $\lambda_\perp \simeq \lambda_{\max} \sim N^{-1}$ and if the total number of cooled particles is the same for both methods, then the time for storing does not depend on the fact whether the beam is cooled at once or by fractions. The condition $\lambda < \lambda_{\max}$ coincides with the condition of coherent-oscillation stability $\lambda_{\text{coh}} < \Delta\omega$ since the decrement of coherent oscillations $\lambda_{\text{coh}} \simeq \lambda N \theta_0^2$.

Let us proceed to discuss energy-spread damping. Recall that the delay times in systems which damp energy spread and transverse oscillations should be different [see (3.2), (3.8)]; therefore, it is reasonable to make these systems independent. In the optimal case, the δ_p that characterizes the rms momentum-spread variation with time is determined with (4.11) by the transverse dimension of the pickup $\Delta\theta \sim l_\perp/R$. Therefore it is desirable to make $l_\perp$ smaller, which requires smaller ψ and β functions in the pickup for a given frequency spread. It is of course necessary to increase the amplification factor K so that $K\psi_p$ and the decrement (3.11) remain constant. In order to avoid any increase of noise influence, it is better to increase the summing amplification factor by an increase in the number of damping systems.

The problem of total phase-volume damping can be solved in two ways, first at a given initial frequency spread $\Delta\omega(0)$ betatron-oscillation damping with a maximum attainable decrement (4.6) $\lambda_{\max} \sim \Delta\omega(0)$ is achieved and then the energy-spread damping according to (4.12) occurs. In the second method, cooling occurs in parallel over both degrees of freedom. In this case, the amplification factor for the system damping betatron oscillations should alter in the process of cooling, so that the decrement $\lambda_\perp$ should have its optimum value $\lambda\Delta(t) \sim \Delta\omega(t)$ at any given moment. In this case, the damping characteristic for transverse oscillations has the form

$$I = I_0 \exp\left\{-2\lambda_{\max}\sqrt{\frac{t}{\delta_p}}\right\},$$

where $\lambda_{\max}$ is determined by (4.6) with $\Delta\omega = \Delta\omega(0)$ and δ_p is determined by (4.11). Damping times for both cases are, generally speaking, of the same order of magnitude.

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