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THE EFFECT OF AN ACCOMPANYING MAGNETIC FIELD ON ELECTRON COOLING

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We consider the cooling of a beam of heavy particles in an accompanying electron flux "frozen" by a longitudinal magnetic field. The velocity distribution of electrons is assumed to be sharply flattened in the longitudinal direction ($\Delta_{e\parallel} \ll \Delta_{e\perp}$). Under these conditions in the region $\Delta_i < \Delta_{e_\perp}$ (where Δ_i is the ion velocity spread) the effective temperature of the electrons is determined by the longitudinal spread of their velocities because of the long-range character of Coulomb forces and because the transverse motion of electrons is magnetized. In this case, the cooling rate increases rapidly compared with the case $\Delta_{e_\parallel} \sim \Delta_{e_\perp}$ and the ion temperature may be decreased down to the level of the longitudinal temperature of the electrons. It is shown that under certain conditions this effect is reduced by deviations of the magnetic-field force lines from the closed-orbit direction in the cooling region.

The electron cooling method is based on the transfer of the heat energy of a heavy-particle beam to an accompanying electron flux by Coulomb collisions.[1] The kinetics of the electron cooling method has some features that distinguish it from the usual relaxation of a two-component plasma. These features are attributed to the cyclic character of particle motion in the beam being cooled and also to the formation conditions of the electron flux.

The main phenomena connected with the character of particle motion in storage rings were considered earlier[2,3] and some principal restrictions were imposed on the possible deviation of the electron flux from the thermodynamic equilibrium state in the reference system moving with the beam. These requirements can be satisfied in practice without any significant decrease in the electron-cooling efficiency. Success of the first experiments on proton-beam cooling[4–6] confirmed the general optimistic viewpoint of the realizability of this method and stimulated further experimental and theoretical studies.

In the work presented here some features are considered of a cooling process in the electron flux with low (with respect to the transverse) longitudinal temperature that is accompanied by a strong magnetic field.

In Ref. 2 the kinetic equation was given taking account of magnetized electrons. In this equation, we used the form of the collision integral in a strong magnetic field that was first derived by S. T. Belyaev,[7] but the analysis of the cooling process was conducted without taking into account the magnetized electrons.[2] The study of electron cooling was primarily concentrated on the cases where the velocity distribution was approximately the same in any direction; under these conditions a magnetic field does not substantially effect the process of relaxation. Further, V. V. Parchomchuk paid attention to a very important fact that, in the case of electron cooling in a passing (not circulating) beam of electrons, their longitudinal temperature becomes much lower than that of the cathode because of electrostatic acceleration. As is shown in Ref. 6, the noise level of accelerating voltage can be lowered down to such a level that the distribution of electrons over velocities in an accompanying system is sharply flattened in the longitudinal direction. The study of magnetization influence on the cooling process was stimulated by the work of N. S. Dikansky and D. V. Pestrikov[8] on the coherent interaction between the proton beam and a magnetized electron flux. In this work, it was found that the decrement of the small coherent oscillations of a short bunch can rapidly increase with decrease of the longitudinal spread for electron velocities.

In the work presented here, it is shown that these two factors, magnetization and smallness of the longitudinal temperature of electron beam, taken together can lead to a quite unexpected phenomenon, a rapid cooling of a beam of heavy particles with the velocity spread lower than the transverse spread of electron velocities ($\Delta_i < \Delta_{e_\perp}$). This can also lead to the fact that the beam temperature is lowered down to the electron longitudinal temperature, which is a few orders of magnitude lower than the cathode temperature.

1) It is known that for the Coulomb interaction the momentum and energy exchange for colliding particles is logarithmically divergent in the region of large impact parameters and should be cut off by a certain macroscopic parameter $\rho_{\max}$ after which the interaction is effectively decreased. It is clear that in collisions between heavy particles and electrons in magnetic fields under the conditions

$$r_L \ll \rho_{\max},$$

where r_L is the Larmor radius of the electron, the region of impact distances ρ

$$r_L < \rho < \rho_{\max} \tag{1.1}$$

can give a substantial contribution to the interaction integral. In this case, if the ion velocity with respect to the Larmor circle is

$$\mathbf{u}_A = \mathbf{v} - \mathbf{v}_{e_\parallel} = \mathbf{v}_\perp + \mathbf{u}_\parallel, \tag{1.2}$$

where $\mathbf{v}$ is the proton velocity in a system accompanying the electron beam, does not exceed the electron velocity $v_{e_\perp}$ transverse to the magnetic field $\mathbf{H}$, then the ions effectively interact with Larmor circles (but not with free electrons) in the region (1.1) since the collision time

$$\tau \simeq \frac{\rho}{u_A}$$

exceeds the Larmor period of the electron.

With a decrease of u_A the exchange intensity grows, together with an increase of the collision time. In this case, only the longitudinal degree of freedom of electron participates in an exchange, because the collisions are adiabatic with respect to the Larmor rotation of the electrons. The effect depends slightly (only logarithmically) on the magnetic-field value and on the transverse electron temperature.

2) The friction force $\mathbf{F}$ and the momentum diffusion tensor $d_{\alpha\beta} = (d/dt)\langle \Delta p_\alpha \Delta p_\beta \rangle$ with collisions in a strong magnetic field can be represented by the sum

$$\mathbf{F} = \mathbf{F}^0 + \mathbf{F}^A,$$
$$d_{\alpha\beta} = d^0_{\alpha\beta} + d^A_{\alpha\beta}, \tag{2.1}$$

where the indices "0" and "A" denote the contributions of the usual (fast) and adiabatic collisions respectively. Expressions for $\mathbf{F}^0$ and $d^0_{\alpha\beta}$ are well known[2,9]

$$\mathbf{F}^0 = -\frac{4\pi n Z^2 e^4}{m} \int L^0(u) \frac{\mathbf{u}}{u^3} f(\mathbf{v}_e) d^3 v_e \tag{2.2}$$

$$d^0_{\alpha\beta} = 4\pi n Z^2 e^4 \int L^0(u) \frac{u^2 \delta_{\alpha\beta} - u_\alpha u_\beta}{u^3} f(\mathbf{v}_e) d^3 v_e, \tag{2.3}$$

where e and m are the charge and mass of the electron, Ze is the ion charge, n is the electron flux density, $\mathbf{u} = \mathbf{v} - \mathbf{v}_e$ is the relative velocity, $f(\mathbf{v}_e)$ is the distribution of electrons over velocities. In the case of relativistic beams, all the considerations are carried out in terms of the accompanying system.

In the Coulomb logarithm

$$L^0(u) = \ln(r_L m u^2 / Z e^2),$$

the average Larmor radius of the electron r_L is taken as the maximum impact parameter $\rho^0_{\max}$ of fast collisions. In the case of the reduced electron velocity distributions, the definition of $\rho^0_{\max}$ requires a correction that will be given later with more detailed consideration of the magnetic-field effects.

The expressions for $\mathbf{F}^A$ and $d^A_{\alpha\beta}$ given in [2] can be obtained by using the known expression for the fast collision diffusion tensor $d^0_{\alpha\beta}$ and also the relation (see Appendix 1)

$$\mathbf{F}_A = \frac{1}{2m} \cdot \frac{\partial}{\partial v_\parallel} \left(\frac{d}{dt} \langle \Delta \mathbf{P} \Delta P_\parallel \rangle \right)^A, \tag{2.4}$$

which is (as is the similar relation $F^0_\alpha = (1/2m) \times (\partial/\partial v_\beta) d^0_{\alpha\beta}$) the consequence of Belyaev's general relations for kinetic momenta.[10]

In the case of Coulomb collisions, in order to find the diffusion tensor it is sufficient to calculate the momentum transfer $\Delta\mathbf{P}$ in first approximation (over the unperturbed trajectories of colliding particles). Then the expression $d^A_{\alpha\beta}$ will be similar to $d^0_{\alpha\beta}$ with the substitution of the relative velocity $\mathbf{u} = \mathbf{v} - \mathbf{v}_e$ by $\mathbf{u}_A = \mathbf{v} - \mathbf{v}_{e_\parallel}$ and the Coulomb logarithm by $L^A(u_A)$.

$$d^A_{\alpha\beta} = 4\pi n Z^2 e^4 \int \frac{u_A^2 \delta_{\alpha\beta} - u_{A\alpha} u_{A\beta}}{u^3{}_A} L^A f(\mathbf{v}_e)\, d^3 v_e, \tag{2.5}$$

where

$$L^A(u_A) = \ln \frac{\rho^A_{\max}}{\rho^A_{\min}}. \tag{2.6}$$

Using (2.5) and (2.4) we obtain

$$\mathbf{F}_\perp^A = -\frac{2\pi n Z^2 e^4}{m}\int \frac{\mathbf{v}_\perp u_\parallel}{u_A^3} L^A \frac{\partial f(\mathbf{v}_e)}{\partial v_{e_\parallel}} \mathrm{d}^3 v_e \quad (2.7)$$

$$F_\parallel^A = \frac{2\pi n Z^2 e^4}{m}\int \frac{v_\perp^2}{u_A^3} L^A \frac{\partial f(\mathbf{v}_e)}{\partial v_{e_\parallel}} \mathrm{d}^3 v_e. \quad (2.8)$$

Note that one cannot derive $\mathbf{F}^A$ in a similar way to $\mathrm{d}_{\alpha\beta}^A$. This is connected with the fact that the friction force occurs principally because the inertia of the scatterer is finite; but with the transition from fast to adiabatic collisions, an electron as a scatterer loses its degrees of freedom transverse to the magnetic field.

In the Coulomb logarithm for adiabatic collisions (2.6), the parameter $\rho_{\max}^A$ should be taken as follows

$$\rho_{\max}^A = \min\left\{r_\perp, \frac{u_A l}{\beta c}, \frac{u_A}{\omega_0}\right\},$$

where $r_\perp$ is the transverse dimension of electron beam, $l/\beta c$ is the time required for particles to pass through the cooling section, ω_0 is the Langmuir oscillation frequency for electrons. The parameter $\rho_{\min}^A$ should be smaller than the impact distance at which the momentum transfer to electron in the longitudinal direction becomes of the order of mu_A, therefore

$$\rho_{\min}^A = \max\left\{r_L, \frac{e^2}{mu_A^2}\right\}.$$

In this work we shall assume that for characteristic velocities u_A the following condition is satisfied:

$$u_A \gg (u_A)_{\min}, \quad (2.11)$$

where

$$(u_A)_{\min} = \max\left\{\frac{r_L}{\tau_{\text{eff}}}, \left(\frac{e^2}{m\tau_{\text{eff}}}\right)^{1/3}\right\}$$

$$\tau_{\text{eff}} = \min\left\{\omega_0^{-1}, \frac{l}{\beta c}\right\}. \quad (2.12)$$

In the experimental conditions at the NAP-M storage ring with electron cooling, in particular, there are electron beam parameter domains and proton velocities where these conditions are well satisfied.

A more general consideration of heavy-particle interaction with magnetized electrons will be given later.

3) Let us discuss the behaviour of $\mathbf{F}^A$ as a function of the ion velocity $\mathbf{v}$ with respect to an electron average velocity. Let $\Delta_{e\parallel}$ be the longitudinal velocity spread of electrons. Let us consider first the situation

$$v \gg \Delta_{e_\parallel}. \quad (3.1)$$

In (2.7), (2.8) it is convenient to carry out an integration by parts. Then the distribution $f(\mathbf{v}_e)$ can be substituted by $\delta(v_{e_\parallel})$. To a correction accuracy of order $1/L^A$ we obtain

$$\mathbf{F}_\perp^A = -\frac{2\pi n Z^2 e^4}{m} L^A(v)\frac{v_\perp^2 - 2v_\parallel^2}{v^2}\cdot\frac{\mathbf{v}_\perp}{v^3}; \quad (3.2)$$

$$F_\parallel^A = -\frac{6\pi n Z^2 e^4}{m} L^A(v)\frac{v_\perp^2}{v^2}\cdot\frac{v_\parallel}{v^3}; \quad (3.3)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \Delta P_\alpha \Delta P_\beta\rangle^A = 4\pi n Z^2 e^4 L^A(v)\frac{v^2\delta_{\alpha\beta} - v_\alpha v_\beta}{v^3}. \quad (3.4)$$

It can be seen that the longitudinal friction originated from collisions with Larmor circles in case (3.1) has the feature that it disappears at $v_\perp \ll |v_\parallel|$. This fact has an evident reason: when an ion moving along the magnetic force line passes by the Larmor circle, the integral momentum transfer in the longitudinal direction is zero.

The transverse friction properties are especially unusual. At $v_\perp < \sqrt{2}|v_\parallel|$, $\mathbf{F}_\perp^A$ is directed along $\mathbf{v}_\perp$ (but not oppositely), i.e. an antifriction occurs. The friction sign alteration at small $v_\perp \ll |v_\parallel|$ (compared with friction on free electrons) can be understood from the following considerations: when an ion is approaching the Larmor circle "force line," the longitudinal velocity of an electron is substantially decreased on the average. At the same time, if an ion is moving away, it is increased. The time difference for effective interaction occurring leads to an ion acceleration. For like charges similar considerations (with evident changes) lead, of course, to the same result.

Let us now estimate the friction and diffusion at velocities

$$v < \Delta_{e_\parallel}. \quad (3.5)$$

To be more definite, let us choose the distribution $f(\mathbf{v}_e)$ in the form

$$f(\mathbf{v}_e) = \left[(2\pi)^{3/2}\Delta_{e_\parallel}\Delta_{e_\perp}^2 \exp\left(\frac{v_{e_\perp}^2}{2\Delta_{e_\perp}^2} + \frac{v_{e_\parallel}^2}{2\Delta_{e_\parallel}^2}\right)\right]^{-1}. \quad (3.6)$$

In this case, from (2.7), (2.8) and (2.5) we obtain

$$\mathbf{F}_{\perp}^{A} = -2\sqrt{2\pi}\frac{nZ^2e^4}{m\Delta_{e_{\parallel}}^3}\mathbf{v}_{\perp}\ln\left(\frac{\Delta_{e_{\parallel}}}{v_{\perp}}\right)L^A(\Delta_{e_{\parallel}}); \quad (3.7)$$

$$F_{\parallel}^{A} = -2\sqrt{2\pi}\frac{nZ^2e^4}{m\Delta_{e_{\parallel}}^3}v_{\parallel}L^A(v_{\perp}); \quad (3.8)$$

$$\frac{d}{dt}\langle(\Delta\mathbf{P}_{\perp})^2\rangle = 8\sqrt{2\pi}\frac{nZ^2e^4}{\Delta_{e_{\parallel}}}\ln\left(\frac{\Delta_{e_{\parallel}}}{v_{\perp}}\right)L^A(\Delta_{e_{\parallel}}); \quad (3.9)$$

$$\frac{d}{dt}\langle(\Delta P_{\parallel})^2\rangle = 4\sqrt{2\pi}\frac{nZ^2e^4}{\Delta_{e_{\parallel}}}L^A(v_{\perp}). \quad (3.10)$$

To within numerical and logarithmic factors, these expressions are like the usual friction and diffusion in non-magnetized electron flux with an isotropic distribution of electron velocities at temperature $T_e = m\Delta_{e_{\parallel}}^2$.

4) The relative role of fast and adiabatic collisions depends on the ion velocity and also on the correlation between the transverse and longitudinal electron temperatures. Let us consider the case of practical interest where $\Delta_{e_{\parallel}} \ll \Delta_{e_{\perp}}$. For comparison with $\mathbf{F}^A$ and $d_{\alpha\beta}^A$, the expressions $\mathbf{F}^0$ and $d_{\alpha\beta}^0$ are given here which are obtained by substitution of (3.6) into (2.2) and (2.3).

a) $v > \Delta_{e_{\perp}}$; in this case

$$\mathbf{F}^0 \approx -\frac{4\pi nZ^2e^4}{m}L^0(v)\frac{\mathbf{v}}{v^3} \quad (4.1)$$

b) $v < \Delta_{e_{\perp}}$; in this region

$$\mathbf{F}_{\perp}^0 \approx -\frac{\pi\sqrt{2\pi}nZ^2e^4}{m}L^0(\Delta_{e_{\perp}})\frac{\mathbf{v}_{\perp}}{\Delta_{e_{\perp}}^3} \quad (4.2)$$

$$F_{\parallel}^0 \approx -\frac{4\pi nZ^2e^4}{m\Delta_{e_{\perp}}^2} \times \begin{cases} \dfrac{v_{\parallel}}{|v_{\parallel}|}L^0(v_{\parallel}) - \dfrac{v_{\parallel}}{\Delta_{e_{\perp}}}\sqrt{\dfrac{\pi}{2}}L^0(\Delta_{e_{\perp}}), & v_{\parallel} > \Delta_{e_{\parallel}} \\ \dfrac{v_{\parallel}}{\Delta_{e_{\parallel}}}\sqrt{\dfrac{2}{\pi}}L^0(\Delta_{e_{\parallel}}), & v_{\parallel} < \Delta_{e_{\parallel}} \end{cases} \quad (4.3)$$

$$\frac{d}{dt}\langle(\Delta\mathbf{P}_{\perp})^2\rangle^0 = \frac{d}{dt}\langle(\Delta P_{\parallel})^2\rangle^0 = \frac{(2\pi)^{3/2}nZ^2e^4L^0(\Delta_{e_{\perp}})}{\Delta_{e_{\perp}}}. \quad (4.4)$$

Let us consider ion-beam damping with the initial velocity spread $\Delta_i > \Delta_{e_{\perp}}$ without taking into account some features determined by the cyclic character of the ion motion in the storage ring and a possible variation in the direction of the magnetic field that "freezes" the electron flux. In the initial stage, when Δ_i is still more than $\Delta_{e_{\perp}}$, the fast and adiabatic collision contributions are related as their corresponding logarithms, so that the dampin decrement is of order of magnitude:

$$\lambda \simeq \frac{4\pi nZ^2e^4L}{3mM\Delta_i^3}; L \simeq L^0(\Delta_i) + L^A(\Delta_i), (\Delta_i > \Delta_{e_{\perp}}).$$

When Δ_i becomes less than the electron transverse spread $\Delta_{e_{\perp}}$, the friction due to fast collisions starts to reduce at the same time as the friction from adiabatic collisions continues to grow rapidly and

$$\lambda \simeq \frac{4\pi nZ^2e^4}{mM}\left[\frac{L^0(\Delta_{e_{\perp}})}{\Delta_{e_{\perp}}^3} + \frac{L^A(\Delta_i)}{\Delta_i^3}\right]; \Delta_{e_{\parallel}} < \Delta_i < \Delta_{e_{\perp}}.$$

The moment comes quite quickly when one can neglect the first term.

Finally, when Δ_i decreases to the value of the electron longitudinal spread $\Delta_{e_{\parallel}}$, the damping decrement achieves its maximum value

$$\lambda_{\max} \simeq \frac{4\pi nZ^2e^4}{mM\Delta_{e_{\parallel}}^3}L^A(\Delta_i < \Delta_{e_{\parallel}});$$

after that, the damping continues with a constant decrement unless the ion temperature over all degrees of freedom becomes the same as the electron longitudinal temperature, that is,

$$T_i|_{st} \simeq T_{e_{\parallel}}$$

(for the electron distribution (3.6), $T_i|_{st} = T_{e_{\parallel}}$).

Note for comparison that without the longitudinal magnetic field (at $\Delta_{e_{\parallel}} \ll \Delta_{e_{\perp}}$) the transverse equilibrium ion temperature is in order of magnitude equal to $T_{e_{\perp}}$ (common relation) and the longitudinal temperature is equal to the geometric mean of $T_{e_{\perp}}$ and $T_{e_{\parallel}}$.[6] For example, for the electron distribution (3.6)

$$T_{i_{\perp}}|_{st} = \tfrac{1}{2}T_{e_{\perp}}, T_{i_{\parallel}}|_{st} = \frac{\pi}{4}\sqrt{T_{e_{\perp}}T_{e_{\parallel}}} \cdot L^0(\Delta_{e_{\perp}})/L^0(\Delta_{e_{\parallel}})$$

Let us remind ourselves that the applicability of expressions (3.7) through (3.10) and (4.5), (4.6) is limited to the domain (sec. 2.12) $\Delta_{e_{\parallel}} > (u_A)_{\min}$.

5) During particle motion in the storage ring, the particle velocity in the cooling section does not

remain constant from turn to turn, but oscillates with a period of the order or less than the rotation period around the average direction corresponding to the closed orbit. Because the friction is small in the electron flux it does not perturb the closed orbit in practice and only leads to a slow change in oscillation amplitudes and longitudinal velocity. With the absence of rf voltage, the longitudinal velocities are damped to their values when the longitudinal friction becomes equal to zero $F_{\parallel}(v_{\parallel}) = 0$. Since the difference between the electron average velocity and the ion equilibrium velocity is not damped in the transverse direction, this value is an independent parameter in the kinetics of electron cooling.

Earlier, in Ref. 2, the effect of the so-called monochromatic instability was considered, i.e., an excitation of ion oscillations that occurs when the average difference in velocities exceeds the velocity spread. The cause of the instability onset is an alteration of the sign of the friction characteristic (decrease of friction force) for velocities $|\langle u_{\parallel}\rangle| > \Delta_{e_\perp}$. It is evident that this effect may occur in collision kinetics with "frozen" electrons not only in the region $v_\perp > \Delta_{e_\perp}$ but also at substantially smaller "detunings" $v_\perp > \Delta_{e_\parallel}$. The detuning of average velocities can be characterized by an angle $\boldsymbol{\alpha}(s) = (\alpha_x, \alpha_z)$ between the direction of the ion closed orbit and that of the magnetic field along the force lines of which the Larmor circles of electrons are moving. That is,

$$\mathbf{v}_\perp = \gamma\beta c[\boldsymbol{\theta} - \boldsymbol{\alpha}(s)];\ \boldsymbol{\theta} = (\theta_x, \theta_z),$$

where s is the coordinate along the closed orbit and $\boldsymbol{\theta}$ is the angular deviation of the ion velocity from the closed orbit, which oscillates from turn to turn with frequencies incommensurable with the rotation frequency, i.e.,

$$\theta_{x,z} = \theta^0_{x,z} \cos \Psi_{x,z};\ \dot{\Psi}_{x,z} = \omega_{x,z}$$

Let us estimate the mean friction power for small amplitudes of vertical and radial oscillations of ions θ^0_z and θ^0_x with the dependence on $\boldsymbol{\alpha}$. We assume that the electron flux is homogeneous in the x, z direction and the ion longitudinal velocities are damped "inside" the spread of the electron longitudinal velocities spread. The average rates of the energy variation for the corresponding oscillators are equal to

$$\dot{\varepsilon}_z = \gamma\beta c\overline{\theta_z F_z},\ \dot{\varepsilon}_x = \gamma\beta c\overline{\theta_x F_x},$$

where the averaging is performed over both the phases Ψ_x, Ψ_z and the ion revolution period in the storage ring. It is evident that small $\alpha \ll \Delta_{e_\parallel}/\gamma\beta c$ (for all s) does not lead to substantial changes in the friction power and decrements and damping generally occurs similarly to that described above in section 4.

The friction effect qualitatively changes in the case when $\alpha(s) \gg \Delta_{e_\parallel}/\gamma\beta c$ along the whole cooling section. Using expressions (3.2), (3.3) the friction force can be represented in the form

$$\mathbf{F}^A_\perp \approx -\frac{2\pi n Z^2 e^4 L^A}{m(\gamma\beta c\alpha)^2} \cdot \frac{\boldsymbol{\theta} - 3\boldsymbol{\sigma}(\boldsymbol{\sigma\theta}) - \boldsymbol{\alpha}}{\alpha}; \quad (5.1)$$

$$F^A_\parallel \approx -\frac{6\pi n Z^2 e^4 L^A}{m(\gamma\beta c\alpha)^2} \cdot \frac{v_\parallel}{\gamma\beta c\alpha}, \quad (5.2)$$

where $\boldsymbol{\sigma} = \boldsymbol{\alpha}/\alpha$. Note that the friction characteristics do not change at reflection $\boldsymbol{\alpha}$.

Let us assume that $\boldsymbol{\alpha}$ is directed along the normal degree of freedom, for example, $\alpha_x = 0$.

Then

$$\dot{\varepsilon}_z \sim 2\overline{\theta^2_z},\ \dot{\varepsilon}_x \sim -\overline{\theta^2_x},$$

i.e., oscillations in the direction of $\boldsymbol{\alpha}$ are excited, while the transverse in direction they are damped, the decrement sum being negative. One can show (see Appendix 2) that under the condition of homogeneity of electron flux in x and z directions near the ion equilibrium orbit, the decrement sum for transverse oscillations at arbitrary coupling of x and z motions does not depend on the direction $\boldsymbol{\alpha}$ and is equal to

$$\lambda^A_1 + \lambda^A_2 = -\frac{1}{2M}\frac{\partial \mathbf{F}^A_\perp}{\partial \mathbf{v}_\perp} = -\frac{\pi Z^2 e^4 L^A}{mM}\overline{(\gamma\beta c)^{-3} n}, \quad (5.3)$$

where the averaging is performed along the closed ion orbit. Thus transverse oscillations generally appear to be unstable if along the whole length of cooling $\alpha(s) \gg \Delta_{e_\parallel}/\gamma\beta c$. In this case, the mean value of the vector $\boldsymbol{\alpha}(s)$ is not significant. As an illustration, we can give some simple examples of $\boldsymbol{\alpha}$ behaviour at $\alpha = \text{const}$.

a) $\boldsymbol{\alpha} = \text{const}$;
b) over the cooling length, $\boldsymbol{\alpha}$ varies in sign in several jumps;
c) $\boldsymbol{\alpha}(s)$ uniformly rotates around the closed-orbit direction.

The total sum of decrements in the conditions under consideration is positive, though, since $\lambda_{\parallel}$ is three times larger in its magnitude than the sum of transverse decrements (5.3),

$$(\lambda_x + \lambda_z + \lambda_{\parallel})^A = -\frac{1}{2M}\frac{\partial \mathbf{F}^A}{\partial \mathbf{v}}$$

$$= \frac{2\pi n Z^2 e^4 L^A}{mM}(\gamma\beta c)^{-3} n. \quad (5.4)$$

According to the general theorem of the total sum of decrements, this value does not depend on the coupling of degrees of freedom and in the general case is determined by the divergence of the friction force as a function of particle velocity (Appendix 2). This property can be used for suppression of the instability of betatron oscillations mentioned above by redistributing decrements between the longitudinal and transverse particle motion by introduction of $z - x$ coupling and a transverse gradient for the longitudinal friction (for example, a gradient of the longitudinal electron velocity $dv_{e_{\parallel}}/dx$. When decrements are positive, the value $\gamma\beta c\alpha$ plays the role of an effective spread of electron velocities and an ion beam will be cooled down to a temperature $\simeq m(\gamma\beta c\alpha)^2$. Here

$$\theta_{st} \simeq \sqrt{\frac{m}{M}}\,\alpha.$$

If decrements are negative, the ion angular oscillations are excited up to

$$\theta^0 \simeq \alpha,$$

i.e., the ion and electron velocity values with respect to the closed orbit are levelled, not the effective temperature as might be expected. This conclusion can be derived on the basis of the study of monochromatic instability carried out in Ref. 2. The dynamics of large amplitudes will be considered later in greater detail.

Let us now consider the case when $\alpha(s)$ oscillates over the cooling section length running through the small values $\alpha \leq \Delta_{e_{\parallel}}/\gamma\beta c$. It is convenient to introduce the distribution $w(\boldsymbol{\alpha})$ of values $\boldsymbol{\alpha}$, $\int w(\boldsymbol{\alpha}) d^2\alpha = 1$. In the general case

$$\langle \mathbf{F}^A \rangle = \int \mathbf{F}^A(\boldsymbol{\alpha}) w(\boldsymbol{\alpha}) d^2\alpha,$$
$$\langle d^A_{\alpha\beta} \rangle = \int d^A_{\alpha\beta}(\boldsymbol{\alpha}) w(\boldsymbol{\alpha}) d^2\alpha, \quad (5.5)$$

where $\mathbf{F}^A(\boldsymbol{\alpha})$, $d_{\alpha\beta}(\boldsymbol{\alpha})$ is derived from (2.7), (2.8), (2.5) with $\mathbf{v}_{\perp} = \gamma\beta c(\boldsymbol{\theta} - \boldsymbol{\alpha})$. We are interested in the case $\theta^2 + (v_n/\gamma\beta c)^2 < \langle \alpha^2 \rangle$; in the opposite case $\mathbf{F}^A$ and $d^A_{\alpha\beta}$ do not depend on $\boldsymbol{\alpha}$ and have the form (2.2)–(3.4). It is easy to estimate that for a two-dimensional distribution $w(\boldsymbol{\alpha})$ of the Maxwellian type having width

$$\alpha_0 = \sqrt{\langle \alpha^2 \rangle / 2} \gg \Delta_{e_{\parallel}}/\gamma\beta c,$$

which is the same along both the transvere directions, the transverse and longitudinal friction forces are equal to ($v_0 = \gamma\beta c\alpha_0$):

$$\langle \mathbf{F}^A_{\perp} \rangle \simeq -\pi\sqrt{\frac{\pi}{2}}\frac{nZ^2e^4L^A(v_0)}{mv_0^2}\cdot\frac{\boldsymbol{\theta}}{\alpha_0}, \quad \theta < \alpha_0 \quad (5.6)$$

$$\langle F^A_{\parallel} \rangle \simeq -\frac{4\pi n Z^2 e^4}{mv_0^2}$$
$$\times\begin{cases} \dfrac{v_{\parallel}}{|v_{\parallel}|} L^A(v_{\parallel}) - \dfrac{3}{2}\sqrt{\dfrac{\pi}{2}}\dfrac{v_{\parallel}}{v_0} L^A(v_0), & v_0 > v_{\parallel} > \Delta_{e_{\parallel}} \\ 2\dfrac{v_{\parallel}}{\Delta_{e_{\parallel}}}\sqrt{\dfrac{2}{\pi}} L^A(\Delta_{e_{\parallel}}). & v_{\parallel} < \Delta_{e_{\parallel}}. \end{cases} \quad (5.7)$$

To within the accuracy of numerical factors, these expressions are analogous to those for the friction force (4.2) and (4.3) for fast collisions, where $\Delta_{e_{\perp}}$ has the same function as a spread v_0 in this case.

Thus, oscillations of $\boldsymbol{\alpha}$ with a "normal" distribution $w(\boldsymbol{\alpha})$ as should be expected, do not lead to instability and create only an effective temperature (included in expressions for decrements) of the transverse motion for Larmor circles. If the condition $\alpha_0 \ll \theta_e \equiv \Delta_{e_{\perp}}/\gamma\beta c$ is satisfied, the contribution of adiabatic collisions to friction and diffusion remains dominant in the region $\theta^2 + (v_{\parallel}/\gamma\beta c)^2 < \theta_e^2$.

Note that the case when $\alpha_0 \ll \theta_e$ is characteristic for experimental conditions. So in the experiments on proton-beam cooling at NAP-M,[11] the angle between directions of an accompanying magnetic field and closed orbit of protons was monitored with an accuracy of several units $\times 10^{-4}$ at an angular spread $\theta_e \simeq 3 \cdot 10^{-3}$. The fast damping of transverse dimensions of a proton beam[11] is possibly accounted for by adiabatic collisions under the conditions

$$\Delta_{e_{\parallel}} \ll \alpha_0 \gamma\beta c \ll \Delta_{e_{\perp}}.$$

We shall give some expressions for the friction force in the case of one-dimensional oscillations α_1. In the direction along these oscillations, the friction force differs from (5.6) only by a logarithmic factor, that is,

$$\langle F_1^A \rangle = -2\sqrt{2\pi}\,\frac{nZ^2e^4L^A(v_0)}{mv_0^2} \cdot \frac{\theta_1}{\alpha_0} \ln\left(\frac{v_0}{\sqrt{v_{tr}^2 + \Delta_{e\parallel}^2}}\right); \quad (5.8)$$

in the directions transverse to α_1 (interpolation formula),

$$\mathbf{F}_{tr}^A = -2\sqrt{2\pi}\,\frac{nZ^2e^4L^A(\sqrt{v_{tr}^2 + \Delta_{e\parallel}^2})}{m(v_{tr}^2 + \Delta_{e\parallel}^2)} \frac{\mathbf{v}_{tr}}{v_0}; \quad \theta < \alpha_0,\, v_{tr} < v_0 \quad (5.9)$$

where $\mathbf{v}_{tr} = (\gamma\beta c\theta_{tr}, v_\parallel)$. From comparison of (5.6), (5.8) and (5.9) with the previous case (formulas (5.1) and (5.3) where only large values $\alpha(s) \gg \Delta_{e\parallel}/\gamma\beta c$ were present), one can derive the qualitative criterion: the ion transverse-oscillation decrements become negative only in the case when the angles of the magnetic field deviation from the closed orbit direction lie around a certain value with relatively small spread $(\langle\alpha^2\rangle - \langle\alpha\rangle^2) \ll \langle\alpha\rangle^2$ (for example, with distribution $w \sim \delta(\alpha^2 - \alpha_0^2)$).

If the monochromatic instability occurs, it can be damped by introducing smoothly oscillating inflection force lines of the magnetic field with an angular amplitude α_0 larger than the uncompensated deviation α_{res}. To this end, another method can be also used, namely, the modulation of the potential accelerating electrons with a relative amplitude $\Delta U/U \gg \gamma\alpha_{res}$. In this case, the modulation frequency should exceed the increment instability.

6) Summarizing the consideration of the magnetized electron influence on the cooling process given above, let us emphasize the dependency differences in the contributions of adiabatic and fast collisions from the transverse and longitudinal spread of electron velocities. Under conditions $\Delta_{e\parallel} \ll \Delta_{e\perp}$ the decrements of the usual (i.e., fast) collisions do not practically depend on $\Delta_{e\parallel}$, but these decrements are very sensitive to the transverse spread of electron velocities. Opposite to this, the decrement of the ion adiabatic collisions with Larmor circles is insensitive to transverse electron temperature and strongly depends on the spread of electron longitudinal velocities (including the spread caused by oscillations of accelerating electrical potentials). Another distinguishing feature is that adiabatic decrements are strongly dependent on the deviations of magnetic force lines of the accompanying magnetic field with respect to the closed ion orbit in the region $\alpha > \Delta_{e\parallel}/\gamma\beta c$, while the decrements of fast collisions become sensitive to this only at angles $\alpha > \Delta_{e\perp}/\gamma\beta c$.

7) The only factor thoroughly studied here is the deflection of the magnetic force lines, which limits the positive effect of a magnetic field on the cooling process. One can also indicate some other factors affecting the kinetics:

a) a drift of the Larmor circles in the field of space charge of the electron beam with a velocity which increases with moving off the beam "center";

b) longitudinal and transverse gradients of electric potential in the cooling section;

c) the Larmor mean radius variation along the beam cross section which occurs, for example, because of imperfections in the electron gun optics;

d) electron–electron interactions leading to the effects like an unstationary shielding of the Coulomb interaction and also to an increase of the longitudinal temperature of electrons;

e) ion–ion interactions, coherent and noncoherent (stochastic) both direct and via an electron beam;

f) multiple collisions of ions with magnetized electrons in the region of impact parameters that are less than the Larmor radius of electrons.

One should consider futher the interaction of a beam of heavy particles with a flux of magnetized electrons without confining oneself to the case of "the Coulomb-logarithm approximation" and taking explicitly into account the factors mentioned above.

Note that one can hope to use in some way the large friction force in the region $v > \Delta_{e\parallel}/\gamma\beta c$ for faster cooling of beams with a large angular spread $\theta > \Delta_{e\perp}/\gamma\beta c$.

The work presented here was performed in conjunction with the experimental studies under way at our Institute. The simultaneous theoretical and experimental analysis and a search of optimal cooling conditions are especially important because of a large number of factors determining the electron beam properties and effectiveness of the method.

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Appendix 1

In order to prove (2.4), we shall give a derivation of a general relation for the kinetic momenta.[10] Let C_ν be the set of canonical integrals of motion, action and phases for two particles in an external field and $V(C, t)$ the interaction between particles. The variation of C_ν with time during collision one can represent in the form of the Poisson bracket

$$C_\nu(t) = \{V; C_\nu\}$$

Let us find $C_\nu(t)$ as a function of the initial conditions at $t = 0$ to the second order in V. Then

$$\dot{C}_\nu^{(2)} = \Delta C_{\nu'} \frac{\partial}{\partial C_{\nu'}} \{V; C_\nu\} = \{\tilde{V}; C_{\nu'}\} \frac{\partial}{\partial C_{\nu'}} \{V; C_\nu\} = \{\tilde{V}; \{V; C_\nu\}\}, \tag{A.1.1}$$

where

$$\tilde{V} = \int_0^t V(C, t')\mathrm{d}t', \; \Delta C_\nu = \{\tilde{V}; C_\nu\}.$$

Splitting (A.1.1) into parts symmetrical and antisymmetrical with respect to V and $\tilde{V}$ and using the Jacobi equation, one can obtain

$$C_\nu^{(2)} = \frac{1}{2} \frac{\partial}{\partial C_{\nu'}} \frac{\mathrm{d}}{\mathrm{d}t} \{\tilde{V}; C_\nu\}\{\tilde{V}; C_{\nu'}\} + \tfrac{1}{2}\{\{\tilde{V}; V\}; C_\nu\}.$$

We are interested in the rate of change of the action variable $\dot{I}_\nu$ averaged over initial phases ψ_ν. The second term of the Poisson bracket with potential $\frac{1}{2}\{\tilde{V}; V\}$ dissappears in averaging and we obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{\Delta I_\nu^{(2)}} = \frac{1}{2} \frac{\partial}{\partial I_{\nu'}} \frac{\mathrm{d}}{\mathrm{d}t} \overline{\Delta I_\nu \Delta I_{\nu'}}.$$

Selecting the coordinates and momenta of ion and electron as the variables of action and phase, after averaging over the initial coordinates we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \overline{\Delta \mathbf{P}} = \frac{1}{2} \frac{\partial}{\partial P_\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \overline{\Delta \mathbf{P} \Delta P_\alpha} + \frac{1}{2} \frac{\partial}{\partial P_{e_\alpha}} \frac{\mathrm{d}}{\mathrm{d}t} \Delta \mathbf{P} \Delta \mathbf{P}_{e_\alpha}. \tag{A.1.2}$$

The friction force corresponds to the second term appearing because of perturbations of the electron motion. In the case of free electrons, the tensor $\Delta \mathbf{P} \Delta P_{e_\alpha}$ depends on the difference of velocities $\mathbf{v} - \mathbf{v}_e$, and $\Delta \mathbf{P}_e = -\Delta \mathbf{P}$; after carring out integration over the electron distribution we obtain

$$\mathbf{F}^0 = \frac{1}{2m} \frac{\partial}{\partial v_\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \langle \Delta \mathbf{P} \Delta P_\alpha \rangle. \tag{A.1.3}$$

In the case of adiabatic collisions in a magnetic field, the electrons have effectively only one degree of freedom, namely, the longitudinal; taking into account that $\Delta P_{e_\parallel} = -\Delta \mathrm{P}_\parallel$, from (A.1.2), we obtain an analogous relation to (A.1.3)

$$\mathbf{F}^A = \frac{1}{2m} \frac{\partial}{\partial v_\parallel} \frac{\mathrm{d}}{\mathrm{d}t} \langle \Delta \mathbf{P} \Delta P_\parallel \rangle.$$

Appendix 2

Let I_ν and ψ_ν ($\nu = 1, 2, 3$) be the action variables and phases of a particle moving in an external magnetic field: $\dot{I}_\nu = \text{const}$, $\dot{\psi}_\nu = \omega_\nu(I) = \text{const}$; they are connected with the generalized momentum $\mathbf{P} = \mathbf{p} + (e/c)\mathbf{A}$ and a coordinate $\mathbf{r}$ by a canonical transformation. Under the effect of the friction force $\mathbf{F}(\mathbf{P}, \mathbf{r})$, the variables I_ν are slowly varied, i.e.,

$$\dot{I}_\nu = \frac{\partial I_\nu}{\partial \mathbf{P}}\mathbf{F} = \frac{\partial I_\nu}{\partial \mathbf{P}}\mathbf{F}.$$

Let us define the decrement sum in the form

$$\Lambda = -\frac{1}{2}\frac{\partial}{\partial I_\nu}\langle \dot{I}_\nu \rangle,$$

where the brackets denote averaging over phases. With averaging, the value Λ can be represented in the form

$$\Lambda = -\frac{1}{2}\left\langle\left(\frac{\partial}{\partial I_\nu}\frac{\partial I_\nu}{\partial \mathbf{P}} + \frac{\partial}{\partial \psi_\nu}\frac{\partial \psi_\nu}{\partial \mathbf{P}}\right)\mathbf{F}\right\rangle. \quad \text{(A.2.1)}$$

Using the canonical relations

$$\frac{\partial I_\nu}{\partial \mathbf{P}} = \frac{\partial \mathbf{r}}{\partial \psi_\nu}, \frac{\partial \psi_\nu}{\partial \mathbf{P}} = -\frac{\partial \mathbf{r}}{\partial I_\nu},$$

(A.2.1) can be transformed to

$$\begin{aligned}\Lambda &= -\frac{1}{2}\left\langle\left(\frac{\partial}{\partial I_\nu}\frac{\partial \mathbf{r}}{\partial \psi_\nu} - \frac{\partial}{\partial \psi_\nu}\frac{\partial \mathbf{r}}{\partial I_\nu}\right)\mathbf{F}\right\rangle \\ &= -\tfrac{1}{2}\langle\{\mathbf{F}; \mathbf{r}\}\rangle.\end{aligned}$$

The decrement sum may thus be expressed through the Poisson bracket of the friction force with the radius vector of a particle; writing the sum explicitly in the variables $\mathbf{P}$, $\mathbf{r}$ we obtain

$$\Lambda = -\langle \partial \mathbf{F}(\mathbf{P}, \mathbf{r})/\partial \mathbf{P} \rangle, \quad \text{(A.2.2)}$$

the very thing required to be proved.

In situations when the longitudinal (with respect to the ion closed orbit) friction force does not depend on the transverse coordinates x and z, the longitudinal decrement is evidently equal to $-\frac{1}{2}\langle \partial F_\parallel/\partial P_\parallel \rangle$; in this case, from (A.2.2), it follows that for other conditions being arbitrary, the decrement sum for betatron oscillations is equal to

$$\lambda_1 + \lambda_2 = -\tfrac{1}{2}\langle \partial \mathbf{F}_\perp/\partial \mathbf{P}_\perp \rangle.$$