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DISTRIBUTION OF THE MAXIMUM ORBIT DISTORTION FOR RANDOMLY DISTRIBUTED MISALIGNMENTS

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Analytic expressions have been developed for the rms value of the distribution of the maximum orbit distortion and for the extreme tail of the distribution, taking into account the possibility of several contributing harmonics of the misalignments. These results are compared with Monte–Carlo computations over a wide range of transverse oscillation frequencies and it is found that the two are in excellent agreement if one includes between three and four harmonics in the analytic results.

I INTRODUCTION

The misalignment of magnets in an AG synchrotron leads to distortion of the closed orbit. When the wave number of the transverse oscillation is close to an integer, this closed-orbit distortion will be dominated by a single harmonic, and its amplitude is easily calculated for a given set of errors.[1] One can also easily obtain the distribution of the maximum orbit distortions for random distributions of the individual magnet misalignments, and thereby establish a "safe" aperture requirement.

In those cases where the wave number of the transverse oscillation is such that more than one harmonic contributes significantly, one finds it necessary to perform Monte-Carlo type numerical simulations.[2,3] Such calculations, which imply a significant increase in the "safe" aperture requirement, have been the basis for selection of aperture size.

The purpose of this paper is to develop approximate analytic expressions for the maximum closed-orbit distortion in the case where more than one harmonic is important. This will provide a better understanding of the computer results, as well as a simple means of estimating the necessary "safe" aperture.

II CLOSED-ORBIT DISTORTION

We shall consider, for simplification, a circular machine with N equally spaced magnets, each of which is displaced randomly and independently. We shall further consider only a "smoothed" machine with wave number ν, such that

$$1 \ll \nu \ll N.$$

Clearly, our calculations can be modified to remove these restrictions, but the primary results contain the features of present interest, namely, the effect of several contributing harmonics.

The basic equation to be solved is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + \nu^2 x = \sum_{i=0}^{N-1} \varepsilon_i \delta(\theta - \theta_i), \quad \theta_i = \frac{2\pi i}{N}. \tag{1}$$

The probability distribution of each orbit error impulse, ε_i is taken to be

$$P(\varepsilon_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\varepsilon_i^2}{2\sigma^2}\right]. \tag{2}$$

A Fourier decomposition of the right side of (1) leads directly to

$$x = \sum_{j=0}^{\infty} p_j \cos j\theta + \sum_{j=1}^{\infty} q_j \sin j\theta, \tag{3}$$

where

$$p_0 = \frac{1}{2\pi\nu^2} \sum_{i=0}^{N-1} \varepsilon_i,$$

$$p_j = \frac{1}{\pi(\nu^2 - j^2)} \sum_{i=0}^{N-1} \varepsilon_i \cos\frac{2\pi ij}{N}, \quad j \geq 1 \tag{4}$$

$$q_j = \frac{1}{\pi(\nu^2 - j^2)} \sum_{i=0}^{N-1} \varepsilon_i \sin\frac{2\pi ij}{N}, \quad j \geq 1$$

Our problem is to determine the distribution of the maximum value of x in (3), for the ε_i distributed according to (2).

A useful index of the size of the closed-orbit distortion is the rms value of the displacement. From (3) this can be written as

$$X^2 \equiv \langle \overline{x^2} \rangle = \langle p_0^2 \rangle + \frac{1}{2} \sum_{j=1}^{\infty} \langle p_j^2 + q_j^2 \rangle,$$

where the bar represents the average over the circumference, and the bracket represents the average over the distribution of errors. Since

$$\langle \varepsilon_i^2 \rangle = \sigma^2$$
$$\langle p_0^2 \rangle = \frac{N\sigma^2}{4\pi^2 \nu^4} \tag{5}$$
$$\langle p_j^2 + q_j^2 \rangle = \frac{N\sigma^2}{\pi^2(\nu^2 - j^2)^2}$$

one finds

$$X^2 = \frac{N\sigma^2}{2\pi^2}\left[\frac{1}{2\nu^4} + \sum_{j=1}^{\infty} \frac{1}{(\nu^2 - j^2)^2}\right]$$
$$= \frac{N\sigma^2}{8\nu^2 \sin^2 \pi\nu}\left[1 + \frac{\sin 2\pi\nu}{2\pi\nu}\right]. \tag{6}$$

For $\nu \gg 1$, one can write for the rms orbit displacement

$$X \cong \frac{\sigma}{\nu|\sin \pi\nu|}\sqrt{\frac{N}{8}}, \tag{7}$$

a result quoted in Refs. 1 and 2.

If the distortion is dominated by a single harmonic, the rms maximum amplitude, R_0, will be $\sqrt{2}X$, or

$$R_0 \simeq \frac{\sigma}{2\nu|\sin \pi\nu|}\sqrt{N}. \tag{8}$$

Actual orbit-distortion amplitudes will be measured in units of R_0 in the present work.

III APPROXIMATIONS

A *One Dominant Harmonic*

If ν is close to the integer M, the harmonic $j = M$ will dominate the closed-orbit distortion and the maximum amplitude will be given by

$$A^2 = p_M^2 + q_M^2. \tag{9}$$

In this case the distribution in amplitude A can be shown to be

$$P_1(A)\mathrm{d}A = \frac{2A\,\mathrm{d}A}{R_M^2} \exp\left[-\frac{A^2}{R_M^2}\right] \tag{10}$$

where

$$R_M = \frac{\sigma\sqrt{N}}{\pi|\nu^2 - M^2|} \cong \frac{\sigma\sqrt{N}}{2\pi M|\nu - M|} \tag{11}$$

is the rms value of the maximum amplitude. It should be noted that R_0 in (8) and R_M in (11) are in agreement for ν close to the integer M.

Another useful index of the distribution is its behavior for large A. If we define $Q(A)$ as the probability of finding an amplitude equal to or greater than A, then

$$Q_1(A) \equiv \int_A^{\infty} \mathrm{d}A\, P_1(A) = \exp\left[-\frac{A^2}{R_M^2}\right] \tag{12}$$

for one dominant harmonic.

B *Two Contributing Harmonics*

If two harmonics ($j = M$ and $M + 1$) are important, one can write

$$\begin{aligned} X &= p_M \cos M\theta + p_{M+1} \cos(M+1)\theta \\ &\quad + q_M \sin M\theta + q_{M+1} \sin(M+1)\theta \\ &= \cos M\theta[p_M + p_{M+1}\cos\theta + q_{M+1}\sin\theta] \\ &\quad + \sin M\theta[q_M - p_{M+1}\sin\theta + q_{M+1}\cos\theta]. \end{aligned} \tag{13}$$

The "maximum amplitude," for large M, is then given by the maximum value of

$$\begin{aligned} &[(p_M + p_{M+1}\cos\theta + q_{M+1}\sin\theta)^2 \\ &\quad + (q_M - p_{M+1}\sin\theta + q_{M+1}\cos\theta)^2]^{1/2} \\ &= [p_M^2 + q_M^2 + 2\cos\theta(p_M p_{M+1} + q_M q_{M+1}) \\ &\quad + 2\sin\theta(p_M q_{M+1} - q_M p_{M+1}) \\ &\quad + p_{M+1}^2 + q_{M+1}^2]^{1/2}. \end{aligned}$$

This expression has a maximum value given by

$$A = r_M + r_{M+1}, \tag{14}$$

where

$$r_M = \sqrt{p_M^2 + q_M^2}, \qquad r_{M+1} = \sqrt{p_{M+1}^2 + q_{M+1}^2}.$$

It can be shown that $1 - \langle r_M r_{M+1} \rangle / \langle r_M \rangle \langle r_{M+1} \rangle$ goes as $1/N$ for large N. Thus, the distributions in r_M and r_{M+1} are essentially uncorrelated. Using (10), one can therefore write the distribution in maximum amplitude with two contributing harmonics,

$$P_2(A) = \int_0^\infty \mathrm{d}r_M \int_0^\infty \mathrm{d}r_{M+1} \frac{4r_M r_{M+1}}{R_M^2 R_{M+1}^2} \exp\left[-\frac{r_M^2}{R_M^2} - \frac{r_{M+1}^2}{R_{M+1}^2}\right] \delta(A - r_M - r_{M+1}) \qquad (15)$$

$$Q_2(A) = \iint_{S_2} \mathrm{d}r_M\, \mathrm{d}r_{M+1} \frac{4r_M r_{M+1}}{R_M^2 R_{M+1}^2} \exp\left[-\frac{r_M^2}{R_M^2} - \frac{r_{M+1}^2}{R_{M+1}^2}\right], \qquad (16)$$

where S_2, the region of integration in (16), is bounded by

$$S_2 = \begin{cases} r_M \geq 0, \\ r_{M+1} \geq 0, \\ r_M + r_{M+1} \geq A. \end{cases}$$

It is a simple matter to calculate the rms value of the maximum amplitude from (15). It is

$$A_2 \equiv \left(\int_0^\infty P_2(A) A^2\, \mathrm{d}A\right)^{1/2} = \left(R_M^2 + \frac{\pi}{2} R_M R_{M+1} + R_{M+1}^2\right)^{1/2}. \qquad (17)$$

It is also possible to express $Q_2(A)$ in terms of the error function defined by

$$\mathrm{Erf}(t) = \int_t^\infty e^{-u^2}\, \mathrm{d}u.$$

Specifically,

$$Q_2(A) = \frac{R_M^2}{R_M^2 + R_{M+1}^2} \exp\left[-\frac{A^2}{R_M^2}\right] + \frac{R_{M+1}^2}{R_M^2 + R_{M+1}^2} \exp\left[-\frac{A^2}{R_{M+1}^2}\right] + \frac{2R_M R_{M+1}}{(R_M^2 + R_{M+1}^2)^{3/2}} A \times \exp\left[-\frac{A^2}{R_M^2 + R_{M+1}^2}\right]$$

$$- \mathrm{Erf}\left(\frac{R_{M+1} A}{R_M \sqrt{R_M^2 + R_{M+1}^2}}\right)\Bigg]. \qquad (18)$$

For large A, the last term dominates (except very close to $\nu = M$ or $M + 1$). Since $\mathrm{Erf}(x) \to 0$ as $x \to \infty$, one can write (18) in the form

$$Q_2(A) \simeq 2\sqrt{\pi} \frac{R_M R_{M+1}}{R^2} \frac{A}{R_0} \exp\left[-\frac{A^2}{R_0^2}\right], \qquad (19)$$

where $R^2 = R_M^2 + R_{M+1}^2$. As we shall see, for additional harmonics, the quantity A always occurs in units of R, which is generalized to include the additional harmonics. For this reason we have replaced A/R by A/R_0 in (19). Here R_0, given by (8), is the generalization (to several harmonics) of

$$R_0^2 = \cdots + R_M^2 + R_{M+1}^2 + \cdots,$$

as illustrated in (6).

C *Several Contributing Harmonics*

If three or more harmonics are important, the maximum amplitude is no longer simply the sum of the amplitudes of each harmonic, since the various harmonics will not necessarily be in phase with one another when each is at its maximum value. Nevertheless, we can obtain an upper bound for the maximum amplitude by writing approximately

$$A = \cdots + r_{M-1} + r_M + r_{M+1} + \cdots \qquad (20)$$

and assuming uncorrelated distributions for the r_j's.

The rms value of the maximum amplitude can then be written, for M harmonics, using the same analysis as that leading from (15) to (17), as a generalization of (17):

$$A_M^2 = \sum R_j^2 + \frac{\pi}{2} \sum\sum_{\substack{\text{all pairs} \\ i \neq j}} R_i R_j \qquad (21)$$

The tail parameter, $Q_m(A)$, can be written as the obvious generalization of (16):

$$Q_m(A) = \iiint_{S_m} \cdots \frac{\mathrm{d}r_{M-1}^2}{R_{M-1}^2} \frac{\mathrm{d}r_M^2}{R_M^2} \frac{\mathrm{d}r_{M+1}^2}{R_{M+1}^2} \cdots \exp\left[-\frac{r_{M-1}^2}{R_{M-1}^2} - \frac{r_M^2}{R_M^2} - \frac{r_{M+1}^2}{R_{M+1}^2}\right] \cdots, \qquad (22)$$

where S_m, the region of integration in (22), is bounded by

$$S_m = \begin{cases} r_j \geq 0, & \text{all } m \text{ terms} \\ \sum r_j \geq A. \end{cases} \qquad (23)$$

For large A and several contributing harmonics, the dominant contribution to the integral in (22) comes in the vicinity of

$$r_j = \frac{R_j^2}{\sum R_j^2} A \tag{24}$$

in which case the region of integration can be extended, for convenience, from $0 \leq r_j \leq \infty$ to $-\infty \leq r_j \leq \infty$.

If one also writes the function

$$f = \begin{cases} 1 & \sum r_j > A \\ 0 & \sum r_j < A \end{cases}$$

as

$$f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{du \exp[iu(\sum r_j - A)]}{u - i\delta}, \tag{25}$$

where δ is a small positive quantity to place the pole in the upper half complex plane, the function $Q_m(A)$ in (22) can be approximated by

$$Q_m(A) \cong \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{du \exp[-iuA]}{u - i\delta} \prod_{j=1}^{m} \times \left[\int_{-\infty}^{\infty} \frac{2r\,dr}{R_j^2} \exp\left[-\frac{r^2}{R_j^2} + iur \right] \right]. \tag{26}$$

Since

$$\int_{-\infty}^{\infty} \frac{2r\,dr}{R_j^2} \exp\left[-\frac{r^2}{R_j^2} + iur \right] = iuR_i\sqrt{\pi} \exp\left[-\frac{u^2 R_j^2}{4} \right],$$

$Q_m(A)$ can be written, letting $\delta \to 0$, as

$$\begin{aligned} Q_m(A) &\cong (2)^{m-1} \pi^{(m/2)-1} \exp\left[-\frac{A^2}{R_0^2} \right] \\ &\quad \times \prod_{j=1}^{M} \left(\frac{R_j}{R} \right) \int_{-\infty}^{\infty} dv \left(\frac{A}{R_0} + iv \right)^{m-1} \exp[-v^2] \\ &\cong (4\pi)^{(m-1)/2} \exp\left[-\frac{A^2}{R_0^2} \right] \prod_{j=1}^{M} \left(\frac{R_j}{R} \right) \\ &\quad \times \sum_{i=0}^{\leq (m-1)/2} \left(\frac{A}{R_0} \right)^{m-1-2i} \frac{(-1)^i (m-1)!}{2^{2i}(m-1-2i)!\,i!}, \end{aligned} \tag{27}$$

where

$$R^2 = \sum R_j^2 \quad (m \text{ terms})$$

is generalized to R_0^2, given in (8), in all terms involving A. This result can easily be seen to agree with (12) for $m = 1$ and with (19) for $m = 2$, and will be the basis for our conclusions about the tail of the distribution. Equation (27) implies that the leading term goes as the $(m - 1)$st power of A, a conclusion which sheds light on the Monte-Carlo results in Ref. 2.

IV COMPARISON WITH CALCULATIONS

The analytic results which contain information about the distribution of maximum amplitude are given in (21) and (27) which are rewritten below

$$\frac{A_m}{R_0} = \left[1 + \frac{\pi}{2} \frac{\sum\sum_{\text{all pairs}}^{i \neq j} R_i R_j}{\sum R_j^2} \right]^{1/2} \tag{28}$$

and

$$Q_m(A) = (4\pi)^{(m-1)/2} \prod_{j=1}^{m} \left[\frac{R_j}{(\sum R_j^2)^{1/2}} \right] \exp\left[-\frac{A^2}{R_0^2} \right] \times \sum_{i=0}^{\leq (m-1)/2} \frac{(-1)^i (m-1)!}{2^{2i} i! (m-1-2i)!} \left(\frac{A}{R_0} \right)^{m-1-2i} \tag{29}$$

We have performed numerical calculations similar to those in Refs. 1 and 2 with which to compare our analytic results. We have taken

$N = 100$
$M = 19, 20$
$\sigma^2 = \frac{1}{3}$ (uniform distribution† of each ε_i between -1 and 1) and have performed calculations for 10,000 machines for each value of v, all having independent, random values of each ε_i. The results are given in Tables I and II, and illustrated in Figures 1 and 2, and are in agreement with the Monte-Carlo results in Refs. 2 and 3 where they can be compared.

Table I summarizes the results for A_m/R_0, the rms value of the distribution of the maximum amplitude in units of the rms value of the one harmonic amplitude distribution given in (8). The second and third columns contain the Monte-Carlo results and the next four columns contain the analytic results obtained from (28) for $m = 1, 2, 3, 4$. In these calculations we have used the second form of (11) for R_j, namely

$$R_j = \frac{\text{const}}{|v - j|}, \tag{30}$$

where we have lumped all other parameters in the

† The analytic results were obtained from a Gaussian distribution in each ε_i. The Monte-Carlo results assume a uniform distribution for each ε_i. The two assumptions are compatible for large N.

TABLE I

RMS value of the distribution of maximum amplitude, in units of the one harmonic result, as given by A_m/R_0 in (28).

	Monte-Carlo Calculations		Analytic calculations[a]				
$\nu - M$	$M = 19$	$M = 20$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 3.7$[b]
0.0	(1.000)	(1.000)	1.00	1.000	1.000	1.000	1.000
0.05	1.098	1.084	1.00	1.040	1.077	1.097	1.091
0.1	1.190	1.179	1.00	1.083	1.152	1.193	1.181
0.2	1.361	1.362	1.00	1.170	1.289	1.370	1.346
0.3	1.496	1.487	1.00	1.252	1.401	1.517	1.482
0.4	1.582	1.575	1.00	1.313	1.475	1.616	1.574
0.5	1.610	1.604	1.00	1.336	1.497	1.652	1.606
0.6	1.578	1.574	1.00	1.313	1.475	1.616	1.574
0.7	1.489	1.485	1.00	1.252	1.401	1.517	1.482
0.8	1.351	1.349	1.00	1.170	1.289	1.370	1.346
0.9	1.178	1.178	1.00	1.083	1.152	1.193	1.181
0.95	1.083	1.087	1.00	1.040	1.077	1.097	1.091
1.0	(1.000)	(1.000)	1.00	1.000	1.000	1.000	1.000

[a] Using Eq. (28).
[b] This column is a linear interpolation between $m = 3$ and $m = 4$.

constant in the numerator which need not be specified since only relative values of R_j are called for in (28) and (29). In all analytic calculations we take the two, three or four nearest harmonics for $m = 2, 3, 4$. For example, for $\nu = 20.6$ and $m = 3$, we take $j = 20, 21, 22$, and add $j = 19$ for $m = 4$.

Comparison of the Monte-Carlo and analytic results in Table I and Figure 1 strongly suggests that as many as four harmonics contribute in a significant way. Although we are conscious of the fact that (20) is not an accurate representation of the maximum amplitude for three or more harmonics, one can assume it is and limit the number of contributing harmonics to three or four. In fact, a linear interpolation for $m = 3.7$, displayed in the last column of Table I and as the dashed curve in Figure 1 duplicates the Monte-Carlo results surprisingly well.

TABLE II

Tail distribution parameter in units of the one harmonic result, as given by $A_t/2R_0$ in (31), (18), and (29).

	Monte-Carlo calculations		Analytic calculations					
$\nu - M$	$M = 19$	$M = 20$	$m = 1$	$m = 2$[a]	$m = 2$[b]	$m = 3$[b]	$m = 4$[b]	$1 + (\nu - M)(1 - \nu + M)$
0.0	1.000	1.000	1.00	1.00	[d]	[d]	[d]	1.000
0.05	1.056	1.056	1.00	—	[d]	[d]	[d]	1.048
0.1	1.099	1.103	1.00	1.045	[d]	[d]	[d]	1.090
0.2	1.172	1.179	1.00	1.092	(1.070)[c]	(1.078)[c]	[d]	1.160
0.3	1.220	1.217	1.00	1.128	1.125	1.174	(1.185)[c]	1.210
0.4	1.247	1.237	1.00	1.155	1.154	1.213	1.258	1.240
0.5	1.256	1.240	1.00	1.164	1.164	1.227	1.281	1.250
0.6	1.246	1.225	1.00	1.155	1.154	1.213	1.258	1.240
0.7	1.218	1.206	1.00	1.138	1.125	1.174	(1.185)[c]	1.210
0.8	1.167	1.161	1.00	1.092	(1.070)[c]	(1.078)[c]	[d]	1.160
0.9	1.100	1.093	1.00	1.045	[d]	[d]	[d]	1.090
0.95	1.055	1.050	1.00	—	[d]	[d]	[d]	1.048
1.0	1.000	1.000	1.00	1.00	[d]	[d]	[d]	1.000

[a] Using Eq. (18).
[b] Using Eq. (29).
[c] Slightly outside the range of validity of (29).
[d] Well outside the range of validity of (29).

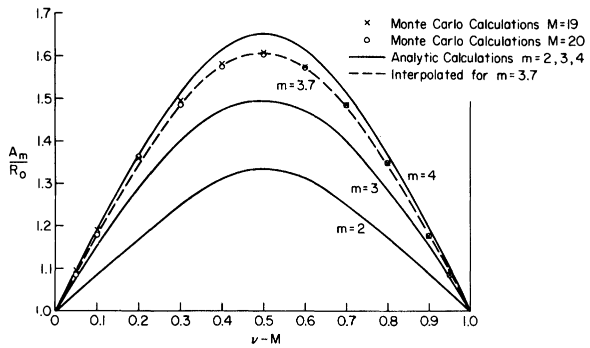

FIGURE 1 RMS value of the distribution of maximum amplitude, in units of the one harmonic result, as given by A_m/R_0 in (28).

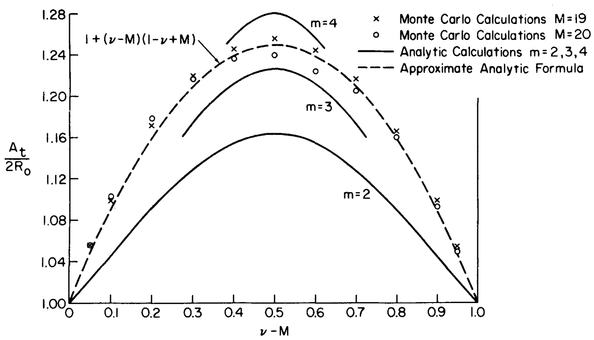

FIGURE 2 Tail distribution parameter in units of the one harmonic result as given by $A_t/2R_0$ in (31), (18), and (29).

The results in Table II and Figure 2 for the tail properties of the amplitude distribution are equally enlightening. Once again, the comparison of Monte–Carlo and analytic calculations for the behavior of the tail of the distribution strongly suggests that between three and four harmonics must be taken into account. In Table II and Figure 2 we have given results for A_t, which is defined as that value of the amplitude for which

$$Q(A_t) = e^{-4} = 0.0183.$$

Thus, the probability that the amplitude in a machine, with a particular set of errors chosen from a random distribution, will exceed A_t is 1.83%.

At this point, it is well to mention that the discussion in Section 4(a) of Ref. 1 addresses the square of the amplitude, averaged around the circumference. The conclusion about a safe aperture applies only if the amplitude does not vary around the circumference, that is, only if one harmonic dominates. Our result is the generalization to several contributing harmonics.

In the comparisons in Table II and Figure 2, we are limited somewhat by the range of applicability of our approximate form in (29), rather than the exact form in (22) which is intractable for $m \geq 3$. This is not serious, however, since the calculations well between resonances, for which (29) is valid, are the ones for which several harmonics are needed. And so it is a simple matter to identify the value for $\nu - M = 0.5$ and $m = 3.5$ and to take the simple parabolic form

$$\frac{A_t}{2R_0} \simeq 1 + (\nu - M)(1 - \nu + M) \qquad (32)$$

to approximate the interpolated analytic result. Once again the simple form in (32) fits the Monte–Carlo results surprisingly well, as can be seen from the last column of Table II and from the dashed curve in Figure 2.

V CONCLUSIONS

We recognize that it is not a difficult matter to perform Monte–Carlo computations to determine the distribution of the maximum amplitude of the orbit distortion for a randomly distributed set of alignment errors. Nevertheless, we felt it is important to explore analytic results for this distribution in order to learn which features are important in different machines and different settings. Simple analytic forms have therefore been developed to calculate the rms value of the maximum amplitude as well as the value of the amplitude for which the probability of exceeding it is $e^{-4} = 1.83\%$. These are contained in (28) and (29).

Monte–Carlo calculations have been performed to check our analytic results and we find that three to four harmonics contribute. In particular, the rms value of the distribution of the maximum amplitude corresponds to taking $m = 3.7$ harmonics in the analytic result in (28). And the tail distribution can be approximated by (29), taking $m = 3.5$ harmonics midway between resonances and using a simple parabolic extrapolation to other frequencies, as illustrated in (32).

The dependence of the tail of the distribution on A in (29) for $m = 3$ or 4 agrees with that found from Monte–Carlo computations in Ref. 3.

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