Particle Accelerators
1978, Vol. 8, pp. 151–160

COHERENT OSCILLATIONS OF A RING OF RELATIVISTIC PARTICLES†

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(*Received December 6, 1976; in final form April 1, 1977*)

The effect of ring curvature on the dynamics of coherent oscillations of a ring of relativistic particles is studied within the framework of the linearized Vlasov equation. The centrifugal force term gives rise to coupling between coherent radial and azimuthal motion, which affects the dispersion relation for the radial mode. In the limit of small-mode frequency ($\omega/\Omega \ll 1$) Landau damping of the resistive wall or electron-ion instability is shown to occur only if the frequency spread exceeds a threshold which depends sensitively on the effect of curvature. This threshold condition and the mode frequency are evaluated for several cylindrical boundary conditions.

1 INTRODUCTION

Longitudinal and transverse coherent oscillations of intense particle beams have been investigated by many authors with respect to the negative mass instability,[1,2] the longitudinal and transverse resistive instabilities[3,4] and the electron-ion instability,[5] which may be important limiting factors for accelerators with pronounced collective behaviour.

In the existing literature the transverse modes are usually treated within the limit of negligible beam curvature. This straight-beam limit is an approximation sufficient to describe many real situations. Rings of highly relativistic electrons, however, like those used in the electron ring accelerator (ERA) devices are significantly influenced by curvature effects, unless there is strong cancellation of the electron space charge by an ion background. In the Garching ERA device[6] axial focussing of the ring in the accelerating structure was found to be possible only if the axially defocussing effect of curvature is compensated by images on an electric image cylinder ("squirrel cage") close to the ring. Not only the equilibrium but also the stability properties of such electron rings may critically depend on curvature effects. In particular one must expect coupling of longitudinal and radial coherent oscillations owing to the centrifugal force term in the single particle equations of motion. This paper treats the analysis of coherent modes within the framework of the relativistic Vlasov equation in cylindrical geometry. In contrast with previous works curvature is correctly taken into account in the Vlasov analysis. The curvature-modified dispersion relations of the most important modes are derived. In special cases (low-frequency radial modes) it is shown that Landau damping by a finite energy spread may be suppressed by curvature effects, which renders the mode (linearly) unstable with respect to the resistive wall and electron-ion instability.

2 BASIC EQUATIONS AND EQUILIBRIUM

The distribution in phase space of one species of relativistic particles with charge e interacting through their collective electric and magnetic field is described by the relativistic Vlasov equation in cylindrical geometry[7,8]:†

$$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial f}{\partial r} + \dot{u}_r\frac{\partial f}{\partial u_r} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{u}_\theta\frac{\partial f}{\partial u_\theta} + \dot{z}\frac{\partial f}{\partial z} + \dot{u}_z\frac{\partial f}{\partial u_z} = 0 \quad (1)$$

† This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM.

† With units such that the speed of light becomes unity

$$
\begin{aligned}
u_r &= \gamma \dot{r} \\
u_\theta &= \gamma v_\theta = \gamma r \dot{\theta} \\
u_z &= \gamma \dot{z} \\
\gamma^2 &= 1 + u_r^2 + u_\theta^2 + u_z^2, \\
\dot{u}_r &= \frac{u_\theta^2}{r\gamma} + \frac{e}{m}\left(E_r + \frac{u_\theta}{\gamma} B_z - \frac{u_z}{\gamma} B_\theta\right); \\
\dot{u}_\theta &= -\frac{u_\theta u_r}{r\gamma} + \frac{e}{m}\left(E_\theta - \frac{u_r}{\gamma} B_z + \frac{u_z}{\gamma} B_r\right) \\
\dot{u}_z &= \frac{e}{m}\left(E_z - \frac{u_\theta}{\gamma} B_r + \frac{u_r}{\gamma} B_\theta\right), \qquad (2)
\end{aligned}
$$

with electric and magnetic fields satisfying the Maxwell equations with appropriate boundary conditions

$$
\begin{aligned}
&\operatorname{div} \mathbf{E} = 4\pi e n; \ \operatorname{curl} \mathbf{B} = 4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \\
&\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \ \operatorname{div} \mathbf{B} = 0
\end{aligned} \qquad (3)
$$

and particle density and current density computed selfconsistently with the Vlasov equation

$$
\begin{aligned}
n(r, \theta, z, t) &= \int f \, du_r \, du_\theta \, du_z \\
\mathbf{j}(r, \theta, z, t) &= e \int \frac{\mathbf{u}}{\gamma} f \, du_r \, du_\theta \, du_z
\end{aligned} \qquad (4)
$$

A rotationally symmetric equilibrium distribution can be described in terms of two constants of the motion, the total energy and the canonical angular momentum[9]

$$H^0 = m\gamma + e\Phi^0 \qquad (5)$$

$$P_\theta = m r u_\theta + e r A_\theta^0 \qquad (6)$$

$$f^0 = f^0(H^0, P_\theta). \qquad (7)$$

Next we replace u_θ by P_θ according to (6). The energy $H^0(r, z, u_r, P_\theta, u_z)$ is in general a complicated function of its variables. With regard to the linearization of the problem it is desirable to expand H^0 about the minimum value of energy for a given P_θ. The variational problem

$$\delta H = 0 \qquad (8)$$

yields a solution

$$u_r \equiv u_z \equiv 0; r \equiv R(P_\theta); z \equiv 0 \qquad (9)$$

if there is symmetry in z and no applied E_z field.

This corresponds to purely circular motion at radius $R(P_\theta)$ with zero-order values $u_{o\theta}(P_\theta)$, $\gamma_o(P_\theta)$, $H_o^0(P_\theta)$ defined by

$$
\begin{aligned}
P_\theta &= m R u_{o\theta} + e R A_\theta^0(R, 0) \\
\frac{m u_{o\theta}^2}{\gamma_o R} &+ \frac{e}{\gamma_o} u_{o\theta} B_z^0(R, 0) + e E_r^0(R, 0) = 0 \\
\gamma_o^2 &= 1 + u_{o\theta}^2 \\
H_o^0 &= m\gamma_o + e\Phi^0(R, 0). \qquad (10)
\end{aligned}
$$

We are now able to expand H^0 about H_o^0 if we define the relative single particle displacement

$$x \equiv r - R(P_\theta) \qquad (11)$$

and obtain

$$H^0 = H_o^0(P_\theta) + H_\perp^0(x, u_r, z, u_z, P_\theta). \qquad (12)$$

$H_\perp^0$ is the transverse energy which can be written in leading order as the sum of the radial and axial oscillation energies, which are constants of the motion in this order

$$
\begin{aligned}
H_\perp^0 &\equiv H_{\perp r}^0 + H_{\perp z}^0 \\
&= \frac{m}{2\gamma_o}(u_r^2 + u_z^2 + \gamma_o^2 \Omega^2 (v_r^2 x^2 + v_z^2 z^2))
\end{aligned}
$$

$$\Omega(P_\theta) \equiv \frac{v_{o\theta}}{R} \equiv \frac{u_{o\theta}}{R\gamma_o} \text{ (zero-order gyrofrequency)}, \qquad (13)$$

$$
\begin{aligned}
v_r^2 &= 1 + \frac{E_r^0 + R(E_r^{0\prime} + v_{o\theta} B_z^{0\prime})}{E_r^0 + v_{o\theta} B_z^0} + \left(\frac{e E_r^0 R}{m v_{o\theta}^2 \gamma_o^2}\right)^2 \\
v_z^2 &= \frac{R(E_z^{0\prime} - v_{o\theta} B_r^{0\prime})}{E_r^0 + v_{o\theta} B_z^0}.
\end{aligned} \qquad (14)
$$

The betratron tunes v_r, v_z agree with the results of a different derivation in Ref. 10.

We can now specify f^0 as a function

$$f^0 = f^0(P_\theta, H_{\perp r}^0, H_{\perp z}^0) \qquad (15)$$

with the following restrictions:

a) f^0 non-negative,

b) f^0 is zero outside a finite range of P_θ values (such that $0 < R_{\min} \leq R(P_\theta) \leq R_{\max}$),

c) f^0 is non-vanishing only within ranges of values of x and z sufficiently small to justify omission of higher than second-order terms in (13). (16)

Although the distributions (15) are only approximate solutions, they give a more realistic description of experimentally observable electron rings than the exact solutions (7), which distribute particles uniformly over energy surfaces in six-dimensional phase space. The experimental observation[11] of energy transfer from the radial to the axial degree of freedom after crossing the Walkinshaw resonance $2v_z = v_r$ under unfavorable conditions suggests that no equidistribution has occurred before resonance crossing. Hence, for weak nonlinearities in the equations of motion (2) the distributions (15) are a convenient starting point for the following linearized analysis of coherent oscillations. Such distributions (15) were also considered in a numerical model.[12]

With (c) a spread of betatron and revolution frequencies due to finite oscillation amplitudes is disregarded in our model. Thus dispersion is introduced in our system of particles only by the spread due to P_θ. It seems reasonable to assume that a spread of frequencies due to a finite range of betatron amplitudes has an effect upon Landau damping and stability which is comparable to the effect of an equally large spread produced by a finite range of P_θ. Moreover, there is some evidence to assume that both effects occur additively in the stability criteria.[4]

3 LINEAR PERTURBATION THEORY AND MOMENT EQUATIONS

Small perturbations about the equilibrium (15)

$$f = f^0 + f^1 \tag{17}$$

are treated with the linearized version of (1).

Using the independence of f^0 of t, θ, we may Fourier-analyze f^1 with respect to these variables. The use of two further constants $H^0_{\perp r}$, $H^0_{\perp z}$ in f^0 suggests to Fourier-analyze also with respect to the phase angles in $x - u_r$ and $z - u_z$. Here, however, we are only interested in modes with density variations in the θ direction and beam displacement in r or z, corresponding to zero or first harmonics in the transverse phase planes. Higher transverse harmonics are neglected here, because they cause oscillations of the beam cross section which in turn have a minor effect on the collective field if γ is large.

With u_θ replaced by P_θ as an independent variable according to (6) we assume

$$f^1 = f^1(r, u_r, z, u_z, P_\theta)e^{i(l\theta - \omega t)} \tag{18}$$

and find from (1) using $\dot{P}_\theta = 0$ in the equilibrium

$$\begin{aligned} il\frac{u_\theta}{r\gamma}f^1 - i\omega f^1 + \frac{u_r}{\gamma}\frac{\partial f^1}{\partial r} &+ \left(\frac{u_\theta^2}{r\gamma} + \frac{e}{m}\left(E_r^0 + \frac{u_\theta}{\gamma}B_z^0\right)\right)\frac{\partial f^1}{\partial u_r} \\ &+ \frac{u_z}{\gamma}\frac{\partial f^1}{\partial z} + \frac{e}{m}\left(E_z^0 - \frac{u_\theta}{\gamma}B_r^0 + \frac{u_r}{\gamma}B_\theta^0\right)\frac{\partial f^1}{\partial u_z} \\ &= -\frac{e}{m}\left(E_r^1 + \frac{u_\theta}{\gamma}B_z^1 - \frac{u_z}{\gamma}B_\theta^1\right)\frac{\partial f^0}{\partial u_r} \\ &\quad - e\left(E_\theta^1 - \frac{u_r}{\gamma}B_z^1 + \frac{u_z}{\gamma}B_r^1\right)r\frac{\partial f^0}{\partial P_\theta} \\ &\quad - \frac{e}{m}\left(E_z^1 - \frac{u_\theta}{\gamma}B_r^1 + \frac{u_r}{\gamma}B_\theta^1\right)\frac{\partial f^0}{\partial u_z}. \end{aligned} \tag{19}$$

Next we derive first-order expressions for the moments $\int f^1\,\mathrm{d}\sigma$, $\int xf^1\,\mathrm{d}\sigma$ and $\int zf^1\,\mathrm{d}\sigma$ with

$$\mathrm{d}\sigma \equiv \mathrm{d}r\,\mathrm{d}u_r\,\mathrm{d}z\,\mathrm{d}u_z \tag{20}$$

which we need for calculating the beam displacement and current modulation in terms of f^1. To this end we expand the coefficients in (19) in terms of x, u_r, z, u_z, carry out appropriate integrations and after partial integrations and elimination of $\int u_r f^1\,\mathrm{d}\sigma$, $\int u_z f^1\,\mathrm{d}\sigma$ we find the following expressions valid up to first order

$$\int xf^1\,\mathrm{d}\sigma = \frac{i(\omega - l\Omega)R\dfrac{\mathrm{d}R}{\mathrm{d}P_\theta}eE_\theta^1 + \dfrac{e}{m\gamma_0}(E_r^1 + v_{0\theta}B_z^1)}{v_r^2\Omega^2 - (\omega - l\Omega)^2}F^0, \tag{21}$$

$$\int f^1\,\mathrm{d}\sigma = \frac{-l\Omega}{\omega - l\Omega}\left(\frac{1}{R} - \frac{eE_r^0}{mv_{0\theta}^2\gamma_0^3}\right)\int xf^1\,\mathrm{d}\sigma - \frac{ie}{\omega - l\Omega}\frac{\mathrm{d}}{\mathrm{d}P_\theta}(E_\theta^1 RF^0), \tag{22}$$

$$\int zf^1\,\mathrm{d}\sigma = \frac{\dfrac{e}{m\gamma_0}(E_z^1 - v_{0\theta}B_r^1)F^0}{v_z^2\Omega^2 - (\omega - l\Omega)^2}, \tag{23}$$

$$F^0(P_\theta) \equiv \int f^0\,\mathrm{d}\sigma. \tag{24}$$

The local radial beam displacement as function of θ, t is defined by

$$\langle r\rangle^1 = \frac{\int rf\,d\sigma\,dP_\theta}{\int f\,d\sigma\,dP_\theta} - \frac{\int rf^0\,d\sigma\,dP_\theta}{\int f^0\,d\sigma\,dP_\theta}$$

$$= \int\left(\int xf^1\,d\sigma\right)dP_\theta + \int(R - \langle R\rangle)\left(\int f^1\,d\sigma\right)dP_\theta \qquad (25)$$

with

$$\int f^0\,d\sigma\,dP_\theta \equiv 1$$

and

$$\langle R\rangle \equiv \int rf^0\,d\sigma\,dP_\theta = \int Rf^0\,d\sigma\,dP_\theta. \qquad (26)$$

The corresponding expression for the axial displacement is

$$\langle z\rangle^1 = \int\left(\int zf^1\,d\sigma\right)dP_\theta \qquad (27)$$

and for the azimuthal line current perturbation we have

$$\langle j_\theta\rangle^1 \equiv \frac{e}{m}\int\frac{u_\theta}{\gamma r}f^1\,d\sigma\,dP_\theta$$

$$= \frac{e}{m}\int\Omega\left(\int f^1\,d\sigma - \left(\frac{1}{R} - \frac{e}{m}\frac{E_r^0}{v_{o\theta}^2\gamma_o^3}\right)\int xf^1\,d\sigma\right)dP_\theta. \qquad (28)$$

Thus we obtain from (21)–(23)

$$\langle r\rangle^1 = \int\left(1 - \frac{R - \langle R\rangle}{R}\frac{l\Omega}{\omega - l\Omega}\left(1 - \frac{eE_r^0R}{v_{o\theta}^2\gamma_o^3}\right)\right)$$

$$\times\frac{i(\omega - l\Omega)R\dfrac{dR}{dP_\theta}eE_\theta^1 + \dfrac{e}{m\gamma_o}(E_r^1 + v_{o\theta}B_z^1)}{v_r^2\Omega^2 - (\omega - l\Omega)^2}F^0\,dP_\theta$$

$$- ie\int\frac{R - \langle R\rangle}{\omega - l\Omega}\frac{d}{dP_\theta}(E_\theta^1RF^0)\,dP_\theta, \qquad (29)$$

$$\langle j_\theta\rangle^1 = -\omega\frac{e}{m}\int\frac{\Omega}{\omega - l\Omega}\frac{1}{R}\left(1 - \frac{eE_r^0R}{v_{o\theta}^2\gamma_o^3}\right)$$

$$\times\frac{i(\omega - l\Omega)R\dfrac{dR}{dP_\theta}eE_\theta^1 + \dfrac{e}{m\gamma_o}(E_r^1 + v_{o\theta}B_z^1)}{v_r^2\Omega^2 - (\omega - l\Omega)^2}F^0\,dP_\theta$$

$$- i\frac{e^2}{m}\int\frac{\Omega}{\omega - l\Omega}\frac{d}{dP_\theta}(E_\theta^1RF^0)\,dP_\theta, \qquad (30)$$

$$\langle z\rangle^1 = \frac{e}{m}\int\frac{1}{\gamma_o}\frac{E_z^1 - v_{o\theta}B_r^1}{v_z^2\Omega^2 - (\omega - l\Omega)^2}F^0\,dP_\theta. \qquad (31)$$

From these formulas we draw the following conclusions:

a) The axial perturbation (31) is in agreement with the corresponding expression in,[4] which was derived there for transverse oscillations of a straight beam.

b) The radial perturbation (29) differs from the straight beam expression by additional terms involving the azimuthal electric field perturbation and the spread in equilibrium radii $R - \langle R\rangle/R$.

c) The azimuthal current perturbation (30) has an additional term which is related to the radial beam displacement and introduces coupling between the radial and azimuthal coherent motions.

The relative importance of the different terms in (29)–(31) depends on the mode under consideration and will be discussed in the next section.

4 DISPERSION RELATIONS FOR COHERENT MODES

The above-established coupling between radial and azimuthal coherent motion depends quantitatively on the frequency ω of the mode. In the limit of vanishing collective field effects and no dispersion the zeros of the denominators in (29)–(31) permit us to classify the modes. For an observer gyrating with the particles a perturbing field $\simeq e^{i(l\theta - \omega t)}$ has a frequency $l\Omega - \omega$. This field may have a resonant action on the particles via

a) their azimuthal motion ($\omega - l\Omega = 0$),

b) their radial betatron oscillation ($(\omega - l\Omega)^2 - v_r^2\Omega^2 = 0$),

c) their axial betatron oscillation ($(\omega - l\Omega)^2 - v_z^2\Omega^2 = 0$).

Next we assume finite, but not too strong self-fields such that for either mode (a), (b) or (c) only those terms have to be maintained in (29)–(31), which are associated with the respective denominator.

We assume a linear dependence of the field perturbations on the quantities $\langle r\rangle^1$, $\langle j_\theta\rangle^1$, $\langle z\rangle^1$. Small variations with R are neglected and the perturbations evaluated at $\langle R\rangle$

$$E_r^1 + v_{o\theta} B_z^1 = F_{r,r}\langle r\rangle^1 + F_{r,\theta}\langle j_\theta\rangle^1 \tag{32}$$

$$E_\theta^1 = F_{\theta,r}\langle r\rangle^1 + F_{\theta,\theta}\langle j_\theta\rangle^1, \tag{33}$$

where the coefficients F depend on ω, l (in general) and follow from the Maxwell equations solved with the respective boundary conditions.

With the leading terms the dispersion relations are in the above defined cases:

a) Azimuthal mode (negative mass mode):
After partial integration in (29), (30) and omission of all terms except those with denominator $(\omega - l\Omega)^2$ we find the dispersion relation for the negative-mass instability

$$1 = iel\left(\frac{e}{m} F_{\theta,\theta} \int \frac{R\Omega \dfrac{d\Omega}{dP_\theta}}{(\omega - l\Omega)^2} F^0 \, dP_\theta + F_{\theta,r} \int \frac{R \dfrac{d\Omega}{dP_\theta}}{(\omega - l\Omega)^2} (R - \langle R\rangle) \cdot F^0 \, dP_\theta\right), \tag{34}$$

with

$$\frac{d\Omega}{dP_\theta} = -\frac{1}{m\gamma_o R^2}\left(\frac{1}{v_r^2} - \frac{1}{\gamma_o^2}\right) \text{ from (10)}$$

The second term expresses coupling to the radial motion through the variation in equilibrium radius and is in general small. The dominant first term agrees with the familiar negative-mass dispersion relation[1,2] with curvature only entering into the expression for v_r and we conclude that curvature has not an important effect on this mode as long as $|\omega - l\Omega| \ll l\Omega$ and $|\omega - l\Omega| \ll v_r\Omega$, otherwise the full expression (30) should be taken into account.

b) Radial-azimuthal mode:
Terms in (29)–(31) with the denominator $v_r^2\Omega^2 - (\omega - l\Omega)^2$ contribute to this mode. Omitting the small term associated with $R - \langle R\rangle/R$ and neglecting

$$\frac{eE_r R}{v_{o\theta}^2 \gamma_o^3}$$

compared with unity, we find

$$1 = \frac{e}{m\gamma_o}\left\{i \frac{\omega - l\Omega}{v_r^2\Omega} F_{\theta,r} + F_{r,r} - \frac{e}{m}\omega \frac{\Omega}{\omega - l\Omega}\frac{1}{R}\left(i\frac{\omega - l\Omega}{v_r^2\Omega} F_{\theta,\theta} + F_{r,\theta}\right)\right\} \cdot D$$

$$D \equiv \int \frac{F^0}{v_r^2\Omega^2 - (\omega - l\Omega)^2} \, dP_\theta. \tag{35}$$

Here we have replaced for simplicity the Ω, R, v_r by their average values, unless they occur in the denominator $v_r^2\Omega^2 - (\omega - l\Omega)^2$, and have used the relation

$$\frac{dR}{dP_\theta} \simeq \frac{1}{m\gamma_o v_{o\theta} v_r^2} \tag{36}$$

which follows from (10).

Equation (35) differs from the straight beam result[4] by two additional force terms in the brackets; the first term is relating the radial displacement with E_θ^1 via the centrifugal force in the radial equation of motion; the second term includes the contribution of the azimuthal current modulation to the collective fields. Such a term appears because the azimuthal component of the particle motion obeys

$$\dot\theta \simeq \Omega\left(1 - \frac{x}{R}\right). \tag{37}$$

Hence it oscillates synchronous with the radial frequency and is therefore driven by the same oscillating collective field that drives the radial coherent motion. This term vanishes for $\omega \to 0$, because in this limit the local beam center moves perpendicular to the collective field which suppresses an azimuthal density modulation. In the limit of vanishing collective effects and dispersion the modes solving (35) are given by

$$\omega = (l \pm v_r)\Omega. \tag{38}$$

The lower sign refers to the "slow wave", the upper sign to the "fast wave". Using this approximation in (35) we find

$$1 = \frac{e}{m\gamma_o}\left\{\pm i \frac{F_{\theta,r}}{v_r} + F_{r,r} - \frac{e}{m}\omega\left(\pm\frac{1}{v_r}\right)\frac{1}{R}\left(\pm i\frac{F_{\theta,\theta}}{v_r} + F_{r,\theta}\right)\right\} \cdot D \tag{39}$$

for either wave.

5 COHERENT FREQUENCY AND THRESHOLD CONDITION FOR DAMPING OF THE SLOW RADIAL MODE FOR DIFFERENT BOUNDARIES

In the low-frequency case, i.e., $l \simeq \nu_r$, the slow wave leads to the dispersion relation

$$1 = \frac{e}{m\gamma_o}\left\{\frac{-i}{l}F_{\theta,r} + F_{r,r}\right\}D \tag{40}$$

which we write in standard notation,[4] with $\langle\ \rangle$ indicating the average value,

$$1 = 2\langle\Omega\nu_r\rangle(U + (1+i)V)D. \tag{41}$$

$V > 0$ is induced by finite resistivity of the walls due to image current damping and is in general small compared with U.

With the realistic distribution

$$F^0(P_\theta) = \frac{3}{2\Delta P_\theta}\left(1 - \left(2\frac{P_\theta - \langle P_\theta\rangle}{\Delta P_\theta}\right)^2\right) \tag{42}$$

for $|P_\theta - \langle P_\theta\rangle| < \Delta P_\theta/2$, otherwise $F^0 = 0$, the evaluation of (41) follows a standard procedure[4] and gives a necessary condition for damping (see Appendix)

$$\Delta S \gtrsim 3U, \tag{43}$$

where ΔS is the full spread in the quantity

$$S \equiv (l - \nu_r)\Omega \tag{44}$$

due to the spread in P_θ. The coefficient U, which is modified here by curvature effects, is the shift of the coherent frequency due to collective effects in the limit of $V = 0$ and negligible spread in S, in which case we write

$$\text{Re}\,\omega = \langle S\rangle + U. \tag{45}$$

Thus criterion (43) requires that the distribution in S be broad enough to cover the shifted frequency ω, otherwise there are no particles in phase with the coherent wave and there is no Landau damping.

With the following definition for the Landau damping coefficient

$$K = \frac{E}{\omega}\frac{dS}{dE}, \quad \Delta S = K\frac{\Delta E}{E}\omega, \tag{46}$$

(43) is converted into a criterion involving the energy spread $\Delta E/E$ and K:

$$|K| \gtrsim \left|\frac{3U}{\frac{\Delta E}{E}\omega}\right| \simeq \left|\frac{3U}{\frac{\Delta E}{E}(\langle S\rangle + U)}\right|. \tag{47}$$

For most of the electron ring applications $|K|$ is in the range $0 \ldots 2$ and we conclude that Landau damping in the low frequency case is possible only if the average value of S obeys

$$\langle S\rangle \gg U, \tag{48}$$

otherwise the mode is unstable ($V > 0$, finite resistivity) or purely oscillatory ($V = 0$). We observe that the same argument holds if the considered mode is driven by resonant interaction with a ring of oppositely charged particles, which renders criterion (47) a necessary criterion for stabilizing the electron-ion instability.†

Ring in Free Space

For a ring in free space with the mode $l = 1$ and $\omega \ll \Omega$ we apply the results for the self-fields of a stationary ring [13] on the coherently shifted ring and find in the unshifted coordinate system (see Appendix)

$$e(E_r^1 + v_{o\theta}B_z^1) = m\gamma_o\Omega^2\mu\left(-\frac{4R^2}{a(a+b)}\frac{1}{\gamma_o^2} + 2\ln\frac{16R}{a+b}\right)\langle r\rangle^1$$

$$eE_\theta^1 = -im\gamma_o\Omega^2\mu\ln\frac{16R}{a+b}\langle r\rangle^1 \tag{49}$$

where a, b are radial and axial semi-axes,

$$\mu = \frac{\nu}{\gamma_o} = \frac{Nr_e}{2\pi R\gamma_o},$$

and r_e is the classical electron radius.

Thus we find for the coherent frequency shift

$$U = \frac{\Omega}{2\nu_r}\mu\left(-\frac{4R^2}{a(a+b)}\frac{1}{\gamma_o^2} + \ln\frac{16R}{a+b}\right). \tag{50}$$

The logarithmic term in U is due to curvature and dominates for large γ_o over the straight beam term. As an example we take

$$\mu = 3\times 10^{-3}, \frac{R}{a} = \frac{R}{b} = 10, \gamma_o = 30$$

$$|K| \lesssim 2, \frac{\Delta E}{E} = 10^{-1}$$

† A sufficient criteriom must involve also dispersion of the ion species as observed already in Ref. 5.

and find that (47) can be satisfied only if

$$(1 - v_r) \gtrsim 10^{-1} \tag{51}$$

whereas omission of the curvature term would allow for the much larger range $(1 - v_r) \gtrsim 5 \times 10^{-3}$.

Ring Close to Cylindrical Conductors

The image contribution to U, if an infinitely long conducting cylinder (radius $\bar{R}$) is present, may be expressed in terms of electric and magnetic coherent image coefficients $\varepsilon_e^{\rm coh}$ and $\varepsilon_m^{\rm coh}$.

$$U = \frac{\Omega}{2v_r}\mu\left(-\frac{4R^2}{a(a+b)}\frac{1}{\gamma_o^2} + \ln\frac{16R}{a+b} + 4\frac{\varepsilon_e^{\rm coh} - \beta^2\varepsilon_m^{\rm coh}}{(1-\bar{S})^2}\right) \tag{52}$$

$$\bar{S} = \frac{\bar{R}}{R},\ \beta = v_{o\theta}.$$

The $\varepsilon_e^{\rm coh}$, $\varepsilon_m^{\rm coh}$ are presented in the Appendix and in Table I. We observe mutual cancellation of electric and magnetic image contributions for an interior cylinder. If an electric image cylinder is considered, which is transparent for the magnetic field perturbations (poor conductor or squirrel cage) we approximate U by putting $\varepsilon_m^{\rm coh} = 0$.

For a combination of a good conductor and an electric image cylinder $\varepsilon_e^{\rm coh}$ follows from Table II, whereas $\varepsilon_m^{\rm coh}$ can be taken from Table I with $\bar{S}$ for the conductor.

The single particle frequency follows from the familiar formula[13,14] using incoherent image coefficients

$$v_r^2 = 1 - n - \mu\left(\frac{4R^2}{a(a+b)}\frac{1}{\gamma_o^2} - \ln\frac{16R}{a+b} + 4\frac{\varepsilon_e^{\rm inc} - \beta^2\varepsilon_m^{\rm inc}}{(1-\bar{S})^2}\right), \tag{53}$$

where for a perfectly conducting cylinder $\varepsilon_e^{\rm inc} \simeq 1/8$ and $\varepsilon_m^{\rm inc} \simeq 1/8 - (1-\bar{S})^2/4$ (for $(1-\bar{S}) \lesssim 0.5$), whereas a typical squirrel cage gives $\varepsilon_e^{\rm inc} \simeq 1/8$, $\varepsilon_m^{\rm inc} \lesssim 0.01(|1-\bar{S}| \gtrsim 0.2)$.[14,15] For the combination of a perfect conductor and an electric image cylinder we find $\varepsilon_e^{\rm inc} \simeq 0.21$ and again $\varepsilon_m^{\rm inc} \simeq 1/8 - (1-\bar{S})^2/4$,[15] if both cylinders are equally far away from the ring.

From (52) and Tables I, II we conclude that the coherent frequency shift U is nearly unchanged with an interior perfect conductor, but an electric image cylinder (useful for axial focusing) increases U considerably above the free space value. There remains a possibility to satisfy the threshold criterion (47) by including image contributions to the Landau damping coefficient, which is readily derived from (46) and the equilibrium equations as

$$K = -\frac{1}{\beta^2}\left(\left(\frac{1}{v_r^2} - \frac{1}{\gamma_o^2}\right)\frac{S}{\omega} + \frac{\Omega}{\omega}\frac{R\cdot\frac{{\rm d}v_r}{{\rm d}R}}{v_r^2}\right). \tag{54}$$

The free space contribution to K was found approximately as

$$Rv_r\frac{{\rm d}v_r}{{\rm d}R}\bigg|_{\rm fr.\,sp.} \lesssim \mu \tag{55}$$

from a numerical solution for the ring self-field[16] with $a/R \sim 0.1$ and $\gamma \gtrsim 20$.

TABLE I
Coherent image coefficients for one perfect conductor

$\bar{S}$	0.5	0.6	0.7	0.8	0.9 — 1.1	1.2	1.3	1.4	1.5
$\varepsilon_e^{\rm coh}$	0.039	0.053	0.069	0.085	0.104 — 0.162	0.206	0.248	0.283	0.312
$\varepsilon_m^{\rm coh}$	0.044	0.057	0.07	0.085	0.104 — 0.137	0.134	0.125	0.115	0.104

TABLE II
Coherent image coefficients for two conductors (electric image cyl.)

$\bar{S}_1(\bar{S}_2)$	1.5(0.5)	1.4(0.6)	1.3(0.7)	1.2(0.8)	1.1(0.9)
$\varepsilon_e^{\rm coh}$	0.37	0.36	0.35	0.34	0.33

More significant may be the image contribution, which for electric images on a cylinder results as

$$R v_r \frac{\mathrm{d}v_r}{\mathrm{d}R}\bigg|_{\text{el. im.}} \simeq 2\mu \frac{1}{8(1-\bar{S})^3}, \qquad (|1-\bar{S}| \ll 1) \tag{56}$$

using an expansion of the integrals involving modified Bessel functions. For magnetic images it has opposite sign.

For a compressed electron ring (field index $n \simeq 0$) the only favorable case with respect to (47) occurs if there is a slightly resistive interior cylinder, which is magnetically transparent for the equilibrium field but screens the field due to the perturbation. Hence $\varepsilon_m^{\text{inc}} \simeq 0$ and K follows from (56), whereas U is independent from $1 - \bar{S}$ and nearly equal to its free space value. It follows from (50), where we use in addition that the correct curvature term in U is about 20% smaller than the approximate value $\ln 16R/(a + b)$.[17] With the dominant electric image contribution to K we find

$$\frac{\Delta E}{E} \gtrsim 18(1-\bar{S})^3 \tag{57}$$

which holds for $\gamma_0 \gtrsim 20$ and $n \simeq 0$ and can be satisfied with $1 - \bar{S} \lesssim 0.2$ and $\Delta E/E \gtrsim 0.15$.

6 CONCLUSIONS

Following a careful analysis of the relativistic Vlasov equations, the effect of curvature on the coupling between the coherent motion in the azimuthal and transverse phase planes was studied. For the negative-mass mode the basic azimuthal density modulation is coupled with coherent radial motion only in the limit of very small $\nu_r \ll 1$. The radial mode is generally accompanied by an azimuthal density modulation which contributes to the perturbing field and should be taken into account in the dispersion relation. This coupling is negligible, however, for the "slow wave" radial mode (in contrast with the "fast wave") in the low-frequency limit. This mode is potentially dangerous because it may give rise to the resistive wall or electron-ion instability. Landau damping of this mode is possible only if the frequency spread exceeds a threshold which depends critically on curvature effects. For a high-γ ring in free space this threshold condition cannot be fulfilled if the field index n approaches zero. This difficulty can be removed with an interior conducting cylinder with a small resistivity such as to increase the Landau damping coefficient and decrease the coherent frequency shift which determines the threshold.

ACKNOWLEDGEMENTS

The author is grateful for critical comments by Dr. P. Merkel and Professor A. Schlüter during the performance of this study.

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Appendix

Evaluation of the Dispersion Integral

For the slow wave we can approximate the denominator in D

$$v_r^2\Omega^2 - (\omega - l\Omega)^2 \simeq -2\langle\Omega v_r\rangle(\omega - S) \quad (A.1)$$

and have

$$D \simeq \frac{3}{\langle\Omega v_r\rangle S' \,\Delta P_\theta} \int_0^1 \frac{y(1-y)}{y-\rho}\,dy \quad (A.2)$$

with

$$S = \langle S\rangle + S'(P_\theta - \langle P_\theta\rangle)$$

$$\rho = \frac{1}{2}\left(1 + 2\frac{\omega - \langle S\rangle}{S'\,\Delta P_\theta}\right)$$

$$y = \frac{1}{2} + \frac{P_\theta - \langle P_\theta\rangle}{\Delta P_\theta}$$

and the path of integration below a singularity, which might eventually occur on the real axis.

We calculate the case of marginal stability, i.e., $\mathrm{Im}\,\omega = \mathrm{Im}\,\rho = 0$. Clearly, for $V \neq 0$ this requires that $\omega - S = 0$ somewhere within the range of the distribution function and a residue contributes to the integral. Hence,

$$D \simeq \frac{3}{\langle\Omega v_r\rangle S'\,\Delta P_\theta}\left(\rho(1-\rho)\ln\frac{1-\rho}{\rho} - \rho + \frac{1}{2} - i\pi\rho(\rho - 1)\right) \quad (A.3)$$

For $V \ll U$ it is required that $|\rho(\rho - 1)| \ll 1/2$ and we get

$$|D| \simeq \frac{3}{2\langle\Omega v_r\rangle S'\,\Delta P_\theta}$$

and thus with (41) and $\Delta S = S'\,\Delta P_\theta$

$$\Delta S \simeq 3U \quad (A.5)$$

as necessary condition for marginal stability, or condition (43) if damping is required.

Derivation of Self-Forces

In the quasistatic approximation the self-forces are calculated as follows

a) Free Space

We start from the well-known expressions[13] for the self-fields of a stationary ring placed at the center of the coordinate system

$$eE_r^0 = m\gamma\Omega^2\mu\left(\frac{4R^2}{a(a+b)}\,\delta r + R\ln\frac{16R}{a+b}\left(1 - \frac{\delta r}{R}\right)\right) \quad (A.6)$$

$$eB_z^0 = m\gamma\Omega^2\mu v_{o\theta}\left(-\frac{4R^2}{a(a+b)}\,\delta r + R\ln\frac{16R}{a+b}\left(1 - \frac{\delta r}{R}\right)\right), \quad (A.7)$$

where $\delta r \equiv r - R$.

The mode $l = 1$ is described by a shifted ring with a local displacement $\langle r\rangle^1 \sim e^{i\theta}$ and an observer in the unshifted coordinate system observes the projections of the fields (A.6, 7)

$$e(E_r + v_{o\theta}B_z) = m\gamma_o\Omega^2\mu\left(\frac{4R^2}{a(a+b)}\frac{1}{\gamma_o^2}(\delta r - \langle r\rangle^1) + 2R\left(1 + \frac{1}{\gamma_o^2}\right)\ln\frac{16R}{a+b} \times \left(1 - \frac{\delta r - \langle r\rangle^1}{R}\right)\right) \quad (A.8)$$

$$eE_\theta = -E_r^0\,\frac{1}{R}\frac{\partial\langle r\rangle^1}{\partial\theta} = iE_r^0\Big|_{\delta r = 0}\cdot\frac{\langle r\rangle^1}{R}. \quad (A.9)$$

The difference between these and the unperturbed quantities gives the perturbations (49).

b) Ring Close to Cylindrical Conductors

The additional fields due to the perturbations of electric and magnetic images on infinitely long cylindrical boundaries close to the ring are conveniently expressed in terms of modified Bessel functions. To this end we start with the familiar expression for the scalar potential of a circular line beam with line charge density en

$$\Phi_s^0(r,z) = 2enR\int e^{ikz} \times I_o\begin{pmatrix}k\cdot r\\ k\cdot R\end{pmatrix}K_o\begin{pmatrix}k\cdot R\\ k\cdot r\end{pmatrix}dk \qquad \begin{matrix}r<R\\ r>R\end{matrix} \quad (A.10)$$

which is invariant if the ring is shifted as a whole according to $r' = r + \langle r\rangle^1$, so that the difference in potential results as

$$\Phi_s^1 = \Phi_s' - \Phi_s^0 = -2enR\langle r\rangle^1\int e^{ikz} \times \begin{pmatrix}I_1(k\ r)K_o(k\ R)\\ -I_o(k\ R)K_1(k\ r)\end{pmatrix} k\,dk \qquad \begin{matrix}r<R\\ r>R\end{matrix} \quad (A.11)$$

The perturbation of the potential due to the image charges can be written as

$$\Phi_c^1 = -2enR\langle r\rangle^1 \int e^{ikz} k a_c(k) I_1(kc) K_1(kr)\, \mathrm{d}k \qquad (r > c) \quad \text{(A.12)}$$

for a cylinder inside the ring at $r = c$ and

$$\Phi_b = -2enR\langle r\rangle^1 \int e^{ikz} k a_b(k) I_1(kr) K_1(kb)\, \mathrm{d}k \qquad (r < b) \quad \text{(A.13)}$$

for an external cylinder at $r = b$.

From the requirement

$$\Phi_s^1 + \Phi_c^1 + \Phi_b^1 = 0 \quad \text{(A.14)}$$

on the boundaries we obtain $a_b(k)$, $a_c(k)$ and thus the electric field perturbations in terms of integrals over modified Bessel functions.

A corresponding procedure leads to the magnetic field quantities with boundary conditions $B_r^1 = 0$. The results of these calculations are summarized in the next section in terms of $\varepsilon_{e,m}^{\mathrm{coh}}$ defined by Eq. (52) together with (32) and (41).

Coherent Image Coefficients

For electric and magnetic images on a conducting cylinder at radius $\bar{R}$ and $\bar{S} \equiv \bar{R}/R$ we find

$$\frac{\varepsilon_e^{\mathrm{coh}}}{(1-\bar{S})^2} = \int_0^\infty y^2 \frac{K_1(\bar{S}y)}{I_1(\bar{S}y)} I_o^2(y)\, \mathrm{d}y \quad \text{(A.15)}$$

$$\frac{\varepsilon_m^{\mathrm{coh}}}{(1-\bar{S})^2} = -\int_0^\infty y^2 \frac{\bar{S}yK_o(\bar{S}y) + K_1(\bar{S}y)}{\bar{S}yI_o(\bar{S}y) - I_1(\bar{S}y)} I_1^2(y)\, \mathrm{d}y \quad \text{(A.16)}$$

if $\bar{S} > 1$, and $I_1 \leftrightarrow -K_1$, $I_o \leftrightarrow K_o$ interchanged if $\bar{S} < 1$. An expansion of the integrals yields

$$\varepsilon_e^{\mathrm{coh}} \simeq \varepsilon_m^{\mathrm{coh}} \simeq \frac{1}{8} - \frac{1-\bar{S}}{4} \quad (\bar{S} \to 1). \quad \text{(A.17)}$$

With two electric image cylinders the corresponding integral is, for $\bar{S}_1 > 1$ and $\bar{S}_2 < 1$

$$-\int_0^\infty y^2 \times \frac{I_1(\bar{S}_1 y) I_1(\bar{S}_2 y) K_o^2(y) + K_1(\bar{S}_1 y) K_1(\bar{S}_2 y) I_o^2(y)}{K_1(\bar{S}_1 y) I_1(\bar{S}_2 y) - I_1(\bar{S}_1 y) K_1(\bar{S}_2 y)}\, \mathrm{d}y \quad \text{(A.15')}$$

which we put equal to $\varepsilon_e^{\mathrm{coh}}/(1-\bar{S})^2$ if the cylinders lie symmetric, i.e., $|1-\bar{S}_1| = |1-\bar{S}_2| = |1-\bar{S}|$.