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LONGITUDINAL COUPLING IMPEDANCE OF A STATIONARY ELECTRON RING IN A CYLINDRICAL GEOMETRY†

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The longitudinal (azimuthal) coupling impedance is investigated for a stationary electron-ring beam circulating between a co-axial pair of conducting tubes. Proximity of the beam to the inner tube is found to be advantageous for reducing $(|Z_n|/n)_{max}$. Similar results are shown to be attainable with operation near the the outer tube, provided the quality factors Q for higher-order resonant modes are deliberately made small. Illustrative computational results are presented graphically and a convenient approximate formula is suggested that may serve to guide the selection of desirable parameters for a typical fully-compressed electron ring.

1. INTRODUCTION

In evaluating the effectiveness of electron rings for the useful acceleration of ions, the requirement of longitudinal (azimuthal) stability appears to constitute a severe constraint. Another paper[1] appearing in this issue is concerned with the selection of parameters for an electron-ring accelerator, and considers explicitly the stability requirements for a fully-compressed loaded ring at the time of release from the magnetic well. That paper reiterates the necessity of strongly limiting the self-generated azimuthal electric fields, that could excite an unstable azimuthal modulation of the electron ring beam, if rings of useful holding power are to be obtained. Such electric fields may be expected to be reduced by the presence of nearby conducting material, that in a magnetic acceleration column might conveniently take the form of conducting tubes co-axial with the electron ring. It may be of interest, therefore, to report in the present paper results from an analysis of the longitudinal coupling impedance of a toroidal electron beam situated co-axially between a pair of infinitely long conducting tubes. The analysis for a pair of tubes constitutes an extension of previous work[2–5] concerned with an electron ring situated interior to a single tube, a ring inside a compressor chamber,[6,7] and of similar work[8] relating to a cylindrical layer of electrons situated between two walls. A two-tube configuration may represent a better approximation to arrangements for magnetic acceleration that would be attractive on other grounds and, in addition, may aid in suppression of the longitudinal instability.

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2. COMPUTATIONAL PROCEDURE

The present analysis is restricted to rings that are essentially stationary with respect to the tube structure, and (when losses are present) would require revision for application to rings with an axial speed comparable to that of light. No dielectric material is considered to be present, and no special frequency-sensitive elements are introduced (save for such as may aid in controlling the 'quality factor', Q).

The longitudinal coupling impedance, associated with an electron-ring beam of major radius R and with a postulated current modulation $I_n = I_0 \exp[j(\omega t - n\phi)]$, is defined in terms of the corresponding longitudinal (azimuthal) electric field E_ϕ as $Z_n = -2\pi R E_\phi / I_n$. Perturbation analyses[9] have suggested the relation between $|Z_n|/n$ and the amount of Landau damping that must be present (*e.g.*, from energy spread) if longitudinal stability is to be assured.

We commence the analysis, for determination of Z_n, by making a formal series development for the steady-state electromagnetic fields associated with a current distribution $I_0\, \delta(r-R)\, \delta(z) \exp[j(\omega t - n\phi)]$, subject to boundary conditions that correspond to outgoing (or damped) waves for $|z|$ large and to conducting surfaces at $r = R_{IN}$, R_{OUT}. Radial coordinates can be expressed conveniently in terms of the radius of the inner tube, so that, in these units, the radial interval of interest extends from unity to $f = R_{OUT}/R_{IN}$ and the ring beam is situated at $R = \rho R_{IN}$ $(1 < \rho < f)$. The angular frequency is $\omega = n\beta c/R = n\beta c/(\rho R_{IN})$. The series development of the electromagnetic field then employs characteristic functions, R_m, and characteristic values, g_m, of the Bessel equation

$$x\frac{d}{dx}\left(x\frac{dR_m}{dx}\right)+(g_m{}^2x^2-n^2)R_m = 0$$

with $R_m|_{x=1} = R_m|_{x=f} = 0$ for the transverse-magnetic (TM) modes—and correspondingly the functions S_m and values h_m, with $S'_m|_{x=1} = S'_m|_{x=f} = 0$ for the transverse-electric (TE) modes.[10] The z-dependence of the fields is contained in factors that, for the TM modes, are circular functions of $\omega t - k_m|z| - n\phi$ with

$$k_m = [(\omega/c)^2 - (g_m/R_{IN})^2]^{1/2}$$

and $\omega = n\omega_{\odot} = n\beta c/R$ for frequencies above cut-off, and are of the form $\exp(-\alpha_m|z|)$ times a circular function of $\omega t - n\phi$ with $\alpha_m = [(g_m/R_{IN})^2 - (\omega/c)^2]^{1/2}$ for frequencies below cut-off—and similarly for the TE modes.

The azimuthal electric field is found in these terms to be such that

$$\frac{1}{n}Z_n = 2\pi Z_0\,\rho\left\{\sum_m [(n/\rho)^2-(g_m/\beta)^2]^{1/2}F_{TM} + \sum_m [(n/\rho)^2-(h_m/\beta)^2]^{-1/2}F_{TE}\right\},$$

where $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 120\pi$ ohms, the 'coupling factors' F_{TM}, F_{TE} are

$$F_{TM} = \frac{[R_m(\rho)]^2}{[fR'_m(f)]^2-[R'_m(1)]^2}$$

and

$$F_{TE} = \frac{[S'_m(\rho)]^2}{(f^2h_m{}^2-n^2)[S_m(f)]^2-(h_m{}^2-n^2)[S_m(1)]^2},$$

and the coefficients before these factors have the character $-j$ and $+j$ respectively when the characteristic values g_m or h_m exceed $n\beta/\rho$. The formal expression written above for Z_n/n clearly requires modification to take account of a non-vanishing transverse extent of the beam and to make allowance for losses that will prevent the factors $[(n/\rho)^2-(g_m/\beta)^2]^{1/2}$ and $[(n/\rho)^2-(h_m/\beta)^2]^{-1/2}$ from becoming exactly zero or infinite under 'resonant' conditions.

With respect to the first of the modifications just mentioned, we may note that for a large m, a combination of two of the terms just written will make a contribution that can be estimated as $-j(Z_0/\beta\gamma^2)(1/m)$, a capacitive impedance by virtue of the factor $-j$. Such terms, summed over large m to a limiting value that will depend on the minor dimensions of the ring, will provide a contribution to Z_n/n of the form $-j(Z_0/\beta\gamma^2)$ times a logarithmic factor in which the minor dimensions of the ring appear in the argument. This result is concordant with the expected low-frequency inductive impedance whose dominant (logarithmic) term is given approximately by $2\pi\beta cj[(Z_0/2\pi c)\ln(8R/a)]$ for a circular ring of round wire (major and minor radii: R, a), combined with the corresponding capacitive contribution (larger, by a factor $1/\beta^2$) of opposite sign—or (more generally) with a contribution to Z_n/n of the form $-j(Z_0/\beta\gamma^2)\ln(D/a)$, where D is related to a major dimension of dominant importance[7] (such as R, λ, or the spacing to the wall). In practice, terms of high m value were diminished by a 'convergence factor' that served to suppress terms for which $m\pi \gg$ outer tube radius/minor radius, and the details of this procedure did not appear to affect the results materially for parameter values of interest in the present work.

With respect to the potentially resonant factors $[(n/\rho)^2-(g_m/\beta)^2]^{1/2}$ and $[(n/\rho)^2-(h_m/\beta)^2]^{-1/2}$, for the TM and TE modes respectively, these were replaced by

$$\{(n/\rho)^2-[(1+j/2Q_{TM})g_m/\beta]^2\}^{1/2}$$

and

$$\{(n/\rho)^2-[(1+j/2Q_{TE})h_m/\beta]^2\}^{-1/2}.$$

Such a replacement, although leading to a typical resonant-factor behavior, perhaps can be justified rigorously only if (i) the boundary condition at a resistive wall can be correctly written in terms of a surface resistance $\mathscr{R}_s$ as $E_\phi/H_z = (1+j)\mathscr{R}_s$ *and* (ii) the corresponding complex characteristic value (g_m or h_m) is given with sufficient accuracy by a first-order development from the case in which $\mathscr{R}_s = 0$. With the exception of cases in which the quality factors are very low (*e.g.*, < 10), however, the resonant factors written above are believed to be suitable in applications of the present work. In performing the computations, one has the option of either (i) computing for each n the Q_{TM} and Q_{TE} values for the m-value lying closest to resonance in each case, using a specified specific volume resistivity ρ_v for the tube material ($\mathscr{R}_s = \sqrt{\mu_0 \omega \rho_v / 2}$), or (ii) simply specifying a single value of Q to be used throughout (thereby permitting the user to represent loss mechanisms deliberately introduced into the structure).

3. EXPECTED CHARACTER OF THE COUPLING IMPEDANCE

The selection of geometrical configurations for which the longitudinal coupling impedance can be expected to be favorable or unfavorable for electron-ring stability may be guided by some general considerations. A ring beam enclosed within a structure with highly-conducting walls potentially can excite resonances that will lead to unacceptably high values of the coupling impedance. Reflections may be expected to be suppressed for certain (high-n) modes, however, if the boundary is poorly reflecting and is situated in the radiation-field zone for such modes—with the result that the corresponding impedance then should be close to that cited for a beam in free space ($Z_n/n \cong 354 i^{1/3} n^{-2/3}$ ohms,[7,12,13] for n well below a critical harmonic number that is of the order of γ^3).[13-15] For lower n, where the free-space coupling impedance would be unacceptably high, a surface of high conductivity close to the beam should serve to lower E_ϕ at the beam and so act to reduce the coupling impedance substantially if resonant responses are avoided for such n-values.

In a computational investigation of coupling impedance for a ring beam in the presence of one or two co-axial tubes, it is of interest, therefore, to include an examination of cases in which the beam is situated only a small distance *outside* an inner conducting tube, in an effort to provide coupling impedances that for low n will be well below the free-space values. If, with such geometry, mechanical considerations require the presence of an additional tube exterior to the beam, one may anticipate that the selection of a suitably large radius for this outer tube will preclude the excitation of dangerous low-order resonances. Under such conditions, the provision of only moderate losses, by any one of several means at the outer radius, may suffice to suppress adequately the contributions of high-order (possibly resonant) TE_{nm} modes (high m)—for which the $m-1$ sign reversals of $E_\phi(r)$ ultimately must serve to reduce the coupling between the electromagnetic field and a beam of appreciable radial extent.

For purposes of comparison, there also is interest in cases in which the beam is located just *inside* an outer tube, with an inner tube either absent or assigned a considerably smaller radius. Under these latter circumstances the surface conductivity of the wall should be high for the low-frequency (low-n) modes, but the resonances that can occur for larger values of n should be suppressed by a deliberate reduction of the quality factor ('de$Q$-ing') for high-frequency fields.

4. COMPUTATIONAL RESULTS

The possibility of attaining undesirably large values of coupling impedance as a result of resonances is illustrated in Fig. 1 for a two-tube structure

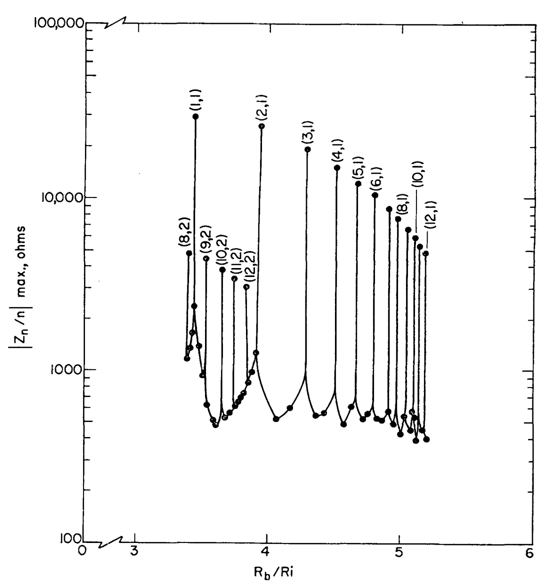

FIG. 1. Maximum longitudinal coupling impedance, divided by n, for the first 12 azimuthal modes for coaxial copper tubes of radius ratio 6. For wall resistivity $\rho_v = 1.8 \times 10^{-6}$ Ω-cm the Q's are in the range of 10^4-10^5. Beam locations were chosen to excite the $TE_{n,1}$ and $TE_{n,2}$ resonances and to exhibit resonant behavior.

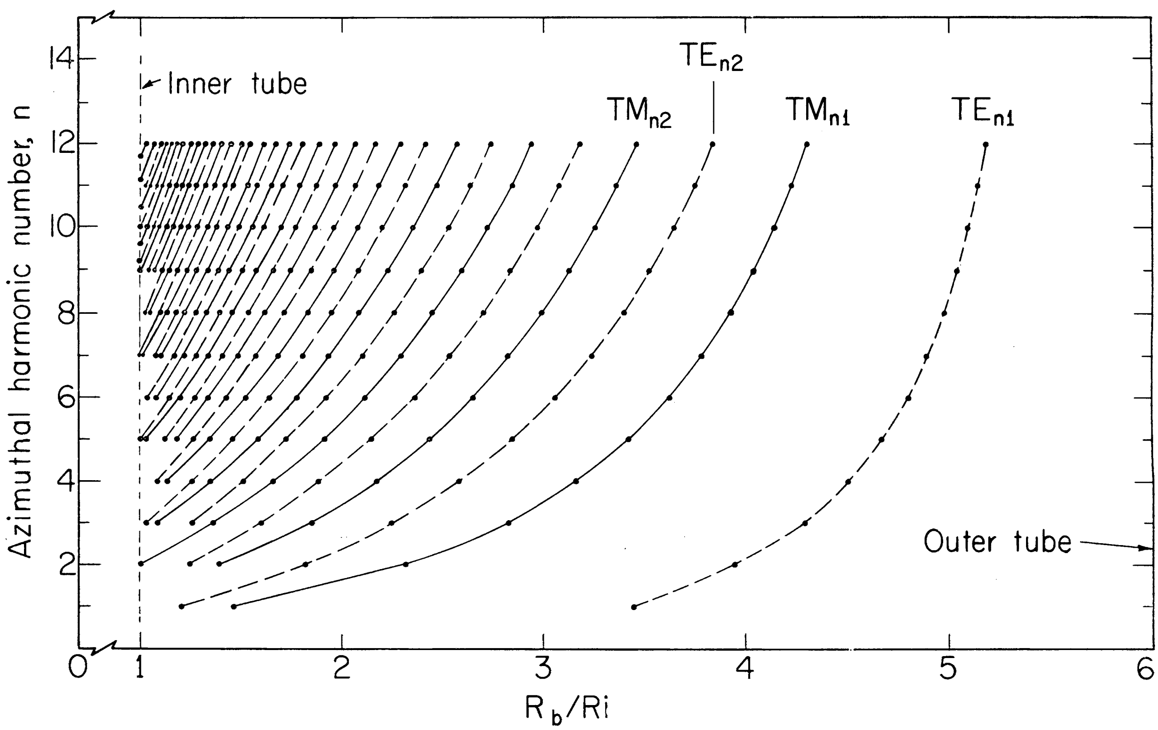

FIG. 2. Resonant beam radii for coaxial tubes with radius ratio 6. For a given n the TE_{nm} and TM_{nm} resonances alternate, with m increasing to the left.

with a radius ratio 1 : 6 and high values of Q which would be appropriate for tubes constructed of a good conductor such as copper. By reference to a mode chart (Fig. 2) that shows the beam radii for resonant excitation of the wave-guide modes in such a geometry (with $\beta = v/c = 0{\cdot}999703$), it is evident that the largest value of $|Z_n|/n$ occurs when the beam is located so as to excite the $TE_{1,1}$ resonance, the second highest value corresponds to excitation of the $TE_{2,1}$ mode, *etc.*, with the low-n $TE_{n,1}$ modes dominating the coupling impedance in the region where the $TE_{n,1}$ modes are excited—that is, from approximately midgap to the outer wall. The $TE_{n,1}$ modes are distinctive[15]—and can be particularly troublesome—because the associated $E_\phi(r)$-field experiences no sign reversal. The $TE_{1,1}$ resonant beam radius for the case of a *single* tube is $R_B \cong \beta R_{OUT}/1.8412$ and the corresponding resonant radius for a *pair* of tubes whose radius ratio does not greatly exceed unity is close to $R_B \cong \beta(R_{IN}+R_{OUT})/2$, while the resonant radii for other $TE_{n,1}$ modes will be progressively larger. The existence of these potentially-resonant modes thus deserves recognition when considering operation with the ring beam fairly close to an outer cylindrical wall.

The curves of Fig. 3 again indicate the behavior of Z_1 *vs.* R_B for $R_{OUT}/R_{IN} = 6$, and illustrate the influence of the quality factor Q. One notes that, as expected, for Q sufficiently large,

(i) For $\omega < \omega_{\text{resonant}}$ [$\rho > n\beta/$(characteristic value, h)]

$Z_{n,\text{Real}}$ is relatively small (in comparison to $Z_{n,\text{Imag.}}$) and is approximately proportional to $1/Q$ (as may be interpreted as due to wall resistance acting on image currents), while $Z_{n,\text{Imag.}}$ is rather insensitive to Q;

(ii) For $\omega = \omega_{\text{resonant}}$

$Z_{n,\text{Real}} \cong |Z_{n,\text{Imag.}}|$, each assuming large values $\propto\sqrt{Q}$;

(iii) For $\omega > \omega_{\text{resonant}}$

$Z_{n,\text{Real}}$ is large (in comparison to $Z_{n,\text{Imag.}}$) and is rather insensitive to Q—corresponding to power

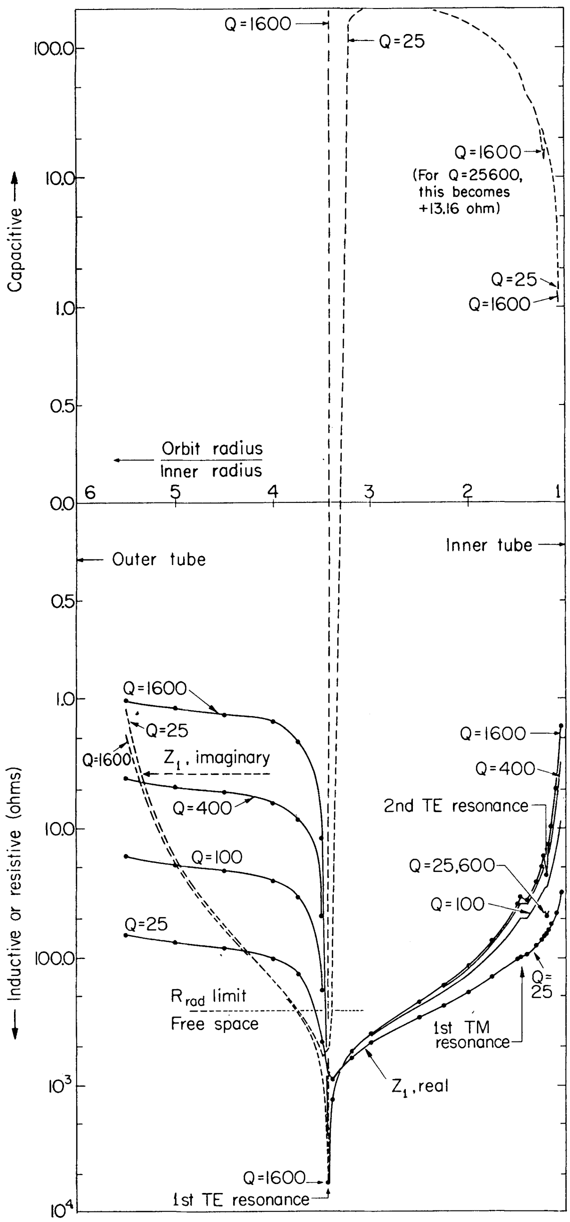

FIG. 3. Longitudinal coupling impedance for $n=1$, for coaxial tubes with a radius ratio of 6. For this case, $\gamma=41$, and the minor dimensions a and b are 0.02 R_{inner}.

radiated down the tubes (and ultimately absorbed, remotely, in the tube walls or emerging from the ends).

The curves of Fig. 4 depict the results of computations intended to indicate how $(|Z_n|/n)_{\max}$ can be held to reasonably low values either (i) by operating with a moderate value of Q and situating the beam close to an inner tube (so that excitation of low-order resonances is precluded), or (ii) by operating with the beam close to an outer tube, with Q deliberately caused to decrease at the higher frequencies in order to reduce the extent to which the higher-order resonances can be excited. In performing computations pertaining to this latter type of operation, the computations (with selected Q values) were extended to sufficiently large values of n that $|Z_n|/n$ appeared to have become distinctly a monotonically decreasing function of n that essentially merged into the free-space curve for this quantity. Two illustrative examples of the computational results for the two cases described are shown in Figs. 5 and 6.

It will be recognized that, in the selection of parameters for an electron-ring device, a decision to operate with a ring situated close to a conducting tube necessarily restricts the amount of energy spread (and attendant radial spread) that can be present and that also could act to suppress the

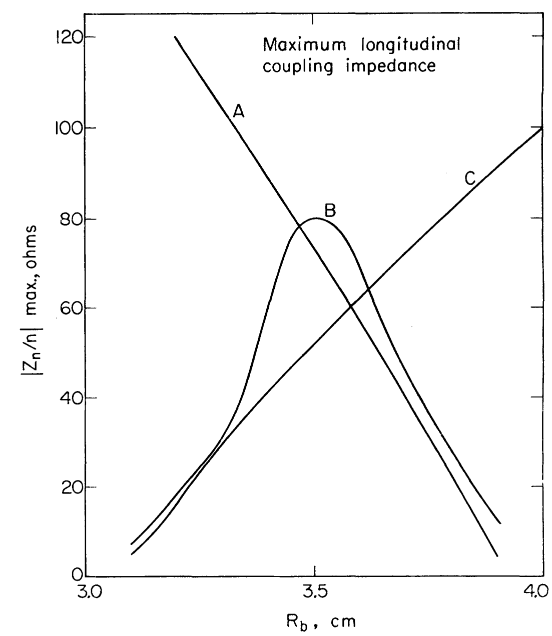

FIG. 4. Maximum longitudinal coupling impedance divided by n. A—Beam located close to an outer wall at 4 cm and $Q(n)$ adjusted to minimize $|Z_n/n|$. B—Beam located between two walls at 3 cm and 4 cm and $Q(n)$ adjusted to minimize $|Z_n/n|$. C—Beam located close to an inner wall at 3 cm and Q made high for all n.

longitudinal instability. As a guide for finding suitable parameters, therefore, it is convenient to have at hand a simple relationship that relates $(|Z_n|/n)_{\max}$ to the 'clearance' throughout the range of possible practical interest for these parameters. The results shown in Fig. 4 for a beam situated at a small distance outside an inner conducting tube suggest that with reasonable accuracy one may write $(|Z_n|/n)_{\max} \cong 300\,(R_B - R_{IN})/R_{IN}$ ohms, for $R_{IN}/30 \leqq R_B - R_{IN} \leqq R_{IN}/3$, under these circumstances. Thus, with $(R_B - R_{IN})/R_{IN} = 0.2$—that should provide sufficient clearance for a beam with a radial spread arising from $\Delta E/E \gtrsim 10$ per cent (full width at half maximum)—we should expect to achieve a longitudinal coupling impedance such that $|Z_n|/n \cong 60$ ohms. The results shown in Fig. 4 are quite insensitive to γ, decreasing typically by about 2 per cent when γ is increased

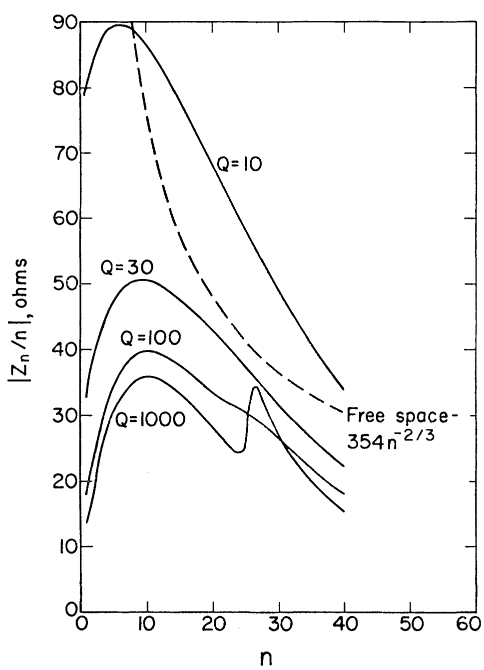

FIG. 5. Typical curves of $|Z_n/n|$ versus n for a beam radius $R_b = 3.4$ cm near an inner tube at 3 cm and an outer tube at 27 cm, with Q as a parameter. $\gamma = 41$ and the minor dimensions a and b are 1 mm. At this radius, resonant behavior (due to the presence of the outer tube) is seen to be developing at $n \simeq 27$, whereas the $|Z_n/n|_{\max}$ of the first peak is at $n \simeq 10$. For min $|Z_n/n|$ for all n at this radius, Q should decrease from ~ 1000 at $n \simeq 10$ to ~ 30 at $n \simeq 27$. $|Z_n/n|_{\max}$ for this case was taken as 40 Ω in the construction of curve C in Fig. 4.

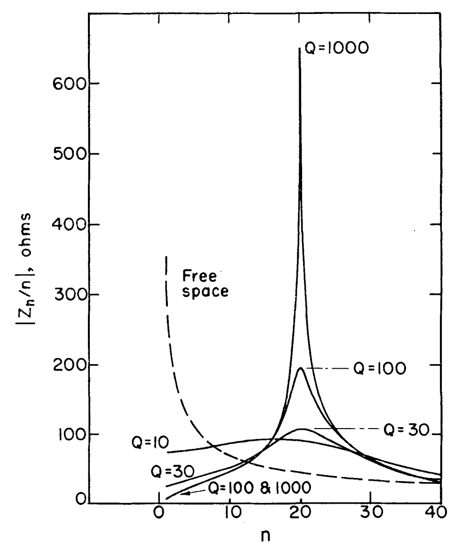

FIG. 6. Typical curves of $|Z_n/n|$ versus n for a beam radius $R_b = 3.6$ cm near an outer tube at 4 cm, and an inner tube at 0.444 cm, with Q as a parameter. The clearance to the wall, 4 mm, is the same as in Fig. 5, $\gamma = 41$, and the minor dimensions a and b are 1 mm. At this radius, resonant behavior is seen at $n = 20$, therefore minimum $|Z_n/n|$ for all n would be obtained by switching from high Q to low Q at $n \simeq 13$. $|Z_n/n|_{\max}$ for this case was taken as 60 Ω in the construction of curve A in Fig. 4.

from 20 to 82. The minor dimensions of the beam have a larger effect on Z_n/n than the γ dependence, because the self field term and the term due to excitation of high m modes decrease as the beam minor dimensions are increased. The curve C in Fig. 4 is moved approximately 0.05 cm to the left as the minor dimensions are decreased from 0.1 cm to 0.05 cm, and approximately 0.15 cm to the right as the minor dimensions are increased to 0.2 cm, for a <10 per cent change of $|Z_n|/n$ in the region around 60 Ω. At much greater spacings from the inner tube the dependence of $|Z_n|/n$ on the minor dimensions becomes negligible because of the dominance of the low n and m modes.

5. CONCLUSIONS

An examination of the longitudinal coupling impedance that can be attained for a ring beam between a pair of co-axial conducting tubes has indicated that low values of $|Z_n|/n$ may be conveniently attained by situating the beam a small distance outside an inner conducting tube. If, alternatively, the beam is close to an outer tube, similar results may be obtained if the quality factors for higher multiples of the circulating frequency are reduced so as to suppress potentially resonant fields. For an electron beam with $\gamma = 41$ and a 3.5-cm orbit radius surrounding a tube of radius 2.9 or 3.2 cm, it should be possible in this way to achieve values of $(|Z_n|/n)_{\max}$ that are approximately 62 or 28 ohms, respectively.

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$$R_m(x) \propto J_n(g_m)\, Y_n(g_m x) - J_n(g_m x)\, Y_n(g_m),$$
with
$$J_n(g_m)\, Y_n(g_m f) - J_n(g_m f)\, Y_n(g_m) = 0,$$
and
$$S_m(x) \propto J_n'(h_m)\, Y_n(h_m x) - J_n(h_m x)\, Y_n'(h_m),$$
with
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