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INFINITE TRANSITION-ENERGY SYNCHROTRON LATTICE USING π-STRAIGHT SECTIONS

L. C. TENG

National Accelerator Laboratory,† Batavia, Illinois, USA

With reverse bends in a synchrotron lattice one can push the transition energy to infinity. We show here that the same can be accomplished by judicious insertion of π-straight sections in a normal periodic lattice with finite transition energy and no reverse bend.

1. INTRODUCTION

It has long been established that in an alternating-gradient synchrotron phase stability can be maintained in crossing the transition energy by simply jumping the phase of the accelerating rf. However, in details, crossing transition has several undesirable features:

(a) The non-linear rf phase-space bucket is inverted across the phase jump. If the bucket is fairly full this change in bucket shape causes a mismatch in phase-space area occupied by the beam in crossing transition.

(b) Over a sizable momentum range on either side of transition there is little or no focusing. Within this range of neutral stability the beam is more susceptible to rf and other noises. These noises and fluctuations lead to dilutions of the longitudinal phase-space density.

(c) Within the range of neutral stability the beam-revolution frequency is sensibly independent of its momentum or radial position. Hence devices such as the radial position rf feedback system are inoperative.

(d) For a short time immediately after transition, the beam is unstable against the 'negative mass' instability due to longitudinal space-charge force. It is, therefore, desirable to always stay below transition where the 'mass' is positive.

(e) At high beam intensity the longitudinal space-charge force which is not reversed as is the rf force in crossing transition causes sizable mismatch in phase-space area. This mismatch leads to a coherent bunch-length oscillation after transition which results in a dilution of phase-space density. Several schemes[1] have been proposed to regain longitudinal matching after transition. But these schemes have only limited effectiveness because of the 'negative mass' instability mentioned above.

In view of these unpleasant features it is, therefore, desirable to resume the early investigation of magnet lattices with high or infinite transition energy to avoid transition crossing. Moreover, for certain applications such as for colliding-beam storage rings there may be specific requirements on beam-bunch length. Again, one would like to be able to manipulate the transition energy to some predetermined value.

It is well known that with reverse bends in a synchrotron lattice one can push the transition energy to infinity.[2] We want to show here that by judicious insertion of π-straight sections[1] in a normal periodic lattice the transition energy can be altered over a broad range of values, specifically this scheme can be employed to obtain infinite transition energy so as to avoid crossing transition. Compared to the scheme with reverse bends this scheme has the practical advantages: (a) No reverse bend is required, resulting in a saving of bending magnets twice those for the reverse bends and (b) Long straight sections in a lattice are always needed for injection, extraction, acceleration, and other beam manipulations.

2. GEOMETRICAL OBSERVATION

In an alternating-gradient lattice if the on-momentum closed orbit is shown stretched out as a straight line L (Fig. 1), the off-momentum closed orbit is oscillatory about a straight line L′ parallel

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to L and having the same length as the off-momentum orbit. Because of the curvature of the on-momentum orbit (say, downward) an outward-swing (relative to L) section of the off-momentum orbit such as AB has a positive first-order length

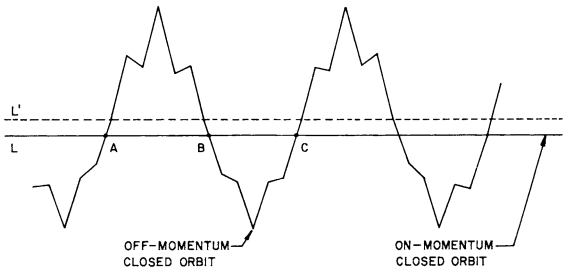

FIG. 1. Closed orbits in a synchrotron lattice.

increment and an inward-swing section such as BC has a negative first-order length increment. For positive momentum increment the net orbit-length increment is generally positive and small.

The scheme of reducing (to zero) the dispersion orbit-length increment by introducing reverse bends is as follows: If the curvature of the on-momentum orbit is reversed over the section AB the first-order orbit-length increment over AB will also reverse sign and become negative. The combined orbit-length increment over AB and BC will, then, have a large excess negative value. By introducing a few such reverse bends it is possible to totally compensate the small net positive orbit-length increment and obtain a zero orbit-length dispersion, hence an infinite transition energy.

An interesting variance of this scheme is the following: Instead of a reverse bend over AB one can make the orbit curvature zero. Since the betatron-oscillation phase advance across AB is approximately π, this can be accomplished by making the section AB into a π-straight section. In this case the first-order orbit-length increment over AB is zero and the combined increment over AB and BC still has a large excess negative value although it is only roughly half that of the case with the reverse bend. Zero orbit-length dispersion and infinite transition energy can similarly be attained.

It is clear from the discussion that *any matched long straight section with a large phase advance close to* π can be used effectively over section AB. But π-straight section is the simplest and can be analyzed most easily. It is also clear that by making the section BC into a π-straight section one can increase the orbit-length dispersion, hence reducing the transition energy.

In a normal periodic lattice the oscillation of the off-momentum orbit about L' is generally much smaller than that shown in Fig. 1 and does not cross the on-momentum orbit L. For this scheme to work the off-momentum orbit must cross L. Hence the resultant infinite-transition lattice can be expected to have a large maximum local position dispersion. This disadvantage is common to all schemes and is the price one has to pay for infinite transition. This can be seen from the general formulas given by Courant and Snyder[4] for the local position dispersion $x/(\Delta p/p)$ (Eq. 5.26 in Ref. 4) and the orbit-length dispersion $\Delta C/(\Delta p/p)$ (Eq. 5.30). To supply a large negative term in $\Delta C/(\Delta p/p)$ one must introduce a large orbit-curvature Fourier coefficient a_k for $k =$ (integer just above ν) which unavoidably adds a large oscillation to $x/(\Delta p/p)$.

3. EFFECT OF π-STRAIGHT SECTION INSERTION

We will write the 3×3 horizontal optics-dispersion transfer matrix as

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f \\ \hline 0 & 0 & 1 \end{array}\right) \equiv (M|D) \tag{1}$$

where

$$\begin{cases} M \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2 \times 2 \text{ optics transfer matrix} \\ D \equiv \begin{pmatrix} e \\ f \end{pmatrix} = \text{dispersion transfer vector.} \end{cases}$$

The dispersion vector† $\begin{pmatrix} x \\ x' \\ 1 \end{pmatrix}$ is written as $X = \begin{pmatrix} x \\ x' \end{pmatrix}$ and we have

$$(M \mid D)X = MX + D. \tag{2}$$

† In this paper we deal only with the dispersion vector and never with the free oscillation vector.

Starting with a lattice without π-insertion, we insert π-straight sections at symmetric locations A, A′, A″, ... around the ring to form superperiods. Without π-insertion the dispersion vector X_A at location A is equal to the dispersion vector $X_{A'}$ at location A′. Hence, X_A is given by

$$X_A = X_{A'} = (M_A \,|\, D_A)X_A = M_A X_A + D_A \quad (3)$$

where $(M_A \,|\, D_A)$ is the optics-dispersion transfer matrix from A to A′.

With the π-insertions the dispersion vector must be inverted between π-insertions (from A to A′). The new dispersion vector $\bar{X}_A$ at location A (end of π-insertion) is, thus, given by

$$\bar{X}_A = -(M_A \,|\, D_A)\bar{X}_A = -M_A \bar{X}_A - D_A. \quad (4)$$

Equations (3) and (4) yield

$$\Delta X_A \equiv \bar{X}_A - X_A = -2(1-M_A{}^2)^{-1} \times D_A = -2(1+M_A)^{-1} X_A. \quad (5)$$

Since the difference between two dispersion vectors transfers like a free oscillation vector, at an arbitrary location P between A and A′ the differential dispersion vector $\Delta X \equiv \bar{X} - X$ is simply given by

$$\Delta X = M \Delta X_A = -2M(1+M_A)^{-1} X_A \quad (6)$$

where M is the 2×2 optics transfer matrix from A to P. We are interested only in the first row Δx of Eq. (6), namely

$$\Delta x = \bar{x} - x = -\frac{\beta^{1/2}}{\cos \mu/2}[(\cos\phi + \alpha_A \sin\phi) x_A \beta_A^{-1/2} + (\sin\phi) x_A' \beta_A^{1/2}] \quad (7)$$

where

α_A, β_A = betatron-oscillation amplitude functions at A,
α, β = betatron-oscillation amplitude functions at P,
μ = betatron-oscillation phase advance from A to A′,
ϕ = (betatron-oscillation phase advance from A to P) $-\mu/2$ = phase measured from 'middle' of superperiod.

all *before* the insertion of the π-straight sections.

The new dispersion orbit-length increment per superperiod with the π-insertions is (the orbit-length increment in the π-straight section is zero)

$$\int_{-\mu/2}^{\mu/2} \frac{\bar{x}}{\rho}\beta \, d\phi = \int_{-\mu/2}^{\mu/2} \frac{x}{\rho}\beta \, d\phi + \int_{-\mu/2}^{\mu/2} \frac{\Delta x}{\rho}\beta \, d\phi = \int_{-\mu/2}^{\mu/2} \frac{x}{\rho}\beta \, d\phi - \frac{1}{\cos \mu/2} \times [(C + \alpha_A S) x_A \beta_A^{-1/2} + S x_{A'} \beta_A^{1/2}] \quad (8)$$

where

$$C \equiv \int_{-\mu/2}^{\mu/2} \frac{\beta^{1/2}}{\rho} \cos\phi \, \beta \, d\phi,$$

$$S \equiv \int_{-\mu/2}^{\mu/2} \frac{\beta^{1/2}}{\rho} \sin\phi \, \beta \, d\phi$$

and $\rho = \rho(\phi)$ is the radius of curvature of the on-momentum orbit. Because of the denominator $\cos \mu/2$, by judicious choice of the phase advance μ between π-insertions we can make the second term on the right-hand side of Eq. (8) cancel the first term to give zero orbit-length dispersion or infinite transition energy.

4. CRUDE ESTIMATE AND EXAMPLE

To give a simple example let us assume that the π-straight sections are inserted at locations where $\alpha_A = 0$, hence $x_A' = 0$. The condition for zero orbit-length dispersion is, then,

$$\int_{-\mu/2}^{\mu/2} \frac{x}{\rho}\beta \, d\phi = \frac{C}{\cos \mu/2} x_A \beta_A^{-1/2}. \quad (9)$$

Now, for a very crude estimate we have

$$C = \int_{-\mu/2}^{\mu/2} \frac{\beta^{1/2}}{\rho} \cos\phi \, \beta \, d\phi \equiv \langle \cos\phi \rangle \int_{-\mu/2}^{\mu/2} \frac{\beta^{1/2}}{\rho} \beta \, d\phi \quad (10)$$

and

$$\int_{-\mu/2}^{\mu/2} \frac{x}{\rho}\beta \, d\phi = \langle x\beta^{-1/2} \rangle \int_{-\mu/2}^{\mu/2} \frac{\beta^{1/2}}{\rho} \beta \, d\phi \quad (11)$$

where $\langle\ \rangle$ denotes an average value as defined. Since $x\beta^{-1/2}$ is more or less constant in a normal lattice, we have

$$\langle x\beta^{-1/2} \rangle \cong x_A \beta_A^{-1/2}. \quad (12)$$

Furthermore, $\langle \cos\phi \rangle$ defined above is roughly

equal to $\cos\phi$ averaged over ϕ from $-\mu/2$ to $\mu/2$, namely

$$\langle\cos\phi\rangle \cong \frac{1}{\mu}\int_{-\mu/2}^{\mu/2}\cos\phi\,d\phi = \frac{2}{\mu}\sin\frac{\mu}{2}. \quad (13)$$

Substituting Eqs. (10), (11), (12), and (13) in Eq. (9) we get

$$\frac{\mu}{2} \cong \tan\frac{\mu}{2}. \quad (14)$$

Since $\mu/2$ has a large positive value, to make $\tan\mu/2$ large and positive $\mu/2$ should be just below an odd multiple of $\pi/2$. Writing

$$\mu = 2\pi\left(\frac{2n-1}{2}-\varepsilon\right) \qquad n = \text{integer},\ \varepsilon = \text{small} \quad (15)$$

we get for the betatron oscillation wave number ν including the π-insertions

$$\nu = \frac{N}{2\pi}(\mu+\pi) = nN-\varepsilon N \quad (16)$$

where N is the number of superperiods. Substituting Eqs. (15) and (16) in Eq. (14) we obtain the following approximate condition for infinite transition energy

$$\nu \cong N\left(\frac{1}{2}+\frac{1}{\pi}\cot\pi\varepsilon\right). \quad (17)$$

The second expression in Eq. (16) shows that (a) the integral part of ν must be a multiple of N and (b) to get the largest μ we must choose the smallest ε under the constraint $\varepsilon N \cong (1/4)$ (for $\nu \cong$ integer $-1/4$) or $\varepsilon \cong (1/4N)$. Therefore, the approximate upper limit of ν for a given number N of π-insertions is

$$\bar{\nu} \cong N\left(\frac{1}{2}+\frac{1}{\pi}\cot\frac{\pi}{4N}\right). \quad (18)$$

Conversely, Eq. (18) states that for a given value of ν there is a minimum number of π-insertions required to make orbit-length dispersion zero. For low values of N we have

N	$\bar{\nu}$	N	$\bar{\nu}$
1	0.82	6	17.5
2	2.54	7	23.3
3	5.06	8	29.9
4	8.40	9	37.2
5	12.5	10	45.4

For an example, we take 90 FODO cells similar to those of the NAL main ring and insert six π-straight sections in the middle of focusing quadrupoles to build up an infinite transition-energy lattice with $\nu \cong 18-(1/4)$. Each cell C has the structure

$$C = \left(\frac{F}{2}\right)T(B)S(B)S(B)S(B)S(D)T(B)S(B)S(B)S(B) \times S\left(\frac{F}{2}\right)$$

and a π-straight section L has the structure

$$L = \frac{V}{2}(2F)U(2D)V(2F)U(2D)\frac{V}{2}$$

where the elements and their lengths are:

Bending magnet:	B	6.0706 m
Focusing (defocusing) quadrupole:	$F(D)$	2.1336 m
Drift spaces:	S	0.3048 m
	T	2.1082 m
	U	4.2672 m
	V adjusted for π-phase advance.	

For 400-GeV protons the field strength of B should be 19.2252 kG, the field gradients of F and D are adjusted to ± 212.85 kG/m, and the proper length of V is 58.4526 m. With these values the computer calculation using the SYNCH program[5] gives

Horizontal phase advance per cell = $0.16401(2\pi)$
Horizontal betatron-oscillation wave number = 17.761
Transition kinetic energy = 905.6 (rest energy)

The phase advance between π-insertions is $\mu = 15 \times 0.16401(2\pi)$ and the approximate Eq. (14) gives $7.73 \cong 7.95$. For higher quadrupole field gradients the orbit-length dispersion becomes negative and the transition energy becomes imaginary.

The maximum β_x in a cell is 103 m, that in a π-insertion is 282 m. This large maximum β_x is inherent in the design of the π-straight section. The maximum local dispersion displacement x is 43.8 m, which is extremely large. This is the result of pushing for the very large ν-value (beyond the approximate upper limit given above) with only six π-insertions. The application of this scheme is more appropriate to lower energy machines for

which ν is smaller but one can still profitably use a fair number of π-straight sections.

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