A RELATIVISTIC ELECTRON RING EQUILIBRIUM WITH THERMAL ENERGY SPREAD

RONALD C. DAVIDSON†

Naval Research Laboratory, Washington, D.C. 20390, USA

AND

SWADESH M. MAHAJAN

Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742, USA

The steady-state ($\partial/\partial t = 0$) Vlasov–Maxwell equations are used to construct a self-consistent relativistic electron ring equilibrium which incorporates a thermal spread in energy of the electrons. The equilibrium properties of the ring are calculated for the specific choice of electron distribution function $f_e^0(H, P_\theta)$ in which all of the electrons have the same value of canonical angular momentum (P_θ) and a Gibbs distribution in total energy (H), i.e., $f_e^0(H, P_\theta) = N_0 \delta(P_\theta - P_0) \exp\{-(H - \gamma_0 mc^2 + e\bar{\phi}_0)/\oplus\}$, where N_0, P_0, γ_0, $\bar{\phi}_0$, and $\oplus$ are constants. The positive ions are assumed to provide a partially neutralizing background with density $n_i^0(r, z) = fn_e^0(r, z)$, where $f =$ const = fractional neutralization. For a thin ring, with $v/\gamma_0 \ll 1$ ($v =$ Budker's parameter, and $\gamma_0 mc^2 =$ electron energy), it is shown that the electrons have a Gaussian density profile, $n_e^0(r, z) = n_0 \exp\{-(r - R_0)^2/2a^2 - z^2/2b^2\}$, where R_0 is the equilibrium radius of the ring. Closed expressions are derived for a and b in terms of properties of the equilibrium distribution function and the external confining field. The temperature profile transverse to the beam is shown to be isothermal with $T_\perp(r, z) = \oplus$.

1. INTRODUCTION AND SUMMARY

In a recent article, Davidson and Lawson have discussed the general procedure for constructing self-consistent relativistic ring equilibria within the framework of the steady-state ($\partial/\partial t = 0$) Vlasov–Maxwell equations.[1] The equilibrium properties of the ring were calculated for the specific choice of electron distribution function $f_e^0(H, P_\theta)$ in which all of the particles have the same value of total energy (H) and the same value of canonical angular momentum (P_θ). Since the equilibrium and stability properties of the ring in general depend on the detailed form of $f_e^0(H, P_\theta)$, in this article we construct a self-consistent ring equilibrium which incorporates a thermal spread in energy H of the electrons. The general procedure, assumptions, and notation employed in the present analysis have been elaborated in some detail in Ref. 1.

In Sec. 2 the steady-state Vlasov–Maxwell description and basic assumptions are briefly summarized. In Sec. 3 the equilibrium properties of the ring are calculated for the specific choice of electron distribution function $f_e^0(H, P_\theta)$ in which

† Alfred P. Sloan Foundation Fellow, 1970–1972. Permanent address: Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742, USA.

all of the electrons have the same value of canonical angular momentum (P_θ) and a Gibbs distribution in total energy (H), i.e.,

$$f_e^0(H, P_\theta) = N_0 \delta(P_\theta - P_0) \exp\{-(H - \gamma_0 mc^2 + e\bar{\phi}_0)/\oplus\},$$

where N_0, P_0, γ_0, $\bar{\phi}_0$, and $\oplus$ are constants (Eq. (7)). The positive ions are assumed to provide a partially neutralizing background with density $n_i^0(r, z) = fn_e^0(r, z)$, where $f =$ const = fractional neutralization. For a thin ring, with $v/\gamma_0 \ll 1$ ($v =$ Budker's parameter, and $\gamma_0 mc^2 =$ electron energy), it is shown in Sec. 4 that the electrons have a Gaussian density profile (Eq. (23)),

$$n_e^0(r, z) = n_0 \exp\left\{-\frac{(r - R_0)^2}{2a^2} - \frac{z^2}{2b^2}\right\},$$

where R_0 is the equilibrium radius of the ring. Closed expressions are derived for a and b (Eqs. (36) and (37)) in terms of properties of the equilibrium distribution function and the external field configuration. In addition, the temperature profile transverse to the beam is shown to be isothermal with $T_\perp(r, z) = \oplus$ (Eq. (17)).

2. STEADY-STATE VLASOV–MAXWELL DESCRIPTION

The equilibrium configuration and co-ordinate system are illustrated in Fig. 1. The ring is located
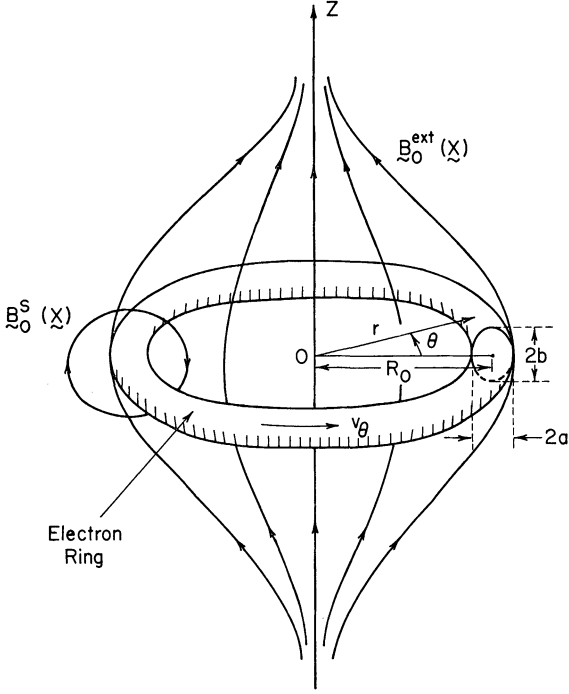

FIG. 1. Equilibrium configuration for relativistic electron ring confined in mirror field.

at the midplane of an external mirror field $\mathbf{B}_0^{\text{ext}}(\mathbf{x})$, which acts to confine the ring both radially and axially. Consistent with the parameters of interest in Electron Ring Accelerator experiments,[2-5] it is assumed that the ring is *thin*,

$$a, b \ll R_0, \qquad (1)$$

and that

$$\frac{\nu}{\gamma_0} = \frac{N_e}{2\pi R_0} \frac{e^2}{mc^2} \frac{1}{\gamma_0} \ll 1, \qquad (2)$$

where ν is Budker's parameter,[6] N_e is the total number of electrons in the ring, $-e$ is the electron charge, m is the electron rest mass, c is the speed of light *in vacuo*, e^2/mc^2 is the classical electron radius, and $\gamma_0 mc^2$ is characteristic energy of an electron in the ring. Equation (1) is simply a statement that the minor dimensions of the ring are much smaller than its equilibrium radius R_0 (see Fig. 1). It is

further assumed that external boundaries are sufficiently far removed from the electron ring that their influence on the equilibrium configuration can be ignored.

For an electron, the single-particle constants of the motion in the equilibrium fields are the total energy, H, and the canonical angular momentum, P_θ, where

$$H = (m^2c^4 + c^2\mathbf{p}^2)^{1/2} - e\phi_0(r, z), \qquad (3)$$

$$P_\theta = rp_\theta - \frac{er}{c}A_\theta^{\text{ext}}(r, z) - \frac{er}{c}A_\theta^{s}(r, z). \qquad (4)$$

In Eqs. (3) and (4), $\phi_0(r, z)$ is the electrostatic potential, $A_\theta^{\text{ext}}(r, z)$ is the θ-component of vector potential for the external confining field, $A_\theta^{s}(r, z)$ is the θ-component of vector potential for the self magnetic field generated by the equilibrium electron current, $\mathbf{p}$ is the mechanical momentum, and $\mathbf{p}^2 \equiv p_r^2 + p_z^2 + p_\theta^2$. As discussed in Ref. 1, any electron distribution function $f_e^0(H, P_\theta)$ which is a function only of the single-particle constants of the motion, H and P_θ, is a solution to the steady-state ($\partial/\partial t = 0$) Vlasov–Maxwell equations. For a partially neutralizing ion background with density $n_i^0(r, z) = fn_e^0(r, z)$ ($f = \text{const} = \text{fractional neutralization}$), $\phi_0(r, z)$ is determined self-consistently in terms of $f_e^0(H, P_\theta)$ from the equilibrium Poisson equation,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\phi_0(r, z) + \frac{\partial^2}{\partial z^2}\phi_0(r, z) = -4\pi\rho^0(r, z), \qquad (5a)$$

where

$$\rho^0(r, z) = -e(1-f)n_e^0(r, z)$$

$$= -e(1-f)\int d^3p f_e^0(H, P_\theta), \qquad (5b)$$

and single ionization has been assumed. Furthermore, the vector potential for the self magnetic field is determined from the θ-component of the $\nabla \times \mathbf{B}^s$ Maxwell equation,

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rA_\theta^{s}(r, z) + \frac{\partial^2}{\partial z^2}A_\theta^{s}(r, z) = -\frac{4\pi}{c}J_\theta^0(r, z), \qquad (6a)$$

where

$$J_\theta^0(r, z) = -en_e^0(r, z)v_\theta^0(r, z)$$

$$= -e\int d^3p v_\theta f_e^0(H, P_\theta). \qquad (6b)$$

In Eq. (6b), $v_\theta \equiv (p_\theta/m)[1 + \mathbf{p}^2/m^2c^2]^{-1/2}$.

3. EQUILIBRIUM PROPERTIES OF THE RING

In the present analysis it is assumed that all of the electrons have the same value of canonical angular momentum (P_θ), and a Gibbs distribution in energy (H), i.e.,

$$f_e^0(H, P_\theta)$$
$$= N_0 \, \delta(P_\theta - P_0) \exp\{-H - \gamma_0 mc^2 + e\bar{\phi}_0)/\oplus\},$$
(7)

where N_0, P_0, γ_0, $\bar{\phi}_0$ and $\oplus$ are constants. Since the δ-function in Eq. (7) selects $P_\theta = P_0$, we note from Eq. (4) that electrons located at (r, z) have azimuthal momentum $p_\theta = P_0/r + eA_0(r, z)/c$, where $A_0(r, z) \equiv A_\theta^{ext}(r, z) + A_\theta^s(r, z)$. Without loss of generality, in Eq. (7) $\gamma_0 mc^2$ is taken to be the azimuthal energy of an electron located at $(r, z) = (R_0, 0)$ with zero transverse momentum ($p_\perp^2 = p_r^2 + p_z^2 = 0$), i.e.,

$$\gamma_0 mc^2 \equiv [m^2 c^4 + c^2(P_0/R_0 + eA_0(R_0, 0)/c)^2]^{1/2}.$$

Furthermore $\bar{\phi}_0$ is taken to be the value of electrostatic potential at $(r, z) = (R_0, 0)$, i.e., $\bar{\phi}_0 \equiv \phi_0(R_0, 0)$. Since the ring is thin, the transverse (r, z) excursions of the electrons composing the ring are small in comparison with R_0 (see Eq. (1)), and the exponent in Eq. (7) can be expanded with $c^2 p_\perp^2 \ll \gamma_\theta^2(r, z)m^2c^4$, where $p_\perp^2 = p_r^2 + p_z^2$ is the transverse momentum-squared, and $\gamma_\theta(r, z)mc^2 \equiv [m^2 c^4 + c^2(P_0/r + eA_0(r, z)/c)^2]^{1/2}$ is the azimuthal energy. Approximating

$$H \simeq \gamma_\theta(r, z)mc^2 + (2m)^{-1} p_\perp^2/\gamma_\theta(r, z) - e\phi_0(r, z)$$

(see Eq. (3)), the equilibrium distribution function defined in Eq. (7) can be expressed as

$$f_e^0(H, P_\theta) = N_0 \, \delta(P_\theta - P_0)$$
$$\exp\left\{\frac{-p_\perp^2}{2\gamma_\theta(r, z)m\oplus}\right\} \exp\left\{-\frac{\psi(r, z)}{\oplus}\right\}, \quad (8)$$

where

$$\psi(r, z) \equiv [\gamma_\theta(r, z) - \gamma_0]mc^2 - e\delta\phi_0(r, z), \quad (9)$$

and

$$\gamma_\theta(r, z)mc^2 \equiv \left[m^2 c^4 + c^2\left(\frac{P_0}{r} + \frac{e}{c}A_0(r, z)\right)^2\right]^{1/2}, \quad (10)$$

$$\delta\phi_0(r, z) \equiv \phi_0(r, z) - \bar{\phi}_0. \quad (11)$$

Equation (8) is a sufficiently accurate representation of $f_e^0(H, P_\theta)$ for present purposes. The approximate form of H used in obtaining Eq. (8) is valid provided the transverse kinetic energy, $p_\perp^2/2m\gamma_\theta(r, z)$ is small in comparison with the azimuthal energy, $\gamma_\theta(r, z)mc^2$. It is straightforward to show that this inequality is satisfied (in an average sense) provided $\oplus \ll \gamma_\theta(r, z)mc^2$, which we assume to be the case.[7] For future reference, we note from Eq. (9) that $\psi(R_0, 0) = 0$, since $\delta\phi_0(R_0, 0) = 0$ and $\gamma_\theta(R_0, 0) = \gamma_0$.

The equilibrium properties of the electron ring, e.g., density profile, azimuthal velocity profile, transverse temperature profile, etc., can be calculated directly from Eq. (8). For example, the electron density $n_e^0(r, z)$ is defined by

$$n_e^0(r, z) = \int d^3 p f_e^0(H, P_\theta). \quad (12)$$

Substituting Eq. (8) into Eq. (12), and representing $\int d^3 p = 2\pi \int_{-\infty}^{\infty} dp_\theta \int_0^\infty dp_\perp p_\perp$, we find that

$$n_e^0(r, z) = n_0 \frac{R_0}{r} \frac{\gamma_\theta(r, z)}{\gamma_0} \exp\left\{-\frac{\psi(r, z)}{\oplus}\right\}, \quad (13)$$

where $n_0 \equiv n_e^0(R_0, 0)$ is the electron density at $(r, z) = (R_0, 0)$. (In terms of n_0, the normalization constant in Eq. (8) is $N_0 = n_0 R_0/(2\pi\oplus\gamma_0 m)$). The azimuthal velocity of an electron fluid element, $v_\theta^0(r, z)$, and the transverse 'temperature' profile, $T_\perp(r, z)$, are defined in terms of the equilibrium distribution function $f_e^0(H, P_\theta)$ by[1]

$$v_\theta^0(r, z) \equiv \frac{\int d^3 p(c^2 p_\theta/[m^2 c^4 + c^2 \mathbf{p}^2]^{1/2}) f_e^0(H, P_\theta)}{\int d^3 p f_e^0(H, P_\theta)}$$
(14)

and

$$T_\perp(r, z) \equiv \frac{\frac{1}{2}\int d^3 p(c^2 p_\perp^2/[m^2 c^4 + c^2 \mathbf{p}^2]^{1/2}) f_e^0(H, P_\theta)}{\int d^3 p f_e^0(H, P_\theta)}$$
(15)

Substituting Eq. (8) into Eqs. (14) and (15), we find

$$v_\theta^0(r, z) = \frac{1}{m\gamma_\theta(r, z)}\left[\frac{P_0}{r} + \frac{e}{c}A_0(r, z)\right], \quad (16)$$

$$T_\perp(r, z) = \oplus. \quad (17)$$

In obtaining Eqs. (16) and (17), we have approximated $[m^2 c^4 + c^2(P_0/r + eA_0(r, z)/c)^2 + c^2 p_\perp^2]^{-1/2} \simeq [\gamma_\theta(r, z)mc^2]^{-1}$ in the integrands of Eqs. (14) and

(15) (see Eq. (10) and the discussion preceding Eq. (8)). The errors incurred in Eqs. (16) and (17) by this approximation are of order $\oplus/\gamma_\theta(r, z)mc^2 \ll 1$. For a thin ring with equilibrium distribution function given by Eq. (8), we conclude from Eq. (17) that the transverse temperature profile is isothermal with temperature $T_\perp(r, z) = \oplus$. We note from Eqs. (13) and (16) that the equilibrium density and velocity profiles, $n_e^0(r,z)$, and $v_\theta^0(r,z)$, are determined explicitly in terms of the self-field potentials, $\phi_0(r, z)$ and $A_\theta^s(r,z)$, and other properties characteristic of the equilibrium configuration, e.g., P_0, $\gamma_0 mc^2$, $A_\theta^{ext}(r, z)$, etc. The self-field potentials may be calculated self-consistently by substituting Eqs. (13) and (16) into Eqs. (5) and (6), and solving the resulting equations for $\phi_0(r, z)$ and $A_\theta^s(r, z)$.

4. ELECTRON DENSITY PROFILE

It is of considerable interest to determine a closed expression for the r–z dependence of the density profile $n_e^0(r, z)$. For a thin ring with $v/\gamma_0 \ll 1$, the variation of $\gamma_\theta(r, z)$ and $1/r$ across the minor dimensions of the ring is small. Therefore, Eq. (13) can be expressed in the approximate form

$$n_e^0(r,z) = n_0 \exp\left\{-\frac{\psi(r,z)}{\oplus}\right\}. \qquad (18)$$

The detailed shape of the density profile is contained in the factor $\exp\{-\psi(r, z)/\oplus\}$ in Eq. (18). Following Davidson and Lawson,[1] we Taylor expand the expression for $\psi(r, z)$ given in Eq. (9) about $(r, z) = (R_0, 0)$ for $a, b \ll R_0$ and $v/\gamma_0 \ll 1$. Introducing $\rho = r - R_0$, and neglecting terms higher than quadratic order, we find

$$\psi(r,z) = \rho\left[\frac{\partial}{\partial r}\psi(r,z)\right]_{R_0,0} + z\left[\frac{\partial}{\partial z}\psi(r,z)\right]_{R_0,0}$$

$$+ \rho z\left[\frac{\partial^2}{\partial r \partial z}\psi(r,z)\right]_{R_0,0} + \frac{\rho^2}{2}\left[\frac{\partial^2}{\partial r^2}\psi(r,z)\right]_{R_0,0}$$

$$+ \frac{z^2}{2}\left[\frac{\partial^2}{\partial z^2}\psi(r,z)\right]_{R_0,0} + \dots, \qquad (19)$$

where use has been made of $\psi(R_0, 0) = 0$ (see the discussion following Eq. (11)). Since the evaluation of the coefficients in Eq. (18) closely parallels the analysis in Sec. 4 and Appendix A of Ref. 1, it is adequate for present purposes to outline the results.

The reader is referred to Ref. 1 for further details on procedure. First, it can be shown from Eq. (9) that $[\partial\psi(r, z)/\partial z]_{R_0,0} = 0$ and $[\partial^2\psi(r, z)/\partial r\,\partial z]_{R_0,0} = 0$ follow directly from axial symmetry of the equilibrium configuration about $z = 0$. (In particular, the coefficients of z and ρz vanish in Eq. (19) since the radial magnetic field and axial electric field are identically zero in the midplane, i.e., $B_{0r}(r, 0) = 0 = E_{0z}(r, 0)$, where $B_{0r}(r, z) = -\partial A_0(r, z)/\partial z$ and $E_{0z}(r, z) = -\partial\delta\phi_0(r, z)/\partial z$). Second, we *impose* the requirement that $[\partial\psi(r, z)/\partial r]_{R_0,0} = 0$, which assures that the term linear in ρ is absent in Eq. (19). This condition effectively *determines* the equilibrium radius R_0 which corresponds to the geometric center of the beam. Making use of Eq. (9), and the definitions of radial electric field $[E_{0r}^s(r, z) = -\partial\delta\phi_0(r, z)/\partial r]$ and axial magnetic field $[B_{0z}(r, z) = \partial A_0(r, z)/\partial r + A_0(r, z)/r]$, we find that the condition $[\partial\psi(r, z)/\partial r]_{R_0,0} = 0$ can be expressed in the equivalent form

$$-\frac{\gamma_0 m\beta_\theta^2 c^2}{R_0} = -eE_{0r}^s(R_0, 0) - e\beta_\theta B_{0z}(R_0, 0), \qquad (20)$$

where

$$\beta_\theta c \equiv v_\theta^0(R_0, 0) = m^{-1}\gamma_0^{-1}[P_0/R_0 + eA_0(R_0, 0)/c]$$

is the azimuthal velocity of an electron located at $(r, z) = (R_0, 0)$ (see Eq. (16)). Equation (20), which is a statement of radial force balance on an electron fluid element at $(r, z) = (R_0, 0)$, effectively determines the equilibrium radius R_0 of the ring. For a thin ring with $v/\gamma_0 \ll 1$, the self-field contributions in Eq. (20) are small in comparison with $B_{0z}^{ext}(R_0, 0)$, and Eq. (20) can be approximated by

$$-\frac{\gamma_0 m\beta_\theta^2 c^2}{R_0} = -e\beta_\theta B_{0z}^{ext}(R_0, 0) \qquad (21)$$

correct to lowest order. Since the terms in Eq. (19) proportional to ρ, z and ρz vanish, Eq. (19) reduces to

$$\psi(r,z) = \frac{\rho^2}{2}\left[\frac{\partial^2}{\partial r^2}\psi(r,z)\right]_{R_0,0} + \frac{z^2}{2}\left[\frac{\partial^2}{\partial z^2}\psi(r,z)\right]_{R_0,0}, \qquad (22)$$

where terms higher than quadratic order have been neglected. Substituting Eq. (22) into Eq. (18), we find that $n_e^0(r, z)$ has a Gaussian profile[8] about

$(r, z) = (R_0, 0)$, i.e.,

$$n_e^0(r, z) = n_0 \exp\left\{-\frac{\rho^2}{2a^2} - \frac{z^2}{2b^2}\right\}, \qquad (23)$$

where $\rho = r - R_0$, and

$$\frac{1}{a^2} \equiv \frac{1}{\oplus}\left[\frac{\partial^2}{\partial r^2}\psi(r, z)\right]_{R_0, 0}, \qquad (24a)$$

$$\frac{1}{b^2} \equiv \frac{1}{\oplus}\left[\frac{\partial^2}{\partial z^2}\psi(r, z)\right]_{R_0, 0}. \qquad (24b)$$

To evaluate a^2 and b^2 we parallel the analysis in Ref. 1. For a thin ring with $\nu/\gamma_0 \ll 1$, it is straightforward to show from Eqs. (9) and (24) that

$$\frac{1}{a^2} = \frac{\gamma_0 mc^2}{\oplus}\frac{\beta_\theta^2}{R_0^2}\left\{1 - n + \frac{R_0^2 e}{\gamma_0 \beta_\theta^2 mc^2}\right.$$

$$\left.\cdot\left[\frac{\partial}{\partial r}E_{0r}^s(r, z) + \beta_\theta \frac{\partial}{\partial r}B_{0z}^s(r, z)\right]_{R_0, 0}\right\}, \qquad (25)$$

$$\frac{1}{b^2} = \frac{\gamma_0 mc^2}{\oplus}\frac{\beta_\theta^2}{R_0^2}\left\{n + \frac{R_0^2 e}{\gamma_0 \beta_\theta^2 mc^2}\right.$$

$$\left.\cdot\left[\frac{\partial}{\partial z}E_{0z}^s(r, z) - \beta_\theta \frac{\partial}{\partial z}B_{0r}^s(r, z)\right]_{R_0, 0}\right\}, \qquad (26)$$

where n is the external field index at $(r, z) = (R_0, 0)$,

$$n \equiv -\left[\frac{r}{B_{0z}^{\text{ext}}(r, z)}\frac{\partial}{\partial r}B_{0z}^{\text{ext}}(r, z)\right]_{R_0, 0}. \qquad (27)$$

Since the self fields are weak for $\nu/\gamma_0 \ll 1$, terms in Eq. (24) which are proportional to $E_{0r}^s(R_0, 0)$ and $B_{0z}^s(R_0, 0)$ have been consistently neglected in obtaining Eqs. (25) and (26). Terms which are proportional to gradients of the self fields, however, are retained in the analysis, e.g., $[\partial B_{0z}^s(r, z)/\partial r]_{R_0, 0}$ and $[\partial E_{0r}^s(r, z)/\partial r]_{R_0, 0}$.

In order to obtain *closed* expressions for a and b it is necessary to evaluate the self-field gradient terms in Eqs. (25) and (26), which requires a self-consistent determination of $\phi_0(r, z)$ and $A_\theta^s(r, z)$ from Eqs. (5) and (6). Considerable simplification occurs in the analysis since $a, b \ll R_0$, $\nu/\gamma_0 \ll 1$, and only the self-field gradients *evaluated at* $(R_0, 0)$ are required in Eqs. (25) and (26). In Poisson's equation (Eq. (5)), we make use of the expression for $n_e^0(r, z)$ given in Eq. (23). Moreover, since $v_\theta^0(r, z)$ varies only a small amount across the minor

dimensions of the ring, we approximate $v_\theta^0(r, z) \simeq \beta_\theta c \equiv v_\theta^0(R_0, 0) = \gamma_0^{-1}m^{-1}[P_0/R_0 + eA_0(R_0, 0)/c]$ (see Eq. (16)) in the $\nabla \times \mathbf{B}^s$ Maxwell equation (Eq. (6)). Introducing the variable $\rho = r - R_0$, Eqs. (5) and (6) can then be expressed as

$$\left\{\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2}\right\}\phi_0(\rho, z)$$

$$= 4\pi e(1 - f)n_0 \exp\left\{-\frac{\rho^2}{2a^2} - \frac{2}{2b^2}\right\}, \qquad (28)$$

$$\left\{\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2}\right\}A_\theta^s(\rho, z)$$

$$= 4\pi e\beta_\theta n_0 \exp\left\{-\frac{\rho^2}{2a^2} - \frac{z^2}{2b^2}\right\}, \qquad (29)$$

where the differential operators on the left-hand sides of Eqs. (5) and (6) have been approximated by their limiting values for b/R_0, $a/R_0 \to 0$. In terms of a Fourier integral representation, the solution to Eq. (28) is

$$\phi_0(\rho, z) = -4\pi e(1 - f)\frac{ab}{2\pi}n_0 \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl$$

$$\cdot \exp\{ikz + il\rho\}\frac{\exp\{-a^2l^2/2 - b^2k^2/2\}}{l^2 + k^2}, \qquad (30)$$

and the solution to Eq. (29) can be expressed in a similar form with $(1 - f)$ replaced by β_θ. Making use of the integral representations of $\phi_0(\rho, z)$ and $A_\theta^s(\rho, z)$, it is straightforward to evaluate the self-field gradient terms in Eqs. (25) and (26). For example, it follows from Eq. (30) that

$$\left[\frac{\partial}{\partial z}E_{0z}^s(r, z)\right]_{R_0, 0} = -\left[\frac{\partial^2}{\partial z^2}\phi_0(\rho, z)\right]_{0, 0}$$

$$= -4\pi e(1 - f)\frac{ab}{2\pi}n_0 \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl$$

$$\cdot \frac{k^2 \exp\{-a^2l^2/2 - b^2k^2/2\}}{l^2 + k^2}. \qquad (31)$$

Carrying out the integrations in Eq. (31), we find

$$\left[\frac{\partial}{\partial z}E_{0z}^s(r, z)\right]_{R_0, 0} = -4\pi e(1 - f)n_0\frac{a}{a + b}. \qquad (32)$$

In a similar fashion it can be shown that

$$\left[\frac{\partial}{\partial r} E_{0r}^s(r,z)\right]_{R_0,0} = -4\pi e(1-f) n_0 \frac{b}{a+b}, \quad (33)$$

$$\left[\frac{\partial}{\partial z} B_{0r}^s(r,z)\right]_{R_0,0} = -4\pi e\beta_\theta n_0 \frac{a}{a+b}, \quad (34)$$

$$\left[\frac{\partial}{\partial r} B_{0z}^s(r,z)\right]_{R_0,0} = 4\pi e\beta_\theta n_0 \frac{b}{a+b}, \quad (35)$$

correct to lowest order in a/R_0, b/R_0. Substituting Eqs. (32)–(35) into Eqs. (25) and (26), we find

$$\frac{1}{a^2} = \frac{\gamma_0 mc^2}{\oplus} \frac{\beta_\theta^2}{R_0^2}$$

$$\cdot \left\{1 - n + 2\frac{\nu}{\gamma_0} \frac{R_0^2}{a(a+b)\beta_\theta^2}(\beta_\theta^2 + f - 1)\right\}, \quad (36)$$

$$\frac{1}{b^2} = \frac{\gamma_0 mc^2}{\oplus} \frac{\beta_\theta^2}{R_0^2}$$

$$\cdot \left\{n + 2\frac{\nu}{\gamma_0} \frac{R_0^2}{b(a+b)\beta_\theta^2}(\beta_\theta^2 + f - 1)\right\}, \quad (37)$$

where $\nu \equiv (N_e/2\pi R_0)(e^2/mc^2)$ is Budker's parameter, and

$$N_e = (2\pi R_0)(2\pi ab)n_0$$

$$= (2\pi R_0)n_0 \int\int d\rho\, dz \exp\left\{-\rho^2/2a^2 - z^2/2b^2\right\}$$

is the total number of electrons in the ring. Equations (36) and (37) constitute closed equations for a and b in terms of properties of the equilibrium distribution function (e.g., $\oplus$, γ_0, β_θ) and the external field configuration (e.g., n).

The expressions for a and b given in Eqs. (36) and (37) correspond to the radial and axial betatron oscillation amplitudes (including equilibrium self-field effects) for an electron with transverse energy equal to the *thermal* energy $\oplus$. Note that the thin ring approximation (Eq. (1)) is valid only if the energy spread of the electrons is small, i.e., $\oplus \ll \gamma_0 mc^2$. Note also that the expressions for a and b given in Eqs. (36) and (37) are similar in form to those derived in Ref. 1 for an electron ring in which all of the electrons have the same value of total energy H and the same value of canonical angular momentum P_θ. The essential difference is that $\beta_0^2 - \beta_\theta^2$ in Eqs. (66) and (67) of Ref. 1 is replaced by $\oplus/\gamma_0 mc^2$.

5. CONCLUSIONS

In this paper the equilibrium properties have been calculated for a relativistic electron ring with equilibrium distribution function $f_e^0(H, P_\theta)$ given by Eq. (7). For a thin ring (Eq. (1)) with $\nu/\gamma_0 \ll 1$ (Eq. (2)), it is found that the transverse temperature profile for the electrons is isothermal with $T_\perp(r, z) = \oplus$ (Eq. (17)), and that the electron density $n_e^0(r, z)$ has a Gaussian profile about $(r, z) = (R_0, 0)$ (Eq. (23)). Closed expressions for a and b, the effective minor dimensions of the ring, are obtained in terms of properties of the equilibrium distribution function and the external field configuration (Eqs. (36) and (37)).

The sensitive dependence of equilibrium properties on the choice of $f_e^0(H, P_\theta)$ should be noted. In Ref. 1, where all of the electrons are assumed to have the same value of total energy H and the same value of canonical angular momentum P_θ, i.e., $f_e^0(H, P_\theta) = N_0\delta(H - \gamma_0 mc^2 + e\bar{\phi}_0)\delta(P_\theta - P_0)$, it is found that the transverse temperature profile is parabolic with $T_\perp(r, z)$ assuming its maximum value at $(r, z) = (R_0, 0)$.[9] Furthermore, in Ref. 1 the electron density is approximately constant in the ring interior, and the minor cross section of the ring has a *sharp* boundary with envelope equation $\rho^2/a^2 + z^2/b^2 = 1$.[10] This is in contrast to the results obtained for the equilibrium distribution function in Eq. (7), where the transverse electron temperature profile is found to be isothermal (Eq. (17)), and the minor cross section of the ring has a *diffuse* boundary (Eq. (23)). It may be anticipated that the stability properties of these two equilibrium distribution functions are also quite different.

In conclusion, it is important to note that the assumption that the ion density profile satisfies $n_i^0(r, z) = fn_e^0(r, z)$ is highly idealized and probably not satisfied in many applications of interest. In this regard it is straightforward to extend the present equilibrium analysis to describe the positive ion background within the framework of the steady-state Vlasov–Maxwell equations.[11] If the ions are described by an equilibrium distribution function $f_i^0(H)$,[12] then the procedure for calculating the equilibrium properties of the electron ring from $f_e^0(H, P_\theta)$ remain essentially the same as that summarized in Secs. 2–4. The only difference is that the electrostatic potential $\phi_0(r, z)$ must be

calculated self-consistently using the ion density computed from $f_i^0(H)$. That is, in Eq. (5) $\rho^0(r, z)$ must be replaced by

$$\rho^0(r, z) = -e[\int d^3p f_e^0(H, P_\theta) - n_i^0(r, z)]$$

where $n_i^0(r, z) \equiv \int d^3p f_i^0(H)$.

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7. The condition for the average transverse kinetic energy to be small in comparison with the azimuthal energy is $\langle p_\perp^2 \rangle / 2m\gamma_\theta(r\,z) \ll \gamma_\theta(r, z)mc^2$. Combining this inequality with Eqs. (15) and (17) gives $\oplus \ll \gamma_\theta(r, z)mc^2$.
8. It should be noted that the general form of $n_e^0(r, z)$ is in good agreement with the Gaussian density profile measured in the Berkeley Compressor II experiments;[2,3] however, the predicted values for a and b (see Eqs. (36) and (37)) do not concur with the experimentally measured values.
9. See Eq. (90) of Ref. 1.
10. Here, a and b are defined in Eqs. (66) and (67) of Ref. 1.
11. See Sec. 6 of Ref. 1.
12. If the ion dynamics are nonrelativistic, then $H = \mathbf{p}^2/2m_i + e\phi_0(r, z)$ where m_i and $+e$ are the ion mass and charge, respectively.