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RESONANCE OF COUPLED TRANSVERSE OSCILLATIONS IN TWO CIRCULAR BEAMS

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It is shown that in a system of two circular beams there exist resonances analogous to the well-known resonances produced by external perturbations in circular beams of singly charged particles. The general resonance condition is $nQ_1 + mQ_2 = k$ where Q_1 and Q_2 represent the natural oscillation frequencies of the particles in the first and second beams, and m, n, and k are integers. Equations are derived for the resonance bandwidth and for a resonance of any order, and the crossing of resonances is investigated. Analysis of these results shows that the two-beam interactions impose a strict upper limit on the frequencies of the self-stabilizing Bennet-Budker beam.

1. INTRODUCTION

In a system consisting of two beams of particles of opposite sign each beam may act upon the other as an 'external force', which gives rise to perturbations of the average field, of the gradient, of the quadratic nonlinearity, etc. affecting the other beam. The 'dipole' two-beam instability ('snake'), which occurs when the beam oscillations modify the average fields acting upon them, has been discussed in the literature.[1–3] It seems that besides dipole resonances higher-order resonances, corresponding to oscillations of the gradient, of the quadratic nonlinearity, of the cubic nonlinearity, etc. are also possible in a two-beam system. These resonances may prove essential in determining the basic parameters of electron-ring accelerators: maximum ion current and maximum accelerating field.

The aim of this paper is a theoretical investigation of the higher-order resonances arising from oscillations of two circular beams.

Investigation of such resonances is difficult, and an exact solution is possible only within the bounds of established models. We consider two models: (1) a beam with constant density over its cross section, and (2) a 'ribbon' beam with constant density in the phase space.

The first of these models enables one to derive and investigate the dispersion equation for the first- and second-order resonances, with the aid of a theory that makes use of the linearized equations for the beam center and the beam envelope. This analysis is particularly instructive and enables one to study both one-dimensional and two-dimensional first- and second-order resonances. However, it is essentially unsuitable for higher-order resonances, for which the equations of motion are nonlinear.

The second model yields the dispersion equations for resonances of arbitrary order by using the kinetic equation. A drawback of this method is the approximate nature of the field calculations. The errors arising from the use of this model can be estimated by comparing the equations for the first- and second-order resonances with those derived earlier.

Note that neither model takes into account the stabilizing effect of the beam's intrinsic nonlinearity; this requires a special study.

2. ANALYSIS OF THE DIPOLE AND QUADRUPOLE RESONANCES

Since the radius of the ring is much larger than its transverse dimensions, the equations for the transverse motion of the electrons and ions with one degree of freedom may be written as follows (neglecting the field exerted by the electrons on electrons and by the ions on ions):

$$\left.\begin{aligned} \ddot{y}_e + \lambda^2 y_e &= \frac{e}{m\gamma}(E^0_{iy} + E'_{iy}) \\ \ddot{y}_i &= \frac{e}{M}(E^0_{ey} + E'_{ey}). \end{aligned}\right\} \tag{1}$$

The dot denotes the total time derivative, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \Omega\partial/\partial\theta$, λ represents the oscillation frequency of the electrons under the influence of the external focusing force, E_{iy} and E_{ey} are the y-components of the electric field strength of the ions and electrons, the superscript 0 refers to the unperturbed field, the prime denotes the additional field produced due to the perturbation, and Ω is assigned the value appropriate to the particle species under consideration.

For microcanonical density distribution of the electrons and ions in the phase space,[4] the electron and ion beams have uniform density across the cross section, and Eqs. (1) become

$$\left.\begin{aligned} \ddot{y}_e+\lambda^2 y_e &= -\frac{x_i^2(y_e-\bar{y}_i)}{g_{yi}(g_{yi}+g_{zi})} \\ \ddot{y}_i &= -\frac{x_e^2(y_i-\bar{y}_e)}{g_{ye}(g_{ye}+g_{ze})}, \end{aligned}\right\} \tag{2}$$

where $x_e^2 = 4N_e e^2/M$, $x_i^2 = 4N_i e^2/m\gamma$, N_i and N_e are the numbers of ions and electrons per unit length of the beam, $\bar{y}_i$ and $\bar{y}_e$ are the centers of mass of the ion and electron beams, and g_y and g_z are the maximum semi-minor dimensions of the beams in the respective degrees of freedom. Equations (1) are basic equations for the analysis of the dipole and quadrupole resonances.

3. DIPOLE RESONANCE

For dipole oscillations of the beams, $g_y = \text{const}$, $g_z = \text{const}$,

$$\begin{aligned} x_i^2[g_{yi}(g_{yi}+g_{zi})]^{-1} &= \Omega_1^2, \\ x_e^2[g_{ye}(g_{ye}+g_{ze})]^{-1} &= \Omega_i^2. \end{aligned} \tag{3}$$

If the momentum spread is neglected and the equations are averaged, which is trivial in this case, we obtain:

$$\left.\begin{aligned} \ddot{\bar{y}}_e+\Omega_e^2\bar{y}_e &= \Omega_1^2\bar{y}_i \\ \ddot{\bar{y}}_i+\Omega_i^2\bar{y}_i &= \Omega_i^2\bar{y}_e, \end{aligned}\right\} \tag{4}$$

$$\Omega_e^2 = \lambda^2+\Omega_1^2. \tag{5}$$

We seek a solution of Eq. (4) in the following form:

$$\left.\begin{aligned} \bar{y}_i &= y_{oi}\exp[i(k\theta-\omega t)], \\ \bar{y}_e &= y_{oe}\exp[i(k\theta-\omega t)]. \end{aligned}\right\} \tag{6}$$

Substituting Eq. (6) into Eq. (4), we obtain the following dispersion equation:

$$[-(k\Omega-\omega)^2+\Omega_e^2](\Omega_i^2-\omega^2) = \Omega_i^2\Omega_1^2. \tag{7}$$

4. QUADRUPOLE RESONANCE

In considering the oscillations of the beam shape, we may disregard the deviation of the center of mass of the clusters. On the other hand, the fluctuations of the dimensions are fundamentally coupled, and therefore the analysis must take both transverse degrees of freedom into account. Equations (2) then become

$$\left.\begin{aligned} \ddot{y}_e+\lambda_y^2 y_e &= -x_i^2 y_e[g_{yi}(g_{yi}+g_{zi})]^{-1}, \\ \ddot{y}_i &= -x_e^2 y_i[g_{ye}(g_{ye}+g_{ze})]^{-1}, \\ \ddot{z}_e+\lambda^2 z_e &= -x_i^2 z_e[g_{zi}(g_{yi}+g_{zi})]^{-1}, \\ \ddot{z}_i &= -x_e^2 z_i[g_{ze}(g_{ye}+g_{ze})]^{-1}. \end{aligned}\right\} \tag{8}$$

To analyze these equations it is convenient to go to the equations for the beam envelopes. This can be done by the well-known substitution

$$\left.\begin{aligned} y_{i,e} &= g_{yi,e}\exp\left(i\int v_{yi,e}\,g_{yi,e}^{-2}\,dt\right), \\ z_{i,e} &= g_{zi,e}\exp\left(i\int v_{zi,e}\,g_{zi,e}^{-2}\,dt\right). \end{aligned}\right\} \tag{9}$$

In Eq. (9), v_i and v_e are the phase-space volumes of ions and electrons, respectively.

By linearizing the resulting equations and by assuming that the unperturbed envelopes of the two beams are identical (this is equivalent to the existence of two circular beams of identical radius), we get

$$\left.\begin{aligned} \ddot{\eta}_i+(2\Omega_i)^2\eta_i-\tfrac{3}{2}\Omega_i^2\eta_e-\tfrac{1}{2}\Omega_i^2\xi_e &= 0, \\ \ddot{\xi}_i+(2\Omega_i)^2\xi_i-\tfrac{3}{2}\Omega_i^2\xi_e-\tfrac{1}{2}\Omega_i^2\eta_e &= 0, \\ \ddot{\eta}_e+(2\Omega_{ey})^2\eta_e-\tfrac{3}{2}\Omega_1^2\eta_i-\tfrac{1}{2}\Omega_1^2\xi_i &= 0, \\ \ddot{\xi}_e+(2\Omega_{ez})^2\xi_e-\tfrac{3}{2}\Omega_1^2\xi_i-\tfrac{1}{2}\Omega_1^2\eta_i &= 0, \end{aligned}\right\} \tag{10}$$

where $\eta_{i,e} = \Delta g_{y^{i,e}}$ and $\xi_{i,e} = \Delta g_{z^{i,e}}$.

In analyzing Eq. (10) we must distinguish between the two cases: (1) $\Omega_{ey} \approx \Omega_{ez} = \Omega_e$ and (2) $\Omega_{ey} \neq \Omega_{ez}$ (this implies that $\lambda_{ey} \neq \lambda_{ez}$).

In the first case, Eqs. (10) are markedly simplified, since only two types of oscillation are possible for ions and electrons: symmetric oscillations ($\xi = \eta$), and antisymmetric oscillations ($\xi = -\eta$). It is clear that no coupling is possible between these two types of oscillation. Introducing the new variables $\beta_{ie} = \eta_{ie} - \xi_{ie}$ and $\alpha_{ie} = \eta_{ie} + \xi_{ie}$, we obtain

$$\left.\begin{aligned} \ddot{\alpha}_i + (2\Omega_i)^2\alpha_i - 2\Omega_i^2\alpha_e &= 0 \\ \ddot{\alpha}_e + (2\Omega_e)^2\alpha_e - 2\Omega_1^2\alpha_i &= 0 \end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned} \ddot{\beta}_i + (2\Omega_i)^2\beta_i - \Omega_i^2\beta_e &= 0 \\ \ddot{\beta}_e + (2\Omega_e)^2\beta_e - \Omega_1^2\beta_i &= 0. \end{aligned}\right\} \tag{12}$$

In the second case the oscillations of the electrons in each degree of freedom are independent. To remove ambiguity, suppose that the electron oscillations are built up in η. By introducing a new variable, $\gamma = \frac{3}{2}\eta_i + \frac{1}{2}\xi_i$, and neglecting the excitation of the second degree of freedom of the electrons, we obtain

$$\left.\begin{aligned} \ddot{\gamma} + (2\Omega_i)^2\gamma - 2{,}5\Omega_i^2\eta_e &= 0 \\ \ddot{\eta}_e + (2\Omega_{ey})^2\eta_e - \Omega_1^2\gamma &= 0. \end{aligned}\right\} \tag{13}$$

Equation (13) was derived for the case $\Omega_{iy} = \Omega_{iz}$. When all four frequencies are different one need only retain those two equations of (10) for which the resonance condition holds.

Equations (11) to (13) are analogous to the equations for a dipole resonance. The dispersion equations for the quadrupole and dipole resonances may be written in the following general form, which is most convenient for subsequent analysis

$$\begin{aligned} F(v, Q_i, Q_e, r) &= (v^2 - Q_i^2)[(v-r)^2 - Q_e^2] \\ &= p^2 Q_i^2 Q_1^2 = \varepsilon, \end{aligned} \tag{14}$$

where $Q_i = \Omega_i/\Omega$, $Q_e = \Omega_e/\Omega$, and $Q_1 = \Omega_1/\Omega$; for an extrinsic resonance $v = \omega/\Omega$, $r = k$, and $p = 1$, and for the quadrupole resonances $v = \omega/2\Omega$, $r = k/2$, and $p < 1$.

5. ANALYSIS OF THE DISPERSION EQUATION

The stability or instability of the oscillations depends on the number of real roots of Eq. (14), which can be determined graphically as the points of intersection of the function $y = F(v)$ with the straight line $y = \varepsilon = p^2 Q_i^2 Q_1^2$. Figure 1 is a qualitative representation of the function for two

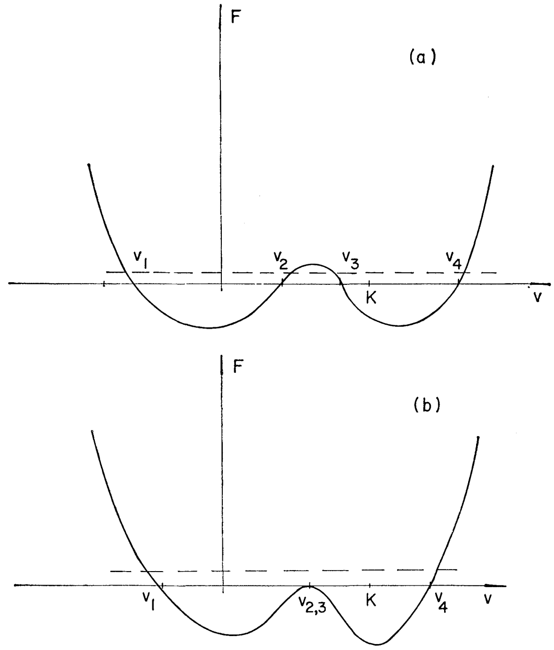

FIG. 1. Diagram of the function $F(v)$.

cases: (a) $v_2 = v_3$, and (b) $v_2 \neq v_3$ (here v_1, v_2, v_3, and v_4 are the roots of the equation $F(v) = 0$ arranged in order of increasing magnitude). In the first case, for any ε there are only two real roots (unstable motion). In the second case, when $\varepsilon \ll \max F$, all four roots of Eq. (14) are real and close to those of the equation $F(v) = 0$ (weak coupling of oscillations). As ε increases, v_2' and v_3' (the second and third roots of Eq. (14)) approach each other, and at the boundary of the stability region $v_2' = v_3'$ and $\varepsilon = \max F$. If $\varepsilon > \max F$, Eq. (14) has only two real roots (instability region).

The boundaries of the instability regions in the Q_i, Q_e plane may be determined by simultaneous solution of Eq. (14) and the equation

$$\frac{dF}{dv} = 0. \tag{15}$$

The solution of Eq. (14) and (15), expressed in parametric form, has the following form:

$$\left.\begin{aligned} Q_e^2 &= \sqrt{\frac{b \pm \sqrt{b^2 - (1-p^2)c}}{1-p^2}} \\ Q_i^2 &= v\left(2v - r + \frac{Q_e^2}{r-v}\right) \end{aligned}\right\} \tag{16}$$

where

$$b = (v-r)[v(1-p^2)-r(1-p^2/2)]-\frac{p^2\lambda_1{}^2}{2},$$

$$c = (v-r)^4-\lambda_1{}^2 p^2(v-r)(2v-r).$$

Figures 2 to 5 show the stability regions in the Q_i, Q_e plane for different values of the parameter $\lambda_1 = \lambda/\Omega$. In calculating the quadrupole resonance bands, we set $p^2 = 0{,}156$, which corresponds to identical oscillation frequencies of the ion beam in the two degrees of freedom and to significantly different betatron oscillation frequencies of the electron beam.

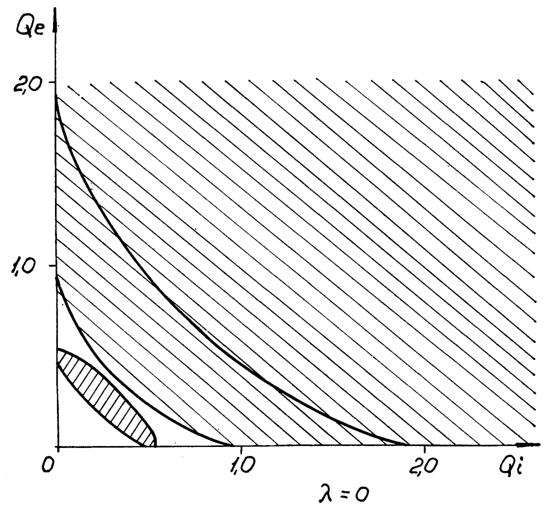

FIG. 2. Instability regions in the Q_e, Q_i plane for the dipole resonance (\\\) and for the quadrupole resonance (///); $\lambda_1 = 0$.

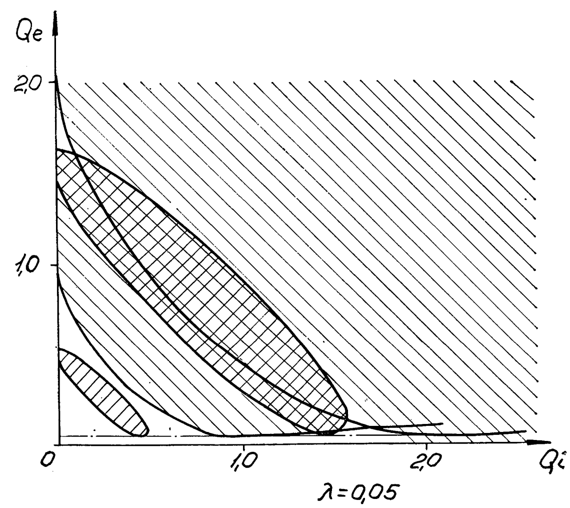

FIG. 3. Same as Fig. 2, $\lambda_1 = 0{,}05$.

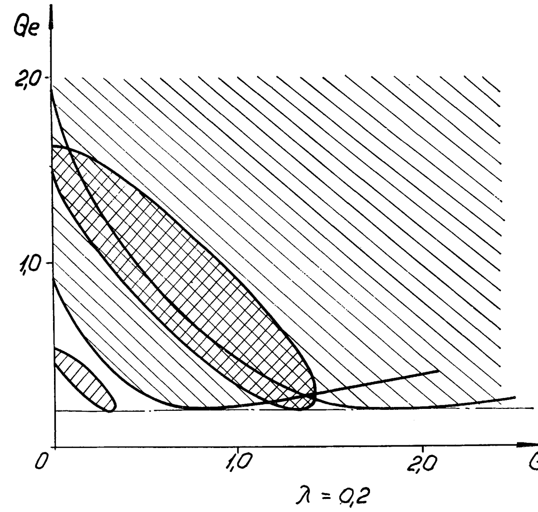

FIG. 4. Same as Fig. 2, $\lambda_1 = 0{,}20$.

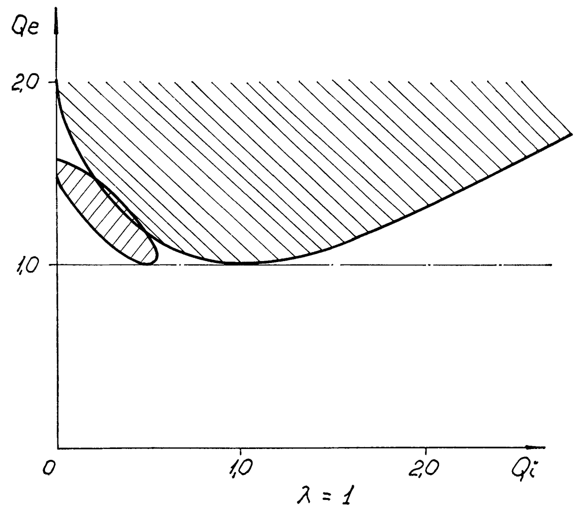

FIG. 5. Same as Fig. 3, $\lambda_1 = 1{,}0$.

The instability region for the quadrupole resonances has resonance bands of a characteristic shape, whereas the instability region for the dipole resonances has only a lower limit. However, it can be shown that the instability has a resonant character in both cases.

We now derive an approximate equation for the complex roots of Eq. (14).

When ε is small, the real parts of the roots of Eq. (14) lie between v_2 and v_3, and their imaginary parts are small. Thus one can approximately substitute $v = v_2 + v_3/2$ in the factors $v - v_1$ and $v - v_4$ in the equation $F(v) = \varepsilon$. Equation (14) then reduces to a quadratic equation, the solution of which is of the form

$$v = \frac{v_2 + v_3}{2} \pm \sqrt{\frac{(v_2 - v_3)^2}{4} - \frac{4\varepsilon}{(v_2 + v_3 - 2v_1)(2v_4 - v_2 - v_3)}}. \quad (17)$$

The center of the instability band is defined by the condition $v_2 = v_3$, which has the following form:

$$Q_i + Q_e = r. \quad (18)$$

If this condition is satisfied,

$$Im(v) = \frac{1}{2}\sqrt{\frac{\varepsilon}{Q_i Q_e}}. \quad (19)$$

The width of the resonance band is defined by

$$\delta v = 2\, Im(v). \quad (20)$$

For the dipole resonance, it follows from Eq. (19) that at the center of the resonance band the expression for the increment $\delta_{\|}$ has the form

$$\delta_{\|} = Im(\omega/\Omega) = \frac{1}{2}\sqrt{\frac{Q_i Q_1{}^2}{Q_e}}. \quad (21)$$

Note that for a dipole resonance these equations are only approximate. As one goes to the high-frequency region, the dipole coupling becomes much stronger; this produces a small shift of the region of maximum growth and makes the upper limit of the resonance go to infinity. For quadrupole resonances, Eqs. (18) to (20) are fairly accurate.

The ratios of the growth rate and width of the various quadrupole resonances to the growth rate and width of the dipole resonance, calculated from Eqs. (17) through (20), are given in Table I.

6. STUDY OF RESONANCES OF ANY ORDER WITH USE OF THE KINETIC EQUATION

In the general case, the fields appearing in the right-hand side of Eqs. (1) are nonlinear and the approximate solution of these equations for $E'_{ey} = E'_{iy} = 0$ and for small nonlinearity is

$$\begin{aligned} y_{i,e} &= \sqrt{I_{i,e}} \cos \chi_{i,e}; \\ y_{i,e} &= -\Omega_{i,e} \sqrt{I_{i,e}} \sin \chi_e; \\ \chi_{i,e} &= \Omega_{i,e} t + \alpha_{i,e}, \end{aligned} \quad (22)$$

where I is the square of the oscillation amplitude and $\Omega = f(I)$.

We restrict ourselves hereafter to the investigation of the 'one-dimensional' resonances in which coupling occurs between the oscillations of the ion and electron beams in one degree of freedom. Moreover, to simplify calculation of the non-equilibrium electric fields we shall use the 'ribbon' beam model, in which it is assumed that the dimensions of the beam in the second degree of

TABLE I

Type of resonance	Condition for the existence of the resonance	$\delta/\delta_{\|}$ (Increment)	$\delta\omega/\delta\omega_{\|}$ (Width)
Symmetric, two-dimensional resonance	$Q_{ye} = Q_{ze};\ Q_{zi} = Q$ $2Q_e + 2Q_i = k$	1	0,5
One-dimensional, uncoupled oscillations	$Q_{ye} \neq Q_{ze};\ Q_{zi} \neq Q_{yi};$ $2Q_{ey,z} + 2Q_{iy,z} = k$	0,75	0,375
One-dimensional-two-dimensional resonance	$Q_{ye} \neq Q_{ze};\ Q_{zi} = Q_{yi};$ $2Q_{ey,z} + 2Q_i = k$	0,78	0,39
Asymmetric, two-dimensional resonance	$Q_{ey} = Q_{ze};\ Q_{zi} = Q_{yi}$ $2Q_e + 2Q_i = k$	0,5	0,29
Two-dimensional, uncoupled oscillations	$Q_{ye} \neq Q_{ze};\ Q_{yi} \neq Q_{zi};$ $2Q_{ey,z} + 2Q_{iz,y} = k$	0,25	0,125

freedom are much greater than those in the degree of freedom under consideration. For a circular beam, this model of course will introduce an error, which we shall estimate below.

To investigate a one-dimensional resonance of any order we use the Vlasov system of self-consistent equations. We seek a solution of the kinetic equation in the following form:

$$f_{i,e} = f_0(I_{i,e}) + f_1(I_{i,e};\, \chi_{i,e};\, \theta, t) \tag{23}$$

where $f_{i,e}$ represents the density of the particles in the space of the canonical variables I and χ, $f_0(I)$ is a steady-state distribution function, and f_1 is a small nonequilibrium component. Substituting Eq. (23) into the kinetic equation and linearizing it by the standard procedure, we obtain

$$\left.\begin{aligned} \frac{\partial f_{1e}}{\partial t} + \Omega_e \frac{\partial f_{1e}}{\partial \chi_e} + \Omega \frac{\partial f_{1e}}{\partial \theta} &= \frac{\partial f_{0e}}{\partial I_e} \frac{2e\sqrt{I_e}}{m\gamma\Omega_e} \sin \chi_e E_i' \\ \frac{\partial f_{1i}}{\partial t} + \Omega_i \frac{\partial f_{1i}}{\partial \chi_i} &= \frac{\partial f_{0i}}{\partial I_i} \frac{2e\sqrt{I_i}}{M\Omega_i} \sin \chi_i E_e'. \end{aligned}\right\} \tag{24}$$

These equations should be supplemented by the equation for the electric field:

$$\frac{\mathrm{d}E'_{i,e}}{\mathrm{d}y} = 4\pi\rho_{i,e}. \tag{25}$$

We seek a solution of Eqs. (24) and (25) under the following assumptions: (1) $f_1 = \exp(ik\theta)\varphi(I, \chi, t)$. (2) The system is situated near the resonance, so that one can retain a single harmonic in the expansion of $f_{1\,i,e}$ in periodic variables χ_1 and χ_e. (3) The frequencies Ω_e and Ω_i are independent of time. (4) The steady-state distribution function has the following form:†

$$f_0 = \begin{cases} \dfrac{N'_{i,e}}{2\pi I_0} & \text{for} \quad I_{i,e} \leqq I_0, \\ 0 & \text{for} \quad I_{i,e} > I_0, \end{cases} \tag{26}$$

where $N' = N/b$, b being the transverse dimension in the second degree of freedom.

Under these assumptions the solution of Eq. (24) has the form

$$\left.\begin{aligned} f_{1e} &= c_l\, \delta(I - I_0) \exp[i(k\theta - \omega t - l\chi_e)], \\ f_{1i} &= c_n\, \delta(I - I_0) \exp[i(k\theta - \omega t + n\chi_i)]. \end{aligned}\right\} \tag{27}$$

† A similar method was used in Ref. 5 in solving a somewhat different physical problem.

Substituting Eqs. (26) and (27) into Eq. (24) and integrating both sides of the equation with respect to I and χ, we obtain

$$\left.\begin{aligned} c_n &= \frac{N_i'}{(2\pi)^2} \int_0^{2\pi} \frac{2e \sin \chi_i E'_{el} \exp(-in\chi_i)\, \mathrm{d}\chi_i}{M\Omega_i(-i\omega + in\Omega_i)\sqrt{I_0}} \Bigg|_{I=I_0} \\ c_l &= \frac{N_e'}{(2\pi)^2} \int_0^{2\pi} \frac{2e \sin \chi_e E'_{in} \exp(il\chi_e)\, \mathrm{d}\chi_e}{m\gamma(-i\omega + ik\Omega - il\Omega_e)\Omega_e\sqrt{I_0}} \Bigg|_{I=I_0} \end{aligned}\right\} \tag{28}$$

Calculating $E_n'(y)$ in terms of the distribution function, we obtain the equation

$$E_n'(y) = \frac{8\pi e}{n} c_n \sin(n \cos^{-1}(y/\sqrt{I_0})]. \tag{29}$$

Thus, $E_n'(y)$ is an even function of y for an odd n and an odd function of y for an even n. On the other hand, oscillation of particles with an odd n can build up only fields that are even functions of y, and vice versa. It follows that for coaxial beams only 'even-even' and 'odd-odd' resonances are possible; when polarization is present there may also appear 'even-odd' resonances which are not considered here.

Substituting Eq. (29) into Eqs. (28) and integrating, we obtain the following dispersion equation:

$$(-\omega + k\Omega - l\Omega_e)(-\omega + n\Omega_i) = -\delta_{\|}^2 \gamma_{ln}^2, \tag{30}$$

where

$$\gamma_{ln} = \frac{\sqrt{|\beta_{nl}\beta_{ln}|}}{\beta_{\|}} = \frac{3}{8}\sqrt{|\beta_{nl} \cdot \beta_{ln}|}, \tag{31}$$

$$\beta_{nl} = \frac{2}{l}\left(\frac{1}{(l+n)^2 - 1} - \frac{1}{(l-n)^2 - 1}\right), \tag{32}$$

$$\delta_{\|}^2 = \left(\frac{16}{3\pi}\right)^2 \frac{N_e N_i}{a^2 b^2} \frac{e^4}{m\gamma M\Omega_i(I_0)\Omega_e(I_0)}. \tag{33}$$

The solution of Eq. (30) has the following form:

$$\omega_{1,2} = \frac{-(k\Omega - l\Omega_e + n\Omega_i) \pm \sqrt{(k\Omega - l\Omega_e - n\Omega_i)^2 - 4\delta_{\|}^2 \gamma_{ln}}}{2}. \tag{34}$$

The parameter $\gamma_{ln}\delta_{\|}$ is thus the ratio of the increment of the instability buildup to the increment of the dipole instability with the indices 1,1. Strictly speaking, this equation is suitable only when $b \gg a$. However, if it is used to calculate the increment of the dipole instability of a circular beam, the resulting increment will be only 20 per cent greater than that obtained from Eq. (21).

Analysis of Eq. (34), taking into account Eq. (32), shows that (1) the increments sharply decrease with increasing distance from the principal diagonal (when $l = n$), (2) the increments of the resonances on the principal diagonal decrease as n^{-1}, and (3) the width of these resonances decreases as n^{-2}.

TABLE II

l	n = 1	2	3	4	5
1	1	0	0,115	0	0,012
2	0	0,42	0	0,100	0
3	0,115	0	0,25	0	0,067
4	0	0,100	0	0,18	0
5	0,012	0	0,067	0	0,15

Table II gives the values of the parameters γ_{ln} for l and $n \leqq 5$.

7. CROSSING OF A RESONANCE OF ANY ORDER

We consider the case in which the system is near a resonance, but the average frequencies vary slowly with time. Using the same initial assumptions as in the preceding section, the solution of the kinetic equation will then have the following form:

$$\left.\begin{aligned} f_{1e} &= f_e(t)\,\delta(I-I_0)\exp(ik\theta)\exp(-il\chi_e) \\ f_{1i} &= f_i(t)\,\delta(I-I_0)\exp(ik\theta)\exp(in\chi_i). \end{aligned}\right\} \tag{35}$$

Substituting Eq. (35) into the kinetic equation and following the same procedure as before, we get the following system of equations:

$$\left.\begin{aligned} \frac{\mathrm{d}f_{1e}}{\mathrm{d}t} + i(k\Omega - l\Omega_e)f_{1e} &= \alpha_{ln}f_{1i} \\ \frac{\mathrm{d}f_{1i}}{\mathrm{d}t} + in\Omega_i f_{1i} &= \alpha_{nl}f_{1e}, \end{aligned}\right\} \tag{36}$$

where $\alpha_{nl} = \beta_{nl}\,\delta_{\|}$ is a constant.

We perform the substitution

$$\begin{aligned} f_{1e} &= c_1 \exp\left[-i\int(k\Omega - l\Omega_e)\,\mathrm{d}t\right], \\ f_{1i} &= c_2 \exp\left[-i\int n\Omega_i\,\mathrm{d}t\right]. \end{aligned} \tag{37}$$

Substituting Eq. (37) into Eq. (36), we obtain

$$\begin{aligned} \frac{\mathrm{d}c_1}{\mathrm{d}t} &= \alpha_{ln}\,c_2 \exp\left(i\int\delta\,\mathrm{d}t\right), \\ \frac{\mathrm{d}c_2}{\mathrm{d}t} &= \alpha_{nl}\,c_1 \exp\left(-i\int\delta t\right), \end{aligned} \tag{38}$$

where $\delta = k\Omega - l\Omega_e - n\Omega_i$.

Eliminating c_2 from Eq. (38) and substituting the values of α_{nl} and α_{ln}, we find that c_1 is defined by the equation

$$\frac{\mathrm{d}^2c_1}{\mathrm{d}t^2} - i\,\delta\frac{\mathrm{d}c_1}{\mathrm{d}t} - \delta_{\|}^2\gamma_{ln}^2\,c_1 = 0. \tag{39}$$

For crossing of a resonance, we can set

$$\delta = \Gamma t, \quad \Gamma = \frac{\mathrm{d}}{\mathrm{d}t}\left(l\Omega_e + n\Omega_i\right). \tag{40}$$

After changing the variables so that $z = i\Gamma t^2/2$, we obtain a hypergeometric equation:

$$z\frac{\mathrm{d}^2c_1}{\mathrm{d}z^2} + (\tfrac{1}{2} - z)\frac{\mathrm{d}c_1}{\mathrm{d}z} - ac_1 = 0, \tag{41}$$

$$a = \frac{\delta_{\|}^2\gamma_{ln}^2}{2i\Gamma}. \tag{42}$$

The solution of this equation is[6]

$$c_1 = AU_1(a, \tfrac{1}{2}, z) + BU_2(a, \tfrac{1}{2}, z) \tag{43}$$

where U_1 and U_2 are the Whittaker functions.

Assume that $c_2 = 0$ and $c_1 \neq 0$ at $t = -\infty$. Then U_1 and U_2 satisfy the following relations:

$$\left.\begin{aligned} U_1(i\infty) &\simeq z^{a-c}\mathrm{e}^z = z^a z^{-\frac{1}{2}}\mathrm{e}^z \to 0 \\ U_2(i\infty) &\simeq \mathrm{e}^{ia\pi}z^{-a} = \mathrm{e}^{ia\pi/2}\left|z\right|^{-i|a|}. \end{aligned}\right\} \tag{44}$$

Hence it can be seen that

$$A = 0, \quad B = \left|c_1(i\infty)\right|\mathrm{e}^{-ia\pi/2}\mathrm{e}^{i\psi}. \tag{45}$$

The function U_2 is analytic everywhere except along the line $\varphi = \pi$. As t varies from $-\infty$ to $+\infty$, the point $z = 0$, which is included in this cut, is encircled. The change of the amplitude can be determined by considering the change of the asymptotic value of U_2 for motion along a circle of infinite radius drawn about the point $z = 0$.

By considering the variation of c_1 as we cross the cut, we find

$$\left|\frac{c_1(t=\infty)}{c_1(t=-\infty)}\right| = \exp(2\pi i a) = \exp\left(\frac{\pi\gamma_{ln}^2\delta_\parallel^2}{\Gamma}\right) \quad (46)$$

The change of the second amplitude is determined by the integral of motion for the system (38):

$$\frac{|c_1|^2}{\alpha_{ln}} - \frac{|c_2|^2}{\alpha_{nl}} = \text{const.} \quad (47)$$

8. LANDAU DAMPING

As a result of the frequency spread of the beam arising from the longitudinal momentum spread, the dispersion Eq. (30) is modified and assumes the following form:

$$1 = -\delta_\parallel^2\gamma_{ln}^2 \int \frac{d\Omega_i f_i(\Omega_i)}{\omega + n\Omega_i} \int \frac{d\Omega_e f_e(\Omega_e)}{k\Omega - l\Omega_e + \omega}, \quad (48)$$

where the functions $f(\Omega_i)$ and $f(\Omega_e)$ are normalized to unity. We look for a solution in the form

$$\omega = \omega_0 - i\delta_{ln}. \quad (49)$$

Substituting Eq. (49) into Eq. (48) and taking into account the fact that $f(\Omega_i)$ and $f(\Omega_e)$ are even functions of $x_1 = \Omega_i - \Omega_{i0}$ and $x_2 = \Omega_e - \Omega_{e0}$, we obtain the following equations:

$$\omega_0 = -\frac{k\Omega - l\Omega_{e0} + n\Omega_{i0}}{2}; \quad (50)$$

$$F(\delta_{ln}) = \delta_\parallel^2\gamma_{ln}^2\left(\frac{\delta^2}{4} + \delta_{ln}^2\right)\int_{-\infty}^{\infty} \frac{f(x_1)\,dx_1}{(-\delta/2 + nx_1)^2 + \delta_{ln}^2} \times \int_{-\infty}^{\infty} \frac{f(x_2)\,dx_2}{(\delta/2 - l'x_2)^2 + \delta_{ln}^2} = 1, \quad (51)$$

where

$$\delta = k\Omega - l\Omega_{e0} - n\Omega_{i0}, \quad l' = k\frac{d\Omega}{d\Omega_e} + l.$$

We confine ourselves to the case of zero detuning ($\delta = 0$). The left-hand side of Eq. (51) is a maximum at $\delta_{ln} = 0$. When $\delta_{ln} \to 0$,

$$F(\delta_{ln}) \to \delta_\parallel^2\gamma_{ln}^2 \frac{\pi^2}{l'n} f_i(\Omega_{i0}) \cdot f_e(\Omega_{e0}). \quad (52)$$

If $F(\delta_{ln} \to 0) < 1$, there is no instability. Since for a Gaussian distribution $f_i(\Omega_{i0}) \sim 1/3\langle\Delta\Omega_{i0}\rangle$ and $f_e(\Omega_{e0}) \sim 1/3\langle\Delta\Omega_{e0}\rangle$, it follows from Eq. (52) that the following is a necessary and sufficient condition for the suppression of instability:

$$\langle\Delta\Omega_{i0}\rangle\langle\Delta\Omega_{e0}\rangle \geqq \frac{\gamma_{ln}^2\delta_\parallel^2}{l'n}. \quad (53)$$

For resonances with $l = n$, $\gamma_{ln} \sim n^{-2}$, and resonances suppressed by Landau damping are determined by the relation

$$n \geqq \left[\frac{\delta_\parallel^2}{\langle\Delta\Omega_i\rangle\langle\Delta\Omega_e\rangle}\right]^{\frac{1}{4}}. \quad (54)$$

Equation (53) implies another characteristic feature of the two-beam resonance: a necessary condition for suppression of instability is that the dispersion of both beams be nonzero. It can be shown that if $f_i(\Omega_i) = \delta(\Omega_i - \Omega_{i0})$, the electron dispersion does not give rise to a threshold, but only reduces the increment, and in that case (for $\langle\Delta\Omega_e\rangle \gg \gamma_{ln}\delta_\parallel$),

$$\delta_{ln} \sim \frac{\gamma_{ln}^2\delta_\parallel^2}{\langle\Delta\Omega_e\rangle}. \quad (55)$$

In the real electron-ion rings, the momentum spread of the ions is close to zero. Thus, to determine the instability threshold in this case it is necessary to solve Eq. (24) for the other unperturbed distributions in I, which would make it possible to take into account the natural frequency spread arising from the nonlinearity of the betatron oscillations; an analysis of this kind falls outside the scope of this paper.

Analyses of instabilities of other types (see, for example, Ref. 7) have shown that the Landau damping is indeed determined by the total frequency spread, irrespective of its nature.

Taking these results into account, we assume that the total frequency spread can be inserted into the equations for the threshold. If we assume that $\delta_\parallel = 0{,}1\,\Omega$ and $\langle\Delta\Omega_i\rangle \cong \langle\Delta\Omega_e\rangle = 0{,}01\,\Omega$, then all the resonances with $n \geqq 3{,}3$ will be suppressed by the Landau damping.

9. CONCLUSION

Let us examine the results obtained qualitatively and quantitatively. It seems that for stability of the rings, a working point outside the two-beam resonance bands must be chosen.

TABLE III

Points	N_i/N_e	$n_e(\text{cm}^{-3})$	a (mm)	E (MeV/m)	N_i
1	$2{,}4\times10^{-2}$	10^{12}	10	48	$2{,}4\times10^{12}$
2	$2{,}7\times10^{-4}$	2×10^{13}	2,2	200	$2{,}7\times10^{10}$

As can be seen from Figs. 2 to 5, the presence of external focusing ($\lambda \neq 0$) substantially increases the capability of the electron ring accelerator. However, a reasonable value cannot be obtained for one degree of freedom, i.e., for the ring axis. The only method capable of achieving this focusing —the image method—does not provide stability for the odd resonances, for which the center of mass of the cluster is displaced. Therefore, the dipole resonances must be considered for $\lambda = 0$. As shown above, at $\lambda = 0$ the first dipole resonance in the high-frequency region is unlimited. Therefore, in practice only two stability regions are feasible: (1) frequencies $Q_i + Q_e < \frac{1}{2}$, and (2) frequencies $1 > Q_i - Q_e > \frac{1}{2}$. The centers of these regions correspond to the points

(1) $Q_{iz} = 0{,}16;\ Q_{ez} = 0{,}16;$

(2) $Q_{iz} = 0{,}7;\ Q_{ez} = 0{,}07.$

For given values of γ, N_e, λ and R the pair of values Q_{iz} and Q_{ez} completely defines the basic parameters of the electron-ion ring: a, E, and N_i. In the calculations, we set $\gamma = 50$, $R = 5$ cm, $\lambda = 0$ and $N_e{}^* = 2\pi R N_e = 10^{14}$. The results are summarized in Table III.

The following equations were used in the calculation.

$$\frac{N_i}{N_e} = \frac{m\gamma}{M}\frac{Q_1{}^2}{Q_i{}^2}, \tag{56}$$

$$a = \frac{1}{Q_i}\sqrt{\frac{r_{pr} R N_e{}^*}{\pi}}, \tag{57}$$

$$E = d \cdot E_{\max} = \frac{1}{2}\frac{Mc^2}{e}\frac{Q_i{}^2 a}{R^2}, \tag{58}$$

where r_{pr} represents the classical proton radius. The question about an acceptable value of the factor d has not been considered. It is evident that the factor $d < 1$. We arbitrarily assume that $d = 0{,}5$.

It seems from Table III that when the external focusing is weak it is difficult to obtain an accelerator capable of competing with a proton synchrotron simultaneously in intensity of the accelerated beam and energy increase per unit length.

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