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DERIVATION AND SOLUTION OF THE SYSTEM OF EQUATIONS DESCRIBING INTENSE BEAMS, FOR NONSTATIONARY DISTRIBUTION OF TRANSVERSE PHASE-SPACE DENSITY

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A system of equations is derived for the dependence of the four-dimensional particle-density distribution function in transverse coordinates and velocities on the longitudinal coordinate of the beam. A numerical solution method is introduced and justified for these equations.

1. INTRODUCTION

Almost all known methods for calculation of intense beams with zero phase-space volume are based on a laminar model for the flux of charged particles. It is assumed that the flux is homocentric, the distribution of charge density over the cross section of the flux is uniform and the forces which act on the particles of the beam are linear in the transverse coordinates. Under these conditions the trajectories of all particles are similar to each other and it is sufficient to deal with one (boundary) particle trajectory for the calculation of the beam. The mathematics of this laminar model of the beam are not essentially different from the mathematics of one-particle approximation schemes.

To take the finite phase-space volume of the beam (i.e., the disordered particle-velocity scatter) into account, one needs a statistical approach which employs the concept of the distribution function of the particle density with respect to coordinates and velocities. Such an approach was first used by I. M. Kapchinskii and V. V. Vladimirskii.[1] In their work a continuous statistical model of the beam was considered and a system of equations solved for the self-consistent field for a four-dimensional distribution function, which depends on the transverse coordinates and velocities of the particles only. Under some simplifying assumptions (for the consistent beam in the smooth approximation) I. M. Kapchinskii[2] found stationary solutions of the above-mentioned system of equations. The latter were also considered by P. Lapostolle.[3] Naturally, the phase-space volume configuration of the beam for the stationary case remains unchanged. The case of a microcanonical distribution was considered in some papers[1,2,4]; this distribution is stationary with respect to six-dimensional distribution functions and nonstationary with respect to four-dimensional ones. The microcanonical phase-space density distribution corresponds to a uniform distribution of the charge density over the cross section of the beam and to linear Coulomb forces, which allows one to derive equations for the envelope of the flux of charged particles in this case. This solution is widely used for the calculation of intense beams with finite phase-space volume.

For real beams, the four-dimensional phase-space density distribution is neither stationary nor microcanonical. Some papers have been devoted to a numerical study of beams with nonstationary phase-space density distribution.[5–13] They showed that, for intense beams, effects connected with the nonlinearity of the Coulomb forces play a dominant role. The nonlinear effects connected with the nonuniform distribution of space charge in intense beams were experimentally and theoretically considered also in Ref. 14.

Discrete models of the flux of charged particles were used in some papers [5–10] for calculation of the dynamics of charged particles in intense beams, while elsewhere[11–13] a continuous statistical model of the beam was used.

In the present work a system of equations is introduced in the framework of the continuous statistical model. This system determines the variation along the beam direction of the four-dimensional density distribution function of the particles (as a function of the transverse coordinates and velocities) for the case where external fields are present and the longitudinal velocities of the particles are modulated. A method for numerical solution of this system of equations is given and justified. In particular, the dependence of the error introduced in evaluation of the transverse density distribution on the numerical-integration step is deduced. The relations obtained are general in nature and are valid also for calculation methods based on discrete models of the charged-particle flux.

2. DERIVATION OF A SYSTEM OF EQUATIONS FOR THE FOUR-DIMENSIONAL DISTRIBUTION FUNCTION

Let us assume once and for all that the transverse and longitudinal motions of the particles are mutually independent (this condition is satisfied for most concrete cases). It follows that both the longitudinal and the transverse phase-space volumes are conserved. However, this fact is insufficient for derivation of a system of equations for the four-dimensional distribution function, which depends only on the transverse coordinates and velocities. Indeed, the conservation of longitudinal phase-space volume implies only that the quantity $N=\iint F\,\mathrm{d}z\,\mathrm{d}w$ is conserved, where F is the six-dimensional distribution function, which depends on both longitudinal and transverse velocities and coordinates; z, w are the longitudinal coordinate and velocity, respectively. At this stage it is still impossible to say anything about the function $f=\int F\,\mathrm{d}w$, which characterizes the density distribution of the beam particles in the transverse phase space. In order to derive a system of equations defining the dependence of f on z, one must allow for the variation of the coordinate part of the longitudinal phase-space.

The quantities responsible for the variation of this part of phase space are: the external longitudinal fields, the longitudinal field of the space charge, the presence in the beam of modulations in the longitudinal velocities and the scatter in the longitudinal velocities of the particles. Obviously, for extended beams the main factors are the external fields and the modulations in the longitudinal velocities of the particles. Only these factors are considered here.

In this paper axially-symmetric beams are considered. It has been shown[13] that, for an axially-symmetric flux, the equation for the distribution function F of the beam particles with respect to coordinates and velocities may be written as follows:

$$\frac{\partial F}{\partial t}+w\frac{\partial F}{\partial z}+w\frac{\partial F}{\partial r}+\dot{w}\frac{\partial F}{\partial w}+\dot{v}\frac{\partial F}{\partial v}+\dot{p}\frac{\partial F}{\partial p}=\frac{\mathrm{d}F}{\mathrm{d}t}=0, \quad (1)$$

where r and θ are the radius and angle in a cylindrical coordinate system, v is the radial velocity and $p=r^2\dot{\theta}$.

If the scatter in the longitudinal velocities of the particles and their dependence on the transverse coordinates are neglected, the initial value of the function F may be written as:

$$F_0=F|_{t=0}=f_0(r_0,v_0,p_0,z_0)\,\delta(w_0-w_0{}^*(z_0)). \quad (2)$$

Along the particles' trajectories, which are the characteristic lines of Eq. (1), the following relation holds:

$$\begin{aligned}F(r,v,p,z,w,t) &= f_0(r_0(r,v,p,z,w,t),v_0(r,v,p,z,w,t),\\ &\qquad p_0(r,v,p,z,w,t),(z_0(z,w,t)\\ &\quad\times\delta(w_0(z,w,t)-w_0{}^*(z,w,t)). \qquad (3)\end{aligned}$$

Let us introduce the function

$$L(w)=w_0(z,w,t)-w_0{}^*(z_0(z,w,t)). \quad (4)$$

Considering z and t as parameters, we get from Eq. (4):

$$\mathrm{d}L=\left(\frac{\partial w_0}{\partial w}-\frac{\mathrm{d}w_0{}^*}{\mathrm{d}z_0}\frac{\partial z_0}{\partial w}\right)\mathrm{d}w. \quad (5)$$

In the general case, the functions $z_0(z,w,t)$ and $w_0(z,w,t)$ are given by the following relations:

$$\int_{z_0}^{z}\frac{\mathrm{d}\xi}{\sqrt{w_0{}^2(z,w,t)+(2e/m)\varphi(\xi)-(2e/m)\varphi(z_0)}}-t=0, \quad (6)$$

$$w^2-w_0{}^2-\frac{2e}{m}\varphi(z)+\frac{2e}{m}\varphi(z_0(z,w,t))=0, \quad (7)$$

where φ is the potential of the electric field.

Equation (6) determines the function $z_0(z,w,t)$, Eq. (7) the function $w_0(z,w,t)$. From (6) and (7) it is possible to find the derivatives $\partial w_0/\partial w$ and $\partial z_0/\partial w$ which appear in Eq. (5). Omitting the intermediate steps, we get the final expression for $\mathrm{d}L$:

$$\mathrm{d}L = \psi^*(w, w_0, z, z_0)\,\mathrm{d}w, \tag{8}$$

where:

$$\psi^* = \frac{w}{w_0} + w\left(w_0 \frac{\mathrm{d}w_0^*}{\mathrm{d}z_0} - \frac{e}{m}\frac{\mathrm{d}\varphi_0}{\mathrm{d}z_0}\right)\int_{z_0}^{z}\frac{\mathrm{d}z}{w^3}. \tag{9}$$

Using Eqs. (8) and (9), one can integrate Eq. (3) with respect to w, giving rise to the following expression for the density function f of particles in the four-dimensional phase space of transverse coordinates and velocities (in the axially-symmetric case f actually depends only on the three variables r, v and p):

$$f(r, v, p, z) = \psi(z) f_0(r_0(r, v, p, z), v_0(r, v, p, z), p_0(r, v, p, z)) \tag{10}$$

where:

$$\psi(z) = \frac{w_0^*(z_0(z))/w^*(z)}{1 + \frac{1}{2} w_0^* \frac{\mathrm{d}}{\mathrm{d}z_0}\left(w_0^{*2} - \frac{2e}{m}\varphi_0\right)\left(\int_{z_0}^{z}\frac{\mathrm{d}z}{w^{*3}(z)}\right)}. \tag{11}$$

In Eq. (10) the independent variable is the longitudinal coordinate z, since for the case considered the relation between z and t is the same for all particles.

Thus we see that in the general case the density of particles in transverse phase-space is not conserved. As shown at the beginning of this section, this is connected with the variation of the coordinate part of the longitudinal phase-space volume Δz. It is the term $\psi(z)$ that takes this variation into account. The numerator of $\psi(z)$ (Eq. (11)) allows for the variation Δz caused by the variation of the longitudinal velocity of the particles under the action of external fields. The denominator of $\psi(z)$ allows for the variation Δz caused by the presence of modulations in the longitudinal velocities.

Equation (10) essentially represents the solution of the one-dimensional (with respect to the coordinate z) continuity equation. This means that the longitudinal motion of the particles in the beam model considered is regarded as the motion of a continuous medium and appears in the equations of transverse motion via the parameter $\psi(z)$. The transverse motion of particles in this model is described statistically by means of the density distribution $f(r, v, p, z)$ of particles in the phase space of transverse coordinates and velocities. The dependence of the distribution function f on the longitudinal coordinate z is given by the equation:

$$\frac{\partial g}{\partial z} + \frac{\mathrm{d}r}{\mathrm{d}z}\frac{\partial g}{\partial r} + \frac{\mathrm{d}v}{\mathrm{d}z}\frac{\partial g}{\partial v} + \frac{\mathrm{d}p}{\mathrm{d}z}\frac{\partial g}{\partial p} = \frac{\mathrm{d}g}{\mathrm{d}z} = 0, \tag{12}$$

where:

$$g = \frac{f(r, v, p, z)}{\psi(z)}. \tag{13}$$

Equation (12) is easily shown to be equivalent to Eq. (10). The latter holds along particle trajectories in the phase space of transverse coordinates and velocities. The equations for trajectories are, in this case:

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \sqrt{\frac{m}{2e}}\frac{v}{\varphi^{1/2}}, \tag{14}$$

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{1}{4\pi\varepsilon_0}\frac{I(r, z)}{r\varphi} + \sqrt{\frac{m}{2e}}\frac{p^2}{r^3\varphi^{1/2}} + \sqrt{\frac{e}{2m}}\frac{1}{\varphi^{1/2}}\left(\frac{\partial\varphi}{\partial r} - \frac{p}{r^2}\frac{\partial}{\partial r}(rA)\right), \tag{15}$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{e}{m}\frac{\mathrm{d}}{\mathrm{d}z}(rA), \tag{16}$$

where:

$$I(r, z) = 2\pi e w^*(z)\psi(z)\int_0^r \mathrm{d}r \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\,\mathrm{d}p\,\mathrm{d}v, \tag{17}$$

A is the azimuthal component of the vector potential of the magnetic field, ε_0 is the dielectric constant of vacuum, e and m are the charge and mass, respectively, of a particle.

Equations (12) to (17) form a complete system of equations for determination of the dependence of the particle density in the transverse phase space of the beam on the longitudinal coordinate.

Let us comment on the conditions under which the system of Eqs. (12) to (17) is valid. The only simplifying assumption made in the derivation of Eqs. (12) to (17) was that in any cross section of the beam the velocities of all particles are equal (usually this condition is fairly well satisfied). Nevertheless, both external fields and the field of the space charge may acquire nonlinearity; the system of Eqs. (12) to (17) remains valid for such cases.

3. SOLUTION OF THE EQUATIONS FOR THE FOUR-DIMENSIONAL DISTRIBUTION FUNCTION

In principle, it is possible to determine the distribution of the transverse phase-space density of particles in any cross section of the beam from Eqs. (12) to (17), provided the distribution f is known in some (initial) cross section. Knowledge of the transverse phase-space density distribution enables one to find all basic characteristics of the flux of charged particles: the distribution of charge density over the cross section of the beam, the magnitude and configuration of the transverse phase-space volume, etc.

A major difficulty in solving the system of Eqs. (12) to (17) is that the unknown function $I(r,z)$, which depends on the required function f (see Eq. (17)), appears in the equation for the radial motion (15). It is possible to solve Eqs. (12) to (17) numerically by replacing the unknown function $I(r,z)$ at each step Δz of the numerical integration by some known function (this will be discussed in more detail below). For a known function $I(r,z)$ the equations of the trajectories, Eqs. (14) to (16), are solvable by known numerical methods. Knowledge of the particle trajectories in phase space, combined with Eqs. (12) and (13), enables one to find the distribution f for the beam cross section considered. We emphasize that when using this solution method for the system (12) to (17) one need not follow trajectories of individual particles during the entire process of calculation. The trajectories of particles at any step are used only for evaluation of the distribution f, which changes from one cross section of the beam to another.

In any numerical method, it is important to estimate the error in the required functions as a function of the integration step. For a known free term, the dependence of the error in the solution of the trajectory equations on the integration step is known for various numerical methods. Nevertheless, it is not obvious that small errors in the coordinates and velocities of the particles will not give rise to a large error in the function f. The point is that, in general, the error in f—the particle density in phase space—is determined by the error in the differences between the coordinates and velocities of particles at neighboring points of phase space.

Let us consider this problem in more detail. Equation (15) may be written in the following form:

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \Phi(r,z). \tag{18}$$

Almost any numerical method for the solution of ordinary differential equations is essentially equivalent to integration of part of the Taylor series of the free term.[15] If the right-hand side of Eq. (18) is replaced by its expansion in Taylor series, then Eq. (18) becomes:

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \Phi(r(r_0, v_0\, z), z), \tag{19}$$

where:

$$\Phi = \Phi_0(r_0) + \Phi_0'(r_0, v_0)\Delta z + \tfrac{1}{2}\Phi_0''(r_0, v_0)(\Delta z)^2 + \ldots, \tag{20}$$

$$\Phi_0' = \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} + \left.\frac{\partial\Phi}{\partial r}\right|_{z=0}\frac{v_0}{w_0{}^*}, \tag{21}$$

$$\Phi_0'' = \left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} + 2\left.\frac{\partial^2\Phi}{\partial z\,\partial r}\right|_{z=0}\frac{v_0}{w_0{}^*} + \left.\frac{\partial^2\Phi}{\partial r^2}\right|_{z=0}\frac{v_0{}^2}{w_0^{*2}} + \left.\frac{\partial\Phi}{\partial r}\right|_{z=0}\Phi_0 - \left.\frac{\partial\Phi}{\partial r}\right|_{z=0}\frac{v_0}{w_0^{*2}}\,w_0^{*\prime}, \tag{22}$$

It follows from Eq. (19) that, when using numerical methods to solve the equations of motion, one is strictly speaking looking at the motion of particles in a field of forces which depend on the initial values of the coordinates and velocities, or, what amounts to the same, in a field of forces which depend not only on the moving coordinates but also on the moving velocities of the particles (since $r_0 = r_0(r,v,z)$, $v_0 = v_0(r,v,z)$).

For this case, instead of Eq. (12) one must use an equation of the following form (see (11)):

$$\frac{\mathrm{d}g}{\mathrm{d}z} = -g\frac{\partial v'}{\partial v}. \tag{23}$$

For $\partial v'/\partial v$, we can write:

$$\frac{\partial v'}{\partial v} = \frac{\partial v'}{\partial r_0}\frac{\partial r_0}{\partial v} + \frac{\partial v'}{\partial v_0}\frac{\partial v_0}{\partial v}. \tag{24}$$

From (18) and (14) we obtain:

$$v = v_0+\int_0^z \Phi(r_0, v_0, z)\,\mathrm{d}z, \tag{25}$$

$$r = r_0+\int_0^z H(r_0, v_0, z)\,\mathrm{d}z, \tag{26}$$

$$H = \frac{v}{w^*}. \tag{27}$$

Using Eqs. (25) and (26) we get the following derivatives:

$$\frac{\partial r_0}{\partial v} = \frac{-(\partial r/\partial v_0)}{(\partial v/\partial v_0)(\partial r/\partial r_0)-(\partial v/\partial r_0)(\partial r/\partial v_0)}, \tag{28}$$

$$\frac{\partial v_0}{\partial v} = \frac{(\partial r/\partial r_0)}{(\partial v/\partial v_0)(\partial r/\partial r_0)-(\partial v/\partial r_0)(\partial r/\partial v_0)}. \tag{29}$$

Substituting (28) and (29) in Eq. (24), we find:

$$\frac{\partial v'}{\partial v} = \frac{-(\partial v'/\partial r_0)(\partial r/\partial v_0)+(\partial v'/\partial v_0)(\partial r/\partial r_0)}{(\partial v/\partial v_0)(\partial r/\partial r_0)-(\partial v/\partial r_0)(\partial r/\partial v_0)}. \tag{30}$$

Since r_0 and v_0 are constant along the trajectories of the particles, we can write down the following equalities:

$$\frac{\partial v'}{\partial v_0} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial \theta}{\partial v_0}\right), \tag{31}$$

$$\frac{\partial v'}{\partial r_0} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial v}{\partial r_0}\right), \tag{32}$$

$$\frac{\partial v}{\partial v_0} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial r}{\partial v_0}\right), \tag{33}$$

$$\frac{\partial v}{\partial r_0} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial r}{\partial r_0}\right). \tag{34}$$

Using Eqs. (31) to (34), we can rewrite Eq. (30) as:

$$\frac{\partial v'}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\partial v}{\partial v_0}\frac{\partial r}{\partial r_0}-\frac{\partial v}{\partial r_0}\frac{\partial r}{\partial v_0}\right). \tag{35}$$

Substituting Eq. (35) in Eq. (23) and integrating the latter, we get:

$$g(r, v, p, z) = \frac{g_0(r_0(r, v, p, z), v_0(r, v, p, z), p_0(r, v, p, z))}{(\partial v/\partial v_0)(\partial r/\partial r_0)-(\partial v/\partial r_0)(\partial r/\partial v_0)}. \tag{36}$$

Substituting the expansion (20) in Eq. (25) and the analogous expansion for H in Eq. (26), we obtain the following expression for v and r:

$$v = v_0+\Phi_0\,\Delta z+\tfrac{1}{2}\Phi_0'(\Delta z)^2+\tfrac{1}{6}\Phi_0''(\Delta z)^3+\ldots, \tag{37}$$

$$r = r_0+H_0\,\Delta z+\tfrac{1}{2}H_0'(\Delta z)^2+\tfrac{1}{6}H_0''(\Delta z)^3+\ldots. \tag{38}$$

Using Eqs. (37), (38) and the relation

$$H' = \frac{\Phi}{w^*}-\frac{v}{w^{*2}}w^{*\prime}$$

which follows from Eq. (27), we have:

$$\frac{\partial v}{\partial v_0}\frac{\partial r}{\partial r_0}-\frac{\partial v}{\partial r_0}\frac{\partial r}{\partial v_0}$$

$$= 1+\frac{1}{2}\left(\frac{\partial \Phi_0'}{\partial v_0}-\frac{1}{w_0{}^*}\frac{\partial \Phi_0}{\partial r_0}\right)(\Delta z)^2$$

$$+\frac{1}{6}\left(\frac{\partial \Phi_0''}{\partial v_0}-2\frac{1}{w_0{}^*}\frac{\partial \Phi_0'}{\partial r_0}+\frac{w_0^{*\prime}}{w_0^{*2}}\frac{\partial \Phi_0}{\partial r_0}\right)(\Delta z)^3+\ldots. \tag{39}$$

If the infinite series (20) is substituted in Eq. (39), then it is easily shown, using Eqs. (21) and (22), that all coefficients of $(\Delta z)^n$ in the right-hand side of Eq. (39) vanish and hence the expression

$$\frac{\partial v}{\partial v_0}\frac{\partial r}{\partial r_0}-\frac{\partial v}{\partial r_0}\frac{\partial r}{\partial v_0}$$

equals unity. If, on the other hand, only finitely many terms of the series (20) are substituted in Eq. (39), then not all coefficients of $(\Delta z)^n$ on the right-hand side of Eq. (39) vanish and the expression

$$\frac{\partial v}{\partial v_0}\frac{\partial r}{\partial r_0}-\frac{\partial v}{\partial r_0}\frac{\partial r}{\partial v_0}$$

is not equal to unity. This means that, in approximate computation of the particles' coordinates and velocities, the function g varies along the trajectories of the particles in phase space.

Let $r_0{}^*(r, v, p, z)$ and $v_0{}^*(r, v, p, z)$ be the true initial values of the coordinates and velocities, respectively, of the particles, r_0 and v_0—the initial values for the parameters of the trajectory obtained by numerical solution of the equations of motion.

Introducing the notation $\Delta r_0 = r_0 - r_0{}^*$, $\Delta v_0 = v_0 - v_0{}^*$, we write Eq. (36) in the following form:

$$g(r, v, p, z) = g_0{}^*(r_0{}^*, v_0{}^*, p_0{}^*) + \frac{\partial g_0{}^*}{\partial r_0{}^*}\Delta r + \frac{\partial g_0{}^*}{\partial v_0{}^*}\Delta v - \Delta g_1, \tag{40}$$

where:

$$\Delta g_1 = \tfrac{1}{2} g_0{}^*\left(\frac{\partial \Phi_0{}'}{\partial v_0} - \frac{1}{w_0{}^*}\frac{\partial \Phi_0}{\partial r_0}\right)\Delta z^2 + \tfrac{1}{6} g_0{}^*\left(\frac{\partial \Phi_0{}''}{\partial v_0} - 2\frac{1}{w_0{}^*}\frac{\partial \Phi_0{}'}{\partial r_0} + \frac{w_0^{*\prime}}{w_0^{*2}}\frac{\partial \Phi_0}{\partial r_0}\right)(\Delta z)^3 + \ldots \tag{41}$$

$g_0{}^*$ is the true value of g. The second and third terms on the right-hand side of Eq. (40) give the error in g due to taking $g_0(r_0, v_0, p_0)$ instead of $g_0{}^*(r_0{}^*, v_0{}^*, p_0{}^*)$ in the approximate computation of r_0 and v_0. The last term Δg_1 on the right-hand side of Eq. (40) gives the error in g due to the approximate calculation of the particles' coordinates and velocities, in which the distance between these particles in phase space (i.e., the particle density in phase space) varies.

If the determination of the particle trajectories involves only one term of the series (20), so that $\Phi = \Phi_0$, then, as follows from (41), no coefficient of $(\Delta z)^n$ on the right-hand side of Eq. (41) vanishes and the error Δg_1 in this case is proportional to $(\Delta z)^2$. If the determination of the particle trajectories involves the relation $\Phi = \Phi_0 + \Phi_0{}' \cdot \Delta z$, then it follows from Eqs. (21) and (22) that the coefficient of $(\Delta z)^2$ in Eq. (41) vanishes and the error Δg_1 is proportional in this case to $(\Delta z)^3$.

Finally, we give expressions for the full error Δg, corresponding to the use of first- and second-order terms of the expansion (20) in evaluation of the trajectories:

$$\Delta g = \begin{cases} \dfrac{1}{2}\left(\Phi_0{}'\dfrac{\partial g_0{}^*}{\partial v_0{}^*} + \dfrac{g_0{}^*}{w_0{}^*}\dfrac{\partial \Phi_0}{\partial r_0}\right)(\Delta z)^2 \text{ for } \Phi = \Phi_0, & (42) \\ \dfrac{1}{6}\left(\Phi_0{}''\dfrac{\partial g_0{}^*}{\partial v_0{}^*} + 2\dfrac{g_0{}^*}{w_0{}^*}\dfrac{\partial \Phi_0{}'}{\partial r_0} - g_0{}^*\dfrac{w_0^{*\prime}}{w_0^{*2}}\dfrac{\partial \Phi_0}{\partial r_0}\right) \\ \cdot(\Delta z)^3 \text{ for } \Phi = \Phi_0 + \Phi_0{}' \cdot \Delta z & (43) \end{cases}$$

Note that Eqs. (42) to (43) are valid for the calculation method described in this paper, as well for other methods based on discrete models of the flux of charged particles. The expansion in Taylor series of the unknown function $I(r, z)$ is shown in the appendix. The use of this expansion allows one to reduce the integro-differential system of Eqs. (12) to (17), to that with the known function $I(r, z)$, i.e. to the case considered in this section.

APPENDIX 1

Taylor formula for the function $I(r(r_0, v_0, z), z)$

In order to construct and investigate the Taylor series of the function $I(r(z), z)$ we make use of the equations for the moments of the function g. The nth moment is defined as:

$$W_n = 2\pi e w^*(z)\psi(z)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v^n g \, \mathrm{d}p \, \mathrm{d}v. \tag{44}$$

In particular:

$$W_0 = 2\pi\rho(r, z) w^*(z) r, \tag{45}$$

$$W_1 = 2\pi\rho(r, z) w^*(z) r\bar{v}(r, z), \tag{46}$$

where ρ is the charge density, $\bar{v}$ the mean radial velocity of particles for given values of r and z.

Multiplying Eq. (12) by v^n and integrating over v and p from $-\infty$ to $+\infty$, we get the following infinite system of equations:

$$\left.\begin{aligned} &\frac{\partial W_0}{\partial z} + \frac{1}{w^*}\frac{\partial W_1}{\partial r} = 0, \\ &\frac{\partial W_1}{\partial z} + \frac{1}{w^*}\frac{\partial W_2}{\partial r} = -W_1{}^*, \\ &\frac{\partial W_2}{\partial z} + \frac{1}{w^*}\frac{\partial W_3}{\partial r} = -W_2{}^* \\ &\quad \cdots\cdots\cdots \\ &\frac{\partial W_n}{\partial z} + \frac{1}{w^*}\frac{\partial W_{n+1}}{\partial r} = -W_n{}^*, \end{aligned}\right\} \tag{47}$$

where:

$$W_n{}^* = 2\pi e w^*\psi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v^n \frac{\mathrm{d}v}{\mathrm{d}z}\frac{\partial g}{\partial v}\mathrm{d}p \, \mathrm{d}v. \tag{48}$$

Since $\mathrm{d}v/\mathrm{d}z$ depends on the particle momentum p (see Eq. (15)), it is impossible to take $\mathrm{d}v/\mathrm{d}z$ out of the integration sign in Eq. (48).

Let us write the system (47) as follows:

$$\left.\begin{aligned}
\frac{dW_0}{dz} &= -\frac{1}{w^*}\frac{\partial W_1}{\partial r}+\frac{v}{w^*}\frac{\partial W_0}{\partial r},\\
\frac{dW_1}{dz} &= -\frac{1}{w^*}\frac{\partial W_2}{\partial r}+\frac{v}{w^*}\frac{\partial W_1}{\partial r}-W_1^*,\\
&\ldots\ldots\ldots\ldots\\
\frac{dW_n}{dz} &= -\frac{1}{w^*}\frac{\partial W_{n+1}}{\partial r}+\frac{v}{w^*}\frac{\partial W_n}{\partial r}-W_n^*,\\
&\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{49}$$

Integrating the first equation of the system (49) over r, we get:

$$\frac{dI}{dz}=\frac{vW_0-W_1}{w^*}. \tag{50}$$

Using the relations (45) and (46), we obtain:

$$\frac{dI}{dz}=2\pi\rho w^* r\frac{v-\bar{v}}{w^*}. \tag{51}$$

The physical meaning of the expression for dI/dz is clear from Eq. (51).

For d^2I/dz^2 we get from Eq. (50):

$$\frac{d^2I}{dz^2}=\frac{1}{w^*}\left(\frac{dv}{dz}W_0+v\frac{dW_0}{dz}-\frac{dW_1}{dz}\right)-\frac{w^{*\prime}}{w^{*2}}(vW_0-W_1). \tag{52}$$

Using Eq. (49), we find:

$$\frac{d^2I}{dz^2}=\frac{1}{w^{*2}}\left(v^2\frac{\partial W_0}{\partial r}-2v\frac{\partial W_1}{\partial r}+\frac{\partial W_2}{\partial r}\right)$$
$$-\frac{w^{*\prime}}{w^{*2}}(vW_0-W_1)+\frac{1}{w^*}\left(W_0\frac{dv}{dz}+W_1^*\right). \tag{53}$$

As follows from Eq. (15), the derivative dv/dz may be written as:

$$\frac{dv}{dz}=f_1(r,z)+f_2(r,z)\cdot p^2, \tag{54}$$

where $f_2(r,z)$ is a function depending only on the magnitude of the potential of the external electric field but not on the derivatives of the potential. This means that $f_2(r,z)$ does not depend on the forces which act on the beam particles.

Using Eq. (54), after some simple manipulations we get the final result for d^2I/dz^2:

$$\frac{d^2I}{dz^2}=\frac{1}{w^{*2}}\left(v^2\frac{\partial W_0}{\partial r}-2v\frac{\partial W_1}{\partial r}+\frac{\partial W_2}{\partial r}\right)$$
$$-\frac{w^{*\prime}}{w^{*2}}(vW_0-W_2)$$
$$+2\pi e\psi f_2\left(p^2\iint g\,dv\,dp-\iint p^2 g\,dv\,dp\right). \tag{55}$$

LIST OF SYMBOLS

- t time
- z longitudinal coordinate
- r radial coordinate
- θ angular coordinate
- w longitudinal velocity
- v radial velocity
- p quantity proportional to angular momentum
- F distribution function
- f distribution function
- g function of coordinates and momentum
- ψ function of z
- ξ integration variable
- φ electric potential
- I current
- A azimuthal component of the magnetic field vector potential
- ρ charge density
- e charge of particle
- m mass of particle
- ε_0 constant of particle
- Φ function symbol
- H function symbol
- W moment of function
- L symbol of function
- N number of particles

REFERENCES

1. I. M. Kapchinskii and V. V. Vladimirskii, *Proc. Intern. Conf. High-Energy Accelerators and Instrumentation, CERN, Geneva, 1959*, p. 274.
2. I. M. Kapchinskii, *Particle dynamics in linear resonating accelerators*, Atomizdat, 1966.
3. P. M. Lapostolle, "Density distribution of intense beams," *Proc. 7th Intern. Conf. High-Energy Accelerators, Erevan, 1969*.

4. A. I. Walsh, *Nucl. Energy* **5,** No. 1, 17 (1963).
5. S. P. Lomnev, *DAN SSSR* **135,** No. 4, 822 (1960).
6. V. A. Enal'skii and V. V. Osipov, *PMTF* **5,** 29 (1967).
7. K. R. Crandall, "A numerical experiment on space-charge effects," *Proc. 1966 Linear Accelerator Conference, Los Alamos*, p. 233.
8. R. Chasman, "Numerical calculations of the effects of space charge on six-dimensional beam dynamics in proton linear accelerators," *Proc. 1968 Proton Linear Accelerator Conference, Brookhaven National Laboratory*, BNL 50120 (C–54), p. 372.
9. M. Martini and M. Promé, "Beam dynamics in the proton linac with consideration of space charge," *Proc. 7th Intern. Conf. High-Energy Accelerators, Everan, 1969.*
10. C. Taylor, A. Davis, and P. Tanguy, "Nonlinear space-charge effects in beam dynamics," *Proc. 7th Intern. Conf. High-Energy Accelerators, Everan, 1969.*
11. V. S. Kuznetsov, *ZhTF* **37,** No. 5, 932 (1967).
12. V. S. Kuznetsov, N. P. Kuznetsova, R. P. Fidel'skaya, A. I. Solnyshkov, and M. A. Abroyan, "Investigation of the redistribution of phase-space density for small beams of charged particles, using numerical methods," *Proc. All-Union Conference on Charged-Particle Accelerators, Oct. 1968, Moscow.*
13. V. S. Kuznetsov, *ZhTF* **38,** No. 2, 274 (1968).
14. P. M. Lapostolle, C. S. Taylor, P. Têtu, and L. Thorndahl, "Intensity dependent effects and space charge limit investigations on CERN linear injector and synchrotron," CERN Report ISR–300/68–35, Oct. 12, 1968.
15. G. N. Lance, *Numerical Methods for High-Speed Computers*, Iliffe & Sons, Ltd., London, 1960.

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