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COMPUTATION OF HIGH-FIELD MAGNETS

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This paper describes a magnet computer program used in the study of high-field superconducting magnets. Some of the techniques developed in this program, and the application of the program to some particular magnets are presented. The topics discussed include the formulation of the problem, the iteration procedure, the current-density calculation, convergence problems, accuracy problems, derivation of the iteration coefficients, and procedures for terminating the mesh.

1. INTRODUCTION

This paper describes a magnet computer program used in the design of high-field superconducting magnets. The program used is GRACY. Some of the techniques developed in the program and its application to some particular magnets will be presented.

An example of a magnet studied using the GRACY program is shown in Fig. 1. This magnet is a high-field dipole with superconducting current windings and surrounded by a circular iron shield. The magnet is excited up to 60 kG. The current is distributed in a cosine dependence, approximated by arranging the current in rectangular blocks. Within each block the current density is uniform and proportional to the cosine of the azimuthal angle. Four current blocks in a quadrant is a possible reasonable choice.

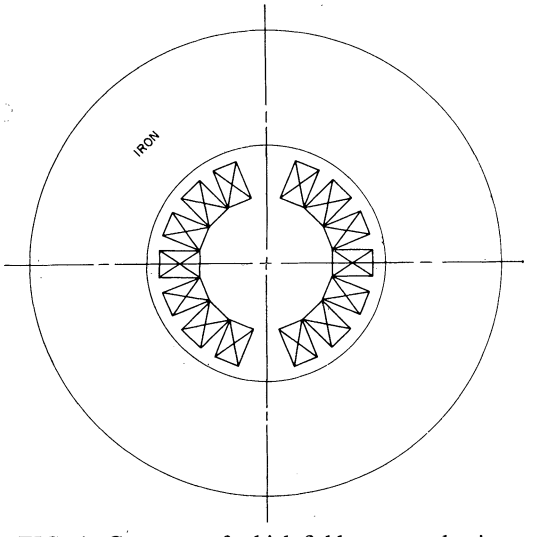

FIG. 1. Geometry of a high-field superconducting magnetic dipole with a cosine current distribution and a circular iron shield.

This kind of magnet differs from conventional low-field magnets in several ways. The iron is severely saturated and considerable field may leak out past the iron. In addition, the field shape on the median plane is primarily determined by the current distribution rather than by the iron shape. If the iron had infinite permeability, and if the current distribution were an exact cosine, then the field inside the magnet would be perfectly uniform. Deviations from a uniform field appear because of the saturation of the iron and because the current is not exactly a cosine distribution. These deviations can be computed using the GRACY program.

The severe tolerances on the allowable field gradient in such accelerator magnets would require the computer program to achieve a relative accuracy of about 2×10^{-4} in the field. This kind of accuracy is difficult to achieve in a mesh-iteration program. It was, however, found possible to develop special techniques to obtain the required accuracy for the dipole problem.

A number of magnet programs have been written. They include LINDA,[1] TRIM,[2] POISSON,[3] NUTCRACKER,[4] MARE,[5] and GRACY.[6] These programs differ in their approach. The different approaches have advantages and disadvantages, and no one approach seems obviously superior to the others. Some consideration of the different approaches is given in this paper.

2. FORMULATION OF THE PROBLEM

The magnetic field can be found from the vector potential $\mathbf{A}(x, y)$ which in air obeys the equation

$$\nabla^2 \mathbf{A} = -4\pi \mathbf{S}, \tag{2.1}$$

where $\mathbf{S}(x, y)$ is the current density. The magnet is assumed to be two-dimensional, which means all field variables are independent of z. The quantity A is in fact the z component of the vector $\mathbf{A}$. The

magnetic flux density **B** can be found from **A** according to **B** = curl **A** or

$$B_x = \frac{\partial A}{\partial y}, \tag{2.2a}$$

$$B_y = -\frac{\partial A}{\partial x}. \tag{2.2b}$$

B and **H** are related by γ, the inverse of the permeability, according to

$$\mathbf{H} = \gamma \mathbf{B}. \tag{2.3}$$

Here γ is assumed to be a function of the magnitude of **B**, and the dependence of γ on **B** is entered in the program by means of an experimentally measured table of γ vs **B**.

In iron, A obeys the equation

$$\frac{\partial}{\partial x}\left(\gamma \frac{\partial A}{\partial x}\right) + \frac{\partial}{\partial y}\left(\gamma \frac{\partial A}{\partial y}\right) = 0, \tag{2.4}$$

where it is assumed that no current flows in the iron.

In addition to the differential Eqs. (2.1) and (2.4), A must satisfy certain boundary conditions. At the iron-air interface, one has

$$\left(\frac{\partial A}{\partial n}\right)_{\text{air}} = \gamma \left(\frac{\partial A}{\partial n}\right)_{\text{iron}}, \tag{2.5}$$

where $\partial A/\partial n$ is the derivative of A normal to the iron surface. Continuity of A across the boundary guarantees that the boundary condition on the normal component of B is satisfied. The question of replacing the differential equations and the boundary conditions by difference equations is treated in Secs. 3 and 7.

It is also possible to describe the magnetic field by a scalar potential $V(x, y)$. One introduces a magnetic-moment vector $\mathbf{M}(x, y)$, which is related to the current-density vector by

$$\mathbf{S} = \text{curl}\, \mathbf{M}. \tag{2.6}$$

Then from curl **H** = 4π**S**, it follows that

$$\text{curl}\,(\mathbf{H} - 4\pi \mathbf{M}) = 0, \tag{2.7}$$

and one can obtain H from a scalar potential $V(x, y)$, according to

$$\mathbf{H} = \text{grad}\, V + 4\pi \mathbf{M}. \tag{2.8}$$

Using div **H** = 0, one finds the differential equation for V in air

$$\nabla^2 V = -4\pi\, \text{div}\, \mathbf{M}. \tag{2.9}$$

The magnetic-moment vector **M** is not uniquely defined. One possible result for **M** is to assume that it has only a y-component, given by

$$M_y = \int_{-\infty}^{x} S(x, y)\, dx, \tag{2.10}$$

where the integral is taken along a path with y held constant.

For the case of infinite permeability, one can find the boundary conditions for V on the iron-air interface from Eq. (2.8). Since in this case **H** is perpendicular to the iron, one finds that V at any point P is given by

$$V(P) - V(P_0) = -4\pi \int_{P_0}^{P} M_l\, dl, \tag{2.11}$$

where the integral is along the iron surface.

In formulating the magnet program, one has to choose between the two-potential method particularly used by R. S. Christian or the vector-potential method which uses the vector potential throughout. In the two-potential method, the scalar potential is used in the air region and the vector potential is used in the iron region. The problem is first solved assuming the iron has infinite permeability. This air-region solution is used to find the vector potential on the iron boundary, which serves as the boundary condition to find the vector potential in the iron. The solution in the iron region is then used to find the scalar potential on the air boundary and the cycle is repeated until the procedure converges to the answer.

The advantages of the two-potential method stem from the superiority of the scalar potential for the infinite-permeability problem. In the case of infinite permeability, the scalar-potential method has a faster convergence rate and a greater accuracy than the vector-potential method. If the saturation of the iron can be considered a small perturbation on the infinite permeability solution, then one can expect the two-potential method to do particularly well. The disadvantage of the two-potential method is the additional complexity in the programming, particularly in the conversion, back and forth, between the scalar and vector potentials. On the other hand, the vector-potential method has the advantages of simplicity and flexibility in treating different magnet geometries.

In the GRACY program, the vector-potential method was used. The primary reason for this choice was that it was feared that the extreme iron saturation occurring in high-field superconducting magnets would lead to convergence difficulties for the two-potential method. In practice, it has also

been found that the disadvantages of the vector-potential method can usually be overcome by special techniques for a particular problem.

3. ITERATION PROCEDURE

The GRACY mesh-iteration program replaces the differential equation for the vector potential by a set of difference equations for each point on an orthogonal mesh. For each interior mesh point, one obtains a difference equation of the form

$$\sum_{i=0}^{4} \alpha_i A_i = -4\pi S_0, \tag{3.1}$$

where the four neighbors of each interior point are indicated by A_1 to A_4; the interior point being considered is indicated by A_0 and has the current density S_0. The derivation of the α_i is given in Sec. 7.

The iteration relation used by GRACY at each interior mesh point is given by

$$A_0^{(n+1)} = A_0^{(n)} - \alpha R^{(n)}, \tag{3.2a}$$

$$R^{(n)} = (\alpha_1 A_1^{(n)} + \alpha_2 A_2^{(n)} + \alpha_3 A_3^{(n+1)} + \alpha_4 A_4^{(n+1)} + \alpha_0 A_0^{(n)} - 4\pi S_0)/\alpha_0 \tag{3.2b}$$

In Eq. (3.2), n is the iteration number, and α is the over-relaxation parameter. The choice of α is discussed in Sec. 5. α_1, α_3 refer to the right and left neighbors and α_2, α_4 refer to the above and below neighbors.

The boundary conditions on the iron-air interface, given by equation (2.5), is also replaced by a difference equation. Using mesh points on the air side, one may write (to first order in the mesh size)

$$\left(\frac{\partial A}{\partial n}\right)_{\text{air}} = \sum_{i=0}^{2} \beta_i A_i, \tag{3.3}$$

where the three mesh points used in equation (3.3) include the boundary point being considered and its two neighbors on the air side that lie closest to the normal. The derivation of the β_i is discussed in Sec. 7. In the same way, using mesh points on the iron side, one may write

$$\left(\frac{\partial A}{\partial n}\right)_{\text{iron}} = \sum_{i=0}^{2} \bar{\beta}_i \bar{A}_i, \tag{3.4}$$

and the boundary condition becomes

$$\sum_{i=0}^{2} \beta_i A_i = \gamma_0 \sum_{i=0}^{2} \bar{\beta}_i \bar{A}_i. \tag{3.5}$$

The vector potential at the air-iron boundary is found in each iteration by using Eq. (3.5). The three-point algorithm used in computing $\partial A/\partial n$ is correct only to first order in the mesh size. Higher-order algorithms are possible but difficult to program. The use of a first-order algorithm for the boundary points is the main reason that the vector potential method is less accurate than the scalar-potential method, which can use a second-order algorithm at boundary points. When the boundary is regular, that is, it lies along one of the coordinate lines, it is not difficult to replace Eq. (3.5) by a second-order algorithm. This algorithm is discussed in Sec. 7.

One also needs a boundary condition at the outer boundary of the mesh. The simplest approach here is to put the outer boundary of the mesh on the outer boundary of the iron and to assume that no magnetic flux escapes from the iron. This means that A is constant along the outer boundary of the iron, and we can for convenience take $A = 0$.

If, however, the magnetic field is high enough that considerable magnetic flux can leak out past the iron, then the assumption of $A = 0$ on the iron outer boundary is not adequate. The treatment of the outer mesh boundary in this case is discussed in Sec. 7.

In the iron region, the iteration of the vector potential requires knowing γ, the inverse permeability, at the mesh points involved. In GRACY, γ is stored on a separate mesh. This γ-mesh is recomputed after each iteration of the vector potential mesh by means of an experimental relation between γ and the magnetic field induction B, which is entered as a table of γ vs B in the program. The γ-mesh is computed according to

$$\gamma^{(n)} = \beta\gamma(B) + (1-\beta)\gamma^{(n-1)}, \tag{3.6}$$

where n is the iteration number, $\gamma(B)$ represents the experimentally found dependence of γ on B, and β is an under-relaxation parameter.

The problem being solved is nonlinear because of the nonlinear relation between γ and B. In order to insure convergence of the iteration procedure in the nonlinear problem, it is necessary to introduce the β parameter in computing the γ-mesh, and to choose for β some small number, $\beta \ll 1$. The choice of β is discussed in Sec. 5.

4. CURRENT-DENSITY CALCULATION

In carrying out the iteration procedure for the vector potential described in Sec. 3, it is necessary to know the current density at each mesh point.

The current distribution is entered into the program by specifying the location of current blocks, within which the current density is uniform. It is easy to compute the current density for mesh points that lie in the interior of a current block, but there is a problem in computing the current density at mesh points that are within a mesh interval of the edge of the current block, where the current density changes suddenly. Indeed, one may wonder whether the program can distinguish between two current blocks whose edges are within a mesh interval of each other.

A procedure that appears to assign the correct current density to each mesh point is to compute the vector potential $\bar{A}$ at each mesh point due to the current with no iron present. This calculation of $\bar{A}$ can be done analytically and the $\bar{A}$ found at each mesh point is exact. The current density at each mesh point can then be computed from $\bar{A}$ according to

$$\nabla^2 \bar{A} = -4\pi S, \tag{4.1}$$

where $\nabla^2 \bar{A}$ is computed numerically using $\bar{A}$ computed at the mesh points. The justification for this procedure is given later in this section.

One can see that the above procedure does distinguish between current blocks whose edges are within a mesh interval of each other. The procedure also has the advantage of avoiding the difficulties involved caused by the discontinuous current density at the current block edges. The vector potential $\bar{A}$ is always a smooth continuous function; the numerical computation of $\nabla^2 \bar{A}$ automatically produces the discontinuities in the current density at the right places. A disadvantage is the time necessary to compute $\bar{A}$.

This method is particularly useful for superconducting high-field magnets. The circular cosine distribution of current required leads to current blocks whose edges are not usually parallel to mesh lines. In this case, the ironless results for the field that can be found from $\bar{A}$ are also of interest.

One may also note that the current density is stored in its own mesh, the S-mesh. It is assumed that there is no current flowing in the iron and thus the γ-mesh and S-mesh do not overlap, and they are both stored in the same bank $S(I, J)$ in the computer. At the iron-air interface, γ and S can overlap, and the two quantities are packed into the same word of the bank $S(I, J)$. There are then two mesh banks in the GRACY program; the $A(I, J)$ bank for the vector potential, and the $S(I, J)$ bank for the current density and γ. Each bank is allotted 10,000 words of memory.

A formulation of the magnet problem that avoids the problem of computing the current density at each mesh point is to introduce the potential $F = A - \bar{A}$, where $\bar{A}$ is the vector potential due to the currents alone with no iron present. It is possible to compute $\bar{A}$ exactly at each mesh point. F then satisfies the equation

$$\nabla^2 F = 0. \tag{4.2a}$$

The boundary conditions then involve $\bar{A}$. For example, at the air-iron interface in the case of infinite permeability, $\partial A/\partial n = 0$ becomes

$$\frac{\partial F}{\partial n} = -\frac{\partial \bar{A}}{\partial n}. \tag{4.2b}$$

If one writes down the difference equations corresponding to Eq. (4.2), one sees they are the same equations one gets using the formulation of the problem that uses the vector potential A and that computes S in the manner discussed earlier in this section. This is the argument for computing S in this way.

5. CONVERGENCE PROBLEMS

The iteration procedure described in Sec. 3 may, in some cases, lead to a very slow convergence rate. A number of techniques may be used to speed up the convergence rate.

The Over-Relaxation Parameter

If the problem is linear, which in magnet problems means that the permeability is assumed to be independent of the magnetic-field strength, then the convergence rate can be increased greatly by the proper choice of α in Eq. (3.2a). There are analytical methods[7] and empirical methods[8] for choosing α. At present the procedure used in the GRACY program is simply to choose α close to 1.9 in linear problems. This procedure, while not optimum, has proved to be satisfactory in many cases.

The Eigencorrection Procedure (Aitken's Procedure)

For problems that have a rectangular geometry, as, for example, a rectangular window magnet whose horizontal width a is much larger than its vertical gap b, the vector-potential method has a much slower rate of convergence than the scalar-potential method. For the case of infinite permeability and with $\alpha = 1$, the convergence rate for the

window magnet is larger using the scalar potential rather than the vector potential by a factor $(a/b)^2$, which may be about 50 for a reasonable bending magnet. Because of the difference in the boundary conditions for the vector-potential and scalar-potential methods, the eigenvalues of the vector-potential problem have a lowest eigenvalue which is smaller by the factor $(b/a)^2$ than the lowest eigenvalue of the scalar-potential problem, which results in the much smaller convergence rate for the vector-potential method. The difference in the convergence rates should become smaller as the geometry becomes less rectangular, that is, as the vertical and horizontal dimensions of the air gap become more equal. High-field superconducting magnets tend to be square or circular, and the difference in the convergence rates for the two methods should not be very great for these magnets.

One method that may be used to speed up the convergence of the vector-potential method for magnets with rectangular geometries is the eigencorrection procedure, or Aitken's procedure.[7] This procedure is based on the assumption that after a sufficient number of iterations, if one expands the error vector, $\epsilon = A - A^{(n)}$ in the eigenfunctions of the iteration matrix, then the only eigenfunction present in this expansion will be the eigenfunction whose eigenvalue is closest to 1. Using this assumption, one can use the results $A^{(n)}$ and $A^{(n+1)}$ found from two successive iterations to compute the error, and obtain the result for A

$$A = A^{(n)} + \frac{1}{1-\lambda}(A^{(n+1)} - A^{(n)}), \qquad (5.1a)$$

where λ is the eigenvalue of the iteration matrix that is closest to 1.

In order to use Eq. (4.1), one needs to know the eigenvalue λ. This can be found by computing the total residuals of the two successive iterations, $\bar{R}^{(n)}$ and $\bar{R}^{(n+1)}$, where $\bar{R}^{(n)}$ is computed by finding the residual at each mesh point $R^{(n)}$ given by Eq. (3.2b), and adding up the absolute value of all the residuals at each mesh point. λ may then be estimated from

$$1 - \lambda = (\bar{R}^{(n)} - \bar{R}^{(n+1)})/\bar{R}^{(n+1)}. \qquad (5.1b)$$

In applying the correction procedure given by Eq. (4.1), one must first iterate long enough until the rate of convergence, RATE $= (\bar{R}^{(n)} - \bar{R}^{(n+1)})/\bar{R}^{(n+1)}$ settles down to an almost constant value. The eigencorrection procedure may then be applied at intervals of a given number of iterations, which might be every 100 iterations.

For some magnet problems, RATE does not appear to settle down but undergoes a slow oscillation. In this case, the eigencorrection procedure leads to poor results. In particular, the procedure is not likely to be successful for magnets having considerable variation in the permeability. This makes the problem nonlinear, and RATE is often found to oscillate.

Initial Guess for A

Another method of improving the convergence rate is to use as an initial input guess for A an analytic function with several parameters. The parameters are then chosen by doing runs of a certain number of iterations and finding the parameters that give the smallest residual at the end of the runs. By this procedure, one is eliminating that part of the error in the initial guess that corresponds to the slow mode, that is, that gives rise to the presence in the error vector of the mode whose eigenvalue is closest to 1. It is often faster to eliminate the slow mode in the error vector by proper choice of the parameters in the initial guess for A than to reduce the slow mode by iterating that may go very slowly.

Convergence of Nonlinear Problems

The dependence of the inverse permeability γ on the strength of the magnetic field makes the problem nonlinear. In order to get convergence, it is necessary to limit the amount that γ can change at each mesh point after one iteration. This is done by introducing the under-relaxation parameter β in the computation of γ according to Eq. (3.6). The choice of β depends on how nonlinear the γ vs B relation is. A common choice of β that gives convergence in iron-magnet problems is $\beta \simeq 0.1$, which means that γ cannot change by more than about 10 per cent in each iteration.

A disadvantage that goes along with such use of the β parameter is that one loses the gain in convergence speed due to using an over-relaxation parameter α of about $\alpha = 1.9$ in Eq. (3.2). One finds that to get convergence with $\beta = 0.1$, one must choose α at about 1 to 1.3. It has been found that for a high-field superconducting dipole, such as shown in Fig. 1, one can regain the ability to run with $\alpha \cong 1.9$, with the accompanying high rate of convergence, by choosing much lower values of β, say, $\beta \simeq 0.008$. If γ does not change after each iteration, that is, $\beta = 0$, then the problem is linear and converges with $\alpha \simeq 1.9$. Thus one may hope that for sufficiently small β, the

iteration process might converge with $\alpha = 1.9$. In choosing these low values of β, one must perform runs with a sufficient number of iterations to allow γ, which cannot change much in each iteration, to reach the correct answer. For a choice of $\beta \simeq 0.01$, a run of over several hundred iterations is required.

6. ACCURACY PROBLEMS

The accuracy of the results obtained by a mesh-iteration program like GRACY is limited by how small one can make the mesh size. Sometimes the accuracy required by the application is more than can be achieved by the program. This situation can arise in the study of high-field superconducting dipoles. The magnetic field on the median plane is very nearly uniform, but the small deviations from uniformity are important. One can expand the median plane field as

$$B = B_0(1 + b_2 r^2 + b_4 r^4 + \ldots), \tag{6.1}$$

where r is the horizontal distance from the center of the dipole. One would like to be able to compute sextupole terms, b_2, of the order of $b_2 \simeq 1 \times 10^{-3}/\text{in.}^2$. With a mesh interval size of the order of 0.125 in., one is trying to see relative field changes between mesh points of the order of 10^{-4}.

In order to improve the accuracy of the program, one observes that the sextupole field is larger at larger values of r, and it is possible to compute the sextupole field at any r that lies in a circle inside the current-carrying conductors, that is, at smaller radius than any current. In the center of the dipole, up to the edge of the current distribution, A satisfies Laplace's equation and may be expanded as

$$A = a_1 r \cos\theta + a_3 r^3 \cos 3\theta + a_5 r^5 \cos 5\theta + \ldots. \tag{6.2}$$

The coefficient a_3, which gives the sextupole term $b_2 = 3a_3$, can be computed using the A found by the program on the largest circle that lies inside the current distribution from the formula

$$a_n = \frac{1}{r^n}\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos n\theta \, A(r, \theta). \tag{6.3}$$

This gives a much better result for b_2 than can be found by simply differentiating the A found by the program near the origin.

The same basic technique can be used to improve the accuracy of the field on the median plane at smaller r, by relating the field at smaller r to the field at larger r by means of a Green's function. One can relate the vector potential on a circle of radius r to the vector potential on a circle of radius R by

$$A(r, \theta) = \int_{-\pi}^{\pi} d\theta' \, G(\theta - \theta') \, A(R, \theta'), \tag{6.3a}$$

$$G(\theta) = \frac{1}{\pi}\frac{r(R\cos\theta - r)}{R^2 - 2rR\cos\theta + r^2}, \tag{6.3b}$$

where $r < R$, and the circle of radius R lies inside the current distribution. The result (6.3b) for the Green's function can be found by substituting Eq. (6.3) for the a_n in Eq. (6.2) for $A(r, \theta)$ and performing the sum over n, which can be done analytically.

In the GRACY program, the integral in Eq. (6.3) is done numerically to find the vector potential at smaller r using the result found by the program for the vector potential at the largest r inside the current distribution. The GRACY program appears in this way to be able to compute sextupole terms b_2 of the order of $10^{-4}/\text{in.}^2$.

7. ITERATION COEFFICIENTS

A mesh-iteration program replaces the differential equations and boundary conditions by difference equations which then are the basis for an iteration procedure. The procedures for obtaining the difference equation at interior mesh points are written down in many textbooks.[7] However, it seems worthwhile to summarize the results for the iteration coefficients, which are the coefficients in the difference equations, for the magnet problem.

Interior Mesh Points

A procedure suggested by R. S. Christian may be used to find the iteration coefficients at an interior mesh point. This consists of applying Ampere's law, $\int H_l \, dl = 4\pi S$, around the rectangle shown in Fig. 2. This rectangle is shown by the dashed lines and surrounds the mesh-point of interest, which is called point 0. The dashed lines lie midway between mesh points. Figure 2 shows both a rectangular and a cylindrical mesh. The cylindrical mesh is advantageous for magnets with cosine current distributions.

We can then find by integrating over the rectangle and using the relation $\mathbf{H} = \gamma \operatorname{curl} \mathbf{A}$ that

$$\int H_l \, dl = \sum_{i=0}^{4} \beta_i A_i, \tag{7.1}$$

where A_0 is the vector potential at the mesh-point 0, and A_1, A_2, A_3, A_4 are the vector potentials of the four nearest neighbor mesh-points.

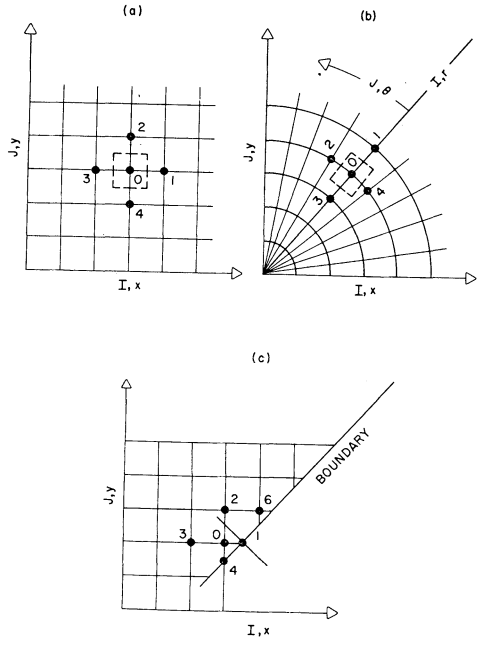

FIG. 2. Examples of a rectangular, and a cylindrical mesh, and of a boundary point.

Results can be obtained for a general orthogonal coordinate system x, y that has the metric

$$ds^2 = Q_x^2\,dx^2 + Q_y^2\,dy^2. \tag{7.2}$$

The β_i are given by

$$\beta_1 = -\frac{Q_{y1}}{Q_{x1}}\frac{\int_1 \gamma\,dy}{h_1}, \tag{7.3a}$$

$$\beta_2 = -\frac{Q_{x2}}{Q_{y2}}\frac{\int_2 \gamma\,dx}{l_2}, \tag{7.3b}$$

$$\beta_3 = -\frac{Q_{y3}}{Q_{x3}}\frac{\int_3 \gamma\,dy}{h_3}, \tag{7.3c}$$

$$\beta_4 = -\frac{Q_{x4}}{Q_{y4}}\frac{\int_4 \gamma\,dx}{l_4}, \tag{7.3d}$$

$$\beta_0 = -(\beta_1+\beta_2+\beta_3+\beta_4). \tag{7.3e}$$

In Eqs. (7.3) Q_{x1} means Q_x evaluated midway between point 0 and point 1. h_1, l_2, h_3, l_4 are the change in coordinates between point 0 and the four neighbors and are positive quantities. $\int_1 \gamma\,dy$ means the integral is taken over the rectangle side near neighbor point 1. This integral is kept because Eqs. (7.3) must hold at a point on the iron-air interface where γ may change suddenly along the rectangle side. Equations (7.3) then hold for any point in the air region or in the iron region or on the iron-air interface.

It is sometimes convenient to know the difference coefficients for $-\text{curl}\ (\gamma\ \text{curl}\ A)$, which is just $\nabla^2 A$ in the air region. If we write at point 0

$$-\text{curl}\ (\gamma\ \text{curl}\ A) = \sum_{i=0}^{4} \alpha_i A_i, \tag{7.4}$$

then the α_i can be related to the β_i by the relation $\int H_l\,dl = \int(\text{curl}\ H)_n\,dA$, and we find

$$\alpha_i = -\beta_i/\text{AREA}, \tag{7.5a}$$

$$\text{AREA} = \tfrac{1}{4}(l_2+l_4)(h_1+h_3)\,Q_xQ_y. \tag{7.5b}$$

AREA is the area of the rectangle, and Q_x, Q_y are evaluated at point 0.

For a Cartesian x, y mesh, at an interior meshpoint not on the iron-air interface, Eq. (7.3) give

$$\beta_1 = -\bar{\gamma}_1\frac{\tfrac{1}{2}(l_2+l_4)}{h_1}, \tag{7.6a}$$

$$\beta_2 = -\bar{\gamma}_2\frac{\tfrac{1}{2}(h_1+h_3)}{l_2}, \tag{7.6b}$$

$$\beta_3 = -\bar{\gamma}_3\frac{\tfrac{1}{2}(l_2+l_4)}{h_3}, \tag{7.6c}$$

$$\beta_4 = -\bar{\gamma}_4\frac{\tfrac{1}{2}(h_1+h_3)}{l_4}, \tag{7.6d}$$

where $\bar{\gamma}_1$ is γ evaluated at a point midway between point 0 and point 1 and can be evaluated as $\bar{\gamma}_1 = \tfrac{1}{2}(\gamma_0+\gamma_1)$.

For a cylindrical r, θ mesh, at an interior point not on the iron-air interface, equations (7.3) give

$$\beta_1 = -\bar{\gamma}_1\frac{\tfrac{1}{2}(l_2+l_4)}{h_1}(r+\tfrac{1}{2}h_1), \tag{7.7a}$$

$$\beta_2 = -\bar{\gamma}_2\frac{\tfrac{1}{2}(h_1+h_3)}{l_2}\frac{1}{r}, \tag{7.7b}$$

$$\beta_3 = -\bar{\gamma}_3\frac{\tfrac{1}{2}(l_2+l_4)}{h_3}(r-\tfrac{1}{2}h_3), \tag{7.7c}$$

$$\beta_4 = -\bar{\gamma}_4\frac{\tfrac{1}{2}(h_1+h_3)}{l_4}\frac{1}{r}. \tag{7.7d}$$

where r is the radial distance to point 0.

Equations (7.3) may also be applied to a point on the air-iron interface. This results in a difference equation that is more accurate than Eq. (3.5). This equation may be used when the interface is regular, that is, it lies along one of the coordinate lines. There are many different cases to be distinguished, depending on whether the boundary is a right, left, up or down boundary or whether the point is a corner point. The results are easily

found from Eqs. (7.3) and will not be given here.

Normal-Derivative Coefficients

For a point on an irregular air-iron interface, the difference Eq. (3.5) is used, which requires a difference result for the normal derivative $\partial A/\partial n$ on the interface. The situation is shown in Fig. 2c for a boundary point obtained when a y-coordinate line intersects a right-hand boundary. Point 1 is the boundary point at which we wish to find $\partial A/\partial n$. Point 0 is the associated interior point. We write

$$\frac{\partial A}{\partial n} = \sum \beta_i A_i. \tag{7.8}$$

It is difficult to use higher-order results for $\partial A/\partial n$ without losing the diagonal-dominance property that guarantees convergence of the iteration scheme. The diagonal-dominance property is present if each diagonal element of the iteration matrix is opposite in sign to each of the off-diagonal elements in the same row of the iteration matrix, and the magnitude of the diagonal element is larger than or equal to the sum of the magnitude of the off-diagonal elements in the same row. For this reason, one uses just the lowest order result for $\partial A/\partial n$ which uses only three points, point 0, 1, 2 for the case shown. Results will be given for the situation shown in Fig. 2c. Results for the many other cases involving the other boundaries may be found in a similar way.

Let the direction cosine of the tangent to the boundary be given by c_x, c_y. For the case shown in Fig. 2c, one has to distinguish between the two cases of whether the normal comes closer to point 0 or to point 6. Assume for the present that point 0 is chosen. Then we find

$$\frac{\partial A}{\partial n} = \frac{A_p - A_1}{\mathrm{d}}, \tag{7.9a}$$

$$A_p = a_2 A_2 + a_0 A_0, \tag{7.9b}$$

$$\mathrm{a}_2 = (y_p - y_0)/r_{20}, \tag{7.9c}$$

$$a_0 = 1 - a_2. \tag{7.9d}$$

Here r_{20} is the positive distance between 2 and 0, x_p, y_p are the coordinates relative to point 1 of point p, where the normal intersects the x-coordinate line between 0 and 2, and x_0, y_0 are the coordinates of point 0 relative to point 1. One can eliminate y_p from Eqs. (7.9) by using $x_p = x_0$ and the relation

$$x_p c_x + y_p c_y = 0. \tag{7.10}$$

We then find

$$a_2 = \frac{-1}{r_{20}}(x_0 c_x + y_0 c_y)\frac{1}{c_y}, \tag{7.11a}$$

$$\mathrm{d} = |x_0/c_y|, \tag{7.11b}$$

$$\beta_2 = a_2/\mathrm{d}, \tag{7.11c}$$

$$\beta_0 = (1 - a_2)/\mathrm{d}, \tag{7.11d}$$

$$\beta_1 = -(\beta_2 + \beta_0). \tag{7.11e}$$

To test whether one should use point 0 or point 6, one tests whether $0 < a_2 < 1$; if true, then one uses point 0; if not true, one uses point 6.

The Outer Mesh Boundary

High-field superconducting magnets can be expected to have a considerable leakage of the field past the iron shield. The iteration mesh cannot extend to infinity, and some boundary condition is required that will allow the mesh to be ended at some reasonable point.

For dipole magnets one may assume that at sufficiently large distance from the magnet center, the vector potential will vary as

$$A \sim f(\theta)/r, \tag{7.12a}$$

from which one obtains the relation at large r

$$\frac{\partial A}{\partial \mathrm{r}} + \frac{1}{r}A = 0. \tag{7.12b}$$

Equation (7.12b) is used to obtain the iteration equation for the points on the mesh boundary. This equation can obviously be generalized for quadrupoles and higher poles.

Using a cylindrical mesh, Eq. (7.12b) gives the difference relation

$$A_0(1 + h/r) - A_3 = 0, \tag{7.12c}$$

where h is the mesh interval in the r-direction, A_0 is the vector potential at the mesh boundary point, and A_3 is the vector potential at the nearest-neighbor mesh point along the r direction. This relationship satisfies the diagonal-dominance requirement for the iteration matrix.

One may also use Eq. (7.12b) for a rectangular mesh. In this case, the mesh boundary is rectangular. Consider that part of the mesh boundary on which x is constant. Then one can replace Eq. (7.12b) by the difference equation

$$A_0\left(\frac{c_x}{h} + \frac{c_y}{l} + \frac{1}{r}\right) - c_x A_3 - c_y A_4 = 0, \tag{7.13}$$

where c_x and c_y are the direction cosines of the radius vector to the mesh boundary point, A_3

is the left horizontal neighbor with a lower x-coordinate, A_4 the lower vertical neighbor with a lower y-coordinate, and h and l are the horizontal and vertical mesh intervals. The choice of which neighbor to use in computing $\partial A/\partial y$ is determined by the diagonal-dominance requirement. A similar difference equation can be found for the other sides of the rectangular mesh boundary.

Applying either Eqs. (7.12) or (7.13) requires that the mesh allow for some air region past the iron shield. How large an air region to allow can be determined by doing computer runs using different air-region extents and noting the effect on the field results. Experience has shown that for magnet dipoles having about 2-in. apertures, an air extent of 3 in. past the iron shield is sufficient, and the field results are not exceptionally sensitive to the choice of the air extent.

8. RESULTS FOR A HIGH-FIELD DIPOLE

As an example of the application of the GRACY program to a magnet problem, results will be given here for a high-field superconducting dipole whose geometry is indicated in Fig. 1.

This dipole is a circular dipole with a cosine current winding. The cosine distribution is approximated by having four blocks of current in each quadrant; within each block the current density is uniform. The inside diameter of the current winding is 2 in., the inside dimension of the iron shield is 3.5 in., and the dipole is referred to as a 2×3.5 CD. The coil thickness is 5/8 in., leaving 1/8 in. between the coils and the iron. The iron-shield thickness here is 2 in.

The B-μ table used here was measured at low temperature by McInturff and Claus.[9]

Figure 3 shows the field at the center of the dipole as a function of the ampere turns. At 40 kG and 60 kG, the presence of iron gives 45 per cent and 30 per cent more field.

A map of the field is indicated in Fig. 4 when the central field is 40 kG. The maximum leakage field just past the iron is 1.34 kG, or about 3 per cent of the main field. At 60 kG, the leakage field is 4.2 kG.

In Fig. 5, the gradient that is present because of saturation is shown as a function of r for various field excitations. The cosine dipole has a remarkably uniform field even up to 60 kG. However, the tolerance on the gradient in the magnet dipole of a high-energy accelerator, 100 GeV or higher, is very severe. The gradients shown in

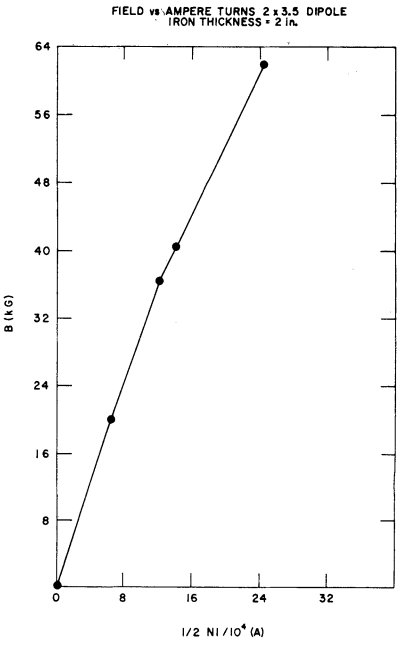

FIG. 3. The central magnetic field in a cosine dipole plotted against the excitation ampere-turns. The iron shield thickness is 2 in.

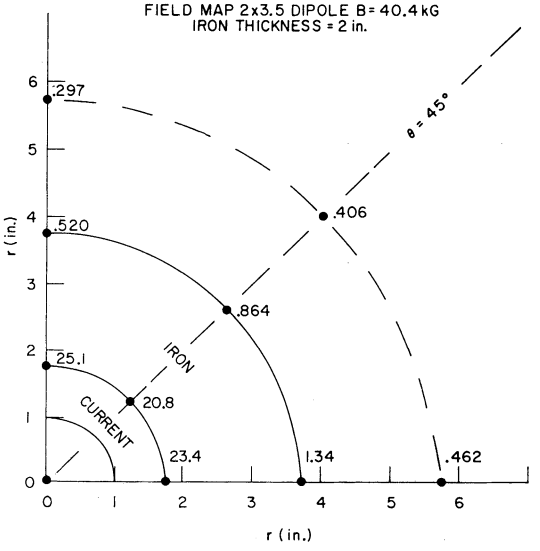

FIG. 4. A map of the magnetic field B in kilogauss for a high-field cosine dipole with an iron shield. The central dipole field is B = 40.4 kG. The iron shield thickness is 2 in.

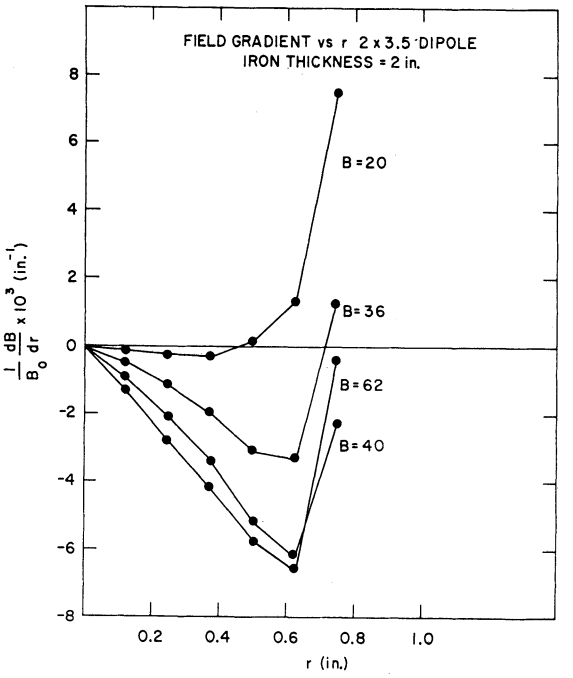

FIG. 5. A plot of the gradient $B^{-1}\,dB/dr$ as a function of r, the distance from the center of the dipole for various levels of excitation of the magnetic dipole. The iron shield thickness is 2 in.

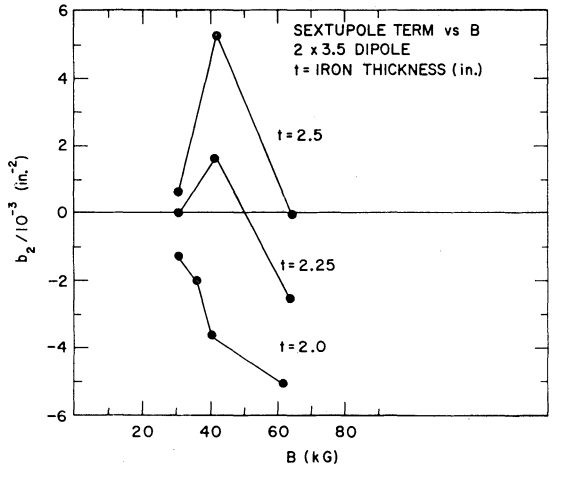

FIG. 6. A plot of the sextupole term, b_2, present in a high-field magnetic dipole against the central magnetic field B, for various thicknesses, t, of the iron shield.

Fig. 4 are probably several times tolerance at 40 kG.

Figure 6 shows the possibility of eliminating the sextupole term in the dipole by properly choosing the thickness of the iron shield. One may write the median plane field as

$$B = B_0(1 + b_2 r^2 + b_4 r^4 \ldots). \tag{8.1}$$

The sextupole term b_2 is plotted in Figure 6 as a function of the main dipole field for various thicknesses t of the iron shield. One sees that b_2 can be minimized by a choice of the iron-shield thickness of about 2.25 in.

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