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IMPOSSIBILITY OF ACHROMATIC FOCUSING WITH MAGNETIC QUADRUPOLES AND SOLENOIDS

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The theorem that a quadrupole focusing system cannot produce a line focus whose location is independent of momentum is generalized to cover any magnetic field distribution in which the central trajectory is a straight line.

It is known[1,2] that when a system of magnetic quadrupole lenses produces a point focus or a line focus from a point or line source, the location of this focus cannot be independent of particle momentum. That is to say, the distance d of the focus from a reference point in the lens system depends on the particle momentum, and

$$\frac{\partial d}{\partial p} \neq 0 \qquad (1)$$

for any focusing system composed entirely of quadrupole magnets.

The theorem is proved[1,2] for systems in which all the quadrupoles are oriented 'normally', i.e. with fields of the form

$$B_x = yG(z), \qquad B_y = xG(z), \qquad (2)$$

where z is the (straight) axis of the beam, and x and y are the transverse coordinates.

Van der Meer[1] has shown that if the central trajectory is not straight, i.e. if the system contains deflecting magnets, sextupole magnets with fields of the form

$$B_y = S(z)(x^2 - y^2), \qquad B_x = 2S(z)xy \qquad (3)$$

can be introduced at suitable locations so as to make the focus achromatic. However, it would often be desirable to produce achromatic focusing in a straight system, without deflecting magnets. One may therefore ask whether by introducing other elements with fields not of the form (2) achromatic focusing can be achieved in a straight system.

The impossibility of producing a focus is related to the fact that the dispersion of a thin quadrupole lens bears a unique relation to its focal length:

$$D = \frac{p}{f}\frac{\partial f}{\partial p} = 1, \qquad (4)$$

where f is the focal length of the lens for a beam of particles of momentum p. This is in contrast to the case of the optics of light beams focused with glass lenses, where the dispersion factor D can have different values for different kinds of glass, making it possible to design achromatic lens systems.

Now a magnetic solenoid, acting as a thin lens, has a focal length

$$f = \frac{4}{l}\left(\frac{pc}{eB}\right)^2, \qquad (5)$$

where B is the field intensity of a solenoid of length l. Therefore $D = 2$ for a solenoid, and it might be thought that an achromatic combination of solenoids and quadrupoles could be devised. However, in optical achromats, the component with the larger dispersion has to be defocusing, while a solenoid is always focusing. Thus the analogy is imperfect, as pointed out by van der Meer.[1] In what follows we shall extend Steffen's derivation[2] of the impossibility of achromatic focusing to cover any arbitrary straight combination of solenoids and quadrupoles.

A magnetic field which permits motion of a particle on the z-axis must have $B_x = B_y = 0$ for $x = y = 0$. Therefore, by Maxwell's equations $\nabla \times B = 0$, $\nabla \cdot B = 0$, the components of the field must be of the form

$$\begin{aligned} B_z &= B(z) \\ B_x &= yG(z) + x\left[F(z) - \tfrac{1}{2}\frac{dB}{dz}\right] \\ B_y &= xG(z) - y\left[F(z) + \tfrac{1}{2}\frac{dB}{dz}\right], \end{aligned} \qquad (6)$$

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where terms of higher than the first order in x and y are neglected, and $B(z)$, $F(z)$, $G(z)$ may be any functions of z. B is the solenoidal field, G is the field gradient of 'normal' quadrupoles (oriented so that the pole asymptotes are in the x and y directions), and F represents 'skew' quadrupoles, oriented at 45° to the normal orientation (any other orientation produces both F and G terms).

The equations of transverse motion for a particle of momentum p are

$$x'' = \frac{e}{pc}(-B_y + y'B_z) = \frac{e}{pc}[-Gx + (F + \tfrac{1}{2}B')y + By'] \qquad (7)$$

$$y'' = \frac{e}{pc}(B_x - x'B_z) = \frac{e}{pc}[Gy + (F - \tfrac{1}{2}B')x - Bx'], \qquad (8)$$

where the prime denotes differentiation with respect to z.

For a particle with momentum $p(1+\delta)$ we write the transverse coordinates as $x + \xi\delta$ and $y + \eta\delta$. Making these substitutions in (7) and (8) and selecting the terms of first order in δ, we find

$$\xi'' = \frac{e}{pc}[-G\xi + (F + \tfrac{1}{2}B')\eta + B\eta'] - x'' \qquad (9)$$

$$\eta'' = \frac{e}{pc}[G\eta + (F - \tfrac{1}{2}B')\xi - B\xi'] - y''. \qquad (10)$$

Multiply (7), (8), (9), (10) respectively by $-\xi$, $-\eta$, x, y and add, obtaining

$$x\xi'' + y\eta'' - (\xi x'' + \eta y'') = -(xx'' + yy'') + \frac{e}{pc}[B'(x\eta - y\xi) + B(x'\eta + x\eta' - y'\xi - y\xi')]. \qquad (11)$$

This may be rewritten

$$\frac{d}{dz}\left[x\xi' + y\eta' - \xi x' - \eta y' + \frac{eB}{pc}(y\xi - x\eta) + xx' + yy'\right] = x'^2 + y'^2. \qquad (12)$$

Integrating from z_1 to z_2 we therefore have

$$M \equiv \left[x\xi' + y\eta' - \xi x' - \eta y' + \frac{eB}{pc}(y\xi - x\eta) + xx' + yy'\right]_{z_1}^{z_2} > 0. \qquad (13)$$

M can never be equal to zero except for the central ray, where x' and y' are identically zero.

A system which is focusing is one where a point or line source at z_1 is converted into a point or line image at z_2. If the focus is achromatic then (ξ, η) must lie on the same line as (x, y) at z_1 and z_2 (or be both zero in case of a point focus), so that $y\xi - x\eta = 0$ at both ends. Furthermore, we can restrict ourselves to a ray with momentum $(1+\delta)p$ leaving the source at the same point and with the same angle as the reference ray with momentum p, which means that $\xi = \eta = \xi' = \eta' = 0$ at $z = z_1$.

We see immediately that $M = 0$ for an achromatic *point* focus, for then $x = y = \xi = \eta = 0$ at both ends, and every term on the left-hand side of (13) vanishes. Since, as we have seen, M must be positive, this establishes that an achromatic point focus is impossible.

To extend this result to the case of a line focus it suffices to show that there is a particular off-axis ray for which $M = 0$ if the focus is achromatic.

Suppose the source is on the line $x\cos\alpha + y\sin\alpha = 0$, and the image on the line $x\cos\beta + y\sin\beta = 0$ (i.e. the focus may be a rotated image of the source). We introduce new coordinates u, v, χ, ζ:

$$\begin{bmatrix} u \\ v \end{bmatrix} = R\begin{bmatrix} x \\ y \end{bmatrix}; \quad \begin{bmatrix} \chi \\ \zeta \end{bmatrix} = R\begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad (14)$$

where R is a rotation through the angle α at z_1 and through the angle β at z_2. We take the rotation angles to be constant over short finite intervals near z_1 and z_2, so that the derivatives also transform by (14). This rotation leaves scalar products invariant, so that M is the same in terms of the new coordinates as in terms of the old ones.

Now the existence of a line focus simply means that whenever $u(z_1) = 0$ then $u(z_2) = 0$; if the focus is achromatic $\chi(z_1)$ and $\chi(z_2)$ are also both zero, and M reduces to

$$M = v\zeta' - \zeta v' + vv' \Big|_{z_1}^{z_2}. \qquad (15)$$

For any initial conditions the displacement vectors of a ray at z_1 and z_2 are connected by a transfer matrix relation. It is convenient to express this in terms of coordinates and canonical momenta rather than coordinates and derivatives:

$$\begin{bmatrix} u \\ p_u \\ v \\ p_v \end{bmatrix}_2 = T\begin{bmatrix} u \\ p_u \\ v \\ p_v \end{bmatrix}_1, \qquad (16)$$

where

$$p_u = pu' - \frac{eB}{2c} v$$
$$p_v = pv' + \frac{eB}{2c} u \tag{17}$$

and T is a 4×4 matrix.

The existence of the line focus means that $u(z_1) = 0$ implies $u(z_2) = 0$ and vice versa; therefore the matrix elements of T and of its inverse T^{-1} must satisfy

$$T_{12} = T_{13} = T_{14} = T^{-1}_{12} = T^{-1}_{13} = T^{-1}_{14} = 0\,. \tag{18}$$

But since the whole motion is governed by an electromagnetic field it is derivable from a Hamiltonian, and therefore the matrix T is symplectic.[3] This means, among other things, that [when the operand components are ordered as in (16)]

$$T^{-1}_{12} = -T_{12}\,, \quad T^{-1}_{13} = T_{42}\,, \quad T^{-1}_{14} = -T_{32} \tag{19}$$

and, by (18), we then have $T_{12} = T_{32} = T_{42} = 0$. Therefore the particular ray with initial conditions $u = v = p_v = 0$, $p_u \neq 0$ ends up at z_2 with $u = v = p_v = 0$, and, by (17), $v = v' = 0$ at both ends of the interval. Therefore $M = 0$ in contradiction to the inequality (13), and consequently the assumption of achromatic focusing is untenable.

We have thus proved that no combination of solenoidal and transverse magnetic fields can produce an achromatic point or line focus in a straight beam system.

One further way of circumventing this result is to use electrostatic quadrupoles. The dispersion factor D [Eq. (4)] for electric quadrupoles equals $2-\beta^2$, and this is equally valid for focusing and defocusing lenses. Thus for slow particles a combination of electric and magnetic quadrupoles can be achromatic.

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