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EFFECTS ON ONE-THIRD INTEGER RESONANT EXTRACTION DUE TO SEXTUPOLE FIELDS IN THE LATTICE BENDING MAGNETS†

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Slow extraction in a circular accelerator can be accomplished by providing a sextupole distribution with a kth harmonic, where the integer k is three times the ν value at resonance. It is pointed out that the size of the separatrix (the triangular phase-space region which separates stable from unstable particles) is extremely sensitive to the ν value. On the other hand, the sextupole components in the bending magnets introduce a ν shift which is a function of the amplitude of the particles being extracted. It thus follows that the separatrix will expand as a function of particle amplitude. Under certain circumstances, which are discussed, particles, that up to a particular amplitude have remained unstable, will become retrapped by the advancing separatrix created by their own amplitude growth and no extraction can occur. The conditions under which this effect occurs are described and the means by which partial correction can be achieved are discussed.

1. INTRODUCTION

Slow extraction in a circular accelerator can be accomplished by providing a sextupole distribution with a kth harmonic. The integer k is three times the ν value at resonance. As an example we consider a case of vertical extraction as is proposed in the cold magnet synchrotron under study at Brookhaven National Laboratory,[1] although the ideas presented here can be applied equally well to normal synchrotrons. Here extraction is achieved with four evenly spaced sextupoles of equal strength and alternating polarity, which will give a 22nd harmonic and can be used for extraction on a $\nu_y = 7\frac{1}{3}$ resonance.[1] If the ν_y value is shifted toward its resonant value, the region of stable phase space decreases. At some point before the resonant ν_y value is reached, the particles of larger amplitude become unstable. When $\nu_y = k/3$, all the particles are unstable and move outward. Their vertical betatron oscillation amplitudes increase with each revolution. At a given amplitude, the particles encounter a septum and are extracted.

Consider an arrangement of four sextupoles, as indicated above, each with integrated strength $g_e = 0.27/\mathrm{m}^2$. If $b = \frac{1}{3}$ cm is the maximum vertical amplitude of the particles and $\hat{\beta}_y = 33$ m is the maximum vertical betatron amplitude function[2] for the lattice, then the deviation of the ν_y value from its value at resonance, for which all the particles of the beam are just stable, is given by[3]

$$|\Delta\nu_y| = \frac{\hat{\beta}_y g_e b}{\sqrt{3}\sqrt{3\pi}} = 0.008. \qquad (1.1)$$

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For this ν_y value the beam is enclosed in a triangular boundary, the separatrix. The three unstable fixed points at the corners of the triangle have an amplitude[3]

$$A_{sp} = \frac{2\pi|\Delta\nu_y|}{\hat{\beta}_y g_e} = b\sqrt{\frac{4\pi}{3\sqrt{3}}} \approx 0.5\ \mathrm{cm}. \qquad (1.2)$$

Thus the separatrix size is extremely sensitive to the value of ν_y. A shift from resonance ($\nu_y = k/3$) by less than 0.01 ν units causes the amplitude of the unstable fixed points, which define the boundary between stable and unstable particles, to change from 0 to 0.5 cm.

2. EFFECTS ON EXTRACTION FROM A SEXTUPOLE COMPONENT IN THE BENDING MAGNETS

A. *Without Compensation of the Bending Magnet Sextupole Component*

Because of the sensitivity of the separatrix size to the deviation of ν from its resonance value, small ν shifts can greatly influence particle extraction. In the case that there is a sextupole component present in the dipole magnets, we find that, via the coupling of vertical and horizontal particle motion, it can produce a shift of the vertical ν value large enough to suppress entirely the vertical extraction process.

The mechanism for this is as follows:

The equations for the horizontal and vertical motions are given by:

$$x'' + K_x x = -2S_E xy + S_D(x^2 - y^2) \qquad (2.1)$$

$$y'' + K_y y = S_E(y^2 - x^2) - 2S_D xy. \qquad (2.2)$$

Here differentiation is with respect to s, the distance around the machine, and K_x and K_y are the horizontal and vertical gradient focusing functions. $S_E(s)$ is the sextupole distribution which is provided for vertical extraction and $S_D(s)$ is the sextupole distribution in the dipole magnets of the lattice. In the particular case considered here, $S_E(s)$ contains all even harmonics which are not divisible by 4, while $S_D(s)$ is largely made up of a 0th harmonic and can be approximated by a constant:

$$S_D(s) = -\frac{\epsilon}{R}. \tag{2.3}$$

ϵ is defined as $\epsilon = \frac{1}{2}B''/B_0$ and R is the average radius of the machine.

It can be shown that the vertical motion at resonance ($\nu_y = k/3$, k is an integer) is essentially a betatron motion with increasing amplitude. This is a reflection of the fact that the particles move out along straight line separatrices in the vertical phase plane. Thus the solution of the equation for the vertical motion can be approximated by $y(s) \approx A(s)\sqrt{\beta_y/\bar{\beta}_y}\cos \nu_y s/R$, where A is a monotonically, but slowly, growing amplitude, and $\bar{\beta}_y = R/\nu_y$. $y^2(s)$ therefore contains an oscillatory (with frequency $2\nu_y$) part and a nonoscillatory part. As particles are extracted, their vertical amplitude, A, grows. This vertical growth causes a shift of the horizontal equilibrium orbit. The part of this orbit shift, δx, which is caused by the nonoscillatory part of $y^2(s)$ in turn produces a shift in the vertical ν value. Using the expression for δx obtained in the appendix one gets that for $\epsilon = 6/\text{m}^2 = 0.0039/\text{in.}^2$, the horizontal orbit shift, originating in the nonoscillatory part of y^2, when the particle vertical amplitude, A, has reached 1 cm is given by

$$\delta x = \frac{\epsilon A^2 R}{4\nu_x^2} \approx 0.018 \text{ cm}. \tag{2.4}$$

$R = 130$ m is the average radius of the accelerator and $\nu_x = 10.45$ is the horizontal ν value. This small orbit shift causes a vertical ν shift ($\nu_y = 22/3$)

$$\Delta\nu_y = -\frac{\epsilon^2 A^2 R^2}{4\nu_y \nu_x^2} \approx -0.019. \tag{2.5}$$

The resulting ν value, being off resonance, corresponds to a separatrix with unstable fixed points at "amplitude"

$$A_{sp} = \frac{2\pi|\Delta\nu_y|}{\hat{\beta}_y g_e} = \frac{2\pi\epsilon^2 A^2 R^2}{4\hat{\beta}_y g_e \nu_y \nu_x^2} = 1.3 \text{ cm}. \tag{2.6}$$

This is large enough to prevent a particle whose vertical amplitude is 1 cm from remaining unstable. The last equation expresses the fact that in the presence of a sextupole component, ϵ, in the dipole magnets, the motion of a particle, which in the extraction process has reached the amplitude A, will be governed by unstable fixed points of amplitude A_{sp}, proportional to A^2. If $A_{sp} \approx A$, the particles which up to A have been unstable will become retrapped by a separatrix created by their own vertical amplitude growth. The situation is thus inherently stable and no extraction can occur. The condition which must be satisfied if extraction is to take place is therefore

$$A_{sp} \ll A \tag{2.7}$$

or

$$\epsilon^2 \ll \frac{2\hat{\beta}_y g_e \nu_y \nu_x^2}{\pi A R^2} \approx \frac{27}{A} \quad (A \text{ in cm}) \tag{2.8}$$

or

$$\epsilon \ll \frac{5.2}{\sqrt{A}}. \tag{2.9}$$

For an extraction amplitude of 1.2 cm, this condition is

$$\epsilon \ll 4.7/\text{m}^2 \approx 0.0035/\text{in.}^2.$$

B. *With Compensation of the Sextupole Component by Discrete Sextupole Magnets*

The effect which a sextupole component in the dipole magnets has on the extraction can be completely eliminated† by the introduction of correcting sextupole windings in the magnets themselves. Partial compensation of the effect can also be achieved by a large number of separate sextupole magnets. In Ref. (1), one compensating sextupole magnet in each of the 48 cells is proposed.‡ This sextupole is placed right next to a vertically focusing quadrupole where β_y has a maximum.

With N separate compensating sextupoles the function $S_D(s)$ in Eqs. (2.1), (2.2) can be approximated by

$$S_D(s) = -\frac{\epsilon}{R} + g_c \sum_{i=1}^{N} \delta(s - s_i), \tag{2.10}$$

† A small residual sextupole component, varying randomly from magnet to magnet, does remain. Because this type of error introduces a resonant harmonic, it is not insignificant and will be dealt with in a separate study.

‡ Actually, two compensating sextupole magnets in each cell is proposed, giving a total of 96. However, the additional 48 are near vertically defocusing quadrupoles and thus have a smaller influence on vertical ν shifts, due to either equilibrium orbit shifts or betatron oscillation amplitudes.

where $\delta(s)$ is the Dirac delta function defined such that $\int_{-\infty}^{\infty} \delta(s)\,ds = 1$, g_c is the integrated strength of each of the N compensating sextupoles, and the s_i's are their positions around the machine. As shown in the appendix, the horizontal orbit shift that will cause a vertical ν shift is now given by

$$\delta x(s) = \tfrac{1}{4}A^2\sqrt{\beta_x}\sqrt{\frac{R}{\nu_x}}\left[\frac{\epsilon}{\nu_x} - \frac{g_c}{\sqrt{2}}\frac{\cos\nu_x(\pi/N - s/R)}{\sin \pi\nu_x/N}\right] \tag{2.11}$$

and the vertical ν shift becomes

$$\Delta\nu_y = \frac{A^2}{8\pi}\frac{R^3}{\nu_x^2\nu_y}\left[-\frac{\epsilon}{R}(2\pi\epsilon - g_c N\sqrt{2}) + \sqrt{2}g_c N \cdot\left(\frac{\epsilon}{R} - \frac{g_c\nu_x}{\sqrt{2}R}\cot\frac{\pi\nu_x}{N}\right)\right]. \tag{2.12}$$

$\hat{\beta}_y = 2R/\nu_y$ and $\beta_x^{\min} = \frac{1}{2}R/\nu_x$ is assumed in Eqs. (2.11) and (2.12). In order to have $\Delta\nu_y = 0$ one obtains the condition

$$g_c = \frac{\sqrt{2}\pi\epsilon}{N}\frac{1}{1+\sqrt{1-(\pi\nu_x/N)\cot(\pi\nu_x/N)}}. \tag{2.13}$$

With $N = 48$ and $\nu_x = 10.45$

$$g_c \approx \frac{\pi\epsilon}{N}. \tag{2.14}$$

If one takes into account the fact that the magnetic radius of curvature ρ is not equal to R, one obtains

$$g_c = \frac{\sqrt{2}\pi\epsilon}{N}\frac{1}{1+\sqrt{1-(\rho/R)(\pi\nu_x/N)\cot(\pi\nu_x/N)}}. \tag{2.15}$$

With $\rho = 90$ m,

$$g_c \approx 0.9\frac{\pi\epsilon}{N}. \tag{2.16}$$

Unfortunately, however, one will not be able to meet this requirement for proper extraction conditions. The presence of the sextupole component in the dipole magnets and the compensating sextupole magnets will also cause another undesirable effect, namely, a variation of ν_y across the aperture of the vacuum chamber of the machine, i.e., $\partial\nu_y/\partial R \neq 0$. Compensating for the latter effect (we neglect here and in what follows the contribution of the gradient forcing function, K_y, to $\partial\nu_y/\partial R$) requires[4]

$$g_c = \sqrt{2}\frac{\pi\epsilon}{N}. \tag{2.17}$$

This condition is not compatible with the requirement of Eq. (2.14) except in the limit $N \to \infty$.

Furthermore, it can easily be shown that even an allowance for $\Delta\nu_y = 0.2$ across the 2-cm-wide aperture of the vacuum chamber will not change the condition of Eq. (2.17) significantly and will not bring the value of g_c much closer to that required by Eq. (2.14). Consequently, one puts a tolerance on the sextupole component in the dipole magnet assuming $\partial\nu_y/\partial R = 0$. Substituting g_c from Eq. (2.17) into Eq. (2.12) one calculates the resulting value of $\Delta\nu_y$ and then gets an upper limit for ϵ using Eqs. (2.6) and (2.7). With the parameters of Ref. (1), one obtains

$$\epsilon \ll \frac{13}{\sqrt{A}}, \tag{2.18}$$

where ϵ is in m^{-2} and A in centimeters. Taking $A = 1.2$ cm

$$\epsilon \ll 12\text{ m}^{-2} = 0.0077\text{ in.}^{-2}. \tag{2.19}$$

3. NUMERICAL RESULTS

Numerical orbit calculations were performed which simulated the resonant extraction process with the parameters of Ref. (1). Runs were made for various sextupole fields in the bending magnets and different degrees of compensation.

The SYNCH program was used. Appropriate subroutines were added to this code in order to integrate the equations of horizontal and vertical motions in the sextupole field.

For each case two particles were traced. The initial conditions for these particles were chosen at the azimuth in the ring where the extraction process starts, i.e., at the septum. One of the particles was initially placed on a separatrix line corresponding to the exact resonance condition, i.e., going through the origin in the y–y' plane. The other particle was made to start on a similar separatrix line, but now corresponding to the off-resonant case for which the area inside the separatrix corresponds to the vertical emittance of the circulating beam. The particle coordinates at the location of the septum after each three revolutions were obtained and plotted in Figs. 1 to 3. If no sextupole fields were present other than those in the extraction magnets the particles would go out along the separatrix lines on which they started. The vertical emittance of the extracted beam corresponds to the area in the y–y' plane which is enclosed by four points, namely the positions of the two particles at the septum and the points which the particles reach after the following three revolutions.[5] These areas are drawn on the graphs.

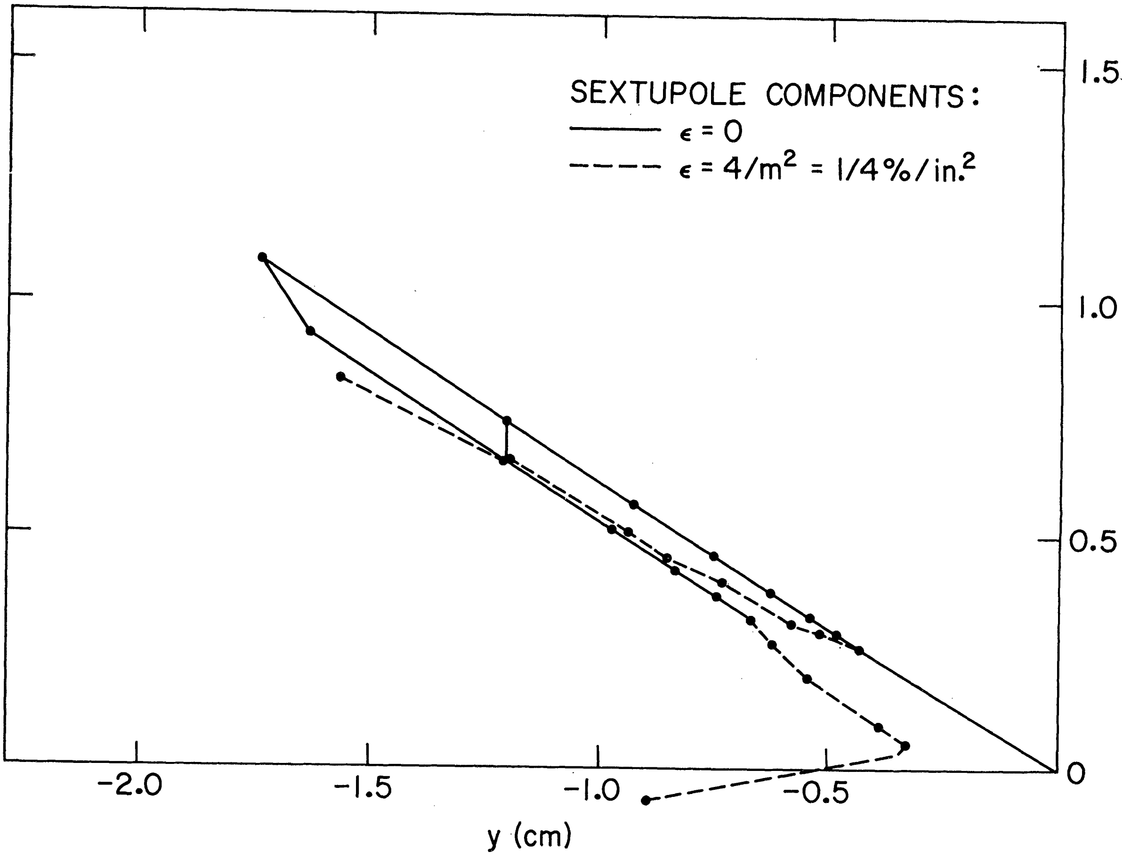

FIG. 1. Vertical phase-space trajectories for one-third integer extraction, no compensation for sextupole components (ϵ) in the dipole magnets.

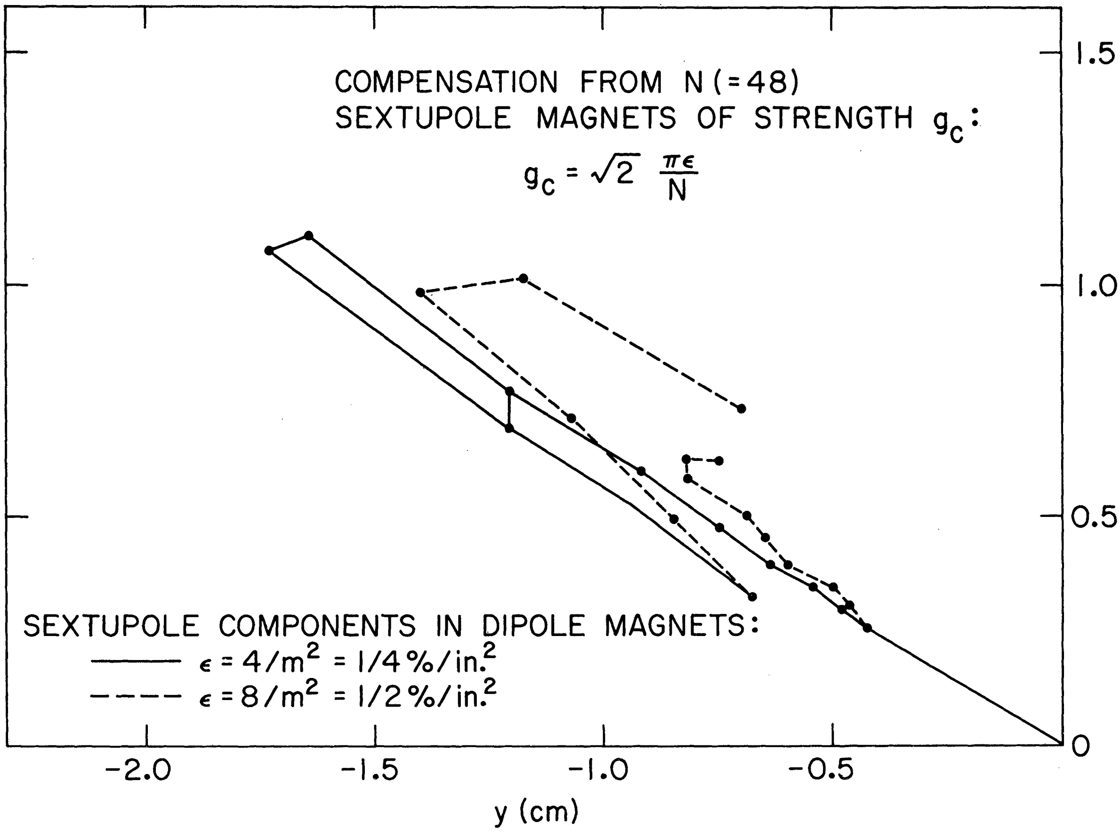

FIG. 2. Vertical phase-space trajectories for one-third integer extraction, with compensation providing for $\partial\nu_y/\partial R = 0$.

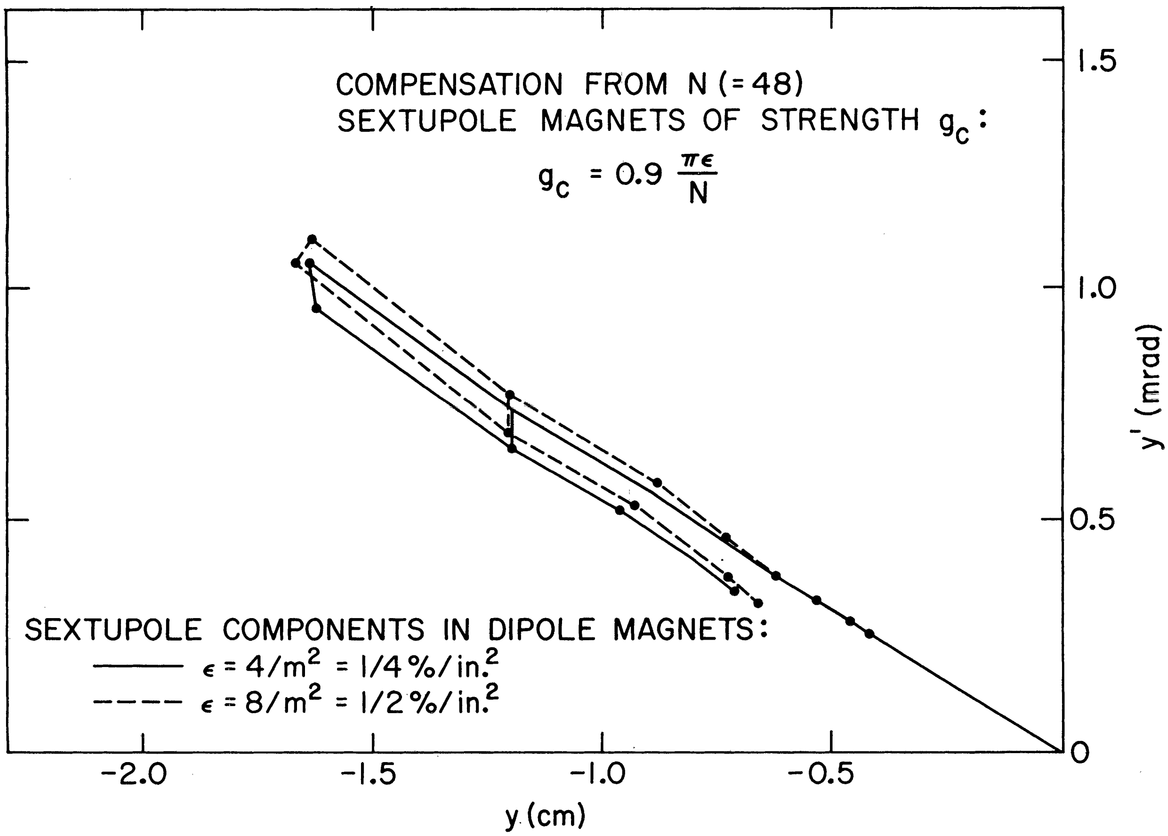

FIG. 3. Vertical phase-space trajectories for one-third integer extraction, with compensation determined by Eq. (2.16).

Figure 1 shows results for the case where no compensation is made for the sextupole fields in the bending magnets. In Fig. 2 data are plotted from runs using compensation according to the requirement of Eq. (2.17), i.e., $g_c = \sqrt{2}\pi\epsilon/N$. Figure 3 shows results which were obtained with $g_c = 0.9\pi\epsilon/N$, i.e., the condition of Eq. (2.16). As can be ascertained from these figures, the extraction quality is badly deteriorated at $\epsilon = 4/\mathrm{m}^2$ if no compensation is made (Fig. 1), while for the case of proper compensation, even $\epsilon = 8/\mathrm{m}^2$ is acceptable (Fig. 3). In the practical case, when the strength of the compensation sextupoles is such that $\partial\nu_y/\partial R = 0$, good extraction quality is obtained for $\epsilon \leq 4/\mathrm{m}^2 = \frac{1}{4}$ per cent/in.^{2}.

4. CONCLUSIONS

Both analytical and numerical results presented in this paper suggest that the presence of sextupole fields in the bending magnets of the lattice causes undesirable effects during one-third integer resonant vertical extraction. These effects can be completely eliminated only if compensation for the sextupole field component is made with pole windings in the magnets themselves. When separate sextupoles are used for compensation, proper resonant conditions (see Eq. (2.15)) cannot be restored completely, because the same sextupole magnets also have to correct for the variation of ν_y across the aperture of the vacuum chamber (see Eq. (2.17)). However, quite acceptable resonant extraction can be attained if the value of ϵ is small enough. With the parameters used in this paper, a tolerance for the sextupole component in the bending magnets of $\epsilon = \frac{1}{2}B''/B_0 \leq \frac{1}{4}$ per cent/in.^{2} was set. Similar conclusions hold for horizontal extraction as well.

APPENDIX

In order to get expressions for the horizonta orbit shift, δx, and the vertical ν shift, $\Delta\nu_y$, the effect of the extraction sextupole distribution will be taken into account by expressing y on the right-hand side of Eq. (2.1) as $y = A(s)\sqrt{\beta_y/\beta}\cos\nu_y s/R$.

Here $A(s)$, the amplitude of the vertical betatron oscillations, is steadily growing because of the existing resonance conditions provided by the extraction sextupoles. Keeping this in mind one can rewrite Eqs. (2.1), (2.2) as

$$x'' + K_x x = S_D(x^2 - y^2) \tag{A.1}$$

$$y'' + K_y y = -2S_D xy, \tag{A.2}$$

where

$$S_D(s) = -\frac{\epsilon}{R} + g_c \sum_{i=1}^{N} \delta(s - s_i). \tag{A.3}$$

In Eq. (A.1) the term $S_D x^2$ on the right-hand side can also be neglected because $x^2 \ll y^2$ during extraction. Following the procedure outlined in Ref. (2) one gets from Eq. (A.1)

$$\delta_x(\theta) = \frac{3\nu_x \sqrt{\beta_x}}{2\sin 3\pi\nu_x} \int_{\theta/3}^{\theta/3+2\pi} \beta_x^{3/2}(-S_D y^2) \cdot \cos 3\nu_x\left(\pi + \frac{\theta}{3} - \varphi\right) d\varphi, \tag{A.4}$$

where

$$\theta = \int \frac{ds}{\nu_x \beta_x}.$$

The integral is taken over three revolutions, which is the period of the sinusoidal part of y^2 in the approximated form of $A^2(s)(\beta_y/\bar{\beta})\cos^2(\nu_y s/R)$, ν_y being one-third of an integer. If one assumes that $y \approx A(s)\sqrt{\beta_y/\bar{\beta}_y}\cos(\nu_y s/R)$, one can replace y^2 by $\frac{1}{2}\bar{A}^2\beta_y/\bar{\beta}_y$. Here $\bar{A}$ is the average amplitude of the vertical oscillations during the three revolutions for which the ν shift is being calculated.

Introducing the expression for $S_D(s)$ from Eq. (A.3) and integrating gives

$$\delta x(\theta) = \frac{\nu_x\sqrt{\beta_x}}{2\sin\pi\nu_x}\frac{\bar{A}^2}{2}\left[\frac{\epsilon}{R}\bar{\beta}_x^{3/2}\frac{2\sin\pi\nu_x}{\nu_x} - \frac{g_c\hat{\beta}_y}{\nu_x\bar{\beta}_y}\sqrt{\beta_x^{\min}}\sum_{l=0}^{N-1}\cos\nu_x\left(\pi - \theta - \frac{2\pi}{N}l\right)\right]. \tag{A.5}$$

It can easily be shown that

$$\sum_{l=0}^{N-1}\cos\nu_x\left(\pi - \theta - \frac{2\pi}{N}l\right) = \frac{\sin\pi\nu_x\cos\nu_x(\pi/N - \theta)}{\sin\pi\nu_x/N}. \tag{A.6}$$

Making the approximations $\hat{\beta}_y = 2\bar{\beta}_y$, $\beta_x^{\min} = \bar{\beta}_x/2$ and $\bar{A}^2 = A^2/2$, where A is the amplitude which the vertical oscillations have reached at the end of the three revolutions under consideration, one gets

$$\delta x(\theta) = \tfrac{1}{4}\sqrt{\beta_x\bar{\beta}_x}A^2\left[\frac{\epsilon}{R}\bar{\beta}_x - \frac{g_c}{\sqrt{2}}\frac{\cos\nu_x(\pi/N - \theta)}{\sin\pi\nu_x/N}\right], \tag{A.7}$$

where $0 \le \theta < 2\pi/N$. If one now replaces x by $x + \delta x$ on the right-hand side of Eq. (A.2), one can calculate the resulting shift in ν_y according to Ref. (2)†:

$$\Delta\nu_y = \tfrac{1}{4}\pi\int_0^{2\pi R}\beta_y(2S_D)\,\delta x(\theta)\,ds. \tag{A.8}$$

Introducing the expressions for S_D and $\delta x(\theta)$ from Eqs. (A.3) and (A.7) into Eq. (A.8) and integrating gives

$$\Delta\nu_y = \frac{A^2}{8\pi}\left[-\bar{\beta}_x^2\bar{\beta}_y\frac{\epsilon}{R}(2\pi\epsilon - \sqrt{2}g_c N) + \frac{\bar{\beta}_x\hat{\beta}_y g_c N}{R} \cdot \sqrt{\beta_x^{\min}\bar{\beta}_x}\left(\epsilon - \frac{g_c}{\sqrt{2}}\nu_x\cot\frac{\pi\nu_x}{N}\right)\right]. \tag{A.9}$$

With $\hat{\beta}_y = 2\bar{\beta}_y$, $\beta_x^{\min} = \frac{1}{2}\bar{\beta}_x$, $\bar{\beta}_x = R/\nu_x$, and $\bar{\beta}_y = R/\nu_y$, there results

$$\Delta\nu_y = \frac{A^2}{8\pi}\frac{R^2}{\nu_x^2\nu_y}\left[-\epsilon(2\pi\epsilon - \sqrt{2}g_c N) + \sqrt{2}g_c N \cdot \left(\epsilon - \frac{g_c}{\sqrt{2}}\nu_x\cot\frac{\pi\nu_x}{N}\right)\right]. \tag{A.10}$$

† The calculation of the vertical ν shift is in fact more complicated. Since the value of ν_y over three revolutions is integer and the introduced 'gradient perturbation', $f(s) = 2S_D(s)\,\delta x(s)$, has no lower symmetry, then to calculate the shift in k, that is, three times the value of ν_y, one cannot use the simple formula given in Ref. (2), Eq. (4.31). It can be shown[(6)] that for an unperturbed ν value exactly an integer the shift Δk is given by

$$(\Delta k)^2 = \frac{1}{16\pi^2}[J_0^2 - |J_2|^2],$$

where

$$J_0 = \int_0^{3C}\beta_y(s)f(s)\,ds,$$

$$J_2 = \int_0^{3C}\beta_y(s)f(s)\exp[i(2k\theta/3)]\,ds,$$

$$\theta = \int_0^s \frac{d\rho}{\nu_y\beta_y(\rho)},$$

and the integration is taken over three revolutions, each of length C. If $|J_0| > |J_2|$, and $J_0 < 0$ as is the case here, then we have[(6)]

$$\Delta\nu_y = \frac{1}{12\pi}J_0\sqrt{1 - \frac{|J_2|^2}{J_0^2}}.$$

Thus the error introduced by using the formula $\Delta\nu_y = (1/12\pi)J_0$ is of the order $\frac{1}{2}|J_2|^2/12\pi J_0$. The quantity $|J_2|$ in the approximations used here is zero. This is because the sinusoidal part of $y^2 = A^2(\beta_y/\bar{\beta}_y)\cos^2(k\theta/3)$ was neglected. That is, J_0 arises from the nonoscillatory part of y^2, while $|J_2|$ comes from the part containing $\cos(2k\theta/3)$ i.e., the oscillatory term of frequency $2k/3$. If this term is included, however, it is found that the correction to the simple formula used in the text is of the order of 10 per cent. Furthermore, the correction to expressions that follow from setting $\Delta\nu_y = 0$ are negligible. We have therefore retained the simpler form for the ν shift.

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