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DISTORTIONS OF THE PHASE SPACE BEHAVIOR OF A PARTICLE DURING ONE-THIRD INTEGRAL RESONANT EXTRACTION†

M. MONTH AND E. D. COURANT‡

Brookhaven National Laboratory, Upton, New York, U.S.A.

Particles can be extracted from a synchrotron continuously over a long period of time, of the order of one second, by using the properties of particle motion in the vicinity of a third-integral resonance (that is, the betatron wave number $\nu = k/3$, with k an integer not divisible by three). This slow extraction can be achieved with an azimuthal sextupole distribution having a k^{th} harmonic. Distortions from the ideal resonant extraction conditions due to the nonresonant harmonics of the sextupole distribution (for example, a 0^{th} harmonic coming from the saturation of the lattice magnets) are considered. The use of an azimuthal octupole distribution to correct for these distortions is suggested.

1. INTRODUCTION

Particles can be extracted from an accelerator continuously over a long period of time, of the order of $\frac{1}{4}$ to 1 second, by using the properties of particle motion in the vicinity of a third-integral resonance, that is, ν = betatron frequency/revolution frequency $= k/3$, with k an integer not divisible by three. This slow extraction can be produced with an azimuthal sextupole distribution having a k^{th} harmonic. The resonance behavior and the extraction procedure have already been extensively studied.(1) Here we will consider distortions from the ideal resonant extraction conditions. We will see how these distortions from the ideal arise, how the extraction mechanism is influenced, and how corrections can be made.

2. EQUATION OF MOTION FOR A PARTICLE WITH NONLINEAR PERTURBATIONS

The motion of a particle under the influence of a gradient forcing function $K(s)$, a sextupole distribution $S(s)$, and an octupole distribution $O(s)$ is, in the one-dimensional case, governed by the Eq.

$$x'' + K(s)x + S(s)x^2 + O(s)x^3 = 0\,, \tag{2.1}$$

where s is the distance measured along the equilibrium orbit,

$$K(s) = \frac{(\partial B/\partial x)}{B\rho}, \tag{2.2}$$

$$S(s) = \frac{\frac{1}{2}(\partial^2 B/\partial x^2)}{B\rho}, \tag{2.3}$$

$$O(s) = \frac{\frac{1}{6}(\partial^3 B/\partial x^3)}{B\rho}, \tag{2.4}$$

B is the magnetic field and ρ is the magnetic radius of curvature. The linear, unperturbed, system can be solved(2) in terms of a betatron amplitude function, $\beta(s)$, and a phase function

$$\phi = \frac{1}{\nu}\int_0^s \frac{\mathrm{d}s'}{\beta(s')}. \tag{2.5}$$

The quantity ν is given by

$$\nu = \frac{1}{2\pi}\int_0^c \frac{\mathrm{d}s'}{\beta(s')},$$

where c is the length of the equilibrium orbit. We introduce normal coordinates η and $\dot{\eta}$ $(= \mathrm{d}\eta/\mathrm{d}\phi)$, related to x and x' by

$$\begin{bmatrix} \eta \\ \dot{\eta}/\nu \end{bmatrix} \begin{bmatrix} \beta^{-1/2} & 0 \\ \beta^{-1/2}\alpha & \beta^{1/2} \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} \tag{2.6}$$

with

$$\alpha = -\tfrac{1}{2}\frac{\mathrm{d}\beta}{\mathrm{d}s}. \tag{2.7}$$

In terms of these coordinates the linear problem reduces to the simple equation $\ddot{\eta} + \nu^2\eta = 0$, while the perturbed equation becomes

$$\ddot{\eta} + \nu^2\eta = \nu\Sigma\eta^2 + \nu\Omega\eta^3, \tag{2.8}$$

where

$$\Sigma = -\nu S(s)\beta^{5/2}, \tag{2.9}$$

and

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‡ Also at the State University of New York at Stony Brook.

$$\Omega = -\nu O(s)\beta^3. \tag{2.10}$$

This differential equation can be written as an integral Eq.

$$\eta(\phi) = \eta_0 \cos\nu\phi + \frac{\dot\eta_0}{\nu}\sin\nu\phi + \int_0^\phi \mathrm{d}\tau \cdot \sin\nu(\phi-\tau)[\Sigma(\tau)\eta^2(\tau) + \Omega(\tau)\eta^3(\tau)] \tag{2.11}$$

which can be solved by iteration. We write the solution

$$\eta = \sum_{i=0}^{\infty} f_i ,$$

where i designates the order of the perturbation, and

$$f_0 = \eta_0\cos\nu\phi + \frac{\dot\eta_0}{\nu}\sin\nu\phi .$$

If we treat the octupole distribution to first order and the sextupole distribution to second order, we obtain

$$\eta = f_0 + f_1^o + f_1^s + f_2^s, \tag{2.12}$$

where

$$f_1^o = \int_0^\phi \mathrm{d}\tau \sin\nu(\phi-\tau)\Omega(\tau)f_0^3(\tau), \tag{2.13}$$

$$f_1^s = \int_0^\phi \mathrm{d}\tau \sin\nu(\phi-\tau)\Sigma(\tau)f_0^2(\tau), \tag{2.14}$$

and

$$f_2^s = 2\int_0^\phi \mathrm{d}\tau \sin\nu(\phi-\tau)\Sigma(\tau)f_0(\tau)f_1^s(\tau). \tag{2.15}$$

If we specialize to the case where ν is near a third integer, $k/3$, where k is an integer not divisible by three:

$$\nu = \frac{k}{3} + \Delta \tag{2.16}$$

then after three revolutions a particle will return close to its phase space position. The deviation $\Delta\eta = \eta - \eta_0$, $\Delta\dot\eta = \dot\eta - \dot\eta_0$ can thus be expanded in terms of Δ, the ν deviation from a third integer, Σ, the sextupole strength, and Ω, the octupole strength. Introducing the vector notation

$$\xi_\mu = \left[\eta_0, \frac{\dot\eta_0}{\nu}\right], \tag{2.17}$$

$$\Delta\xi_\mu = \left[\Delta\eta, \frac{\Delta\dot\eta}{\nu}\right], \tag{2.18}$$

$$\lambda_\mu(\phi) = [\cos\phi, \sin\phi], \tag{2.19}$$

and

$$f_0(\phi) = \xi\cdot\lambda(\phi), \tag{2.20}$$

then using Eq. (2.12) and $\Delta\xi_\mu$, the change in phase space $(\eta, \dot\eta/\nu)$ position after three revolutions, can be written,

$$\Delta\xi_\mu = \Delta_1^\Delta\xi_\mu + \Delta_1^o\xi_\mu + \Delta_1^s\xi_\mu + \Delta_2^s\xi_\mu . \tag{2.21}$$

Here we have retained the first order in the ν shift from $k/3$, the first order in the octupole strength and up to the second order in the sextupole strength. The ν value in the expansion terms is taken at resonance, $\nu = k/3$. We find, using Eqs. (2.12) to (2.15) that

$$\Delta_1^\Delta\xi_\mu = 6\pi\Delta\epsilon_{\mu\nu}\xi_\nu, \tag{2.22}$$

$$\Delta_1^o\xi_\mu = -\int_0^{6\pi}\mathrm{d}\tau\Omega(\tau)\epsilon_{\mu\nu}\lambda_\nu(\tau)[\xi\cdot\lambda(\tau)]^3, \tag{2.23}$$

$$\Delta_1^s\xi_\mu = -\int_0^{6\pi}\mathrm{d}\tau\Sigma(\tau)\epsilon_{\mu\nu}\lambda_\nu(\tau)[\xi\cdot\lambda(\tau)]^2, \tag{2.24}$$

$$\Delta_2^s\xi_\mu = -2\int_0^{6\pi}\mathrm{d}\tau\int_0^\tau \mathrm{d}\omega\Sigma(\tau)\Sigma(\omega) \cdot\sin\frac{k}{3}(\tau-\omega)\epsilon_{\mu\nu}\lambda_\nu(\tau)[\xi\cdot\lambda(\tau)][\xi\cdot\lambda(\omega)]^2, \tag{2.25}$$

where repeated indices are summed, and $\epsilon_{\mu\nu}$ is the two-dimensional antisymmetric tensor, defined by

$$\epsilon_{12} = -\epsilon_{21} = 1,$$
$$\epsilon_{11} = \epsilon_{22} = 0.$$

Here the superscripts o and s refer to octupole and sextupole contributions, respectively. We will also use the superscript Δ to refer to contributions due to the deviation of ν from resonance.

3. THEORY OF THE PHASE SPACE INVARIANT NEAR A ONE-THIRD INTEGER RESONANCE

In the previous section, we found an expression for the change in the particle phase space coordinates, $\Delta\xi$, near a one-third integer ν value due to three influences: (1) the ν deviation from $k/3$, termed Δ, to first order; (2) the octupole distribution, $\Omega(s)$, to first order; and (3) the sextupole distribution, $\Sigma(s)$, to second order. We wish now to find an invariant function,[3] $I(\xi)$, which after three revolutions has the same value, i.e.,

$$I(\xi + \Delta\xi) = I(\xi). \tag{3.1}$$

If we expand the left side of this equation in a Taylor series, we have to second order in $\Delta\xi$,

$$\frac{\partial I}{\partial \xi_\mu} \Delta\xi_\mu + \tfrac{1}{2}\frac{\partial^2 I}{\partial \xi_\mu \partial \xi_\nu} \Delta\xi_\mu \Delta\xi_\nu = 0. \tag{3.2}$$

Writing I as an expansion in the perturbing parameters Δ, Ω, and Σ:

$$I = I_1^\Delta + I_1^o + I_1^s + I_2^s \tag{3.3}$$

and using the expansion for $\Delta\xi$ in Eq. (2.21) we obtain, after equating to zero separately the terms due to the three different perturbation parameters,

$$\frac{\partial I_1^\Delta}{\partial \xi_\mu} \Delta_1^\Delta \xi_\mu = 0, \tag{3.4}$$

$$\frac{\partial I_1^o}{\partial \xi_\mu} \Delta_1^o \xi_\mu = 0, \tag{3.5}$$

$$\frac{\partial I_1^s}{\partial \xi_\mu} \Delta_1^s \xi_\mu = 0, \tag{3.6}$$

$$\frac{\partial I_2^s}{\partial \xi_\mu} \Delta_1^s \xi_\mu + \frac{\partial I_1^s}{\partial \xi_\mu} \Delta_2^s \xi_\mu + \frac{1}{2}\frac{\partial^2 I_1^s}{\partial \xi_\mu \partial \xi_\nu} \Delta_1^s \xi_\mu \Delta_1^s \xi_\nu = 0. \tag{3.7}$$

To obtain Eq. (3.4), for example, we put $\Omega = \Sigma = 0$. Equations (3.6) and (3.7) are obtained by setting $\Delta = \Omega = 0$ and equating to zero separately the two orders in Σ. Equation (3.7) can be simplified if we differentiate Eq. (3.6) and use the result in Eq. (3.7). We obtain in place of Eq. (3.7),

$$\frac{\partial I_2^s}{\partial \xi_\mu} \Delta_1^s \xi_\mu + \frac{\partial I_1^s}{\partial \xi_\mu}\left[\Delta_2^s \xi_\mu - \frac{1}{2}\frac{\partial \Delta_1^s \xi_\mu}{\partial \xi_\nu} \Delta_1^s \xi_\mu\right] = 0. \tag{3.8}$$

In arriving at these equations for the invariants I_1^Δ, I_1^o, and I_1^s, we have made the physical assertion that they be independent. That is, since they are of the first order in the perturbation parameters then I_1^s, for example, cannot depend on either Δ or $O(s)$. However, when the expansions for $I(\xi)$ and $\Delta\xi$ are substituted into Eq. (3.2), cross terms do arise. We have thus implicitly assumed that the solution for $I(\xi)$ will lead to no such cross terms. This provides us with the following subsidiary equations on the solutions for the partial invariants:

$$\frac{\partial I_1^s}{\partial \xi_\mu} \Delta_1^o \xi_\mu + \frac{\partial I_1^o}{\partial \xi_\mu} \Delta_1^s \xi_\mu = 0, \tag{3.9}$$

$$\frac{\partial I_1^s}{\partial \xi_\mu} \Delta_1^\Delta \xi_\mu + \frac{\partial I_1^\Delta}{\partial \xi_\mu} \Delta_1^s \xi_\mu = 0, \tag{3.10}$$

and

$$\frac{\partial I_1^o}{\partial \xi_\mu} \Delta_1^\Delta \xi_\mu + \frac{\partial I_1^\Delta}{\partial \xi_\mu} \Delta_1^o \xi_\mu = 0. \tag{3.11}$$

The coordinate changes satisfy certain conditions, integrability conditions, which enable us to solve the partial differential Eqs. (3.4) to (3.8). These are:

$$\frac{\partial \Delta_1^\Delta \xi_\mu}{\partial \xi_\mu} = 0, \tag{3.12}$$

$$\frac{\partial \Delta_1^o \xi_\mu}{\partial \xi_\mu} = 0, \tag{3.13}$$

$$\frac{\partial \Delta_1^s \xi_\mu}{\partial \xi_\mu} = 0, \tag{3.14}$$

$$\frac{\partial \Delta_2^s \xi_\mu}{\partial \xi_\mu} = \frac{1}{2}\frac{\partial \Delta_1^s \xi_\mu}{\partial \xi_\nu}\frac{\partial \Delta_1^s \xi_\nu}{\partial \xi_\mu}. \tag{3.15}$$

Using these conditions we can show that the following functions satisfy the partial differential Eqs. (3.4) to (3.8):

$$I_1^\Delta = \tfrac{1}{2}\xi_\mu \epsilon_{\mu\nu} \Delta_1^\Delta \xi_\nu, \tag{3.16}$$

$$I_1^o = \tfrac{1}{4}\xi_\mu \epsilon_{\mu\nu} \Delta_1^o \xi_\nu, \tag{3.17}$$

$$I_1^s = \tfrac{1}{3}\xi_\mu \epsilon_{\mu\nu} \Delta_1^s \xi_\nu, \tag{3.18}$$

$$I_2^s = \tfrac{1}{4}\xi_\mu \epsilon_{\mu\nu} \Delta_2^s \xi_\nu. \tag{3.19}$$

Furthermore, they satisfy the subsidiary conditions, Eqs. (3.9) to (3.11). The sum

$$I(\xi) = I_1^\Delta + I_1^o + I_1^s + I_2^s$$

satisfies Eq. (3.2) and is therefore the desired invariant of the motion. With the expressions, Eqs. (2.22) to (2.25), for the coordinate changes after three revolutions, we obtain

$$I_1^\Delta = -\tfrac{1}{2}6\pi\Delta(\xi \cdot \xi), \tag{3.20}$$

$$I_1^o = \tfrac{1}{4}\int_0^{6\pi} d\tau\,\Omega(\tau)[\xi \cdot \lambda(\tau)]^4, \tag{3.21}$$

$$I_1^s = \tfrac{1}{3}\int_0^{6\pi} d\tau\,\Sigma(\tau)[\xi \cdot \lambda(\tau)]^3, \tag{3.22}$$

$$I_2^s = \tfrac{1}{2}\int_0^{6\pi} d\tau \int_0^{\tau} d\omega\,\Sigma(\tau)\Sigma(\omega)$$

$$\sin\frac{k}{3}(\tau - \omega)[\xi \cdot \lambda(\tau)]^2[\xi \cdot \lambda(\omega)]^2. \tag{3.23}$$

The invariant functions can be written in terms of the Fourier harmonics of the sextupole and octupole distributions. Expanding these, we have

$$\Omega(\phi) = \sum_{\kappa=-\infty}^{\infty} \Omega_\kappa e^{i\kappa\phi} \tag{3.24}$$

$$\Sigma(\phi) = \sum_{\kappa=-\infty}^{\infty} \Sigma_\kappa e^{i\kappa\phi}, \tag{3.25}$$

where

$$\Omega_\kappa = \frac{1}{2\pi}\int_0^{2\pi} \Omega(\phi)\, e^{-i\kappa\phi}\, d\phi\,, \tag{3.26}$$

$$\Sigma_\kappa = \frac{1}{2\pi}\int_0^{2\pi} \Sigma(\phi)\, e^{-i\kappa\phi}\, d\phi\,. \tag{3.27}$$

We introduce the complex variable

$$z = \xi_1 + i\xi_2 = \eta_0 + \frac{i\eta_0'}{\nu}. \tag{3.28}$$

The invariants I_1^0 and I_1^s are then given by

$$I_1^o = \frac{9\pi}{16}\Omega_0 z^2\bar{z}^2, \tag{3.29}$$

and

$$I_1^s = \frac{\pi}{2} Re[z^3\Sigma_k], \tag{3.30}$$

where $\bar{z}$ is the complex conjugate of z. The second order invariant is more complicated. However, if we neglect the resonant k^{th} harmonic, it simplifies to

$$I_2^s \simeq -\frac{9\pi\nu}{16} z^2\bar{z}^2 \sum_\kappa |\Sigma_\kappa|^2 \left[\frac{1}{\kappa^2-(k/3)^2} + \frac{1}{\kappa^2-k^2}\right]. \tag{3.31}$$

This is a reasonable approximation to make in that we are looking for terms which distort the ideal resonance behavior. It is the resonant harmonic, Σ_k, which produces this ideal behavior and it is assumed that it is small enough so that in the region of phase space up to the amplitude where extraction occurs, terms of the order of $|\Sigma_k|^2$ do not distort the resonant behavior. If there is some harmonic, say Σ_l, that does distort the ideal phase space resonance behavior, then terms of the order $|\Sigma_l\Sigma_k|$ may have some influence, but certainly the dominant influence will come from terms of the order $|\Sigma_l|^2$, which are included in Eq. (3.31).

4. DISTORTION OF THE IDEAL INVARIANT AND ITS CORRECTION

The part of the invariant which represents the ideal resonance behavior of a particle in phase space is given by I_1^s in Eq. (3.30) and is derived from the k^{th} harmonic of the sextupole distribution. Distortions of this ideal particle behavior will come from other harmonics of the sextupole distribution and contribute to I_2^s, as in Eq. (3.31). A term like I_2^s may produce three stable fixed points in the phase space and, in any case, if it is large enough, inhibits the particles from being extracted by curving them backward toward the central orbit.[4] Keeping in mind the approximation used in deducing Eq. (3.31), i.e., small Σ_k, we have that these distortions can be eliminated by adding a set of octupole magnets with a 0^{th} harmonic given by (see Eq. 3.29):

$$\Omega_0 = \nu\sum_\kappa |\Sigma_\kappa|^2\left\{\frac{1}{\kappa^2-(k/3)^2} + \frac{1}{\kappa^2-k^2}\right\}. \tag{4.1}$$

In cases where one harmonic, Σ_l, is the dominant distorting influence, we can remove this harmonic with an additional sextupole distribution, thus reducing the cross terms of the order $|\Sigma_l\Sigma_k|$. Then we can apply Eq. (4.1) to the total remaining sextupole distribution. This will be the situation in the case of saturation of the lattice magnets of the accelerator, where the dominant distorting influence is a 0^{th} harmonic.

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