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INVARIANCE PROPERTIES OF SOME BILINEAR AND QUADRATIC FORMS IN CORPUSCULAR OPTICS

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The purpose of this paper is to point out the invariance properties of certain combinations of the matrix elements of a beam transport channel.
These properties can be used, for example, in the analytic determination of the phase acceptance of a quadrupole multiplet.

1. INVARIANCE PROPERTIES RELATED TO A FOCUSING LENS

1.1. *Backward Invariance*

Let F be an arbitrary focusing lens located in a beam transport channel (Fig. 1). If l is the length of the lens and $k = \sqrt{Gq/p}$ its focusing parameter (G stands for the magnetic gradient, q for the charge of the particle, and p for its momentum), its transfer matrix can be written

$$F = \begin{Vmatrix} \cos\theta & (1/k)\sin\theta \\ -k\sin\theta & \cos\theta \end{Vmatrix} \tag{1}$$

where $\theta = kl$.

Let m_e now be the transfer matrix from some arbitrary origin to the entrance of F, and M_e the transfer matrix from the same origin to *some arbitrary location* inside F. We put

$$m_e = \begin{Vmatrix} a_e & b_e \\ c_e & d_e \end{Vmatrix} \tag{2}$$

and

$$M_e = \begin{Vmatrix} A_e & B_e \\ C_e & D_e \end{Vmatrix} \tag{3}$$

(the subscript e stands for entrance).
It is readily seen that:

$$\begin{aligned} k^2A_e^2 + C_e^2 &= k^2a_e^2 + c_e^2 \\ k^2B_e^2 + D_e^2 &= k^2b_e^2 + d_e^2 \\ k^2A_eB_e + C_eD_e &= k^2a_eb_e + c_ed_e \end{aligned} \tag{4}$$

Although the matrix coefficients A_e, B_e, C_e and D_e are all functions of the location $\theta_e = kl_e$ inside the lens (Fig. 1), the quantities

$$\begin{aligned} &k^2A_e^2 + C_e^2 \\ &k^2B_e^2 + D_e^2 \\ &k^2A_eB_e + C_eD_e \end{aligned}$$

are invariants with respect to location inside the lens and independent of the length of the lens.

1.2. *Forward Invariance*

As before we consider the lens F to be part of a beam channel and define the two matrices m_s and M_s (Fig. 2), extending respectively from the exit of the lens to some arbitrary termination and from *some arbitrary location* inside the lens to the same termination.

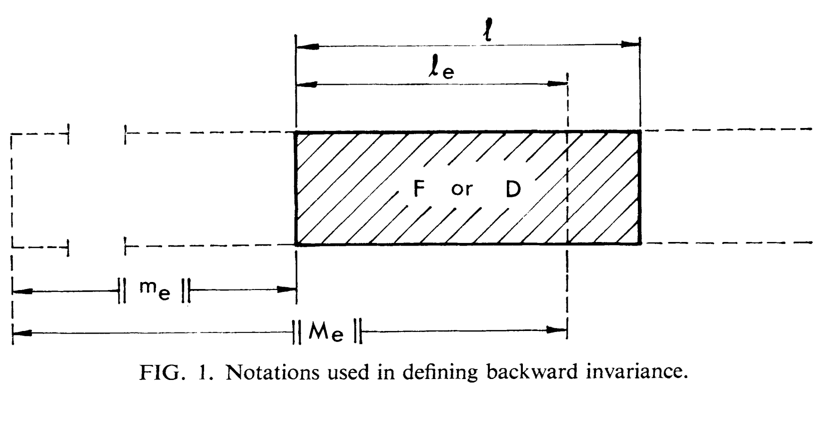

FIG. 1. Notations used in defining backward invariance.

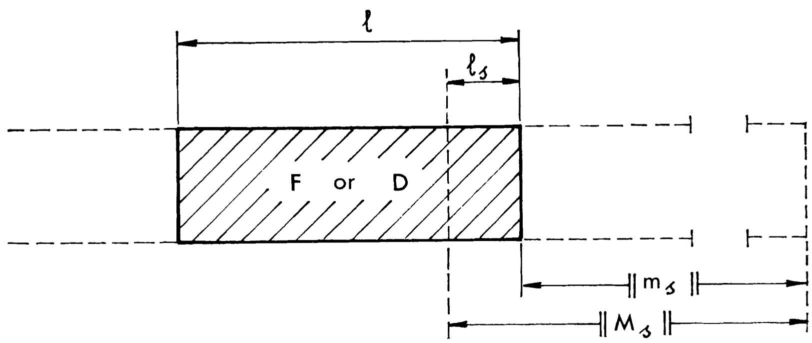

FIG. 2. Notations used in defining forward invariance.

Putting

$$m_s = \begin{Vmatrix} a_s & b_s \\ c_s & d_s \end{Vmatrix} \tag{5}$$

and

$$M_s = \begin{Vmatrix} A_s & B_s \\ C_s & D_s \end{Vmatrix} \tag{6}$$

(s stands for exit (sortie)), we have

$$\begin{aligned} k^2 D_s^2 + C_s^2 &= k^2 d_s^2 + c_s^2 \\ k^2 B_s^2 + A_s^2 &= k^2 b_s^2 + a_s^2 \\ k^2 B_s D_s + A_s C_s &= k^2 b_s d_s + a_s c_s \end{aligned} \tag{7}$$

These relations can be obtained without calculation, merely by interchanging A and D in Eqs. (4), by virtue of mirror symmetry properties.

As before, while the matrix coefficients A_s, B_s, C_s and D_s are all functions of the location $\theta_s = kl_s$ inside the lens (Fig. 2), the quantities

$$\begin{gathered} k^2 D_s^2 + C_s^2 \\ k^2 B_s^2 + A_s^2 \\ k^2 B_s D_s + A_s C_s \end{gathered}$$

are invariants with respect to location inside the lens and independent of the length of the lens.

1.3. *Mixed Invariance*

We now consider the two situations (Fig. 3) and calculate

$$\begin{aligned} & B_e A_s + B_s D_e \\ & \quad = (b_e a_s + b_s d_e) \cos\theta + \left(\frac{a_s d_e}{k} - k b_e b_s\right) \sin\theta \\ & C_e D_s + C_s A_e \\ & \quad = (c_e d_s + c_s a_e) \cos\theta + \left(\frac{c_e c_s}{k} - k a_e d_s\right) \sin\theta \\ & k^2 B_e B_s - A_s D_e \\ & \quad = (k^2 b_e b_s - a_s d_e) \cos\theta + k(b_e a_s + b_s d_e) \sin\theta \\ & k^2 A_e D_s - C_e C_s \\ & \quad = (k^2 a_e d_s - c_e c_s) \cos\theta + k(c_e d_s + c_s a_e) \sin\theta \end{aligned} \tag{8}$$

The quantities

$$\begin{gathered} B_e A_s + B_s D_e \\ C_e D_s + C_s A_e \\ k^2 B_e B_s - A_s D_e \\ k^2 A_e D_s - C_e C_s \end{gathered}$$

are therefore invariants with respect to location inside the lens; however, they depend on the length of the lens. The same property holds for the expressions:

$$(B_e \pm B_s)^2 + \frac{1}{k^2}(A_s \mp D_e)^2.$$

On the contrary, the quantities

$$\begin{gathered} k^2(B_e A_s + B_s D_e)^2 + (k^2 B_e B_s - A_s D_e)^2 \\ k^2(C_e D_s + C_s A_e)^2 + (k^2 A_e D_s - C_e C_s)^2 \end{gathered}$$

are invariants with respect to location inside the lens and do not depend on the length of the lens, as can also be seen from Eqs. (4) and (7).

2. INVARIANCE PROPERTIES RELATED TO A DEFOCUSING LENS

2.1. *Backward Invariance*

Using again Fig. 1, we consider a defocusing lens placed in a beam transport channel. With the same notations as before, its matrix will now be:

$$D = \begin{Vmatrix} \operatorname{ch}\theta & (1/k)\operatorname{sh}\theta \\ k\operatorname{sh}\theta & \operatorname{ch}\theta \end{Vmatrix} \tag{9}$$

Calling again m_e the transfer matrix from some arbitrary origin to the entrance of D, and M_e the transfer matrix from the same origin to *some arbitrary location* inside D, we have

$$\begin{aligned} k^2 A_e^2 - C_e^2 &= k^2 a_e^2 - c_e^2 \\ k^2 B_e^2 - D_e^2 &= k^2 b_e^2 - d_e^2 \\ k^2 A_e B_e - C_e D_e &= k^2 a_e b_e - c_e d_e. \end{aligned} \tag{10}$$