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SOME LIMITATIONS IN RELATIVISTIC CHARGED PARTICLE BEAMS

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Some of the basic characteristics of linear and ring-shaped particle beams are reviewed, with the aid of a simple model and suitable dimensionless parameters. The limiting behaviour as the parameter ν/γ approaches unity (where $\nu = Ne^2/m_0c^2$) is discussed in detail. Both static and dynamic systems are considered, and a simple derivation is given of the equilibrium conditions of Budker's self-constricted ring beam.

Emphasis is placed on the essential physical characteristics of the beams rather than accurate mathematical description; many of the results represent order of magnitude calculations to show the relation between parameters rather than precise formulae on which to base design. In this respect the paper forms a sequel to a previous one.[1]

1. INTRODUCTION

The limitation to the beam currents which can be held in particle accelerators and storage rings is set either by instabilities which arise from collective interactions, or by modifications to the external focusing field which arise because of static space-charge forces.[2] In the present review the ultimate limits of this second type are considered, even though in practice the beams considered might well be unstable. Although the treatment may for this reason appear at times to be academic, it shows clearly what the limiting physical features are, and leads naturally to a discussion of what is meant by a 'beam' and a 'plasma'.

A complete description of a typical beam in a particle accelerator is complicated; the beam is in general bunched, it contains particles of different energies, distributed in radius, which move at various angles to some mean equilibrium orbit. Different models are used to describe the beam according to what needs to be calculated; in the present paper an extremely simple though somewhat unrealistic model suffices. By 'unrealistic' it is meant that such a beam is unlikely to be produced in practice, not that the model is not properly self-consistent. Lack of realism is compensated for by the clarity with which essential features relevant to the present study are exhibited; similar features apply of course to actual beams, but form factors of order perhaps 2 may be needed in the equations, and single valued qualities need in some cases to be replaced by distributions.

2. DESCRIPTION OF MODEL TO BE STUDIED

Although the arguments in this paper may be applied to beams of any type of charged particle, attention will be directed towards electron beams partly neutralized with positive ions. To begin with, a beam with the following properties is postulated:

1. All the electrons have the same momentum $\beta\gamma mc$.
2. The current density is uniform for $r < a$, and zero for $r > a$.
3. The beam is partially neutralized by positive ions, which are uniformly distributed over its cross section.
4. The ion motion is transverse, giving zero contribution to the current.
5. Collisions between electrons and ions may be neglected.
6. The transverse velocity of the electrons is small compared with their velocity along the beam.
7. Any external focusing force is a linear function of radius measured from the beam axis.

The conditions under which a beam can possess these properties (with or without external focusing) will become apparent as the analysis proceeds.

The essential properties of such a beam in the absence of external focusing may be found quite simply. It is more convenient to work with the basic dimensionless quantities β, γ, ν, f (defined in the list of symbols, Sec. 10), than the more usual current, energy, number density, etc. In terms of these, and the 'generalized perveance' defined as

$$K = \frac{2\nu}{\beta^2\gamma}\left(\frac{1}{\gamma^2} - f\right) \tag{1}$$

the following properties may readily be established[3]:

A. Since B_θ and E_r are proportional to r, the projections of the electron trajectories on a plane through the beam axis are sinusoidal with wavelength

$$\lambda\!\!\!^{-}_s = a(-K)^{-1/2}, \text{ for } (-K)^{-1/2} \gg 1 \text{ and real.} \tag{2}$$

B. The ions move in a transverse parabolic potential well of depth

$$W = (1 - f)\,\nu m_0 \beta^2 c^2. \tag{3}$$

C. The ratio of transverse oscillation frequencies of electrons and ions is

$$\omega_e{}^2/\omega_i{}^2 = M(f - \gamma^{-2})/\gamma m_0(1 - f) \tag{4}$$

where M is the ion mass per unit charge.

From Eq. (2) it is evident that for a beam to exist $-K$ must be positive; this implies, as may be seen from Eq. (1), that $f > \gamma^{-2}$. Furthermore, property 6 attributed to the beam (drift velocity much greater than transverse velocity) can only be satisfied if $\lambda\!\!\!^{-} \gg a$, or $-K \ll 1$. Indeed, by suitable averaging, it is not difficult to show[3] that

$$\frac{\langle \beta^2{}_T \rangle}{\langle \beta_L{}^2 \rangle} \approx -\tfrac{1}{2}K \tag{5}$$

where β_T and β_L are the normalized transverse and longitudinal velocities. Thus in a sense $K = -1$ denotes the demarcation between a beam, where the particle motion is predominantly longitudinal, and a plasma in which the transverse velocity exceeds the drift velocity.[1] For the special case of a completely neutralized beam, $K = -2\nu/\gamma$.

3. RELATION TO BENNETT'S MODEL

It is interesting to see the relationship between the uniform model and Bennett's original model of the self-constricted electron beam.[4] Bennett's model refers only to completely neutralized beams, but it is slightly more general than the model described above in that he assumes also a drift velocity for the ions. Otherwise however it is in essence very similar, despite the fact that his choice of a Maxwellian transverse velocity distribution and subsequent use of the 'temperature' concept gives it a superficial appearance of being different. Since, as in the uniform model, collisions are neglected, the choice of a Maxwellian distribution is somewhat arbitrary, though, as he explains, it is not altered if a few collisions do in fact occur. Bennett also assumes that the drift velocity greatly exceeds the tranverse velocity, though he does not explicitly state that this implies the constraint $-K \ll 1$. His well-known Pinch Relation

$$I^2 = 2Nk(T_e + T_i) \tag{6}$$

relates the transverse velocities to the drift velocities and N through the relations $\frac{1}{2}m_0\langle\beta_T^2\rangle = kT$ and $I = Ne\beta_L$; Eq. (6) is in fact completely equivalent to Eq. (5) (and the corresponding relation for ions) as is explained in Ref. 3.

It is a remarkable fact tha Eq. (6), though derived originally for a collision free system with drift velocity greatly exceeding the transverse thermal velocity, is also true when collisions are present and the thermal velocity greatly exceeds the drift velocity.[5]

The radial density function has a more natural shape in Bennett's model, being proportional to $(a^2 + r^2)^{-2}$; this is fairly sharply defined distribution, but does not have the discontinuity at $r = a$ exhibited by the uniform model. The value of $\lambda\!\!\!^{-}_s$ is not the same for all electrons in the beam, but depends on the oscillation amplitude. For small amplitudes it has the same value as the uniform model with the same perveance.

4. LIMITATION TO THE IDEA OF A BEAM

It was suggested in Sec. 2 that $-K = 1$ marks the division between self-constricted beams and plasmas. With this definition there is no limit to the current obtainable in a beam provided that $f = \gamma^{-2}$. In other respects, however, the parameter ν/γ which equals $-K/2$ for the special case of $f = 1$, marks a more significant transition point. This will be seen in Sec. 5, but it is also apparent from the following argument.

It is possible to consider an electron beam as a special case of a collisionless plasma, in which the beam radius is less than the skin depth. This implies that when an external accelerating field is applied, the inner electrons are not shielded by those on the outside, a condition clearly necessary in an accelerator. The collisionless skin depth is given by

$$\delta = c/\omega_p = (\gamma m_0 c^2/4\pi n e^2)^{1/2} = a(\gamma/4\nu)^{1/2}. \quad (7)$$

Setting this much greater than a gives

$$4\nu/\gamma \ll 1. \quad (8)$$

For a neutral beam it is evident that $\delta = \lambda\!\!\!^{-}_s/\sqrt{2}$.

5. EFFECT OF EXTERNAL FOCUSING FIELD

The simplest type of external focusing field is a betatron field with $n = -d(\ln B)/d(\ln r) = \frac{1}{2}$. In a low current beam ($K \approx 0$) this produces an oscillation wavelength $\lambda\!\!\!^{-}_e = R/\sqrt{n} = \sqrt{2}R$, where R is the radius of the equilibrium orbit. The wavelength $\lambda\!\!\!^{-}$ in the presence of self and external fields is given by

$$\frac{1}{\lambda\!\!\!^{-2}} = \frac{1}{\lambda\!\!\!^{-}_s{}^2} + \frac{1}{\lambda\!\!\!^{-}_e{}^2} = -\frac{K}{a^2} + \frac{1}{2R^2}. \quad (9)$$

$\lambda\!\!\!^{-}$ is real even for positive values of K provided that $K < a^2/2R^2$. Setting these two quantities equal for the case of $f = 0$ yields the standard formula for the maximum current which can be held in a betatron (neglecting image effects):

$$\nu = \frac{a^2}{4R^2}\beta^2\gamma^3. \quad (10)$$

If n is not equal to $\frac{1}{2}$, the radial and vertical forces are not equal, and the beam assumes an elliptical shape. In practice the walls of the vacuum chamber need to be taken into account, but in an idealized system in which they are absent, the radial and axial Q-values, where $Q = R/\lambda\!\!\!^{-}$, can quite simply be shown to be[6]

$$\left.\begin{aligned} Q_R{}^2 &= 1 - n - \frac{2R^2K}{a(a+b)} \\ Q_A{}^2 &= n - \frac{2R^2K}{b(a+b)} \end{aligned}\right\} \quad (11)$$

where a and b are the radial and axial semi-axes.

6. STORED MAGNETIC AND ELECTRIC ENERGY

The stored magnetic and electric energy associated with an electron beam increase as the square of the beam current and charge respectively. As N increases therefore there comes a point at which they can no longer be neglected in comparison with the kinetic energy, which increases linearly. For a straight beam the stored energy is logarithmically infinite; for a ring on the other hand the divergence is removed, and the stored energy is proportional to $\frac{1}{2}(LI^2 + Q^2/C)$, where the symbols have their usual meaning.

Denoting the inductance per unit length by l and the reciprocal of the capacity per unit length by k, the various components contributing to the energy may readily be seen to be in the following proportions:

Rest energy:	1.
Kinetic energy:	$\gamma - 1$, ($\approx \frac{1}{2}\beta^2$ for $\gamma - 1 \ll 1$).
Magnetic energy:	$\beta^2 l\nu$.
Electrostatic energy:	$(1-f)\,k\nu$.

The energy of trapped positive ions is not included, since this is in general small. For a ring beam the dimensionless quantities k and l are of order $2\ln R/a$; this quantity, which will be denoted by Λ, might typically be 5–10. Very roughly therefore the ratio of stored energy U to kinetic energy T may be written

$$U/T = 2\{(1-f)/\beta^2 + 1\}\Lambda\nu \approx \tfrac{1}{2}K\Lambda \quad \text{(Non Relativistic).}$$

$$\approx (2-f)\Lambda\nu/\gamma \quad \text{(Extreme Relativistic).}$$

$$\approx -\tfrac{1}{2}K\Lambda \quad \text{(Extreme Relativistic and } f = 1\text{).}$$

It is interesting to see again the role of the factor ν/γ in the extreme relativistic case.

As might be expected, the equilibrium configuration of a ring beam in a betatron type magnetic field is affected when the parameter U/T becomes significant. The radius of such a ring at low currents is determined by the balance between the centrifugal and Lorentz forces; as the charge and current increase, however, the mutual interaction of the different elements of the ring produces additional outward forces which increase the radius. In the absence of vacuum chamber walls it is not difficult to show that the radius of the ring is increased, for the same field at the orbit, by the factor[7]

$$R/R_0 = 1 + \tfrac{1}{2}\{1 + (1-f)/\beta^2\}\Lambda\nu/\gamma. \quad (12)$$

The energy of such a ring can be increased by raising the field as in a betatron; the behaviour in a varying guide field will be treated in the next section.

7. BEHAVIOUR IN A TIME VARYING GUIDE FIELD

In a normal betatron, in which ν/γ is negligible, the orbit radius remains constant if the '2:1 condition' $A_\theta = BR$ is maintained. If on the other hand the field obeys a scaling law, $B(r, t) = F(t)r^{-n}$, the equilibrium orbit and momentum of a particle vary in such a way that

$$\beta\gamma m_0 c \propto B^{1/2},\ R \propto B^{-1/2} \tag{13}$$

where the value of B is that at the equilibrium orbit.[8] The change in R for a fractional change of B at the original orbit radius depends on n.

When ν/γ is not negligible the value of A_θ at the orbit is increased by just $\Lambda Ne\beta$. A detailed analysis[7] shows that in the relativistic limit ($\beta \approx 1$) an un-neutralized beam ($f = 0$) behaves as though the electrons are independent, but possess an inertial mass $(\gamma + \nu\Lambda)m_0$. The 2:1 condition still holds provided that the small change in Λ, which arises from the shrinkage of the minor radius a due to adiabatic damping, is neglected. This damping is not the same as in a conventional betatron, since Q varies; it can be found, however, by writing down the usual condition that the ratio of the energy to the frequency of the transverse oscillations is constant. This yields (for the general case of R varying also)

$$\beta\gamma Q a^2/R = \text{const.} \tag{14}$$

For a normal betatron $a \propto (\beta\gamma)^{-1/2}$, for contraction of a low current beam in a scaling field, however, it is evident from Eq. (13) that $a \propto R$.

If ions are present in the beam the dynamic behaviour is in general complicated. Changing the flux through the orbit accelerates ions in the opposite direction to the electrons, and since their radius of curvature is different they tend to leave the beam; in some cases, however, they are retained by electrostatic attraction, though the adiabatic damping rates will be different from that of the electrons if the latter are relativistic.[9] Some simplification can be introduced by assuming that the ions have zero transverse inertia and infinite longitudinal inertia. Though this is a reasonable approximation in some circumstances, it is by no means always so, and its validity must be checked in individual cases.

As indicated before, the betatron 2:1 condition does not in general hold, nor do Eqs. (13) if the field does not scale. In such a field however, the radius of the ring still decreases and ν increases as the field is raised. For a ring containing some ions the stored electric and magnetic energy would equal the total energy of the electrons when ν reached about γ/Λ. Although instabilities would amost certainly prevent such a high value of ν from being attained, it is in principle possible since if Λ is large, ν/γ is much less than unity. In such a ring the electric and magnetic fields would thus contribute a mass equal to the relativistic mass $2\pi RN\gamma$. Further, the 'classical radius' of such a ring is of order

$$r_R = 2\pi\nu R/(\gamma + \nu\Lambda). \tag{15}$$

As ν increases, R/r_R decreases. It will be seen, however, that the ratio cannot decrease to unity without violating the criterion that ν/γ should be much less than unity. If, however, $f = \gamma^{-2}$ so that $K = 0$, and also $\Lambda < 2\pi$ it is possible to visualize a self-consistent ring structure with R/r_R less than unity. Such a ring would have some curious properties. Unfortunately however, the prospects of ever forming one are very remote rndeed.

8. BUDKER'S RADIATION LIMITED BEAM AND THE PEASE-BRAGINSKY PLASMA

So far, scattering of the electrons by the ions in a neutralized beam has been ignored, and energy loss arising from radiation from the electrons has been neglected. Under extreme conditions both of these must be taken into account.[10] The effect of ions is essentially to produce gas scattering which increases the betatron oscillation amplitude and hence the cross section of the beam. In a very dense neutralized relativistic beam this expansion is opposed by damping of the oscillations which arises from synchrotron radiation of the electrons in the self-field of the beam.

In his original analysis of the problem Budker considered a steady-state system in which a longitudinal electric field along the beam is maintained constant by betatron acceleration, but acceleration of the ions in the contrary direction is neglected. A Bennett distribution of charge density was assumed and the steady state calculated with the aid of a Lorentz transformation to a frame of reference with zero mean electron velocity. Here an order of magnitude estimate is made using the uniform model for the beam and working entirely in the laboratory reference frame.

The beam is specified by four parameters, E, ν, γ and a (with $\nu \ll \gamma$), and relations between them are determined by writing down equations for energy and momentum and balance. Extreme relativistic conditions are assumed, so that $\beta \approx 1$. The power loss per electron Eec is given by the standard formula

$$W_R = \tfrac{2}{3} c r_0^2 \gamma^2 B^2. \tag{16}$$

Replacing B by an average value Ne/a yields (ignoring the $\frac{2}{3}$)

$$\frac{E}{e} = \frac{\gamma^2 \nu^2}{a^2}. \tag{17}$$

The momentum transfer per electron per unit time Ee can be equated to the sum of the momentum transferred to the ions by scattering ($\dot{p}_s$) and that carried away by the radiation ($\dot{p}_r$). The first term can easily be found with the aid of the Rutherford scattering formula

$$\begin{aligned} \dot{p}_s &= \gamma m_0 c^2 \cdot \frac{N}{\pi a^2} \cdot \left(\frac{r_0}{2\gamma}\right)^2 \int \frac{2\pi \sin\theta\,(1-\cos\theta)d\theta}{(\sin\frac{1}{2}\theta)^4} \qquad (18) \\ &= 4\nu L_c e^2/\gamma a^2 \end{aligned}$$

where L_c denotes the 'coulomb logarithm'

$$L_c = \int_{\theta_{\min}}^{\theta_{\max}} d\theta/\theta = \ln(\theta_{\max}/\theta_{\min});$$

θ is small over the range which contributes most to the integral. The radiation term can be written down immediately from Eq. (17), remembering (from Sec. 2) that the electrons make an average angle of order $(\nu/\gamma)^{1/2}$ to the axis, and setting $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ yields

$$\dot{p}_r = \gamma^2 \nu^2 e^2 (1 - \nu/2\gamma) \tag{19}$$

so that, for the total momentum balance,

$$Ee = \frac{\gamma^2 \nu^2 e^2}{a^2}\left(1 - \frac{\nu}{2\gamma}\right) + \frac{4\nu L_c e^2}{a^2 \gamma}. \tag{20}$$

From Eqs. (17) and (20) it follows that

$$\left.\begin{aligned} \gamma\nu &= (8L_c)^{1/2} \\ a &= (8L_c e)^{1/2}/E^{1/2}. \end{aligned}\right\} \quad (21)$$

It is remarkable that each equation contains only two of the four variables, so that in the equilibrium state the energy of the electrons depends only on the line density N, and the beam radius depends only on the applied field.

Taking $L_c = 20$, a typical set of parameters for an equilibrium beam is, in practical units: energy = 20 MeV, current = 5000 A, $E = 1$ V/cm, $a = 5 \times 10^{-3}$ cm. The self magnetic field of such a beam would be 200 kG, a very high value. It is this property which led Budker to propose the use of such beams to provide the guide field for particle accelerators.

The numerical factors in Eq. (20) are somewhat different from those given by Budker. Some confusion exists in the literature due to failure to distinguish between the values of γ associated with individual electrons, and the value associated with the frame of reference with respect to which the mean longitudinal velocity of the electrons is zero. Although for ν/γ small the values of β in both cases are near to unity, the values of $1-\beta^2$ and hence γ can be very different. (See Appendix to Ref. 1.)

The sketchy calculation above illustrates the essential physics of the steady-state relativistic beam; it is greatly oversimplified however, and, even if such beams did not suffer from instabilities, they would be very difficult to set up. Their transient behaviour has been discussed briefly by Lawson[11] and more carefully by Capps.[12]

It is implicit in the above analysis that $\nu/\gamma \ll 1$. Toroidal self-constricted ring currents in nonrelativistic plasma in which $\nu/\gamma \gg 1$ have been studied in connection with fusion research. In such plasmas collisions are important, and the electron velocity distribution is essentially isotropic and Maxwellian but with a small drift component. For a typical value $\nu = 10^4$, β_L is of order 1% of β_T. Energy and momentum balance calculations analogous to those for the Budker beam have been carried out independently by Pease[13] and Braginsky.[14] There is a difference, however, in that the energy loss occurs mainly by bremsstrahlung, and the momentum carried by it is negligible since it is no longer peaked in the forward direction but is emitted more or less isotropically. A detailed analysis is complicated, as reference to the above papers shows, nevertheless it is again possible to simplify very considerably to obtain an order of magnitude result. The only additional physical fact required is the nonrelativistic cross section for bremsstrahlung[15]

$$\sigma = \frac{16}{3}\alpha r_0^2 \tag{22}$$

where α is the fine structure constant. Proceeding as before and making use of this result the equations corresponding to (21) become

$$\left.\begin{aligned} \nu\beta &= \left(\frac{3\pi}{4}\frac{L_c}{\alpha}\right)^{1/2} \\ a &= \left(\frac{16\alpha}{3\pi}\frac{e}{E}\right)^{1/2} \nu^{3/4}. \end{aligned}\right\} \quad (23)$$

The variables do not separate out into pairs as before, the second equation contains a, E and ν. The first of these equations is the more interesting, in practical units it is simply $I \approx 10^6$ A. It must be emphasized again that this is a highly idealized result, and such plasmas are in practice limited by many types of instability. Some further comments are given in Ref. 1.

9. CONCLUSION

The limitations both of linear and ring beams with zero, partial, or complete space-charge neutralization have been discussed in terms of dimensionless parameters. The ratio ν/γ is found to be important in several respects; not only, as explained in Ref. 1, does it form a transition region between a beam and a plasma, but it also indicates when the self-fields substantially affect the equilibrium conditions in an external field.

It is interesting to note that the $\nu/\gamma = 1$ limit to a neutralized electron beam appears first to have been realized by Alfvén when studying streams of cosmic particles.[18] The parameter ν was introduced by Budker,[10] and in his papers many of the concepts discussed in the present paper are implied. Kapitza[17] has recently also discussed the parameter ν, and shown that it belongs to a wider class of 'natural' parameters significant in classical electrodynamics.

10. LIST OF SYMBOLS

Only symbols used in more than one place are included; those which occur only once are defined locally. Gaussian mixed units are used.

a = radius of beam.
B = magnetic field.
c = velocity of light.
E = electric field.
e = electronic charge.
f = ratio of ion charge to electron charge per unit length of beam.
I = current.
K = generalized perveance [Eq. (1)].
k = reciprocal of capacity per unit length of ring beam (Sec. 5).
l = inductance per unit length of beam.
L_c = coulomb logarithm [defined after Eq. (18)].
m_0 = electron rest mass.
N = number of electrons per unit length of beam.
r = radial co-ordinate.
R = equilibrium orbit radius of ring beam.
β = velocity of electron normalized to that of light; subscripts T and L denote transverse and longitudinal components.
γ = $(1-\beta^2)^{-1/2}$.
θ = angle to beam axis.
Λ = $2 \ln R/a$.
$\lambdabar$ = wavelength/2π. Subscripts s and e are explained before Eq. (9).
ν = Budker's parameter = Ne^2/m_0c^2.

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