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One-loop effective action in orbifold compactifications

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ABSTRACT: We employ the covariant background formalism to derive generic expressions for the one-loop effective action in field theoretic orbifold compactifications. The contribution of each orbifold sector is given by the effective action of its fixed torus with a shifted mass matrix. We thus study in detail the computation of the heat kernel on tori. Our formalism manifestly separates UV sensitive (local) from UV-insensitive (nonlocal) renormalization. To exemplify our methods, we study the effective potential of 6d gauge theory as well as kinetic terms for gravitational moduli in 11d supergravity.

KEYWORDS: Field Theories in Higher Dimensions, Supersymmetric Effective Theories.

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Contents

1. Introduction	1
2. Toroidal compactifications	3
3. Orbifold compactification	7
4. Examples	11
4.1 6d gauge theory	11
4.2 11d supergravity	13
5. Conclusions	19
A. Mass matrices	20
A.1 Gauge theory	21
A.2 Gravity	21
B. Coincidence limits	22
C. Zeta regularization	24

1. Introduction

Orbifolds play a prominent role in both field and string theory compactifications to four dimensions. They provide the simplest geometries allowing for four-dimensional (4d) chiral fermions and $\mathcal{N} = 1$ supersymmetry, offer a plethora of symmetry breaking possibilities, and at the same time possess a high degree of calculability. They are also particular limits of more general backgrounds such as Calabi-Yau manifolds, and thus provide a useful tool to understand these more involved geometries.

Over the last decade or so, field theories with extra dimensions have become one of the most popular ideas for theories beyond the Standard Model. Consequently, many papers deal with radiative corrections in these kind of models [1–6]. An intriguing feature of these models is that many operators in the 4d effective action are independent of the UV completion, as they do not correspond to local counterterms in the higher dimensional theory, and the UV sensitivity is cut off by the inverse size of the internal space. The majority of the literature on radiative corrections deals with particular orbifolds and applications, and either sums over the whole Kaluza Klein tower or restrict to the effective 4d theory. The latter procedure is very simple but discards part of the UV completion and thus sacrifices some of the calculability. The former procedure grasps the higher dimensional structure

of the theory but can quickly become rather complicated, especially if one is interested in renormalization of operators beyond the effective potential.

Without doubt, the most efficient method to calculate the one-loop effective action (OLEA) is the manifestly covariant method by DeWitt [7] and Gilkey [8]. External lines in Feynman diagrams are traded for a field-dependent mass matrix that is totally covariant in the background fields. For a noncompact d -dimensional theory, closed expression for the OLEA (up to a fixed order in the dimension of the operators) can be obtained, that are valid for particles of any spin. The effort for particular applications then consists in determining the background dependence of the mass matrix of the dynamical particles and using this particular form in the general expressions. Since this can be found by a linearization of the equations of motion, this method provides an extremely simple and efficient way to calculate the OLEA. The central quantity in the calculation of the effective action is the Schwinger proper-time propagator, or heat kernel,

$$K(T) = \exp[-T(-D^2 + E)], \tag{1.1}$$

where D^2 is the background covariant d’Alambertian and E the background dependent mass matrix. The standard evaluation of K proceeds through an expansion in powers of T or, equivalently, in the dimension of local operators. The goal of this paper is to apply the covariant background method to orbifold compactifications.¹ Our formalism avoids the summation over KK modes altogether and shows the local and nonlocal structure of these models in a particularly clear way.

In this paper, we will make the assumption that the fields occurring in the covariant derivatives as well as the mass matrix E in eq. (1.1) are independent of the extra dimensional coordinates. Backgrounds of this type allow one to study the effective action of the light degrees of freedom of most orbifold compactifications, i.e., whenever the zero modes have flat profiles.² While this assumption greatly simplifies the results, one loses some of the invariances inherited from the higher dimensional theory. As is well known [3], gauge invariances related to normal derivatives lead to a larger invariance group at the orbifold fixed point than one would naively expect. These additional symmetries are not manifest in our approach. We will come back to this issue in the examples and in the conclusions.

The organization of this paper is as follows. In section 2 we study the simple case of a toroidal geometry. We will show that the one-loop trace involves a summation over the torus lattice and propose a further expansion of the heat kernel coefficients in powers of the lattice vectors. We explicitly evaluate the coefficients up to operators of dimension four. In section 3 we proceed to orbifold geometries of the type T^n/Z_N . We show that each of the N orbifold sectors generates a contribution that corresponds to the heat kernel of the sector’s fixed torus with a shifted mass matrix. We also discuss the presence of discrete Wilson lines which modifies the individual contributions in an interesting way. In section 4

¹Heat kernel techniques have previously been applied to orbifolds in ref. [9] in the context of anomalies. Heat kernel coefficients on boundaries have been calculated in refs. [10], see also ref. [11] and references therein. The case of conical singularities has been discussed in refs. [12].

²Let us stress though that this assumptions only applies to background fields, i.e. we retain the full tower of Kaluza Klein excitations in the loop.

we present two applications of our formalism, the calculation of the effective potential in 6d T^2/Z_N gauge-Higgs unification, and the one loop kinetic terms for the gravitational moduli in 11d supergravity compactified on an orbifold.

2. Toroidal compactifications

In this section we would like to analyze the one-loop effective action of zero modes of the compactification on a torus.³ As it turns out, the case of the torus provides all the technical tools for the orbifold compactifications, to be considered in section 3. The contribution to the OLEA from a generic field can be written as

$$S_{\text{eff}}[A, g, \dots] = -(-)^F \frac{1}{2} \int_0^\infty \frac{dT}{T} \text{Tr} \exp[-T(-D^2 + E)]. \quad (2.1)$$

On the right hand side we have included a field-dependent mass-matrix E whose form will depend on the particle circulating in the loop. The derivative is covariant with respect to all gauge and gravitational symmetries. We will assume that the zero modes have a flat profile⁴ in the extra dimension, but will take into account an arbitrary dependence on the 4d coordinates, i.e. $A_M(x^\mu)$, $g_{MN}(x^\mu)$ etc. This particular background allows one to extract information on renormalization of operators containing derivatives, such as kinetic terms. We will frequently use d -dimensional covariant quantities which should be decovariantized at the end. It is worth noticing that the inverse propagator can always be cast in the form $-D^2 + E$, at least for a suitable choice of gauge [11]. The mass matrices for a fairly generic class of theories are reviewed in appendix A. The trace Tr includes an integration over spacetime as well as a summation over all internal indices; in the following we will denote the internal trace by tr . The exponential in eq. (2.1) is called the heat kernel of the differential operator $-D^2 + E$,

$$K(x, x', T) \equiv \langle x | \exp[-T(-D^2 + E)] | x' \rangle. \quad (2.2)$$

From its definition it satisfies the differential equation

$$-\partial_T K(x, x', T) = (-D^2 + E)K(x, x', T), \quad (2.3)$$

and the initial condition

$$K(x, x', 0) = \delta(x - x') \mathbb{1}. \quad (2.4)$$

Let us now consider the internal space to be an n -dimensional torus T^n defined by a lattice Λ whose elements we will denote by λ . The trace can then be written as

$$\text{Tr} K(T) = \text{tr} \int d^d x \sum_{\lambda \in \Lambda} K(x, x - \lambda, T). \quad (2.5)$$

³In this paper we will be interested only in the effective action of the light modes, but will take into account all the heavy KK states in the loop.

⁴We would like to stress that in principle there exists no conceptual difficulty in incorporating non-flat profiles in our formalism. However, the zero modes of many applications do have flat profiles and we will restrict to these cases in this paper. The case of nontrivial wave functions, e.g. warped compactifications, quasi-localized fields, or massive KK modes is left to future work.

The term in this sum corresponding to $\lambda = 0$ will just give rise to the usual d -dimensional one-loop effective action. It describes particles traveling on closed loops that can be contracted to a point. On the other hand, the terms with nonzero lattice vectors describe closed loops that cannot be contracted and, hence, have finite length. These contributions can never lead to ultraviolet divergent amplitudes. To see this, it is instructive to directly evaluate eq. (2.5) under the assumption that the background fields are constant. In other words, we calculate the operators in the OLEA that do not contain any derivatives, i.e the effective potential. Introducing a complete set of momentum states⁵ one finds

$$\begin{aligned} K(x, x - \lambda, T) &= \int \frac{d^d p}{(2\pi)^d} \exp(i p \cdot \lambda - T[(p - A)^2 + E]) \\ &= \frac{1}{(4\pi T)^{d/2}} \exp\left(i\lambda \cdot A - \frac{\lambda^2}{4T} - T E\right). \end{aligned} \quad (2.6)$$

Here we have also assumed that $[A_i, A_j] = 0$ so that we can diagonalize the A_i simultaneously and perform the shift in the momentum variable. The ultraviolet region of the T integration corresponds to small T which is thus strongly suppressed for nonzero λ . We will refer to the contributions of non-vanishing λ as nonlocal throughout this paper, while the $\lambda = 0$ term is called local and leads to renormalization of all local d -dimensional operators in the effective action that are compatible with the symmetries of the theory. All nonlocal operators in the effective action (including the ones containing derivatives of fields) will come with an exponential suppression factor as well as the Wilson line present in eq. (2.6).⁶ In the following we will evaluate the heat kernel on the torus for more general backgrounds.

The standard evaluation of the heat kernel proceeds through an expansion in powers of T or, equivalently, dimension of the local operators. To satisfy the initial condition eq. (2.4), one introduces the following ansatz

$$K(x, x', T) = \frac{1}{(4\pi T)^{d/2}} \bar{\Delta}(x, x')^{1/2} e^{-\sigma(x, x')/2T} \sum_{r \geq 0} T^r a_r(x, x'), \quad (2.7)$$

with $a_0(x, x) = \mathbb{1}$. Here, $\sigma(x, x')$ is the so-called geodesic biscalar function which, by definition, equals one half the geodesic distance squared between the points x and x' . It satisfies the following differential equations and initial conditions.

$$\frac{1}{2} \sigma_{;M} \sigma^{;M} = \sigma, \quad (2.8)$$

$$[\sigma_{;M}] = 0, \quad [\sigma_{;MN}] = [\sigma_{;M'N'}] = -[\sigma_{;MN'}] = g_{MN}, \quad (2.9)$$

where the semicolon denotes covariant differentiation with respect to unprimed or primed coordinates, and the brackets stand for the coincidence limit $x' \rightarrow x$. The vector $\sigma^{;M}$ has

⁵Note that the momentum variable is continuous as we are working on the covering space.

⁶A perhaps more physical interpretation can be obtained by representing the propagator or, equivalently, the heat kernel by a classical path integral. One has to sum over all closed paths of periodicity T with a weight (action) given by their geodesic length squared and the associated Wilson line phase. The non-contractible loops have nonzero length and are always weighted by $\exp(-\lambda^2/4T)$. This approach has, e.g., been followed in ref. [13].

length equal to that of the geodesic from x' to x , is tangent to the geodesic at x and points in the direction from x' to x . The so-called van Vleck determinant is defined as

$$\bar{\Delta} \equiv \det(-\sigma_{;MN'}). \quad (2.10)$$

$\bar{\Delta}$ is a biscalar density with the coincidence limit $[\bar{\Delta}] = \det g_{MN} \equiv g$. The quantities a_r are called the heat kernel coefficients, they should be considered as bitensors, gauge transforming at x from the left and at x' from the right. In particular, the coefficient a_0 is the operator of parallel transport (the Wilson line) connecting the two fibers at x' and x along the geodesic between these two points. The ansatz eq. (2.7) is designed to satisfy the initial condition eq. (2.4). In the standard evaluation of the heat kernel, the quantities needed are the coincidence limits $[a_r]$, as only these enter in the local renormalization. There exists various ways to obtain these quantities, the most straightforward being the recursive procedure [7] which we briefly review in appendix B. On the torus, one needs in addition the periodic coincidence limits

$$[a_r]_\lambda \equiv \lim_{x' \rightarrow x - \lambda} a_r(x, x'), \quad (2.11)$$

that enter in the nonlocal renormalization. The OLEA can then be written as

$$S_{\text{eff}} = -(-)^F \int d^d x \sum_{r,\lambda} \alpha_{d,r} \frac{[\bar{\Delta}^{\frac{1}{2}}]_\lambda \text{tr}[a_r]_\lambda}{|\lambda|^{d-2r}}, \quad \alpha_{d,r} = \frac{\Gamma(\frac{d}{2} - r)}{2^{2r+1} \pi^{\frac{d}{2}}}. \quad (2.12)$$

As expected, only the term with $\lambda = 0$, corresponding to local bulk renormalization, is UV divergent. Introducing, for simplicity, a Schwinger cutoff $\exp(-\frac{1}{4\Lambda_{\text{UV}}^2 T})$ in eq. (2.1), we can write the local and nonlocal renormalizations in a similar way

$$S_{\text{eff, loc}} = -(-)^F \int d^d x \sqrt{g} \sum_r \alpha_{d,r} \Lambda_{\text{UV}}^{d-2r} \text{tr}[a_r], \quad (2.13)$$

$$S_{\text{eff, fin}} = -(-)^F \int d^d x \sum_{r,\lambda \neq 0} \alpha_{d,r} \frac{[\bar{\Delta}^{\frac{1}{2}}]_\lambda \text{tr}[a_r]_\lambda}{|\lambda|^{d-2r}}. \quad (2.14)$$

Eqs. (2.12) to (2.14) are only valid for $r < d/2$ because of infrared (IR) divergences in the T integration that are hidden in the poles of the Gamma-function present in $\alpha_{d,r}$. Introducing an IR cutoff μ , one can see that at $r = \frac{d}{2}$ one needs to make the replacements

$$\alpha_{d,\frac{d}{2}} \Lambda_{\text{UV}}^0 \rightarrow -(4\pi)^{-\frac{d}{2}} \log\left(\frac{\mu}{\Lambda_{\text{UV}}}\right), \quad (2.15)$$

$$\alpha_{d,\frac{d}{2}} |\lambda|^0 \rightarrow -(4\pi)^{-\frac{d}{2}} \log(\mu|\lambda|). \quad (2.16)$$

The IR regulated result valid for arbitrary r is derived in appendix C, where we also apply the well-known zeta-regularization technique to the UV divergences in the local part of the OLEA.

It remains to calculate the periodic coincidence limits, eq. (2.11). From the interpretation of a_0 as the operator of parallel transport it is clear that we must have

$$[a_0]_\lambda = W(\lambda) \equiv \exp(i\lambda^m (A_m + \omega_m)). \quad (2.17)$$

The Wilson line $W(\lambda)$ contains both gauge and spin connection parts, denoted by A and ω respectively. For the remaining coefficients, we expand $a_r(x, x')$ in a covariant Taylor series around $x' = x$. To this end, we multiply a_r by the Wilson line $a_0(x', x)$ from the right, so the coefficients are polynomials of gauge covariant objects at x . Then

$$a_r(x, x')a_0(x, x')^{-1} = [a_r] + [a_{r;M'}]\sigma_{;M'}^{M'} + \frac{1}{2}[a_{r;(M'N')}]\sigma_{;M'}^{M'}\sigma_{;N'}^{N'} + \dots \quad (2.18)$$

Eq. (2.18) can easily be proven order by order by differentiating w.r.t. x' , taking the coincidence limit $x' \rightarrow x$, and using the identities

$$[\sigma_{(N'R'\dots)}^{M'}] = 0, \quad [a_{0;(M'\dots)}] = 0, \quad (2.19)$$

that can also be proven with the methods reviewed in appendix B. The result can be expressed as

$$[a_r]_\lambda = [e^{-D_{\lambda'}} a_r]W(\lambda) = \left([a_r] - [a_{r;\lambda'}] + \frac{1}{2}[a_{r;\lambda'\lambda'}] + \dots \right) W(\lambda), \quad (2.20)$$

where $D_{\lambda'} \equiv \lambda^m D'_m$. Eq. (2.20) is the main result of this section. We have employed a covariant Taylor expansion despite the fact that we are breaking covariance by the backgrounds. It proves more efficient to keep the covariant notation and insert the explicit background at the end of the calculation. The calculation of the coefficients is more compact since no distinction is made between the different types of indices. We will show in appendix B that calculating, e.g., $[a_{1;\lambda'\lambda'}]$ in the covariant way already provides all necessary information to find $[a_2]$ etc. Moreover, we can always use the classical (tree-level) equations of motion in the on loop corrected terms, as this is equivalent to a field redefinition [14] (see also ref. [15]). For instance, we can always replace the d -dimensional curvature scalar R by the trace of the energy momentum tensor, the corresponding field redefinition being a simple Weyl rescaling.

The mass dimension of the local operators in the parenthesis in eq. (2.20) is given by $-4 + 2r + s$ with s the order in the λ expansion. As an example, let us calculate $[a_1]_\lambda$ and $[a_2]_\lambda$ up to dimension four operators, i.e. we have to evaluate the quantities $[a_1]$, $[a_2]$, $[a_{1;\lambda'}]$ and $[a_{1;\lambda'\lambda'}]$. The evaluation can be done in the well-known manner by DeWitts recursive procedure [7]. We perform this evaluation in appendix B, one finds

$$[a_1]_\lambda = \left\{ \frac{1}{6}R - E - \frac{1}{12}R_{;\lambda} + \frac{1}{2}E_{;\lambda} - \frac{1}{6}\Omega^M_{\lambda;M} + \frac{1}{40}R_{;\lambda\lambda} + \frac{1}{120}R_{\lambda\lambda;M}^M \right. \\ \left. - \frac{1}{90}R^M_{\lambda}R_{M\lambda} + \frac{1}{180}R^{MN}R_{M\lambda N\lambda} + \frac{1}{180}R^{MNL}_{\lambda}R_{MNL\lambda} - \frac{1}{6}E_{;\lambda\lambda} \right. \\ \left. + \frac{1}{24}R^M_{\lambda}\Omega_{M\lambda} + \frac{1}{12}\Omega_{M\lambda}\Omega^M_{\lambda} + \frac{1}{12}\Omega_{M\lambda;M}^M + \mathcal{O}(\lambda^3) \right\} W(\lambda), \quad (2.21)$$

$$[a_2]_\lambda = \left\{ \frac{1}{2}\left(\frac{1}{6}R - E\right)^2 + \frac{1}{6}\left(\frac{1}{5}R - E\right)_{;M}^M - \frac{1}{180}R_{MN}R^{MN} \right. \\ \left. + \frac{1}{180}R^{MNLS}R_{MNLS} + \frac{1}{12}\Omega_{MN}\Omega^{MN} + \mathcal{O}(\lambda) \right\} W(\lambda). \quad (2.22)$$

Here, Ω is the field strength of the gauge and spin connections

$$\Omega_{MN} = [D_M, D_N] = -iF_{MN} + \frac{i}{2}\Sigma^{AB}R_{ABMN}. \quad (2.23)$$

We will also need the corresponding expansion of the determinant $\bar{\Delta}$,

$$[\bar{\Delta}^{\frac{1}{2}}]_{\lambda} = \sqrt{g} \left(1 + \frac{1}{12}R_{\lambda\lambda} + \mathcal{O}(\lambda^3) \right). \quad (2.24)$$

The next step is to set $\partial_i = 0$ and decovariantize these expressions. We will leave this step to the explicit examples and end this section by making a few comments on the form of eq. (2.21) and Eq (2.22). First of all, notice that they contribute to the 4d effective potential for the 4d scalar zero modes. There are contributions both from the Wilson line as well as the prefactors $[A_i, A_j]^2$, $[A_i, E]$ etc. However, restricting to the tree level flat directions, $[A_i, A_j] = 0$ (i.e. the moduli space), we see that many contributions vanish. This is the expansion resulting from eq. (2.6). If, in addition, we assume that A_i and E commute, then the only constant terms left in a_r result from the expansion of the field dependent mass suppression, e^{-TE} .

The next comment concerns the IR divergences of the double expansion in T and λ . The integration over T is IR divergent when $2r \geq d$, as is evident from the presence of the Γ function in eq. (2.12). On the other hand, the summation over λ produces IR divergences once the dimension of the operator exceeds four. It is worth noticing that if the matrix E is positive definite its smallest eigenvalue provides an effective d -dimensional IR cutoff, in which case one gets a good approximation if one includes in the summation over Λ only the terms with small $|\lambda|$. This corresponds to closed loops that only wind a few times around the torus, which dominate the IR behavior.

3. Orbifold compactification

In order to obtain phenomenologically more interesting models, we would like to orbifold the toroidal geometries considered in the previous section. The Z_N orbifold⁷ is constructed by identifying points that are related by a rotation of the torus:

$$x \sim P^k x - \lambda, \quad \lambda \in \Lambda, \quad P^N = 1. \quad (3.1)$$

This operation is well defined on the torus only if the Z_N action defines an automorphism of the torus lattice, i.e., maps Λ to itself. This property, also known as the crystallographic principle, greatly restricts the allowed lattices and values for N . These are well known and classified for the dimensions most interesting for phenomenological applications, see e.g. ref. [16] for the 10d case. The Z_N group acting as rotations is known as the point group G , while the one generated by both lattice translations and rotations is called the space

⁷Here we consider only orbifolds with one Z_N factor. The generalization to several factors, or even nonabelian groups, is straightforward. We also rewrite orbifolds that involve non-integral lattice shifts as integral shifts with discrete Wilson lines.

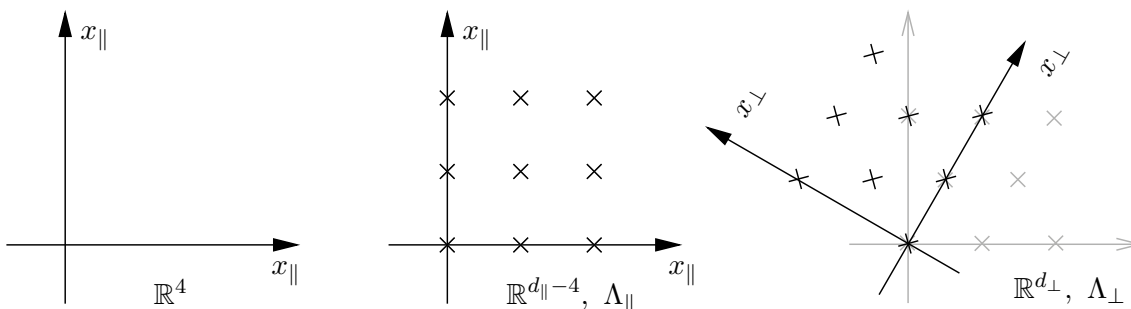


Figure 1: In each k sector of the orbifold, the coordinates split into fixed (x_{\parallel}) and rotated (x_{\perp}) under the action of P^k . The crosses indicate the lattice Λ of the underlying torus, which also splits into the direct sum $\Lambda = \Lambda_{\parallel} + \Lambda_{\perp}$. Notice that either torus can be trivial for particular sectors.

group S . We can decompose each $g \in S$ as in eq. (3.1) and accordingly write $g = (k, \lambda)$. The space group is represented on the fields as

$$\phi(gx) = W_0(\lambda) (P_L \otimes P_G)^k \phi(x), \quad (3.2)$$

where P_L is the representation of P on the Lorentz group and P_G acts on all internal indices (in particular the gauge group \mathcal{G}). We have also included a discrete Wilson line W_0 . Discrete Wilson lines commute with P_G and satisfy

$$W_0(P\lambda) = W_0(\lambda), \quad W_0^N = \mathbb{1}. \quad (3.3)$$

For reasons of clarity we will present the calculations of this section for $W_0 = \mathbb{1}$ and only give the relevant results for nontrivial W_0 at the end.

In order to calculate the effective action in the orbifolded theory, we make use of the fact that any field satisfying the point group constraint

$$\phi_{\text{orb}}(Px) = (P_L \otimes P_G)\phi(x)_{\text{orb}} \quad (3.4)$$

can be obtained from the fields on the torus by applying the linear projection

$$\phi_{\text{orb}}(x) = \frac{1}{N} \sum_{k=0}^{N-1} (P_L \otimes P_G)^{-k} \phi_{\text{tor}}(P^k x) \quad (3.5)$$

on any torus field. Consequently, we can evaluate the trace on the orbifold as⁸

$$\text{Tr } K(T) = \frac{1}{N} \int dx \text{tr } K(x, P^k x - \lambda) (P_L^\dagger \otimes P_G^\dagger)^{-k}. \quad (3.6)$$

Following the notation of ref. [5], for a given point group element P^k we split the covering space according to $\mathbb{R}^d = \mathbb{R}^{d_{\parallel}} \oplus \mathbb{R}^{d_{\perp}}$, where by definition the d_{\parallel} coordinates x_{\parallel} are left fixed by P^k , see the illustration in figure 1. This splitting obviously depends on k , in order to avoid a cumbersome notation such as $x_{k,\parallel}$ etc. we will omit the index k when no confusion

⁸The projection method was first developed for the codimension-one case [10].

can arise. In the same way we split the torus $\Lambda = \Lambda_{\parallel} + \Lambda_{\perp}$. For the orbifold we need to evaluate the matrix element

$$K(x, P^k x - \lambda) = \langle x | \exp(-T[-D^2 + E]) | P^k x - \lambda \rangle. \quad (3.7)$$

Using the splitting just introduced, one finds

$$K(x, P^k x - \lambda) = \int dp_{\perp} \exp\left(ip_{\perp}(P^k - 1)[x_{\perp} - x_f(\lambda_{\perp})]\right) \times \\ \times \langle x_{\parallel} | \exp\left(-T\left[-D_{\parallel}^2 + (p_{\perp} - A_{\perp} - \omega_{\perp})^2 + E\right]\right) | x_{\parallel} - \lambda_{\parallel} \rangle, \quad (3.8)$$

where we have used that the nonsingular matrix $P^k - 1$ provides a one-to-one map from the set of fixed points on the transverse space to the lattice vectors in Λ_{\perp} . The next thing we would like to do is to perform the trace over the transverse torus, i.e. we would like to perform the integration/summation

$$\text{Tr}_{\perp} = \sum_{\Lambda_{\perp}} \int_{\mathcal{F}_{\perp}} dx_{\perp}, \quad (3.9)$$

where the integration is over the fundamental domain \mathcal{F}_{\perp} of the torus. We now replace the sum over Λ_{\perp} by the sum over all fixed points in the covering space, again by virtue of the map. We then can write

$$\text{Tr}_{\perp} \exp\left(ip_{\perp}(P^k - 1)[x_{\perp} - x_f(\lambda_{\perp})]\right) = \sum_{x_f} \int_{\mathcal{F}_{\perp}} dx_{\perp} \exp(\dots) \\ = \sum_{x_f \in \mathcal{F}_{\perp}} \int dx_{\perp} \exp(\dots) \quad (3.10)$$

where the integration is now over the whole covering space whereas the summation over the fixed points is restricted to the fundamental domain. From now on all summations over fixed points are implicitly assumed to be only over \mathcal{F}_{\perp} . The integration over x_{\perp} gives $|\det(1 - P^k)|^{-1} \delta(p)$. According to Lefschetz' formula, the determinant equals the number of fixed points in \mathcal{F}_{\perp} , leading to

$$\text{Tr}_{\perp} K(T) = \exp\left(-T[-D_{\parallel}^2 + E + (A_{\perp} + \omega_{\perp})^2]\right) \equiv K_{\parallel}(T). \quad (3.11)$$

The final result on the orbifold without discrete Wilson lines is thus

$$S_{\text{eff}} = -(-)^F \frac{1}{2N} \int \frac{dT}{T} \sum_{k=0}^{N-1} \text{Tr}(P_{\mathcal{G}} \otimes P_L)^k K_{\parallel}(T). \quad (3.12)$$

The trace in eq. (3.12) includes an integration over the d_{\parallel} dimensions x_{\parallel} as well as a summation over the lattice Λ_{\parallel} of the fixed torus of the k^{th} sector. One concludes that the renormalization of the k^{th} orbifold sector (i.e., the k^{th} term in the sum) is localized on the corresponding fixed torus. Moreover, the UV sensitive contribution, $\lambda_{\parallel} = 0$, is the local

renormalization at the fixed points. The evaluation of eq. (3.12) now proceeds precisely as described in section 2 in $d = d_{\parallel}$ dimensions, the only difference being the shifted mass matrix and the orbifold twists inside the trace. In particular, eqs. (2.17), (2.21), and (2.22) are still valid. Notice, however, that the original mass matrix E is still the one obtained in the d -dimensional theory. For instance, a non-minimally coupled scalar has $E = \eta R_d$, the d dimensional curvature scalar, the mass matrix for a vector particle is still a $d \times d$ matrix etc.

In case there are discrete Wilson lines, we first perform the splitting

$$W_0(\lambda) = W_0(\lambda_{\perp} + \lambda_{\parallel}) = W_0(\lambda_{\perp})W_0(\lambda_{\parallel}). \tag{3.13}$$

The Wilson line $W_0(\lambda_{\parallel})$ just multiplies the background (continuous) Wilson line $W(\lambda)$ occurring in the periodic coincidence limit of the heat kernel coefficients, i.e. eqs. (2.17), (2.21) and (2.22). To take into account the effect of the orthogonal Wilson line, one has to introduce the following matrix in the trace in eq. (3.12)

$$Q_{\perp} = |\det(1 - P^k)|^{-1} \sum_{x_{f,k}} W_0(\lambda_{\perp}(x_{f,k})), \tag{3.14}$$

leading to⁹

$$S_{\text{eff}} = -(-)^F \frac{1}{2N} \int \frac{dT}{T} \sum_{k=0}^{N-1} \text{Tr}(P_G \otimes P_L)^k Q_{\perp} K_{\parallel}(T). \tag{3.15}$$

As emphasized earlier, the splitting into x_{\parallel} and x_{\perp} depends on the orbifold sector (i.e. on k), and, as a consequence, the same holds true for the quantities Q_{\perp} and K_{\parallel} . One can interpret this result by noting that the quantity Q_{\perp} is nothing but the projector onto zero modes on the transverse torus defined by the lattice Λ_{\perp} , i.e., the zero modes that would be obtained from compactification on the torus Λ_{\perp} with the discrete Wilson lines $W_0(\lambda_{\perp})$ present. These projectors actually take very simple forms in concrete examples, as the possible Wilson lines are very restricted. We will give the explicit forms of Q_{\perp} for the T^2/Z_N orbifolds in section 4.

Let us emphasize an important point. The contribution with $\lambda_{\parallel} = 0$ corresponds to a local renormalization at the fixed points of the k -sector of the orbifold. As expected, these are UV divergent and should respect all symmetries preserved at the fixed point. As discussed in the literature [3], the gauge symmetries actually further constrain the allowed operators because of shift symmetries related to normal derivatives. These remnant gauge symmetries are not manifest in our formalism due to the fact that we have only considered backgrounds with vanishing normal derivatives.¹⁰ With some effort one can set up a fully covariant heat kernel expansion that manifestly displays the surviving symmetries at a given fixed point. However, the resulting formulae are considerably more involved and we will leave this to future research.

Finally, localized matter (twisted sectors) can appear on the fixed tori. In the trivial case of a 4d fixed point, their contribution is just the usual 4d one

$$S_{\text{eff}}^{\text{twisted}} = -(-)^F \frac{1}{2} \int d^4x \text{tr} K_{4d}(x, x, T). \tag{3.16}$$

⁹In deriving eq. (3.15) one has to use eq. (3.3).

¹⁰We will come back to this issue in section 4.

For higher dimensional fixed points, the geometry seen by these fields is again an orbifold, of dimension $d' < d$ and order $N' < N$ which can be treated as before.

4. Examples

4.1 6d gauge theory

As our first example for the use of our methods, we consider the effective action in 6d gauge-Higgs unification models on the orbifold T^2/Z_N . In these type of models, the bulk gauge group \mathcal{G} is broken to a subgroup \mathcal{H} by the orbifold twist $P_{\mathcal{G}}$. The Higgs for the further breaking of \mathcal{H} then resides in the $A_{4,5}$ components of the gauge field belonging to the coset \mathcal{G}/\mathcal{H} . The tree-level potential derives from the F_{MN}^2 kinetic term in the action. It is important to distinguish two kinds of 4d scalar fields resulting from the compactification: generic massless ones (orbifold invariant states) and flat directions (a subset of the zero modes corresponding to the condition $[A_i, A_j] = 0$). Not all zero modes correspond to flat directions.

We will consider pure gauge theory and calculate the contribution from gauge and ghost loops, the corresponding mass matrices are given in appendix A. In flat gravitation background they read

$$E_{1,MN} = 2iF_{MN}, \quad E_{1,gh} = 0. \quad (4.1)$$

The result for the $k = 0$ sector can be read off from eqs. (2.13) to (2.17) as well as (2.21) and (2.22).

$$S_{\text{eff, loc}}^{k=0} = -\frac{1}{N} \int d^6x \left\{ \frac{4 \dim(\mathcal{G}) \Lambda_{\text{UV}}^6}{\pi^3} + \frac{5C_2(\mathcal{G}) \Lambda_{\text{UV}}^2}{96\pi^3} F_{MN}^a F^{a,MN} \right\} \quad (4.2)$$

$$S_{\text{eff, fin}}^{k=0} = -\frac{1}{N} \int d^6x \sum_{\lambda \neq 0} \text{tr} \left\{ \frac{4}{\pi^3 |\lambda^6|} W(\lambda) + \frac{1}{12\pi^3 |\lambda^4|} W(\lambda) [iF_{\lambda;M}^M \right. \\ \left. - \frac{1}{2} F_{M\lambda} F^{M\lambda} - \frac{i}{2} F_{M\lambda; \lambda}^M] + \frac{5}{96\pi^3 |\lambda|^{-2}} W(\lambda) F_{MN} F^{MN} \right\}, \quad (4.3)$$

where the integration is over the volume of the torus, the trace in the adjoint representation, and we recall from section 2 our shorthand notation $X_\lambda \equiv \lambda^i X_i$. Eq. (4.2) is the renormalization of the bulk cosmological constant and the bulk gauge kinetic term. Eq. (4.3) contains the Hosotani potential [1], kinetic terms for A_μ , as well as potential and kinetic terms for $A_{4,5}$.

The contributions from the sectors with $k \neq 0$ correspond to the renormalizations at the fixed points. The fixed points are four dimensional and contain no further toroidal dimensions, so there is only a local renormalization. We define the orbifold action on the coordinates to be a counterclockwise rotation of angle $2\pi k/N$. The action on the gauge fields thus reads

$$P_L^k = \begin{pmatrix} \mathbb{1}_4 & & & \\ & c_k & s_k & \\ & -s_k & c_k & \end{pmatrix}, \quad c_k = \cos\left(\frac{2\pi k}{N}\right), \quad s_k = \sin\left(\frac{2\pi k}{N}\right). \quad (4.4)$$

The gauge loop gives

$$S_{\text{eff, loc}}^{k \neq 0, \text{vector}} = -\frac{1}{N} \int d^4x \operatorname{tr} \left\{ (P_L \otimes P_G)^k \times \right. \\ \left. \times \left(\frac{\Lambda_{\text{UV}}^4}{2\pi^2} - \frac{\Lambda_{\text{UV}}^2}{8\pi^2} E'_1 - \frac{\log \frac{\mu}{\Lambda_{\text{UV}}}}{192\pi^2} [6E_1'^2 - F_{\mu\nu} F^{\mu\nu}] \right) \right\}, \quad (4.5)$$

where the shifted mass matrix $E'_1 \equiv E_1 + A_1^2$ reads

$$E'_{1, MN} = 2iF_{MN} + A_k A^k \delta_{MN}, \quad (4.6)$$

with A_k and F_{MN} considered as matrices in the adjoint representation. The ghosts correspond to two scalars. Their contribution is thus obtained by setting $P_L = 1$ in eq. (4.5), multiplying by -2 , and using the mass matrix

$$E'_{1, gh} = A_k A^k. \quad (4.7)$$

Adding up the contribution of the gauge fields and the ghosts and performing the trace over the Lorentz indices one obtains

$$S_{\text{eff, loc}}^{k \neq 0} = -\frac{1}{N} \int d^4x \operatorname{tr} \left\{ P_G^k \left(\frac{(c_k + 1)\Lambda_{\text{UV}}^4}{\pi^2} + \frac{(c_k - 3)(1 + c_k - is_k)\Lambda_{\text{UV}}^2}{8\pi^2} BB^\dagger \right. \right. \\ \left. \left. - \frac{(c_k + 5)(1 + c_k - is_k) \log \frac{\mu}{\Lambda_{\text{UV}}}}{64\pi^2} BB^\dagger BB^\dagger + \frac{(3c_k - 1) \log \frac{\mu}{\Lambda_{\text{UV}}}}{32\pi^2} B^\dagger B^2 B^\dagger \right. \right. \\ \left. \left. - \frac{\log \frac{\mu}{\Lambda_{\text{UV}}}}{4\pi^2} F_{\mu i} F^{\mu i} + \frac{(c_k - 11) \log \frac{\mu}{\Lambda_{\text{UV}}}}{96\pi^2} F_{\mu\nu} F^{\mu\nu} \right) \right\}, \quad (4.8)$$

where we have defined $B = A_4 + iA_5$ and made use of the fact that the orbifold boundary condition implies $BP_G^k = P_G^k B(c_k - is_k)$. As stressed earlier, the result is not covariant w.r.t. the remnant gauge symmetry related to the normal derivatives [3]. This is obviously so, as we have explicitly set to zero all normal derivatives in order to obtain the simple result in eq. (3.12). In the present case it is, however, easy to reconstruct the covariant structure as follows. The potential should result from the following operators [3]

$$\operatorname{tr} P_G^k F_{45}, \quad \operatorname{tr} P_G^k (F_{45})^2, \quad \operatorname{tr} P_G^k F_{45; i}^i. \quad (4.9)$$

Using again the orbifold boundary conditions, we can write

$$\operatorname{tr} P_G^k F_{45} = \frac{1 - c_k + is_k}{2} \operatorname{tr} P_G^k BB^\dagger \quad (4.10)$$

$$\operatorname{tr} P_G^k (F_{45})^2 = \frac{1 + c_k - is_k}{4} \operatorname{tr} P_G^k BB^\dagger BB^\dagger - \frac{1}{2} \operatorname{tr} P_G^k B^\dagger B^2 B^\dagger \quad (4.11)$$

$$\operatorname{tr} P_G^k F_{45; i}^i = -(1 - c_k + is_k) \operatorname{tr} P_G^k BB^\dagger BB^\dagger + is_k \operatorname{tr} P_G^k B^\dagger B^2 B^\dagger \quad (4.12)$$

These relations can clearly be inverted and used to replace the operators occurring in eq. (4.8) by the covariant ones. One finds:

$$S_{\text{eff, loc}}^{k \neq 0} = -\frac{1}{N} \int d^4x \text{tr} \left\{ P_{\mathcal{G}}^k \left(\frac{\Lambda_{\text{UV}}^4 (c_k + 1)}{\pi^2} - i \frac{\Lambda_{\text{UV}}^2 (c_k - 3)(c_k + 1)}{4\pi^2 s_k} F_{45} \right. \right. \\ \left. \left. - i \frac{\log \frac{\mu}{\Lambda_{\text{UV}}} (c_k - 3)(c_k + 1)}{16\pi^2 (c_k - 1) s_k} F_{45; i} - \frac{\log \frac{\mu}{\Lambda_{\text{UV}}} (c_k^2 + 7)}{16\pi^2 (c_k - 1)} (F_{45})^2 \right. \right. \\ \left. \left. - \frac{\log \frac{\mu}{\Lambda_{\text{UV}}}}{4\pi^2} F_{\mu i} F^{\mu i} + \frac{\log \frac{\mu}{\Lambda_{\text{UV}}} (c_k - 11)}{96\pi^2} F_{\mu\nu} F^{\mu\nu} \right) \right\}. \quad (4.13)$$

It is, however, not clear if this procedure can be generalized to higher dimensional fixed points and gravitational symmetries. First, one would need to find an independent set of covariant operators, as in eq. (4.9), suitable for the surviving symmetries at the fixed point. Given this set, it is not clear whether there is a one-to-one correspondance to the operators obtained with the simpler background constant in the normal directions. We believe that the better approach is to directly compute a fully covariant heat kernel expansion that manifestly displays all gauge symmetries inherited from the higher dimensional theory. This approach will be presented elsewhere [17], along the lines presented in section 5.

Including discrete Wilson lines is simple. First note that eq. (3.3) implies that the two discrete Wilson lines have to be of order 2, 3, 2 and 1 for $N = 2, 3, 4$ and 6 respectively, they also have to coincide for $N \neq 2$. Eq. (4.2) remains unaltered in the presence of discrete Wilson lines, while in eq. (4.3) the background Wilson lines become multiplied by $W_0(\lambda)$. Finally, the localized renormalizations now include the projectors Q_{\perp} . The explicit forms of these projectors are

$$Q_{\perp}^{N=2} = \frac{1}{4}(\mathbb{1} + W_{0,1})(1 + W_{0,2}), \quad Q_{\perp}^{N=3} = \frac{1}{3}(\mathbb{1} + W_0 + W_0^2), \quad (4.14)$$

$$Q_{\perp}^{N=4} = \frac{1}{2}(\mathbb{1} + W_0). \quad (4.15)$$

For $N = 6$ one necessarily has $W_0 = \mathbb{1}$ and hence the projector is trivial.

4.2 11d supergravity

In this section we would like to calculate the one-loop corrections to the kinetic terms of the gravitational moduli in an orbifold compactification of 11d supergravity. This is an important quantity as it largely determines the one-loop renormalization of the Kähler potential which, in turn, determines the scalar potential once supersymmetry is broken. The perturbative scalar potential is of high relevance due to the large number of moduli that need to be stabilized in these models.

We will consider an $\mathcal{N} = 1$ compactification on the space $T^6/Z^3 \times S^1/Z_2$. The Z_3 action is given by the $U(3) \subset SO(6)$ preserving shift vector $\phi = (1, 1, -2)$. The T^6 complex torus coordinates transform under Z_3 as

$$z_1 \rightarrow e^{2\pi i/3} z_1, \quad z_2 \rightarrow e^{2\pi i/3} z_2, \quad z_3 \rightarrow e^{-4\pi i/3} z_3. \quad (4.16)$$

The Z_2 action is given by $x^{10} \rightarrow -x^{10}$. To keep a compact notation we will write everything in terms of Z_6 generated by $P = P_{Z_2}(P_{Z_3})^{-1}$.

		$h = 2$	$h = 1$	$h = 0$	
Z_2	+1	1	6	20'+2×1	
	-1		1	6	
Z_3	θ^0	1₀	1₀	8₀+2×1₀	
	θ^1		3₁	6₋₂	3₁
	θ^2		3₋₁	6₂	3₋₁

Table 1: The orbifold parities of the metric (44) of the supergravity multiplet. Here, h denotes the 4d helicity and $\theta = e^{2\pi i/3}$. For the Z_2 parities, we label fields by SO(6) irreducible representations. For Z_3 we write fields in terms of representations of the surviving U(3) with the U(1) generator normalized as $Q = \Sigma^{45} + \Sigma^{67} + \Sigma^{89}$.

		$h = 1$		$h = 0$	
Z_2	+1	6		15+1	
	-1	15		20'+6	
Z_3	θ^0	1₀+8₀		8₀+2×1₀	8₀
	θ^1	3₁	3₋₂	3₋₂	6₋₂
	θ^2	3₋₁	3₂	3₂	6₂

Table 2: The orbifold parities of the 3-form (84) of the supergravity multiplet. See explanations below table 1.

The field content of our 11d theory is a bulk supergravity multiplet consisting of the graviton, a Majorana gravitino and an antisymmetric three-form B . To cancel localized anomalies, we would introduce E_8 and E'_8 gauge multiplets at the two fixed points of the S^1/Z_2 orbifold [18]. In this paper, we will restrict ourselves to the supergravity sector only. The parities of these fields are as follows. Each vectorial index on the metric, the gravitino, and the three-form transform as the coordinates. There is an additional overall minus sign for the B -field w.r.t. Z_2 , i.e. $B_{\mu\nu\rho}$ has negative parity under reflection of x^{10} . Finally, the spinor indices transform with γ^{10} under Z_2 .¹¹ This assignment results in the orbifold twists displayed in tables 1 to 3.

We will focus on the following background:

$$g_{MN} = \text{diag} (g_{\mu\nu}, \rho_1^2, \rho_1^2, \rho_2^2, \rho_2^2, \rho_3^2, \rho_3^2, \sigma^2), \quad B_{MNR} = 0, \quad (4.17)$$

where all fields $g_{\mu\nu}$, ρ_I and σ are assumed to be independent of the internal coordinates x^i . This does not cover all zero modes in the supergravity multiplet: From table 1, 2 and 3 for instance one can see that the $\mathcal{N} = 1$ chiral superfields come in SU(3)-multiplets: There are two singlets as well as one octet. The two singlets correspond to the two volume moduli of T^6 and S^1 respectively, whereas the octet describes the precise shape of the T^6 torus. However, it turns out that it is sufficient to consider the simplified background, eqs. (4.17),

¹¹Recall that we use Euclidean conventions with $\{\gamma^A, \gamma^B\} = 2\delta^{AB}$.

		$h = 3/2$		$h = 1/2$			
Z_2	+1	4		$2 \times \bar{4}$		$\bar{20}$	
	-1		$\bar{4}$		2×4		20
Z_3	θ^0	$\mathbf{1}_{3/2}$	$\mathbf{1}_{-3/2}$	$2 \times \mathbf{1}_{-3/2}$	$2 \times \mathbf{1}_{3/2}$	$\mathbf{8}_{-3/2}$	$\mathbf{8}_{3/2}$
	θ^1	$\mathbf{3}_{-1/2}$			$2 \times \mathbf{3}_{-1/2}$	$\mathbf{3}_{5/2}$	$\mathbf{3}_{-1/2} + \bar{\mathbf{6}}_{-1/2}$
	θ^2		$\bar{\mathbf{3}}_{1/2}$	$2 \times \bar{\mathbf{3}}_{1/2}$		$\bar{\mathbf{3}}_{1/2} + \mathbf{6}_{1/2}$	$\bar{\mathbf{3}}_{-5/2}$

Table 3: The orbifold parities of the gravitino (128) in the supergravity multiplet. See explanations below table 1.

and reconstruct the full kinetic terms by SU(3) invariance. Before doing any detailed calculation, let us summarize the different places where contributions to the kinetic terms of the gravitational moduli can arise. After restricting to the background eqs. (4.17) the heat kernel coefficients quadratic in the 4d derivatives are

$$[\bar{\Delta}^{\frac{1}{2}}]_{\lambda} = \frac{1}{12} \sqrt{g} R_{\lambda\lambda}, \quad (4.18)$$

$$[a_0]_{\lambda} = -\frac{1}{2} (\lambda \cdot \omega_{\parallel})^2, \quad (4.19)$$

$$[a_1]_{\lambda} = \frac{1}{6} R - E - \omega_{\perp}^2. \quad (4.20)$$

In eq. (4.19) we have expanded the Wilson line to second order in the spin connection, which is linear in the 4d derivative. We now parametrize the full result as follows. Let us combine the sectors according to their fixed tori. There are thus 4 sectors, corresponding to the Z_N elements with $N = 1, 2, 3, 6$. They possess fixed tori T^7, T^6, T^1, T^0 and have $\mathcal{N} = 8, 4, 2, 1$ supersymmetry respectively. Then the one-loop kinetic terms are

$$\mathcal{K} = \sum_{\substack{d=11,10,5,4 \\ r=0,1}} (\mathcal{K}_d^{r,\text{loc}} + \mathcal{K}_d^{r,\text{fin}}) \quad (4.21)$$

where the finite nonlocal contributions result from the terms with $\lambda \in \Lambda_{\parallel}$ nonvanishing, and the local UV-sensitive ones from the term with $\lambda = 0$. Recall that Λ_{\parallel} by definition is the lattice of the fixed torus associated to each orbifold sector. Clearly, $\mathcal{K}_d^{0,\text{loc}} = 0$ as this contribution occurs only for nonzero λ . Moreover, $\mathcal{K}_4^{r,\text{fin}} = 0$ as the fixed torus is trivial. We also expect that all \mathcal{K}_{11}^r and \mathcal{K}_{10}^r vanish from supersymmetry, as we will explicitly verify below. Furthermore, one can see that all contributions from eq. (4.18) as well as from the curvature term in eq. (4.20) vanish: they are proportional to $\text{str } P_L$, which is just the sum over bosonic minus fermionic degrees of freedom, weighed by their orbifold phases. Since we have at least $\mathcal{N} = 1$ supersymmetry everywhere, this term vanishes for all sectors. The

nonzero terms in eq. (4.21) are thus $\mathcal{K}_5^{r,\text{fin}}$ and $\mathcal{K}_{4,5}^{1,\text{loc}}$. Writing $\lambda^{10} = 2\pi n$, we have

$$\mathcal{K}_5^{0,\text{fin}} = \frac{\alpha_{5,0}}{6} (2\pi\sigma) \sum_{n \neq 0} |2\pi n \sigma|^{-5} \text{str} \left[\frac{1}{2} (2\pi n \omega_{10})^2 (P_L^2 + P_L^4) \right], \quad (4.22)$$

$$\mathcal{K}_5^{1,\text{fin}} = \frac{\alpha_{5,1}}{6} (2\pi\sigma) \sum_{n \neq 0} |2\pi n \sigma|^{-3} \text{str} \left[(E + \omega^\ell \omega_\ell) (P_L^2 + P_L^4) \right], \quad (4.23)$$

$$\mathcal{K}_5^{1,\text{loc}} = \frac{\alpha_{5,1}}{6} (2\pi\sigma) \Lambda_{\text{UV}}^3 \text{str} \left[(E + \omega^\ell \omega_\ell) (P_L^2 + P_L^4) \right], \quad (4.24)$$

$$\mathcal{K}_4^{1,\text{loc}} = \frac{\alpha_{4,1}}{6} \Lambda_{\text{UV}}^2 \text{str} \left[(E + \omega^\ell \omega_\ell + \omega^{10} \omega_{10}) (P_L^1 + P_L^5) \right], \quad (4.25)$$

where $\ell = 4 \dots 9$ and the constants $\alpha_{d,r}$ have been defined in eq. (2.12). The symbol str denotes the supertrace. In the following we will calculate these terms and also verify the cancellations for the sectors with $\mathcal{N} \geq 4$ supersymmetry. It is convenient to define the following combinations of kinetic terms

$$\mathcal{O}_1 = \sum_{I < J=1}^3 \partial_\mu \log \rho_I \partial^\mu \log \rho_J \quad (4.26)$$

$$\mathcal{O}_2 = \sum_{I=1}^3 (\partial_\mu \log \rho_I)^2, \quad (4.27)$$

$$\mathcal{O}_3 = \partial_\mu \log \sigma \sum_{I=1}^3 \partial^\mu \log \rho_I, \quad (4.28)$$

$$\mathcal{O}_4 = (\partial_\mu \log \sigma)^2. \quad (4.29)$$

For the contribution $\mathcal{K}_5^{0,\text{fin}}$ we need to evaluate the supertrace over the spin connection in eq. (4.22). The spin connection along the fixed torus S^1 is given by

$$\omega_{10} = -\Sigma^{10\beta} \partial_\beta \sigma \quad (4.30)$$

where the Σ^{AB} are the $\text{SO}(11)$ generators. Let us define the quantity

$$C_k^{AB,CD} = \text{str} \Sigma^{AB} \Sigma^{CD} P_L^k. \quad (4.31)$$

For the 5d sector we are interested in calculating C_2 and C_4 . Since the generators in eq. (4.30) are in fact generators of $\text{SO}(5) \subset \text{SO}(11)$, P_L^k commutes with the Σ^{AB} and we can symmetrize in the two generators. Each representation of $\text{SO}(5)$ has a definite phase p under the orbifold action. The quantity C^{ab} above is then given by

$$C_k^{AB,CD} = C \delta^{AB,CD}, \quad C_k = \sum_p p \sum_{r_p} (-)^F C_{r_p} \quad (4.32)$$

where r_p label the different $\text{SO}(5)$ -representations of a given parity p and C_{r_p} is the corresponding Dynkin index. The easiest way to calculate the Dynkin indices is to consider the $\text{SO}(2)$ helicity group, which is a subgroup of $\text{SO}(5)$. This choice has the advantage that one can restrict to physical states only and discard any unphysical and ghost states that have to cancel each other. The Dynkin index of an $\text{SO}(2)$ representation of helicity h is simply¹² $C_h = 2h^2$ and all one needs to know to evaluate C is which 4d fields have a given

¹²The factor of 2 arises from the normalization: the $\text{SO}(5)$ vector representation has $C_5 = 2$ in the standard convention for the generators.

parity. According to tables 1 to 3 we obtain

$$C_2 = 2 \left[4 \cdot 1 + 1 \cdot (10 + 9\theta + 9\bar{\theta}) - \frac{9}{4} \cdot (2 + 3\theta + 3\bar{\theta}) - \frac{1}{4} \cdot (20 + 18\theta + 18\bar{\theta}) \right] = \frac{27}{2} \quad (4.33)$$

The result for C_4 is the same. It remains to be shown that in the 11d and 10d sectors there occur cancellations, as required by $\mathcal{N} \geq 4$ supersymmetry. The spin connection now transforms in $SO(11)$ ($SO(10)$) but we can again apply our trick of calculating the Dynkin indices from the $SO(2)$ subgroup. From the tables one finds

$$C_{k=0} = 2 \left[4 \cdot 1 + 1 \cdot 28 - \frac{9}{4} \cdot 8 - \frac{1}{4} \cdot 56 \right] = 0 \quad (4.34)$$

$$C_{k=3} = 2 \left[4 \cdot 1 + 1 \cdot (12 - 16) - \frac{9}{4} \cdot (4 - 4) - \frac{1}{4} \cdot (28 - 28) \right] = 0 \quad (4.35)$$

Let us then turn to the kinetic terms generated by the moduli dependence of the mass matrices. According to our discussion in section 2, we can use the tree-level equations of motion in the one-loop correction to the effective action, since — up to higher order terms in the loop expansion parameter — this simply corresponds to a field redefinition. For the background we are considering here, the equations of motion simply read $R_{MN} = 0$.¹³ Notice that this procedure also takes care of any additional Weyl rescalings arising at one-loop order. On-shell, the only nonzero mass matrices are

$$E_{2,t}{}^{MN}{}_{PQ} = -2R_{(P}{}^{(M}{}_{Q)}{}^{N)} \quad (4.36)$$

$$E_{a3}{}^{MNL}{}_{PQR} = -6R_{[P}{}^{[M}{}_{Q]}{}^{N]}{}^L{}_{R]} \quad (4.37)$$

$$E_{a2}{}^{MN}{}_{PQ} = -2R_{[P}{}^{[M}{}_{Q]}{}^{N]} \quad (4.38)$$

$$E_{3/2}{}^A{}_B = -\frac{1}{2}R^A{}_{BMN}\gamma^{MN} \quad (4.39)$$

(Recall that the ghosts for the gauge symmetries of the antisymmetric three-form contain two real antisymmetric two-forms). Clearly, for the 11d sector, the trace over any of these matrices is proportional to R and hence again vanishes by the equations of motion. For the 10d sector, notice that any trace $\text{tr } P_L E$ can generally be written in terms of $R_{(10)}$. However, the equations of motion also imply $R_{(10)} = 0$ and there are no Kähler corrections, as required by $\mathcal{N} = 4$ supersymmetry. For the 5d and 4d sectors, notice that P_L always acts trivially on the 4d indices. The equations of motion then allow one to make the replacements

$$R^\mu{}_{i\mu j} = -R^k{}_{ikj}, \quad R^\mu{}_{\nu\mu} = R^i{}_{j i}. \quad (4.40)$$

It should be clear at this point why the use of the equations of motion can drastically simplify the analysis. In particular, there are no one-loop terms proportional to the 4d

¹³Had we been interested in the kinetic terms for the moduli originating from the gauge sector or the B field we would have to take into account terms proportional to the energy momentum tensor when using the equations of motion.

curvature scalar and hence no additional Weyl rescalings are necessary. The curvature tensor with all compact indices can be expressed as

$$R^{\ell r}{}_{sk} = (\partial_\mu \log \bar{\rho}_\ell)(\partial^\mu \log \bar{\rho}_r)(\delta^\ell_k \delta^r_s - \delta^\ell_s \delta^r_k) \quad (4.41)$$

where $\bar{\rho}_4 = \bar{\rho}_5 = \rho_1$, $\bar{\rho}_6 = \bar{\rho}_7 = \rho_2$, $\bar{\rho}_8 = \bar{\rho}_9 = \rho_3$ and $\bar{\rho}_{10} = \sigma$. It is now straightforward to evaluate the traces. One finds for the 5d sector

$$\text{tr } E_t P_L^2 = -18\mathcal{O}_1 \quad (4.42)$$

$$\text{tr } E_{a3} P_L^2 = 6\mathcal{O}_2 \quad (4.43)$$

$$-2 \text{tr } E_{a2} P_L^2 = -36\mathcal{O}_1 - 12\mathcal{O}_2 \quad (4.44)$$

$$-1/2 \text{tr } E_{3/2} P_L^2 = 6\mathcal{O}_2 \quad (4.45)$$

There is an identical contribution from the element P_L^4 in the sum eq. (3.12). Adding all contributions, one finds

$$\text{str } E(P_L^2 + P_L^4) = -108\mathcal{O}_1, \quad (4.46)$$

In a similar manner, for the 4d sector one finds

$$\text{str } E(P_L^1 + P_L^5) = 108\mathcal{O}_1 + 36\mathcal{O}_2 + 48\mathcal{O}_3 \quad (4.47)$$

To evaluate the traces over the square of the spin connection occurring in eq. (4.23) to (4.25), notice that ω_ℓ is given by

$$\omega_\ell = -\Sigma^{a\beta} \delta_{a\ell} \partial_\beta \bar{\rho}_i, \quad (4.48)$$

with the index $i, \ell = 4 \dots 9$. The Σ^{AB} in eq. (4.48) are now generators that are broken by the Z_N action, which changes the evaluation of $C^{AB,CD}$. Using the SO(11) commutation relations as well as the orbifold transformations of the generators one can write

$$C_k^{\alpha\alpha, \beta\beta} = \delta^{\alpha\beta} C_k^{ab}, \quad C_k^{ab} = i(1 - P^{-1})^{-1} a_c \text{str } \Sigma^{cb} P_L^k, \quad (4.49)$$

Again, this vanishes for the 10d sector ($c = b = 10$). For the 5d and 4d sectors, Σ^{cb} is a generator of SO(6) or SO(7) respectively, and the trace projects onto the U(1) generator of the surviving U(3) subgroup, so we can write

$$\text{str } \Sigma^{cb} P_L = \frac{i}{3} Q^{cb} \text{str } Q P_L = \frac{i}{3} Q^{cb} \sum_p p \sum_{q_p} (-)^F q_p. \quad (4.50)$$

where $Q = \Sigma^{12} + \Sigma^{34} + \Sigma^{56}$. The charges can be read off from Tabs 1 to 3. Without loss of generality we can symmetrize C^{ab} in the two indices, so we finally obtain

$$C_k^{(ab)} = \delta^{ab} C_k \quad (4.51)$$

With

$$C_1 = -C_5 = \frac{9i}{2\sqrt{3}}, \quad C_2 = C_4 = 4, \quad (4.52)$$

This concludes the evaluation of the traces in eq. (4.22) to eq. (4.25). The result is thus

$$\mathcal{K}_5^{0, \text{fin}} = \frac{27\zeta(3)}{32\pi^4} \sigma^{-2} \mathcal{O}_4, \quad \mathcal{K}_5^{1, \text{fin}} = \frac{\zeta(3)}{4\pi^4} \sigma^{-2} \left[-\frac{9}{4} \mathcal{O}_1 + \frac{1}{3} \mathcal{O}_2 \right], \quad (4.53)$$

$$\mathcal{K}_5^{1, \text{loc}} = \frac{1}{\pi} \Lambda_{\text{UV}}^3 \sigma \left[-\frac{9}{4} \mathcal{O}_1 + \frac{1}{3} \mathcal{O}_2 \right], \quad (4.54)$$

$$\mathcal{K}_4^{1, \text{loc}} = \frac{1}{4\pi^2} \Lambda_{\text{UV}}^2 [9\mathcal{O}_1 + 3\mathcal{O}_2 + 4\mathcal{O}_3]. \quad (4.55)$$

As in the previous subsection, one can recovariantize these terms in order to make manifest the higher-dimensional invariances preserved by the orbifolding. To this end, one should identify the $\text{SO}(4) \times \text{U}(3)$ and $\text{SO}(5) \times \text{U}(3)$ singlets that one can form from the curvature tensor and use them to replace the operators \mathcal{O}_i .¹⁴ A direct evaluation of the covariant result will be presented elsewhere [17].

5. Conclusions

In this paper we have analyzed the one-loop effective action on orbifolds. We have shown how the evaluation of the heat kernel in each sector of the orbifold can be reduced to the one for the corresponding fixed torus with a shifted mass matrix. We have proposed a further expansion of the heat kernel coefficients in powers of the lattice vectors defining the tori, and explicitly evaluated the expansion of the coefficient a_1 to second order. Our formalism is carried out entirely in position space, avoiding KK decomposition and displaying very clearly the separation between local (UV sensitive) renormalization and nonlocal (UV-finite) one. The main results of the paper can be found in eqs. (2.13), (2.14), (2.17), (2.21), and (2.22) for the torus, and eqs. (3.11), (3.12), (3.14), and (3.15) for the orbifold.

To exemplify our methods we have calculated the effective potential in 6d gauge theory on T^2/Z_N and the corrections to moduli kinetic terms in 11d supergravity on $T^6/Z_3 \times S^1/Z_2$. In particular, the latter example shows how Kähler corrections can be computed in orbifold compactifications. This is extremely useful as it allows one to analyze the moduli effective potential in a way that is independent on the supersymmetry breaking mechanism.

Our results are restricted to operators that do not contain extra dimensional derivatives. For some applications (e.g. effective operators involving KK modes, warped backgrounds or otherwise nontrivial profiles) one might wish to study backgrounds including such normal derivatives. While the evaluation of the toroidal heat kernel can be straightforwardly extended to this case,¹⁵ the orbifold heat kernel receives further corrections. These can be computed along the following lines. Notice that, as a consequence of eq. (3.2), the

¹⁴The $\text{SO}(5) \times \text{U}(3)$ singlets are, in the usual complex basis, $\mathcal{C}_1 = R^{IJ}{}_{IJ}$ and $\mathcal{C}_2 = R^{I\bar{J}}{}_{I\bar{J}}$. For $\text{SO}(4) \times \text{U}(3)$ one can, in addition, form the invariant $\mathcal{C}_3 = R^{10I}{}_{10I}$. All other possible invariants are either related to these by the equations of motion or by the symmetries of the curvature tensor. One can then immediately verify that $\mathcal{C}_1 \sim \mathcal{O}_1$, $\mathcal{C}_2 \sim \mathcal{O}_1 + \mathcal{O}_2$, and $\mathcal{C}_3 \sim \mathcal{O}_3$. Note that the operator \mathcal{O}_4 originated from the expansion of the Wilson line which, as a nonlocal object, does not correspond to any local operator.

¹⁵Gravitational backgrounds depending on the extra-dimensional coordinate will require to replace the straight lattice vectors λ by the corresponding geodesics.

heat kernel coefficients occurring in the renormalization at, say, the fixed point $x_f = 0$ satisfy the following identity

$$\text{tr } a_r(x, Px)(P_L \otimes P_G) = \text{tr} [a_0(x_f, x)a_r(x, Px)a_0(Px, x_f)](P_L \otimes P_G). \quad (5.1)$$

The quantity in square brackets is a covariant object at the fixed point, i.e., it transforms at the fiber at x_f from both sides. There exists thus a covariant Taylor expansion in the geodesic distance from the fixed point that has as coefficients gauge-covariant operators at $x_f = 0$. After integrating over x , the powers in this expansion are replaced by powers of T . In this way one can obtain a fully covariant fixed point action that takes into account the complete invariance surviving the local projection. The explicit calculation and evaluation of the expansion will be left to future research [17].

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A. Mass matrices

In this appendix we would like to summarize the background dependence of the inverse propagators, or fluctuation operators, for fields of various spins, see for instance refs. [11, 20]. We work in Euclidean spacetime with the following conventions. The Christoffel connection is given by $\Gamma_{SN}^M = -\frac{1}{2}\partial^M g_{SN} + \dots$ and the curvature by $R^M{}_{NRS} = \partial_R \Gamma_{SN}^M - \dots$. The covariant derivative is $D_M = \nabla_M - iA_M - i\omega_M$ with hermitian gauge and spin connections, the latter being related to the Christoffel connection by $\omega_M = -\frac{1}{2}\Sigma^{AB} e_A^N \nabla_M e_{NB}$. The conventions for the vector generators of the Lorentz group is $(\Sigma^{AB})^C{}_D = -i(\delta^{AC}\delta_D^B - \delta^{BC}\delta_D^A)$.

For bosonic fields, the inverse propagator \mathcal{F} is obtained by linearizing the equations of motion in the fluctuations around a generic background. For fermions, one takes the absolute square of that operator. For a suitable choice of gauge, \mathcal{F} can be cast in the form

$$\mathcal{F} = -D^2 + 2i B^N D_N + i D_N B^N + E, \quad (A.1)$$

where the covariant derivative D contains all background gauge and spin connections, and E and B_M are matrices depending on the background fields. The parametrization of eq. (A.1) is such that, with B_M and E hermitian, \mathcal{F} is hermitian. The matrices B_M and E can mix different fields, in particular particles of different spin. Note that we can formally redefine the connection and the mass matrix to absorb the terms linear in the derivative:

$$\mathcal{F} = -(D - iB)^2 + (E - B^2). \quad (A.2)$$

Whereas off-diagonal elements in E are relatively easy to deal with,¹⁶ a nontrivial B poses a bigger challenge from a computational point of view. For the examples in this paper, we will restrict to backgrounds that have $B = 0$. In the following, we give the mass matrices for gauge theory and gravity.

¹⁶Note that, in calculating $\text{tr } E$, the off-diagonal terms in E do not contribute and only show up at $\mathcal{O}(E^2)$ [20].

A.1 Gauge theory

We take as quantum fields all particles with spin ≤ 1 , but will include a general gravitational background. We use the following gauge fixing function in R_ξ gauge with $\xi = 1$.

$$\mathcal{G} = D_M \mathcal{A}^M + iG^\dagger \phi - iG^T \phi^* . \quad (\text{A.3})$$

where $G_b^A = T_{bc}^A \phi_0^c$. Here, ϕ and \mathcal{A} are the dynamical fields and G and D only contain backgrounds. All covariant derivatives as well as field strengths, curvatures etc. are to be evaluated at the background.

For complex scalar particles ϕ one finds

$$E_0 = \partial_{\phi^*} \partial_\phi V + GG^\dagger + \eta R , \quad (\text{A.4})$$

where V is the scalar potential and η is an arbitrary constant. For minimally coupled fields we have $\eta = 0$ while for conformally coupled ones we have $\eta = (d-2)/4(d-1)$. The second term in eq. (A.4) comes from the gauge fixing. For fermions one finds

$$E_{1/2} = -\Sigma^{AB} F_{AB} + \frac{1}{4} R \quad (\text{A.5})$$

with F denoting the field strength of the gauge connection. The overall result has to be multiplied by -1 , $-1/2$, or $-1/4$ for Dirac, Majorana or Weyl, and Majorana-Weyl fermions respectively. Note that $E_{1/2}$ is a $2^{[d/2]}$ dimensional matrix. For the gauge fields themselves, one finds

$$E_{1MN} = 2i F_{MN} + G^\dagger G g_{MN} + R_{MN} , \quad (\text{A.6})$$

$$E_{1,gh} = G^\dagger G . \quad (\text{A.7})$$

The second equation is the mass matrix for the ghost, whose contribution to the effective action has to be multiplied by -2 . Notice that the matrix F acts in the adjoint representation. The matrix E actually contains off-diagonal mixing terms as discussed above. They are given by

$$\Delta \mathcal{L} = \frac{1}{2} \phi^T G^* G^\dagger \phi + i \phi^\dagger (D_M G) \mathcal{A}^M + \text{h.c.} \quad (\text{A.8})$$

but do not contribute at $\mathcal{O}(E^1)$.

A.2 Gravity

Including dynamical gravitational fields is more involved, as now a generic background generates mixing terms linear in derivatives as discussed after eq. (A.1). For instance, a nonzero background for the gauge field induces terms such as

$$(B_M)_{N,(PQ)} \sim F_{M(P} g_{Q)N} , \quad (\text{A.9})$$

that mixes spin-one and spin-two fluctuations. For the sake of simplicity, we shall consider purely gravitational backgrounds, in which case one finds $B_M = 0$. Although slightly less general, this background allows us, e.g., to calculate the effective action of the gravitational

moduli. The gauge fixings for the various gauge symmetries are taken as in ref. [20]. The mass matrices for the fields with spin ≤ 1 can be taken from the previous subsection. The mass matrix of a rank- p antisymmetric tensor field is given by [21]

$$E_{ap}^{M_1 \dots M_p}_{N_1 \dots N_p} = p R^{[M_1}_{[N_1} \delta^{M_2}_{N_2} \dots \delta^{M_p]}_{N_p]} - p(p-1) R^{[M_1 M_2}_{[N_1 N_2} \delta^{M_3}_{N_3} \dots \delta^{M_p]}_{N_p]} \quad (\text{A.10})$$

where the square brackets on the indices denote antisymmetrization. There are p' -form ghosts of any $0 \leq p' < p$ that are fermions (bosons) for $p - p'$ odd (even) and that occur in multiplicities of $p - p' + 1$. Their contribution to the effective action has thus to be multiplied by $(p - p' + 1)(-)^{p-p'}$. For a Rarita-Schwinger field (gravitino) one has

$$E_{3/2 AB} = \frac{1}{4} R g_{AB} - i R_{ABMN} \Sigma^{MN} \quad (\text{A.11})$$

$$E_{3/2, gh} = \frac{1}{4} R \quad (\text{A.12})$$

The first line corresponds to the spin 3/2 field, its contribution to the effective action has to be multiplied by $-1/2$ ($-1/4$) for Majorana or Weyl (Majorana-Weyl) fermions. There are three spinor ghosts in total. Having bosonic statistics, the result has to be multiplied by $+3/2$ ($3/4$). The dimension of the matrices $E_{3/2}$ and $E_{3/2, gh}$ are $d \cdot 2^{[d/2]}$ and $2^{[d/2]}$ respectively. For the symmetric traceless part of the graviton one finds

$$E_{2,t}^{MN}{}_{PQ} = R \left[\delta^{(M}_{(P} \delta^{N)}_{Q)} - \left(\frac{4}{d^2} + \frac{1}{d} \right) g^{MN} g_{PQ} \right] - 2 R^{(M}{}_{(P}{}^{N)}{}_{Q)} + \frac{4}{d} (R^{MN} g_{PQ} + R_{PQ} g^{MN}) - 2 R^{(M}{}_{(P} \delta^{N)}_{Q)} \quad (\text{A.13})$$

where the parenthesis on the indices stand for their symmetrization. Furthermore, the canonically normalized trace part and the fermionic vector ghosts give a contribution

$$E_{2,s} = \frac{d-4}{d} R, \quad (\text{A.14})$$

$$E_{2, gh}{}_{MN} = -R_{MN}. \quad (\text{A.15})$$

The ghosts contribute with a factor -2 to the effective action. Let us remark that there are also mass mixings between the tensor and scalar modes of the metric [20].

B. Coincidence limits

In this section we would like to review DeWitt's recursive procedure to calculate the coincidence limits of heat kernel coefficients and their covariant derivatives,

$$[a_{r; \dots}] = \lim_{x' \rightarrow x} a_{r; \dots}(x, x'), \quad (\text{B.1})$$

where the dots stand for any combination of primed and unprimed indices and the semicolon denotes covariant differentiation. The ansatz eq. (2.7) is inserted in the differential equation eq. (2.3) to derive the recursion relations

$$\sigma_{; M} a_{0; M} = 0, \quad (\text{B.2})$$

$$\sigma_{; M} a_{r; M} + r a_r = \Delta^{-1} (\Delta a_{r-1})_{; M}^M - E a_{r-1}. \quad (\text{B.3})$$

where $\Delta = \bar{\Delta}^{1/2}(gg')^{-1/4}$ is a biscalar (as opposed to $\bar{\Delta}$ which is a biscalar density). It is now easy to derive expressions for the coincidence limits needed in the evaluation of the local part of eq. (2.1). This is done by taking repeated covariant derivatives of eqs. (2.8), (2.10) (B.2) and (B.3), making use of the commutation relations for covariant derivatives, and taking coincidence limits [7]. We will first calculate the quantities with derivatives w.r.t. x only, the ones w.r.t. x' can then easily be derived from Synge's rule

$$[X\dots]_{;M} = [X\dots;M] + [X\dots;M'] . \quad (\text{B.4})$$

In particular, one can show that¹⁷

$$[\sigma;_M] = [\sigma;_{MNR}] = [\Delta;_M] = [a_0;_M] = 0 . \quad (\text{B.5})$$

Recall that the boundary condition eq. (2.4) implies $[a_0] = \mathbb{1}$. Hence,

$$[a_1] = [\Delta;_M^M + a_0;_M^M - E] , \quad (\text{B.6})$$

$$2[a_1;_S] = [\Delta;_{MS}^M + a_0;_{MS}^M - E;_S] , \quad (\text{B.7})$$

$$3[a_1;_{(ST)}] = [\Delta;_{(ST)}^{-1} \Delta;_M^M + \Delta;_{M(ST)}^M + \Delta;_{(ST)} a_0;_M^M + \Delta;_M^M a_0;_{(ST)} + 4\Delta;_{\lambda}^M a_0;_{M\lambda} + a_0;_{M(ST)}^M - E;_{(ST)}] , \quad (\text{B.8})$$

$$2[a_2] = [\Delta;_M^M a_1 + a_1;_M^M - E a_1] , \quad (\text{B.9})$$

and so on. Notice that $[a_1;_{(ST)}]$ is needed both for the evaluation of $[a_2]$ as well as $[a_1;_{\lambda\lambda'}$ which enters in eq. (2.21). The covariant expansion introduced in section 2 is thus quite economic in that most of the algebra for $[a_1;_{\lambda\lambda'}$ is the same as for $[a_2]$. To evaluate eq. (B.6) to (B.9) one needs to know the coincidence limits of a_0 and Δ with two, three and four derivatives which are again obtained by differentiation of eq. (2.8), eq. (2.10) and eq. (B.2). The result is expressed in terms of field strength and curvature tensors:

$$[a_0;_{MN}] = -\frac{1}{2}\Omega_{MN} , \quad (\text{B.10})$$

$$[a_0;_{MN}^M] = -\frac{1}{3}\Omega^M_{N;M} , \quad (\text{B.11})$$

$$[a_0;_{M(ST)}^M] = \frac{1}{2}\Omega_{M(S}\Omega^M_{T)} - \frac{1}{2}\Omega_{M(S;T)}^M + \frac{7}{12}R_{M(S}\Omega^M_{T)} , \quad (\text{B.12})$$

$$[\Delta;_{(ST)}] = \frac{1}{6}R_{ST} , \quad (\text{B.13})$$

$$[\Delta;_{MS}^M] = \frac{1}{6}R;_S , \quad (\text{B.14})$$

$$[\Delta;_{M(ST)}^M] = \frac{3}{20}R;_{ST} + \frac{1}{20}\square R_{ST} - \frac{1}{15}R_{MS}R^M_T + \frac{1}{30}R^{MN}R_{MSNT} + \frac{1}{36}RR_{ST} + \frac{1}{30}R^{MNL}_S R_{MNLT} \quad (\text{B.15})$$

where Ω is the field strength of the gauge and spin connections

$$\Omega_{MN} = [D_M, D_N] = -iF_{MN} + \frac{i}{2}\Sigma^{AB}R_{ABMN} . \quad (\text{B.16})$$

¹⁷An extensive discussion of the quantities σ and Δ as well as their derivatives can be found in ref. [19].

Inserting these expressions into eqs. (B.6) to (B.9) one gets

$$[a_1] = \frac{1}{6}R - E, \quad (\text{B.17})$$

$$[a_{1;\lambda}] = \frac{1}{12}R_{;\lambda} - \frac{1}{6}\Omega^M{}_{\lambda;M} - \frac{1}{2}E_{;\lambda}, \quad (\text{B.18})$$

$$[a_{1;\lambda\lambda}] = \frac{1}{60}\square R_{\lambda\lambda} + \frac{1}{20}R_{;\lambda\lambda} - \frac{1}{3}E_{;\lambda\lambda} - \frac{1}{45}R^M{}_{\lambda}R_{M\lambda} + \frac{1}{90}R^{MN}R_{M\lambda N\lambda} \\ + \frac{1}{90}R^{MNL}{}_{\lambda}R_{MN L\lambda} + \frac{1}{12}R^M{}_{\lambda}\Omega_{M\lambda} + \frac{1}{6}\Omega_{M\lambda}\Omega^M{}_{\lambda} - \frac{1}{6}\Omega_{M\lambda}{}^M{}_{\lambda}, \quad (\text{B.19})$$

$$[a_2] = \frac{1}{2}\left(\frac{1}{6}R - E\right)^2 + \frac{1}{6}\square\left(\frac{1}{5}R - E\right) - \frac{1}{180}R_{MN}R^{MN} \\ + \frac{1}{180}R^{MNLS}R_{MNLS} + \frac{1}{12}\Omega_{MN}\Omega^{MN}, \quad (\text{B.20})$$

Finally, using Synge's rule, one finds

$$[a_{1;\lambda'}] = -[a_{1;\lambda}] + [a_1]_{;\lambda} \quad (\text{B.21})$$

$$[a_{1;\lambda'\lambda'}] = [a_{1;\lambda\lambda}] - 2[a_{1;\lambda}]_{;\lambda} + [a_1]_{;\lambda\lambda} \quad (\text{B.22})$$

leading to eq. (2.21).

C. Zeta regularization

In performing the proper time integration of the heat kernel, zeta-function regularization-techniques are often used [22] (see also refs. [11, 23]). In this scheme, one exploits the fact that the zeta function

$$\zeta(s) = \text{Tr}(-D^2 + E)^{-s}, \quad (\text{C.1})$$

is UV convergent for $s > \frac{d}{2}$ and has an analytic continuation that is regular at $s = 0$. One then writes formally

$$S_{\text{eff}} = (-)^F \frac{1}{2} \sum \text{Tr} \log(-D^2 + E) = -(-)^F \frac{1}{2} \lim_{s \rightarrow 0} \zeta'(s), \quad (\text{C.2})$$

and uses the relation

$$\zeta(s) = \Gamma(s)^{-1} \int dT T^{s-1} K(T), \quad (\text{C.3})$$

to write the renormalized effective action as

$$S_{\text{eff}} = -(-)^F \frac{1}{2} \lim_{s \rightarrow 0} \frac{d}{ds} \left(\Gamma(s)^{-1} \int dT T^{s-1} \text{Tr} K(T) \right). \quad (\text{C.4})$$

All relations in eqs. (C.1) to (C.4) are well defined at large s and at $s = 0$ after analytic continuation. Of course, if the integral in eq. (C.4) is UV finite for $s = 0$ one just recovers the old expression, eq. (2.1), by means of the expansion $1/\Gamma(s) = s + \mathcal{O}(s^2)$. IR divergences have to be treated separately. We do this by introducing an explicit mass μ to all fields

such that a suppression factor of $\exp(-\mu^2 T)$ is present in the integrals. Let us define $\nu = r + s - \frac{d}{2}$, then we can write the integral appearing in eq. (C.4) as

$$\int \frac{dT}{T^{1-\nu}} \exp \left[-T\mu^2 - \frac{\lambda^2}{4T} \right] = 2(2\mu^2)^{-\nu} x^\nu K_\nu(x), \quad (\text{C.5})$$

where $K_\nu(x)$ are the modified Bessel functions of the second kind and we have defined $x = |\lambda|\mu$.

For the nonlocal contributions, $\lambda \neq 0$, the integral is convergent at $s = 0$. Using that $1/\Gamma(0) = 0$ and $[1/\Gamma(0)]' = 1$ one finds

$$S_{\text{eff, fin}} = -(-)^F 2^{-r-\frac{d}{2}} \pi^{-\frac{d}{2}} \sum_{r, \lambda \neq 0} \left(\frac{|\lambda|}{\mu} \right)^{r-\frac{d}{2}} K_{r-\frac{d}{2}}(|\lambda|\mu) [\bar{\Delta}]_\lambda \text{tr}[a_r]_\lambda. \quad (\text{C.6})$$

Being both IR and UV finite, this result is valid for all r and d . For $r < \frac{d}{2}$, the integration over T is IR convergent and we can take the limit $\mu \rightarrow 0$. Using the small- x asymptotic expansion

$$K_{r-\frac{d}{2}}(x) = K_{\frac{d}{2}-r}(x) \sim 2^{\frac{d}{2}-r-1} \Gamma\left(\frac{d}{2}-r\right) x^{r-\frac{d}{2}} \quad (\text{C.7})$$

we precisely recover eq. (2.14). However, the summation over λ is still IR sensitive as soon as the dimension of the operator exceeds 4. It is reassuring that the presence of the IR cutoff also takes care of the divergences for large λ as the Bessel functions are exponentially suppressed at large argument. Similarly, for $r = \frac{d}{2}$ one uses

$$K_0(|\lambda|\mu) \sim -\log(|\lambda|\mu') \quad \mu' = \frac{e^{\gamma_E}}{2} \mu \sim 0.89 \mu, \quad (\text{C.8})$$

leading to

$$S_{\text{eff, fin}}^{r=d/2} = (-)^F (4\pi)^{-\frac{d}{2}} \sum_{\lambda \neq 0} \log(|\lambda|\mu') \int d^d x [\bar{\Delta}]_\lambda \text{tr}[a_{\frac{d}{2}}]_\lambda. \quad (\text{C.9})$$

For $|\lambda| = 0$, the local contribution, the integral eq. (C.5) diverges at $s = 0$ for $r < \frac{d}{2}$. Applying the prescription of analytic continuation from the large s -region one finds

$$\lim_{x \rightarrow 0} x^\nu K_\nu(x) = 2^{\nu-1} \Gamma(\nu). \quad (\text{C.10})$$

Since the zeta function, eq. (C.1), and its derivative are analytic at $s = 0$ we expect that the poles of the Gamma function in eq. (C.10) cancel with the pole of the Gamma function in eq. (C.4). Let us define

$$\beta_{d,r}(\mu/Q) \equiv \frac{1}{2} (4\pi)^{-\frac{d}{2}} \lim_{s \rightarrow 0} \frac{d}{ds} \left((\mu/Q)^{-2s} \frac{\Gamma[r + s - \frac{d}{2}]}{\Gamma[s]} \right), \quad (\text{C.11})$$

where a renormalization scale Q has been introduced to account for the correct dimension. It can immediately be verified that this quantity is finite for any r . Explicitly, one finds

$$\beta_{d,r}(\mu/Q) = \frac{1}{2} (4\pi)^{-\frac{d}{2}} \begin{cases} \frac{(-)^\nu}{(-\nu)!} \left[-\log\left(\frac{\mu^2}{Q^2}\right) + H_{-\nu} \right] & \nu = r - \frac{d}{2} \leq 0, \text{ } d \text{ even} \\ \Gamma(\nu) & \text{else.} \end{cases} \quad (\text{C.12})$$

Here, H_n are the harmonic numbers defined as $H_n = \sum_1^n k^{-1}$ with the convention $H_0 = 0$. The local part of the OLEA thus reads

$$S_{\text{eff, loc}}^{\zeta\text{-reg}} = -(-)^F \int d^d x \sqrt{g} \sum_r \beta_{d,r}(\mu/Q) \mu^{d-2r} \text{tr}[\alpha_r]. \quad (\text{C.13})$$

As with dimensional regularization, zeta function regularization does not capture power-like divergences but only logarithmic ones (if present).

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