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Spectral Methods

Evolution to Complex Geometries
and Applications to Fluid Dynamics

With 183 Figures and 11 Tables

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