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Boundary Askey-Wilson Symmetry and Related Spectral Problem Exact Solution

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Abstract

The sine-Gordon model and affine Toda field theories on the half-line, on the one hand, the XXZ spin chain with nondiagonal boundary terms, and interacting many-body lattice systems with a flow, on the other, have a common characteristic. They possess nonlocal conserved boundary charges, generating a coideal subalgebra of the bulk quantized affine symmetry. We consider a related spectral problem for a system with boundary symmetry based on the Askey-Wilson algebra. The importance of our study is motivated by the representation of one of the boundary algebra generators as the second order difference operator for the Askey-Wilson polynomials, known to be exactly solvable. The difference equation for the Askey-Wilson polynomials becomes equivalent to the diagonalization problem for a general quadratic form in the quantum group generators, interpreted in a proper way as the Hamiltonian of a physical system with boundary Askey-Wilson symmetry. We argue that the boundary Askey-Wilson symmetry is the deep algebraic property allowing for integrability of the physical system in consideration.

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1 Introduction

The definition of integrability in the classical mechanics is due to Liouville. It says, that the classical Hamiltonian of a finite-dimensional system is completely integrable if there exists a set of independent integrals of motion commuting with respect to the Poisson brackets $\{I_i, I_j\}$, their number (including the Hamiltonian) being half of the dimension $2N$ of the phase space. The system is superintegrable if it possess more that N integrals of motion. To generalize the Liouville's definition for complete integrability to the quantum case one requires the existence of N commuting operators $[I_i, I_j] = 0$ (including the Hamiltonian). Accordingly, a quantum system is called integrable if it is possible to exactly calculate some quantities of physical interest, namely the common spectrum of commuting quantum integrals of motion and some correlators. This is the basic idea of the developed algebraic approach, known as the quantum inverse scattering method [1] to integrable models. It uses new algebras, the R matrix algebras, and the existence of a family of commuting transfer matrices, depending on a spectral parameter, to describe the dynamical symmetry of integrable systems. The transfer matrices give rise to infinitely many mutually commuting conservation laws. This is the abelian symmetry of the system. The infinitely many commuting conserved charges can be diagonalized simultaneously and their common eigenspace is finite-dimensional in most cases. Thus the abelian symmetry reduces the degeneracies of the spectrum from infinite to finite which is the reason for integrability. In addition many systems possess nonabelian symmetries. They determine the R matrix operator, a solution of the Yang-Baxter equation, up to an overall scalar factor and are identified as the quantum bulk symmetries. In the presence of general boundaries the quantum symmetry, and the integrability of the model as well, are broken. However with suitably chosen boundary conditions [2, 3] a remnant of the bulk symmetry may survive and the system possesses hidden boundary symmetries, which determine a K-matrix, a solution of a boundary Yang-Baxter equation and allow for the exact solvability. Such non local boundary symmetry charges were originally obtained for the sine Gordon model [4] and generalized to affine Toda field theories [5], and derived from spin chain point of view as commuting with the transfer matrix for a special choice of the boundary conditions [6] or analogously as the one boundary Temperley-Lieb algebra centralizer in the "nondiagonal" spin 1/2 representation [7]. The derivation of the nonlocal charges used the algebraic technique based on the quantum affine symmetry in the bulk and the known boundary reflection K-matrices. They were obtained as coideals of the bulk quantum symmetry and interpreted as generating a new symmetry.

In a recent paper [8] we have defined the Askey-Wilson (AW) algebra as a coideal subalgebra of the quantum affine $U_q(\hat{sl}(2)$. (See also [9] and [10, 11, 12] for previous discussions of these problems.) We have constructed a K-matrix in terms of the Askey-Wilson algebra generators, which satisfies a boundary Yang-Baxter equation (known as a reflection equation). As an example of an Askey-Wilson boundary symmetry we have considered a model of nonequilibrium physics, the open asymmetric exclusion process with most general boundary conditions. This model is exactly solvable in the stationary state within the matrix product ansatz to stochastic dynamics and it can be shown that the boundary operators generate the Askey-Wilson algebra. The model is equivalent to the integrable spin 1/2 XXZ chain with most general boundary terms, whose bulk Hamiltonian (infinite chain) possesses the quantum affine symmetry $U_q(\hat{sl}(2))$.

There is a natural homomorphism of the Askey-Wilson algebra to a tridiagonal algebra, known as deformed Dolan-Grady relations or deformed Onsager algebra. Recently Baseilhac and Koizumi [13] have explicitly constructed the deformed analogue of the Onsager algebra for the XXZ spin chain. Using the representation theory of the q-Onsager they diagonalize the transfer matrix of the spin 1/2 XXZ chain of L sites with general integrable boundary conditions and generic anisotropy parameter q with $|q| = 1$. They argue to have obtained the complete exact spectrum from the roots of the characteristic polynomial of dimension 2^L.

Even though the ASEP is equivalent to the integrable spin 1/2 XXZ chain with most general boundary terms, yet the two models describe different physics. The XXZ chain is a Hamiltonian system, while the ASEP is a dynamical process, which has a nonvanishing current in the steady state. The exact solution of the model in the stationary state was achieved in terms the Askey-Wilson polynomials, the irreducible modules of the hidden boundary AW (and tridiagonal) symmetry. It has been emphasized that the steady state solution of ASEP is ultimately related to the AW polynomials. In this paper we obtain the exact spectrum of the transition rate matrix Γ (equivalently the Hamiltonian) of the ASEP on a one-dimensional lattice of L site. We construct the complete set of 2^L eigenvalues whose eigenspaces have dimension one. The unique ground state of the stationary ASEP belongs to the complete set of eigenvectors. This is done in the auxiliary space with discrete basis, constructed with the help of the roots of the AW polynomial of order L. There is a representation in this space of the tridiagonal algebra in the form of the deformed Dolan-Grady relations which can be interpreted as the exact q-analogue of the Dolan-Grady relation for the Ising model.

In our opinion, it is quite remarkable that affine Toda field theory (the special case of which is the sine Gordon model), the XXZ spin chain, and the interacting lattice many-body system with a flow, through its equivalence to the XXZ spin chain, have the common characteristic to possess a quantum affine symmetry $U_q(\hat{sl}(2))$ (or $U_q(\hat{su}(2))$ in the bulk and boundary nonlocal charges generating an Askey-Wilson algebra, a coideal subalgebra of the quantum group bulk symmetry. The existence of an operator-valued reflection matrix, expressed in terms of the AW algebra generators and satisfying a boundary Yang-Baxter equation is the deep algebraic property behind these models allowing for integrability. This puts forward a connection to a Bethe Ansatz solution for the spectrum of the relevant physical quantities.

As known the nondeformed analogue of the tridiagonal algebra are the celebrated Dolan-Grady [14] relations

$$\begin{aligned} [A^*, [A^*, [A^*, A]]] &= k^*[A^*, A], \\ [A, [A, [A, A^*]]] &= k[A, A^*] \end{aligned} \tag{1}$$

It has been proved (see [14] for the details) that if the above relations hold the Hamiltonian

$$H = f^* A^* + f A \tag{2}$$

where f^*, f are some (coupling) constants, belongs to an infinite family of mutually commuting operators - integrals of motion of H. It was shown in [14] that for the Ising model only the first relation in (1) was satisfied, the other one followed from duality properties of the operators A^*, A. In our study we observe the same property for the deformed algebra of the ASEP.

In this paper we first define the AW algebra with N generators as a coideal subalgebra of the quantum affine $U_q(\hat{sl}(N))$ by a generalization of the homomorphism in the $U_q(\hat{sl}(2))$ case. We define the tridiagonal algebra with N generators through the natural homomorphism to the Askey-Wilson algebra.

We then consider a related spectral problem exact solution of the asymmetric simple exclusion process as a consequence of its boundary symmetry based on the Askey-Wilson algebra. The importance of the AW algebra related spectral problem is motivated by the identification of one of the generators with the second order difference operator for the AW polynomials in the basic representation. The difference equation for the AW polynomials [15] becomes equivalent to the diagonalization problem for a general quadratic form in the quantum group generators (commonly interpreted as the Hamiltonian of a proper physical system). We present a diagonalization of the transition matrix by Bethe Ansatz procedure for the second order difference operator for the Askey-Wilson polynomials and obtain the complete spectrum for the lattice system with boundary AW symmetry, namely the asymmetric simple exclusion process. The AW polynomials are the infinite-dimensional irreducible modules of the boundary AW algebra, as well as of the boundary TD algebra. The finite-dimensional irreducible modules can be naturally obtained through a constraint on the boundary parameters. This corresponds to the known XXZ Bethe ansatz solution of Nepomechie et. al and Cao et al. However this is not applicable to the open asymmetric exclusion process since any relation between the boundary parameters spoils the nonequilibrium behaviour. We construct the finite-dimensional irreducible module of the tridiagonal algebra for arbitrary range of the parameters. From the properties of the tridiagonal algebra representation we derive the complete set of states and exact spectrum of the ASEP transition matrix of dimension 2^L, such that all the eigenvalues are real and distinct. In view of the importance of the AW algebra for the ASEP stationary exact solution we consider the proposed method for diagonalization of the transition matrix the most efficient for this model of nonequilibrium physic. The algebraic scheme can be applied to other systems with boundary Askey-Wilson algebra upon the identification of the second order difference operator with a proper Hamiltonian and with the specific properties of the model taken into account.

2 The quantum affine $U_q(\hat{sl}(N))$

In this section we recall the definition of the affine $U_q(\hat{sl}(N))$ [16, 17, 18, 19]. We fix a real number $0 < q < 1$ (in the general case q is complex) and use the q-symbol in the form

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \equiv [x]_{q^{1/2}} \tag{3}$$

The quantum affine $U_q(\hat{sl}(N))$ is defined as the associative algebra with a unit with generators $E_i^{\pm}$ and q^{H_i}, $i = 0, 1, ..., N-1$ in the Chevalley basis and defining relations

$$\begin{aligned} q^{H_i} q^{-H_i} &= q^{-H_i} q^{H_i} = 1 \\ q^{H_i} q^{H_j} &= q^{H_j} q^{H_i} \end{aligned} \tag{4}$$

$$
\begin{aligned}
q^{H_i} E_j^{\pm} q^{-H_i} &= q^{\pm a_{ij}} E_j^{\pm} \\
[E_i^{\pm}, E_j^{\mp}] &= \delta_{ij} \frac{q^{H_i} - q^{-H_i}}{q^{1/2} - q^{-1/2}}
\end{aligned} \tag{5}
$$

together with the q-Serre relations

$$
\sum_{0}^{1-a_{ij}} (-1)^r \left[\begin{array}{c} 1 - a_{ij} \\ r \end{array} \right]_q (E_i^{\pm})^r E_j^{\pm} (E_i^{\pm})^{1-a_{ij}-r} = 0 \tag{6}
$$

where

$$
\left[\begin{array}{c} m \\ r \end{array} \right]_q = \frac{[m]!}{[r]![(m-r)]!} \tag{7}
$$

and a_{ij} is the extended Cartan matrix of type A_{N-1}, i.e. $a_{ii} = 2, a_{ij} = -1 (i = j+1), a_{0N} = -1, a_{ij} = 0$ (otherwise).

For $U_q(\hat{sl}(2))$ the simple roots can be chosen such that $\vec{\alpha}_0 = -\vec{\alpha}_1, |\vec{\alpha}_1|^2 = 2$. Hence $a_{01} = -2$.

Let a_i^V , $i = 0, ..., N-1$ be the labels of the dual Dynkin diagram [20]. The element $c = \sum_{i=0}^{i=N-1} a_i^V H_i$ is central and its value is the level of the affine $U_q(\hat{s}l_N)$. The algebra is endowed with the structure of a Hopf algebra. Namely, the coproduct Δ, the counit ϵ and the antipode S are defined as

$$
\begin{aligned}
\Delta(E_i^+) &= E_i^+ \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^+ \\
\Delta(E_i^-) &= E_i^- \otimes q^{-H_i/2} + q^{H_i/2} \otimes E_i^- \\
\Delta(H_i) &= H_i \otimes I + I \otimes H_i
\end{aligned} \tag{8}
$$

$$
\epsilon(E_i^+) = \epsilon(E_i^-) = \epsilon(H_i) = 0, \qquad \epsilon(I) = 1 \tag{9}
$$

$$
S(E_i^{\pm}) = -q^{\mp 1/2} E_i^{\pm}, \quad S(H_i) = -H_i, \quad S(I) = 1 \tag{10}
$$

Let α_0 denote the longest root and ρ be $1/2$ the sum of positive roots. We consider the $U_q(\hat{sl}(N))$ algebra with a scaling element d, defined by $(d, \alpha_0) = 1$ ($(,)$ is the nondegenerate bilinear form on the Cartan subalgebra) and denote $h = (\alpha_0, \alpha_0) + 2(\rho, \alpha_0)$. With a finite-dimensional representation π_V of $U_q(\hat{sl}(N))$ one associates the quantum R-matrix $R^{VV}(\lambda)$ which acts in $V \otimes V$ and satisfies the Yang-Baxter equation. One has also $[d \otimes d, R] = 0$. The universal R-matrix $R(\lambda)$ is uniquely defined [21] by the first terms in its expansion in powers of the Chevalley generators of $U_q(\hat{sl}(N))$

$$
R = q^{c \otimes d + d \otimes c + \sum_{i=0}^{N-1} H_i \otimes H^i} \left(1 \otimes 1 + (q^{1/2} - q^{-1/2}) \sum_{i=0}^{N-1} q^{H_i/2} E_i^+ \otimes q^{-H_i/2} E_i^- + ... \right) \tag{11}
$$

One can define an automorphism T_λ

$$
T_\lambda H_i = H_i, \qquad T_\lambda E_i^{\pm} = \lambda^{\pm} E_i^{\pm}, \qquad i = 0, ..., N-1 \tag{12}
$$

and put $R(\lambda) = (T(\lambda) \otimes id) R$. For a fixed finite-dimensional representation (π, V) of the quotient algebra $U_q(\hat{sl}(N))$, obtained by setting $c = 0$, one has $R^{VV}(\lambda) = (\pi \otimes \pi) R(\lambda)$. Following

[21], one introduces the currents $L^{\pm}(\lambda) \in EndV \otimes U_q(\hat{sl}(N))$, given by $L^+ = (id \otimes \pi_V)(R), L^- = (id \otimes \pi_V)(R^t)$, which are explicitly expressed in terms of the Chevalley generators of $U_q(\hat{sl}(N))$. The operators $L^{\pm}(\lambda)$ generate a Hopf algebra $A(R)$ and their matrix coefficients generate an algebra $A_0(R) \subset A(R)$. Let $L(\lambda)$ denote the quantum current

$$L(\lambda) = L^+(\lambda q)(L^-(\lambda)^{-1} \tag{13}$$

with a finite Laurent series expansion

$$L(\lambda) = \sum_n l_V(n)\lambda^{-n-2}, \qquad n \in Z \tag{14}$$

A theorem (by Reshetikhin and Semenov-Tian-Shansky [21]) states that the element $t(\lambda) = tr_q(L(\lambda))$ lies in the center of the quotient algebra of $A(R)$, obtained by setting $c = -h$. Hence from the explicit expressions of the currents $L^{\pm}$ in terms of the Chevalley generators follows that $t(\lambda)$ is the generating function of the Casimir elements of the quotient algebra of $U_q(\hat{sl}(N))$, obtained by setting $c = -h$.

For our purposes we will need a slightly different realization of the algebra in terms of the Chevalley generators and following [9] for $N = 2$ we define a new basis in $U_q(\hat{sl}(N))$ generated by $H_i, Q_i^s, \bar{Q}_i^s$

$$Q_i^s = (q^{1/2} - q^{-1/2})\sqrt{[2]}E_i^+ q^{-H_i/2} + q^{-H_i}, \qquad \bar{Q}_i^s = -(q^{1/2} - q^{-1/2})\sqrt{[2]}E_i^- q^{-H_i/2} + q^{-H_i} \tag{15}$$

In terms of the operators $Q_i = E_i^+ q^{-H_i/2}$ and $\bar{Q}_i = E_i^- q^{-H_i/2}$ the deformed q-Serre relations can be expressed in a very simple form

$$adj_{Q_i}^{1-a_{ij}}(Q_j) = adj_{\bar{Q}_i}^{1-a_{ij}}(\bar{Q}_j) = 0 \tag{16}$$

Let now $u_i, v_i, i = 0, ...N-1$ be some scalars (in general complex). We denote

$$U_i = u_i Q_i^s, \qquad V_j = v_j \bar{Q}_j^s, \tag{17}$$

There is no summation over repeated indices in eq.(17). Then we have

$$qV_iU_i - q^{-1}U_iV_i = (q - q^{-1})v_iu_i \tag{18}$$

$$q^{1/2}U_iV_{i+1} - q^{-1/2}V_{i+1}U_i = (q^{1/2} - q^{-1/2})u_iv_{i+1}q^{-H_i/2-H_{i+1}/2} \tag{19}$$

$$q^{1/2}U_{i+1}V_i - q^{-1/2}V_iU_{i+1} = (q^{1/2} - q^{-1/2})u_{i+1}v_iq^{-H_i/2-H_{i+1}/2} \tag{20}$$

The operators U_i, V_j satisfy the following relations which are the direct consequence of (16)

$$U_i^3U_{i+1} - [3]U_i^2U_{i+1}U_i + [3]U_iU_{i+1}U_i^2 + U_{i+1}U_i^3 = 0, \qquad i = 0, 1, ..., n-1 \tag{21}$$

$$U_{i+1}^3U_i - [3]U_{i+1}^2U_iU_{i+1} + [3]U_{i+1}U_iU_{i+1}^2 + U_iU_{i+1}^3 = 0 \tag{22}$$

$$V_i^3V_{i+1} - [3]V_i^2V_{i+1}V_i + [3]V_iV_{i+1}V_i^2 + V_{i+1}V_i^3 = 0 \tag{23}$$

$$V_{i+1}^3V_i - [3]V_{i+1}^2V_iV_{i+1} + [3]V_{i+1}V_iV_{i+1}^2 + V_iV_{i+1}^3 = 0 \tag{24}$$

$$[U_i, U_j] = 0, \qquad [V_i, V_j] = 0, \quad |i-j| \geq 2 \tag{25}$$

We can now form the linear combinations

$$A_i = U_i + V_i \tag{26}$$

It can be verified directly by using the q-commutation relations (18-20) between the operators U_i, V_j and the relations (21-25) that the operators A_i satisfy the following relations, which are a set of tridiagonal relations for the pairs A_i, A_{i+1}

$$\begin{aligned} A_i^3A_{i+1} - [3]_qA_i^2A_{i+1}A_i + [3]_qA_iA_{i+1}A_i^2 + A_{i+1}A_i^3 &= u_iv_i(q-q^{-1})^2[A_i, A_{i+1}] \\ A_{i+1}^3A_i - [3]_qA_{i+1}^2A_iA_{i+1} + [3]_qA_{i+1}A_iA_{i+1}^2 + A_iA_{i+1}^3 &= u_{i+1}v_{i+1}(q-q^{-1})^2[A_{i+1}, A_i] \end{aligned} \tag{27}$$

where $[3]_q = q + q^{-1} + 1$ and $[X, Y] = XY - YX$. In the following section we are going to consider a general realization of the Askey-Wilson algebra (AW) and a tridiagonal algebra (TD) with N generators as coideal subalgebras of the quantum affine $U_q(\hat{sl}(N))$.

3 The Askey-Wilson algebra with N generators

We consider linear combinations of the generators of the quantized affine algebra $U_q(\hat{sl}(N))$, $N \geq 3$.

Proposition I: Let $u_i, , v_i, , k_i, i = 0, ..., N-1$ be some scalars. The operators $A_i, i = 0, ..., N-1$ defined by

$$A_i = u_iE_i^+q^{-H_i/2} + v_iE_i^-q^{-H_i/2} + k_iq^{-H_i}, \quad i = 0, ..., N-1 \tag{28}$$

(It is assumed that $E_i^{\pm}$ in (28) are rescaled by $\pm(q^{1/2} - q^{-1/2})$ according to (15).) satisfy the following algebraic relations

$$[A_i, A_j] = 0, \quad |i-j| \geq 2, \quad i, j = 1, ..., N-1 \tag{29}$$

$$\begin{aligned} [[A_i, A_{i+1}]_q, A_i]_q &= -\rho_iA_{i+1} - \omega_{i+1}A_i - \eta_i^{i+1} \\ [A_{i+1}, [A_i, A_{i+1}]_q]_q &= -\rho_{i+1}A_i - \omega_{i+1}A_{i+1} - \eta_{i+1}^i \end{aligned} \tag{30}$$

with $i = 0, ..., N-2$ and

$$\begin{aligned} [[A_0, A_{N-1}]_q, A_0]_q &= -\rho_0A_{N-1} - \omega_0A_0 - \eta_0^{N-1} \\ [A_{N-1}, [A_0, A_{N-1}]_q]_q &= -\rho_{N-1}A_0 - \omega_0A_{N-1} - \eta_{N-1}^0 \end{aligned} \tag{31}$$

where $[A_i, A_j]_q$ is the q-commutator

$$[A_i, A_j]_q = q^{1/2}A_iA_j - q^{-1/2}A_jA_i, \quad 0 \leq i, j \leq N-1 \tag{32}$$

and the structure constants on the RHS are representation dependent

$$-\rho_i = u_i v_i (q - q^{-1})^2, \tag{33}$$

$$\omega_{i+1} = (q^{1/2} - q^{-1/2})^2 k_i k_{i+1} q^{-\mu(i)-\mu(i+1)} \tag{34}$$

$$\eta_i^{i+1} = (q - q^{-1})^2 u_i v_i k_{i+1} q^{-2\mu(i)-\mu(i+1)}, \qquad \eta_{i+1}^{i} = (q - q^{-1})^2 u_{i+1} v_{i+1} k_i q^{-2\mu(i+1)-\mu(i)} \tag{35}$$

with $\omega_0, \eta_0^{N-1}, \eta_{N-1}^0$ following from (34) and (35) by replacing on the RHS $i \to 0, (i+1) \to (N-1)$. The representation module is type 1 [22] $V = \oplus_\mu V_\mu$ with weight space $V_\mu = (\nu \in V | q^{H_i}\nu = q^{\mu(i)}\nu)$, a joint eigenspace of the commuting operators q^{H_i},$i = 0, ..., N-1$.

The homomorphism to the level zero quantum affine $U_q(\hat{sl}(N))$ algebra is straightforward to verify using Jimbo evaluation homomorphism [16] for $U_q(\hat{sl}(N))$ where it becomes transparent in terms of the basis given by the elements $\hat{E}_{ij} (i \neq j)$ such that

$$\hat{E}_{ij} = \hat{E}_{ik}\hat{E}_{kj} - q^{\pm}\hat{E}_{kj}\hat{E}_{ik} \quad i \lesseqgtr k \lesseqgtr j \tag{36}$$

These elements have the property of the simple roots. Then

$$\hat{E}_{i-1i} = E_i^+, \quad \hat{E}_{ii-1} = E_i^- \tag{37}$$

Let $(\hat{e}_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The evaluation representation π_λ is defined as

$$\pi(q^{\pm H_i}) = q^{\pm(\hat{e}_{ii} - \hat{e}_{i+1i+1})}, \quad \pi(E_i^+) = \hat{e}_{ii+1} \quad \pi(E_i^-) = \hat{e}_{i+1i} \tag{38}$$

for $\quad i = 1, ..., N-2 \quad$ and $\quad \pi(q^{\pm H_0}) = q^{\pm(\hat{e}_{N-1N-1} - \hat{e}_{00})}$,

$$\pi((E_0^+) = \lambda q^{\hat{e}_{00} + \hat{e}_{N-1N-1} - 1}\hat{e}_{N-10}, \qquad \pi((E_0^-) = \lambda^{-1} q^{-\hat{e}_{00} - \hat{e}_{N-1N-1} + 1}\hat{e}_{0N-1} \tag{39}$$

Using the evaluation representation it is readily verified that for the particular values of the structure constants, namely

$$\rho_0 = \rho_1 = ... = \rho_{N-1} = -1, \tag{40}$$

$$\omega_i = 0, \quad \eta_i = 0, \quad i = 0, ..., N-1 \tag{41}$$

which is achieved by rescaling the generators A_i and setting

$$k_i = 0, \tag{42}$$

then the algebra (29 - 31) is equivalent to the algebra $U_q'(so_N)$ introduced by Gavrilik and Klimyk [23] as the nonstandard deformation of the universal enveloping algebra of the Lie algebra of $so(N)$.

Formula (28) defines the homomorphism of the AW algebra with N generators to the quantum affine $U_q(\hat{sl}(N))$. The AW algebra with N generators obeying the relations (29) and (30, 31) is an associative algebra with a unit, depending on the structure constants determined by eqs.(33-35).

Definition 1: The AW algebra (29-31) with N generators $A_i, i = 0, ..., N-1$ defined by the homomorphism (28) is a deformation in the parameters $\rho_i, \omega_i, \eta_i^{i+1}, \eta_{i+1}^i, \eta_0^{N-1}, \eta_{N-1}^0, i = 0, ..., N-1$ of the q-Serre relations of level zero quantum affine $U_q(\hat{sl}(N))$.

Taking the commutator of both sides of the first and second line in (30), (31), with A_i and A_{i+1}, A_0 and A_{N-1}, respectively, we obtain the relations

$$\begin{aligned} [A_i[A_i[A_i, A_j]_q]_{q^{-1}} &= \rho_i[A_i, A_j], \\ [A_j[A_j[A_j, A_i]_q]_{q^{-1}} &= \rho_j[A_j, A_i] \end{aligned} \tag{43}$$

where $j = i+1$ or $i = 0, j = N-1$ and

$$[A_i, A_j] = 0, \qquad |i-j| \geq 2, \quad i, j = 0, ..., N-1 \tag{44}$$

Relations (43) define a tridiagonal algebra (TD) with N generators through a natural homomorphism to the Askey-Wilson algebra.

Taking the limit $q \to 1$ in (43) one obtains relations that generalize the well known Dolan-Grady relations [14] for the case $N = 2$. Eq.(43) can be interpreted as tridiagonal relations following from a deformation of the Serre relations of level zero affine $sl(N)$.

From the explicit realization of the operators A_i it follows that they generate a linear covariance algebra for the $U_q(\hat{sl}(N))$ which has the property of a coideal subalgebra. Let $B_q(\hat{sl}(N))$ denote the algebra generated by A_i.

Proposition II: The Askey-Wilson algebra defined by the homomorphism (28) is a coideal subalgebra of $U_q(\hat{sl}(N))$. The proof is straightforward by using the comultiplication (8). One has

$$\Delta(A_i) = I \otimes A_i + (A_i - k_i I) \otimes q^{-H_i} \quad i = 0, ..., N-1 \tag{45}$$

where the expressions on the RHS of (45) obviously belong to $B_q(\hat{sl}(N)) \otimes U_q(\hat{sl}(N))$.

The chain of homomorphisms $TD \to AW \to U_q(\hat{sl}(N))$ determines the tridiagonal algebra as a coideal of $U_q(\hat{sl}(N))$.

Definition 2: The tridiagonal algebra with N generators is the associative algebra with a unit, a coideal subalgebra of $U_q(\hat{sl}(N))$ and defining relations (43, 44) where the N structure constants are given by eq. (33).

The Askey-Wilson algebra with two generators A, A^* was first considered in the works of Zhedanov [24, 25] who showed that the Askey-Wilson polynomials give raise to two infinite-dimensional matrices satisfying the AW relations. In [26] the AW algebra has been equivalently described as an algebra with two generators and with structure constants determined in terms of the elementary symmetric polynomials in four parameters $a, b, c, d, abcd \neq q^m, m = 0, 1, 2...; q \neq 0, q^k \neq 1, k = 1, 2, ...$ The tridiagonal relations have recently been discussed in a more general framework [27, 28] where tridiagonal pairs A, A^* have been classified according to their dependence on the sequence of scalars [27, 28, 29] and a correspondence to the orthogonal polynomials in the Askey-Wilson scheme was given. The coideal tridiagonal subalgebra was considered in [11] for a particular case of the homomorphism to $U_q(\hat{sl}(2))$.

The Zhedanov algebra was introduced as a linear covariance algebra for $U_q(sl(2))$. In [8] the general homomorphism of the AW algebra with generators A, A^* to the quantum affine algebra $U_q(\hat{sl}(2))$ was studied. Special cases of this homomorphism is the representation considered by Terwilliger [27] with $\rho = \rho^* = 0$ and the one by Baseilhac and Koizumi [12] with $u = v^*$ and $v = u^*$. The Zhedanov realization [25] corresponds to the evaluation representation for the $U_q(\hat{sl}(2))$ generators

$$\pi_\nu(E_1^\pm) = E^\pm, \quad \pi_\nu(E_0^\pm) = \nu^{\pm 1} E^\mp, \quad \pi_\nu(q^{H_1}) = q^H, \quad \pi_\nu(q^{H_0}) = q^{-H} \tag{46}$$

where $E^\pm, H$ are the $U_q(sl(2))$ generators.

4 Boundary Askey-Wilson symmetry

We can now summarize the known results on the nonlocal conserved charges for the models in consideration.

A. Two-dimensional field theory models
The sine-Gordon model can be viewed as a perturbation of a free bosonic conformal field theory, with the action on the whole line (with the condition $\phi(-\infty, t) = 0$)

$$S = \frac{1}{4\pi} \int d^2 z \partial\phi \bar{\partial}\phi + \frac{\lambda}{2\pi} \int d^2 z \Phi^{pert}(x, t), \tag{47}$$

with the perturbing operator

$$\Phi^{pert}(x, t) = e^{i\hat{\beta}\phi(x,t)} + e^{-i\hat{\beta}\phi(x,t)}, \tag{48}$$

where $\hat{\beta}$ is the Toda coupling constant. The $U_q(\hat{s}l_2)$ symmetry of the sine-Gordon model with deformation parameter

$$q = \exp\left(\frac{2i\pi(1-\bar{\beta}^2)}{\bar{\beta}^2)}\right) \tag{49}$$

is generated by the charges

$$Q_\pm = \frac{1}{4\pi} \int_{-\infty}^{\infty} (J_\pm - H_\pm), \qquad \bar{Q}_\pm = \frac{1}{4\pi} \int_{-\infty}^{\infty} (\bar{J}_\pm - \bar{H}_\pm) \tag{50}$$

together with the topological charge

$$T = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \phi \tag{51}$$

The explicit expressions of the currents $J_\pm, \bar{J}_\pm, H_\pm \bar{H}_\pm$ are given by formulae (3.5), (3.6) and (3.7) in [5]. The charges are related to the conventional basis in $U_q(\hat{s}l_2)$ $Q_\pm = E^\pm q^H, \bar{Q}_\pm =$

$E^{\pm}q^{-H}, T = H$. To restrict the sine-Gordon model to the half line $x \geq 0$ one imposes the Neumann boundary condition $\partial_x \phi = 0$ at $x = 0$. The sine-Gordon model was found to be classically integrable with rather more general boundary conditions

$$\partial_x \phi = i\beta\lambda_b \left(\epsilon_- e^{i\hat{\beta}\phi(0,t)} - \epsilon_+ e^{-i\hat{\beta}\phi(0,t)} \right) \tag{52}$$

which can be considered as a perturbation to the Neumann boundary conditions with a perturbed operator

$$S_\epsilon = S_{Neumann} + \frac{\lambda}{2\pi} \int dt \Phi^{pert}_{boundary}(t), \tag{53}$$

with the boundary perturbing operator

$$\Phi^{pert}_{boundary}(t) = \epsilon_- e^{i\hat{\beta}\phi(0,t)} + \epsilon_+ e^{-i\hat{\beta}\phi(0,t)} \tag{54}$$

It has been shown in [5] that with these boundary conditions the nonlocal charges,

$$\begin{aligned} \hat{Q}_- &= Q_- + \bar{Q}_+ + \hat{\epsilon}_- q^{-T} \\ \hat{Q}_+ &= Q_+ + \bar{Q}_- + \hat{\epsilon}_+ q^{T} \end{aligned} \tag{55}$$

where

$$\hat{\epsilon}_\pm = \frac{\lambda_b \epsilon_\pm}{2\pi} \frac{(1 - \bar{\beta}^2)}{\bar{\beta}^2)} \tag{56}$$

are conserved and generate a coideal subalgebra of $U_q(\hat{sl}_2)$. It is now straightforward to show that the algebra of the charges $\hat{Q}_\pm$ is the Askey-Wilson algebra of the pair

$$A = Q_+ + \bar{Q}_- + \hat{\epsilon}_+ q^T, \qquad A^* = Q_- + \bar{Q}_+ + \hat{\epsilon}_- q^{-T} \tag{57}$$

with structure constants corresponding to the value $l^0_V = q^{1/2} + q^{-1/2}$

$$\rho = \rho^* = -(q - q^{-1})^2, \tag{58}$$

$$\omega = (q^{1/2} - q^{-1/2})^2 \left(\hat{\epsilon}_+ \hat{\epsilon}_- + (q^{1/2} + q^{-1/2})^2 \right) \tag{59}$$

$$\eta = \eta^* = (q - q^{-1})^2 (\hat{\epsilon}_+ + \hat{\epsilon}_-) \tag{60}$$

The tridiagonal boundary algebra of the sine-Gordon model on the half-line is defined by the structure constants (58).

A Hamiltonian, describing a sine-Gordon model on the half-line coupled to a non-linear oscillator at the boundary, was proposed in [30] and has been shown to be integrable at the classical level. The model was then studied at the quantum level [31] and nonlocal charges $\hat{\mathcal{E}}_\pm$, corresponding to the dynamical case, were constructed, which were natural extensions to the known nondynamical ones. These nonlocal charges have been shown to generate a coideal subalgebra of $U_q(\hat{sl}_2)$, defined by the algebraic relations

$$\begin{aligned} (q + q^{-1})\hat{\mathcal{E}}_+ \hat{\mathcal{E}}_- \hat{\mathcal{E}}_+ - \hat{\mathcal{E}}_+^2 \hat{\mathcal{E}}_- - \hat{\mathcal{E}}_- \hat{\mathcal{E}}_+^2 &= -c^2 (q^{1/2} + q^{-1/2})^2 \hat{\mathcal{E}}_- \\ (q + q^{-1})\hat{\mathcal{E}}_- \hat{\mathcal{E}}_+ \hat{\mathcal{E}}_- - \hat{\mathcal{E}}_-^2 \hat{\mathcal{E}}_+ - \hat{\mathcal{E}}_+ \hat{\mathcal{E}}_-^2 &= -c^2 (q^{1/2} + q^{-1/2})^2 \hat{\mathcal{E}}_+ \end{aligned} \tag{61}$$

where $c^2 = i2\mu(q-1)/\lambda^2$, $\lambda = 2/\hat{\beta}^2 - 1$, μ is the boundary perturbation parameter and $q \equiv \exp -2\pi i/\hat{\beta}^2$. These relations define the AW boundary algebra of the dynamical model with structure constants

$$\rho = \rho^* = -c^2(q^{1/2} + q^{-1/2})^2 \tag{62}$$

$$\omega = 0, \qquad \eta = \eta^* = 0 \tag{63}$$

The more general class of models is the affine Toda field theory associated to every affine Lie algebra of rank N and defined by the Euclidean action for an N-component boson field in two dimensions

$$S = \frac{1}{4\pi}\int d^2z\partial\phi\bar{\partial}\phi + \frac{\lambda}{2\pi}\int d^2z\sum_{j=0}^{N-1}\exp\left(-i\hat{\beta}\frac{1}{|\alpha_j^2|}\alpha_j\cdot\phi\right) \tag{64}$$

where the exponential interaction potential is expressed by the simple roots $\alpha_j, j = 0, ..., N-1$, λ is the mass parameter and $\hat{\beta}$ is the coupling constant. The quantum symmetry $U_q(\hat{sl}(N))$ is generated by the topological charges

$$T_j = \frac{\hat{\beta}}{2\pi}\int_{-\infty}^{\infty} dx\alpha_j\partial_x\phi \tag{65}$$

and the nonlocal conserved charges

$$Q_j = \frac{1}{4\pi}\int_{-\infty}^{\infty}(J_j - H_j), \qquad \bar{Q}_j = \frac{1}{4\pi}\int_{-\infty}^{\infty}(\bar{J}_j - \bar{H}_j) \tag{66}$$

where $j = 0, ..., N-1$, $Q_i \equiv E_i^+ q^{H_i/2}, \bar{Q}_i \equiv E_i^- q^{H_i/2}, T_i \equiv H_i$. The explicit expressions for $J_j, \bar{J}_j, H_j, \bar{H}_j$ are given by the formulae (4.4), (4.5) and (4.6) in [5]. The linear combinations $Q_j + \bar{Q}_j$ are parity invariant and conserved on the half-line with Neumann boundary conditions.

Adding to the action a boundary perturbation

$$S_\epsilon = S_{Neumann} + \frac{\lambda}{2\pi}\int dt\Phi^{pert}_{boundary}(t), \tag{67}$$

where

$$\Phi^{pert}_{boundary}(t) = \sum_{j=0}^{N-1}\epsilon_j\exp\left(-\frac{i\hat{\beta}}{2}\alpha_j\cdot\phi(0,t)\right) \tag{68}$$

we obtain a more general boundary condition

$$\partial_x\phi = -i\hat{\beta}\lambda_b\sum_{j=0}^{N-1}\epsilon_j\alpha_j\exp\left(-\frac{i\hat{\beta}}{2}\alpha_j\cdot\phi(0,t)\right), \qquad x = 0 \tag{69}$$

The new conserved charges are

$$\hat{Q}_i = Q_i + \bar{Q}_i + \hat{\epsilon}_i q^{T_i}, \qquad \hat{\epsilon}_i = \frac{\lambda_b\epsilon_i}{2\pi c}\frac{(1-\hat{\beta}^2)}{\hat{\beta}^2}, \qquad i = 0, ..., N-1 \tag{70}$$

The nonlocal charges of affine Toda field theory generate the AW algebra with N generators and with structure constants, depending on the weights, are given by (no summation over repeated indices)

$$\rho_i = \rho_{i+1} = -(q-q^{-1})^2, \tag{71}$$

$$\omega_{i+1} = (q^{1/2}-q^{-1/2})^2\hat{\epsilon}_i\hat{\epsilon}_{i+1}q^{-\mu(i)-\mu(i+1)} \tag{72}$$

$$\eta_i^{i+1} = (q-q^{-1})^2\hat{\epsilon}_{i+1}q^{-2\mu(i)-\mu(i+1)}, \qquad \eta_{i+1}^{i} = (q-q^{-1})^2\hat{\epsilon}_i q^{-2\mu(i+1)-\mu(i)} \tag{73}$$

The nonlocal charges of affine Toda field theory generate the tridiagonal algebra with structure constants ρ_i given by eq.(71).

B. The XXZ spin chain and the open asymmetric simple exclusion process (ASEP)

The ASEP is an interacting many-body system with wide range of applications [32, 33, 34]. It is described in terms of a master equation for the probability distribution $P(s_i,t)$ of a stochastic variable $s_i = 0,1$, at a site $i = 1,2,....L$ of a linear chain. In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, or $s_i = 1$ if the site i is occupied. On successive sites particles hop with probability $g_{01}dt$ to the left, and $g_{10}dt$ to the right. The event of hopping occurs if out of two adjacent sites one is a vacancy and the other is occupied by a particle. The symmetric simple exclusion process is the lattice gas model of particles hopping between nearest-neighbour sites with a constant rate $g_{ik} = g_{ki} = g$. The asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density and additional processes can take place at the boundaries. Namely at the left boundary $i = 1$ a particle can be added with probability αdt and removed with probability γdt, and at the right boundary $i = L$ it can be removed with probability βdt and added with probability δdt. Without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The totally asymmetric process corresponds to $q = 0$. The time evolution of the system is governed by the master equation $\frac{dP(s,t)}{dt} = \sum_{s'}\Gamma(s,s')P(s',t)$ which is mapped to a Schroedinger equation in imaginary time $\frac{dP(t)}{dt} = -HP(t)$ for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms. The ground state of this Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary. A relation to the integrable spin 1/2 XXZ quantum spin chain is obtained through the similarity transformation $\Gamma = -qU_\mu^{-1}H_{XXZ}U_\mu$ (see [35] for details); H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain [36] with anisotropy Δ and with added non diagonal boundary terms B_1 and B_L

$$H_{XXZ} = -1/2\sum_{i=1}^{L-1}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y - \Delta\sigma_i^z\sigma_{i+1}^z + h(\sigma_{i+1}^z - \sigma_i^z) + \Delta) + B_1 + B_L \tag{74}$$

The transition rates of the ASEP are related to the parameters Δ and h, and the boundary terms in the following way (μ is free parameter, irrelevant for the spectrum)

$$\begin{aligned}\Delta &= -1/2(q+q^{-1}), \quad h = 1/2(q-q^{-1}) \\ B_1 &= \frac{1}{2q}\left(\alpha+\gamma+(\alpha-\gamma)\sigma_1^z - 2\alpha\mu\sigma_1^- - 2\gamma\mu^{-1}\sigma_1^+\right) \\ B_L &= \frac{\left(\beta+\delta-(\beta-\delta)\sigma_L^z - 2\delta\mu q^{L-1}\sigma_L^- - 2\beta\mu^{-1}q^{-L+1}\sigma_L^+\right)}{2q}\end{aligned} \tag{75}$$

Bethe Ansatz solution (BA) was possible [37, 38] through the mapping to the XXZ chain Hamiltonian with most general nondiagonal boundary terms whose diagonalization on its turn was recently achieved [39, 40] (by means of algebraic Bethe ansatz) provided the parameters satisfy a condition which in terms of the ASEP notations reads

$$(q^{L+2k}-1)(\alpha\beta - q^{L-2k-2}\gamma\delta) = 0 \tag{76}$$

In (76) k is an integer such that $|k| \leq L/2$. Given k, the first factor zero in (76) assumes q to be a root of unity and the second factor zero imposes a relation between the bulk and the boundary parameters. For generic q, however, one can satisfy the constraint [37] by choosing k to be $k = -L/2$.

Matrix Product State Ansatz (MPA): The idea is that the steady state properties of the ASEP can be obtained exactly in terms of matrices obeying a quadratic algebra [41, 42]. For a given configuration $(s_1, s_2, ..., s_L)$ the stationary probability is defined by the expectation value $P(s) = \frac{\langle w|D_{s_1}D_{s_2}...D_{s_L}|v\rangle}{Z_L}$, where $D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0 + D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. The operators $D_i, i = 0, 1$ satisfy the quadratic (bulk) algebra

$$D_1D_0 - qD_0D_1 = x_1D_0 - D_1x_0, \qquad x_0 + x_1 = 0 \tag{77}$$

with boundary conditions of the form

$$\begin{aligned}(\beta D_1 - \delta D_0)|v\rangle &= x_0|v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= -\langle w|x_1.\end{aligned} \tag{78}$$

The exact solution in the stationary state was related to Askey-Wilson polynomials [43, 44]. Emphasizing the equivalence of the open ASEP to the $U_q(\hat{su}(2))$ XXZ invariant quantum spin chain with added general boundary terms we have shown [45] that the boundary operators generate the AW algebra with the structure constants ($x_1 = -x_0$)

$$\rho = x_0^2\beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \qquad \rho^* = x_0^2\alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2 \tag{79}$$

$$-\omega = x_0^2(\beta-\delta)(\gamma-\alpha) - x_0^2(\beta\gamma+\alpha\delta)(q^{1/2}-q^{-1/2})Q \tag{80}$$

$$\begin{aligned}\eta &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\beta\delta(\gamma-\alpha)Q + \frac{(\beta-\delta)(\beta\gamma+\alpha\delta)}{q^{1/2}-q^{-1/2}}\right) \\ \eta^* &= q^{1/2}(q^{1/2}+q^{-1/2})x_0^3\left(\alpha\gamma(\beta-\delta)Q + \frac{(\alpha-\gamma)(\alpha\delta+\beta\gamma)}{q^{1/2}-q^{-1/2}}\right)\end{aligned} \tag{81}$$

where Q is the central element of the finite-dimensional $U_q(su(2))$ representation. The corresponding AW algebra for the XXZ spin chain, proposed in [10], is a particular case of the ASEP boundary AW algebra. It is generated by the operators

$$A = \frac{1}{c_0} Q_+ + \bar{Q}_-, \qquad A^* = Q_- + \frac{1}{c_0} \bar{Q}_+ \tag{82}$$

with c_0 an arbitrary parameter and structure constants are given by

$$\rho = \rho^* = \frac{(q^{1/2} + q^{-1/2})^2}{c_0} \tag{83}$$

$$\omega = -\frac{\omega^{(j)}}{c_0(q - q^{-1})}, \qquad \eta = \eta^* = 0 \tag{84}$$

where $\omega^{(j)} = (q^{j+1/2} + q^{-j-1/2})$ denotes the eigenvalue of the $U_q(sl_2)$ Casimir in the spin j representation. The tridiagonal algebra as a coideal subalgebra of the $U_q(\hat{s}l(2))$, explored for the exact spectrum of the XXZ chain with general boundary terms in [13] has also equal structure constants $\rho = \rho^* = k_+ k_-$, where k_+, k_- belong to the boundary parameters at the left end of the chain.

We emphasize the different form of the structure constants in the boundary AW algebras for the ASEP and XXZ spin chain. Despite the equivalence of the ASEP to the XXZ spin chain through a similarity transformation they describe different physics. A relation among the structure constants of the type $\rho = \rho^*$ is unacceptable for a model of nonequilibrium physics as the ASEP is because it will restrict the physics of the system.

5 Representation of the Askey-Wilson algebra

We summarize the most important formulae and notations about the representations of the AW algebra with two generators A, A^* which we will need to present the related spectral problem. Let $p_n = p_n(x; a, b, c, d)$ denote the n-th Askey-Wilson polynomial [46] depending on four parameters a, b, c, d

$$p_n = {}_4\Phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix} \; |q; q \right) \tag{85}$$

with $p_0 = 1$, $x = y + y^{-1}$ and $0 < q < 1$. There is a basic representation of the AW algebra [28, 26] in the space of symmetric Laurent polynomials $f[y] = f[y^{-1}]$ with a basis $(p_0, p_1, ...)$ as follows

$$Af[y] = (y + y^{-1}) f[y], \qquad A^* f[y] = \mathcal{D} f[y] \tag{86}$$

where $\mathcal{D}$ is the second order q-difference operator [46] having the Askey-Wilson polynomials p_n as eigenfunctions. It is a linear transformation given by

$$\begin{aligned} \mathcal{D}f[y] = (1 + abcdq^{-1}) f[y] \;&+\; \frac{(1-ay)(1-by)(1-cy)(1-dy)}{(1-y^2)(1-qy^2)} (f[qy] - f[y]) \\ &+\; \frac{(a-y)(b-y)(c-y)(d-y)}{(1-y^2)(q-y^2)} (f[q^{-1}y] - f[y]) \end{aligned} \tag{87}$$

with $\mathcal{D}(1) = 1 + abcdq^{-1}$. The eigenvalue equation for the joint eigenfunctions p_n reads

$$\mathcal{D}p_n = \lambda_n^* p_n, \qquad \lambda_n^* = q^{-n} + abcdq^{n-1} \tag{88}$$

and the operator A^* is represented by an infinite-dimensional matrix diag$(\lambda_0^*, \lambda_1^*, \lambda_2^*, ...)$. The operator $Ap_n = xp_n$ is represented by a tridiagonal matrix. Let $\mathcal{A}$ denote the matrix whose matrix elements enter the three-term recurrence relation for the Askey-Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \tag{89}$$

where $p_{-1}(x) = 0, p_0(x) = 1$ and

$$\mathcal{A} = \begin{pmatrix} a_0 & c_1 & & \\ b_0 & a_1 & c_2 & \\ & b_1 & a_2 & \cdot \\ & & \cdot & \cdot \end{pmatrix} \tag{90}$$

The explicit form of the matrix elements of A reads

$$a_n = a + a^{-1} - b_n - c_n \tag{91}$$

$$b_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})} \tag{92}$$

$$c_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})} \tag{93}$$

The basis is orthogonal with the orthogonality condition for the Askey-Wilson polynomials $P_n = a^{-n}(ab, ac, ad; q)_n p_n$

$$\int_{-1}^{1} \frac{w(x)}{2\pi\sqrt{1-x^2}} P_m(x; a, b, c, d|q) P_n(x; a, b, c, d|q) dx = h_n \delta_{mn} \tag{94}$$

where $w(x) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}$ and $h(x, \mu) = \prod_{k=0}^{\infty}[1 - 2\mu x q^k + \mu^2 q^{2k}]$, and

$$h_n = \frac{(abcdq^{n-1}; q)_n (abcdq^{2n}; q)_\infty}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty} \tag{95}$$

6 Related spectral problem

The importance of the AW algebra related spectral problem is motivated by the identification of the generator A^* with the second order difference operator for the AW polynomials in the basic representation. It has been known [47] for quite a long time that the standard quantum algebra $U_q(sl(2))$ can be obtained as a contraction of a degenerate Sklyanin algebra [48] and the diagonalization problem for a general quadratic form in the generators (commonly interpreted as the Hamiltonian of a proper physical system) is equivalent to the difference equation for the AW polynomials [15].

We consider the eigenvalue equation (88) for the operator $\mathcal{D}$ in (87) for a polynomial of a given finite degree n

$$\varphi(y)(p_n(qy)-p_n(y))+\varphi(y^{-1})(p_n(q^{-1}y)-p_n(y))=(q^{-n}-1)(1-q^{n-1}abcd)p_n(y) \tag{96}$$

$$\varphi(y)=\frac{\prod_{\nu=1}^{4}(1-w_\nu y)}{(1-y^2)(1-qy^2)}, \qquad w_1=a, w_2=b, w_3=c, w_4=d \tag{97}$$

and use the procedure of algebraic Bethe Ansatz [15]. Expanding the function p_n as a product of its zeros

$$p_n(y)=\prod_{m=1}^{n}(y-y_m)(y-y_m^{-1}) \tag{98}$$

we plug it in eq.(96) dividing thereafter both sides by $p_n(y)$. Since the LHS is a meromorphic function, the RHS is a constant. One needs to cancel the singularities which at LHS are located at $y=0,\infty,y_m$. The singular part at $y=0$ vanishes identically. (It is assumed that $y_m\neq 0$.) When one implements $U_q(su(2))$ algebraic Bethe-Ansatz to quasi-exactly solvable operators, i.e. operators in a linear or bilinear form in terms of the $U_q(su(2))$ generators, the vanishing of the residue at infinity determines the degree of the polynomial $n\equiv L=2j$. The second order AW difference operator is exactly solvable and the LHS is regular at ∞ from the beginning so that no restriction for the degree of the polynomial follows. Annihilation of poles at $y=y_m$ gives the Bethe-Ansatz equation [15] for the zeros of the Askey-Wilson polynomials

$$\prod_{\nu=1}^{4}\frac{y_k-w_\nu}{w_\nu y_k-1}=\prod_{l=1,l\neq k}^{L}\frac{(qy_k-y_l)(qy_ky_l-1)}{(y_k-qy_l)(y_ky_l-q)} \tag{99}$$

These equations are valid for any $L<2j+1$, so that for any L there is exactly one polynomial (98). This means that for each $L(<2j+1)$ the Bethe equations have exactly one solution for the set $y_k, k=1,...,L$.

We will use the unique solution for the AW zeros Bethe-Ansatz equation to construct a finite-dimensional representation of the AW algebra in the space of Laurent polynomials of a given degree.

The operator $Ap_n=xp_n$ is represented by a tridiagonal matrix $\mathcal{A}$ whose matrix elements are obtained from the three-term recurrence relation (89) for the Askey-Wilson polynomials. A natural way to obtain the finite-dimensional representations of the AW algebra is to set $b_n=0$ which is achieved by the vanishing of one of the factors e.g. $(1-abq^n)$, $(1-adq^n)$ or $(1-abcdq^{n-1})$ in the numerator of (92). We need find a way to terminate this sequence at p_L without using conditions, such that the parameters a,b,c,d become q^L because for some models this might impose a restriction on the physics of the system.

Proposition III: For any finite $n=L$ the tridiagonal algebra, obtained through the natural homomorphism $TD\to AW$ to the Askey-Wilson algebra has an irreducible finite-dimensional representation in the space of Laurent polynomials of degree $L-1$, related to the polynomial representation, associated with the highest weight vector of the irreducible finite-dimensional representation of the quantum affine $U_q(\hat{s}l(2))$.

In the following we present our argumentation for the proof of this proposition.

The theory of the orthogonal polynomials, and in particular the Askey-Wilson polynomials, is based on the three-term recurrence relation, eq.(89) with the initial conditions $p_0(x) = 1$, $p_{-1}(x) = 0$. There is a characterization theorem (see [49, 50]) concerning the orthogonality of the polynomials with respect to some measure. Namely, when the matrix elements are real, then the measure can be chosen real-valued and nondecreasing and the integration in the orthogonality condition is the real line (as it is the case with Askey-Wilson polynomials with real parameters, or if complex, in conjugate pairs). Then all zeros $x_s, s = 0, 1, ...L-1$ of any polynomial are real and simple [46]. Hence, these zeros can be used to construct a discrete orthogonality relation for polynomials of degree lower than L. In the finite discrete case the three-term recurrence relation is a discrete analogue of the Sturm-Liouville two-point boundary value problem with boundary conditions $p_{-1}(x) = 0, p_L(x) = 0$. If all the zeros $x_0, ...x_{L-1}$ are real and distinct (see a theorem by Atkinson [51] for a complete proof) then the orthogonality condition can be written in the from

$$\sum_{s=0}^{L-1} P_m(x_s)P_n(x_s)w_s = h_n\delta_{mn}, \qquad m, n = 0, 1, ..., L-1 \tag{100}$$

and the weight function is

$$w_s = \frac{h_{L-1}}{p_{L-1}(x_s)p'_L(x_s)} \tag{101}$$

where the prime indicates the first derivative. The theorem was proved in [51] for real a_n and positive b_n, c_n.

As mentioned above, to terminate the three-term recurrence relation at any finite $(n+1) \equiv L$ for a discrete set of AW polynomials $(p[y] = p[y^{-1}])$ due to

$$p_L[y] = 0 \tag{102}$$

we have to find a way to set $b_n = 0$ in the matrix representing the operator A without imposing a restrictive conditions on the model parameters. We note that by directly setting (in the numerator of b_n in (92)) e.g. $1 - abcdq^{n-1} = 0$, we obtain the second factor in the BA condition (76) for the XXZ spin chain (with the proper identification of the AW parameters a, b, c, d with the parameters of the boundary terms in (75)). For the ASEP we recall the presence of a parameter $\zeta \equiv x_0$ in the stationary state which is associated with an abelian symmetry of the bulk operators D_0, D_1. Making use of this parameter we can obtain a discrete set of AW polynomials in the following steps:

1. We first expand p_L as a product of its zeros obtaining the Bethe Ansatz equation (99) for the L zeros $y_0, y_1, ...y_{L-1}$.
2. Then we rescale

$$a \to \zeta a, \quad b \to \zeta b, \quad c \to \zeta c, \quad d \to \zeta d \tag{103}$$

With $|\zeta| \leq 1$ we have a representation in terms of a', b', c', d', which has no effect on the Bethe equations and will not change the identification of the parameters a, b, c, d with the boundary probability rates. In the numerator of the matrix elements b_n, e.g. in the factor $((1 - abq^n)$ we can redefine $\zeta^2 ab$ as a new parameter t treating it independent on a, b. The condition to terminate the AW algebra ladder representation due to $b_n = 0$ becomes

$$tq^n = 1 \tag{104}$$

We thus obtain a discrete set of AW polynomials

$$p_n(x_k, a, b, c, d|t, q), \qquad n = 0, 1, ..., L-1 \tag{105}$$

such that

$$\sum_{k=0}^{L-1} w_k p_n(x_k) p_m(x_k) = 0 \tag{106}$$

for distinct n, m. Then we will have

$$x p_{L-1}(x) = a_{L-1} p_{L-1}(x) + c_{L-1} p_{L-2}(x), \qquad if \quad x = x_k \tag{107}$$

For general x the relation

$$x p_{L-1}(x) - (a_{L-1} p_{L-1}(x) + c_{L-1} p_{L-2}(x)) \tag{108}$$

will define a polynomial

$$p_L(x) = const \prod_{k=0}^{L-1} (x - x_k) \tag{109}$$

Thus the condition (104) with $n = L-1$ determines an irreducible finite-dimensional representation W of the tridiagonal algebra for $x = x_k$. The matrices, representing A, A^*, in the tridiagonal, diagonal representation are finite $L^2 \times L^2$ square matrices. They are block-tridiagonal and block-diagonal respectively, where each block is an $L \times L$ square matrix. It is important to emphasize that the representation W of the tridiagonal algebra on the space with basis, the discrete set of AW polynomials, is such, that the spectrum of the diagonal operator A^* is degenerate. Each eigenvalue λ^* has an eigenspace $p_n(x_k)$, with $k = 0, ..., L-1$ of dimension L.

For each fixed x_k, however, there is a finite dimensional subrepresentation V, with basis $p_n(x_k)$, $n = 0, ..., L-1$, which is not an invariant subspace of W. The vectors $|\nu_n\rangle = |p_n\rangle$ form an orthogonal basis for this representation $\langle \nu_m | \nu_n \rangle = \delta_{mn}$. The tridiagonal matrix representing A is irreducible tridiagonal, while the diagonal is such that each eigenvalue λ_n has dimension one. We have

$$\begin{aligned} [A, [A, [A, A^*]_q]_{q^{-1}}] &= \rho [A, A^*], \\ [A^*, [A^*, [A^*, A]_q]_{q^{-1}}] &= \rho^* [A^*, A] \end{aligned} \tag{110}$$

with ρ, ρ^* given by (79).

We want to relate this representation to a highest weight irreducible finite-dimensional representation of the $U_q\hat{s}u(2)$ with deformation parameter q. (Note the change of the deformation parameter from $q^{1/2}$ to q. We have defined the Askey-Wilson algebra as a coideal subalgebra of $U_{q^{1/2}}\hat{s}u(2)$ [8], however we relate the corresponding tridiagonal algebra finite-dimensional representation with a discrete set of AW polynomials (96) to $U_q\hat{s}u(2)$.)

To proceed further we first need some definitions from [19]. Let $V_n(a) \equiv V_n(a;\nu_0,...,\nu_n)$ ($a \subset \mathbf{C^x}$) denote the finite-dimensional representation (dimension $(n+1)$) of $U_q\hat{s}u(2)$ with basis $\nu_i, i = 0,...n$, and highest weight vector ν_0 (and $V_n(qa;\nu_0,...,\nu_n)$ denotes its dual). $V_n(a)$ may be regarded as representations of $U_q(L(su(2))$, where $U_q(L(su(2))$ denotes the quotient of $U_q\hat{s}u(2)$ by the two-sided ideal generated by the central element C. $U_q(L(su(2))$ is a Hopf algebra which is a deformation of the universal enveloping algebra of the loop algebra $L(su(2)) = su(2)[s,s^{-1}]$ (with $a = s$ for $V_n(a)$). There is a unique polynomial with constant coefficient 1 associated to any finite dimensional highest weight representation $V_n(a,\nu_i)$ of $U_q\hat{s}u(2)$, given by

$$P(u) = (1 - q^{n-1}au)(1 - q^{n-2}au)...(1 - q^{-n+1}au) \tag{111}$$

The tensor product of an r-tuple of $V_1(a_r)$ irreducible highest weight finite-dimensional representations

$$V = V_1(a_1) \otimes V_1(a_2) \otimes ...V_1(a_r) \tag{112}$$

has a highest weight vector which is the tensor product of the highest vectors in each factor. The associated polynomial is

$$\prod_{i=1}^{r}(1 - a_i u) = \prod_{i=1}^{r}(1 - \xi_i^{-1}u) = P(u) \tag{113}$$

where $\xi_1, \xi_2, ..., \xi_r$ are the roots of the polynomial $P(u)$. A non-empty finite set of (complex) numbers is said to be a q-string if it is of the form $\xi, q^{-2}\xi, q^{-4}\xi, ...q^{-2r}\xi, r \in Z_{>0}$. The roots of the polynomial associated to an evaluation representation $V_n(a)$ are of the form $\xi, q^{-1}\xi, ..., q^{-(n-1)}\xi$ with $\xi = q^{n-1}a$ and form a q-string $S_n(a)$ with $r = n - 1$. Two q-strings $S_n(a)$ and $S_n(b)$ are said not to be in general position iff

$$\frac{b}{a} = q^{\pm(m+n-2p-2)} \tag{114}$$

for some $0 < p \leq min\{m,n\}$. A theorem by Chari and Pressley [19] states that a tensor product $V_{n_1} \otimes ... \otimes V_{n_r}$ is irreducible iff the q-strings $S_{n_1}(a_1)...S_{n_r}(a_{n_r})$ are in general position.

Given the finite-dimensional representation of the TD algebra we relate it with the irreducible highest weight representation of $U_q\hat{s}u(2)$. For each fixed zero $y_i, i = 1,2,3...,L$, from the zeros of the solution to the Bethe-Ansatz equation we identify the set $p_0(x_i), p_1(x_i)$ (where $x_i = y_i + y_i^{-1}$) with the irreducible highest weight evaluation module $V_1(\nu_0(x_i), \nu_1(x_i))$ defined by [19]

$$q^H\nu_0 = q\nu_0, \quad q^H\nu_1 = q^{-1}\nu_1 \qquad E^+\nu_0 = 0, \quad E^+\nu_1 = \nu_0 \qquad E^-\nu_0 = p\nu_1, \quad E^-\nu_1 = 0 \tag{115}$$

We have $V_1(a_i) = V_1(x_i)$ and $V_1(qa_i) = V_1(qx_i)$. The polynomial associated with the representation $V_1(p_0(x_i), p_1(x_i))$ is

$$P(u) = (1 - x_i u) \tag{116}$$

We now consider the tensor product of any two evaluation representation $V_1(x_i) \otimes V_1(x_j)$. By construction this representation is of dimension 2^2. If we assume that the two q-strings $S_1(x_i)$ and $S_1(x_j)$ are not in general position, then we will have

$$\frac{x_i}{x_j} = q^2 \tag{117}$$

Hence it follows

$$y_i + y_i^{-1} = q^2(y_j + y_j^{-1}) \tag{118}$$

which contradicts the Bethe-Ansatz equation for the zeros y_j of the AW polynomials. Therefore the tensor product of any two irreducible highest weight evaluation modules $V_1(x_i)$ is irreducible. We further consider the tensor product of L irreducible modules V_1

$$V_1(x_1) \otimes V_1(x_2) \otimes ... \otimes V_1(x_L) \tag{119}$$

A theorem by Chari and Pressley [19] states that the tensor product (107) is irreducible iff the q-strings $S_1(x_1), ..., S_1(x_L)$ are in general position. Since any two q-strings $S_1(x_i)$ and $S_1(x_j)$ for $i \neq j$ are in general position, the proof can be generalized to the set of the q-strings $S_1(x_1), ..., S_1(x_L)$, which means that the tensor product (119) of L irreducible highest weight evaluation modules, each of dimension 2, is a finite dimensional irreducible highest weight $U_q\hat{s}u(2)$ module of dimension 2^L. The tensor product contains up to a scalar factor, a unique highest weight vector Ω of weight L,

$$K \equiv q^H \Omega = q^L \Omega \tag{120}$$

The subrepresentation generated by Ω is the $L+1$-dimensional irreducible representation of the $U_q(su(2))$ subalgebra of $U_q(L(su(2)))$. The polynomial $P(u)$, associated with the subrepresentation generated by Ω, , with $x_k \to x_k^{-1}$ and $u \equiv x$, coincides with the polynomial (98), for the choice $const = (-1)^L \prod_{k=0}^{L-1} x_k^{-1}$. (To simplify notations in what follows we keep x_k to denote $x_k = (y_k + y_k^{-1})^{-1}$.)

We thus obtain a finite-dimensional irreducible polynomial representation of $U_q(\hat{s}u(2))$ on the tensor product of dimension 2^L. We will prove further that the module (119) forms an irreducible finite-dimensional module of the tridiagonal algebra.

We recall, that for suitable choice of the structure constants in the AW algebra with two generators A, A^*, the Askey-Wilson polynomial $p_n(x)$ is kernel of the intertwining operator between a representation by a difference operators on the space of polynomials in x and a representation by tridiagonal operators on the space of infinite sequences $(c_n)_{n=1,2,...}$ [24, 26]. In the first representation A is multiplication by x and A^* is the second order q-difference operator for which the Askey-Wilson polynomials are eigenfunctions with explicit eigenvalues λ_n^*. In the second representation A^* is the diagonal operator with diagonal elements λ_n^* and A is the tridiagonal operator corresponding to the three-term recurrence relation for the Askey-Wilson polynomials.

Let us consider the action of A, A^* on the module $V_1(x_k)$. In the c-number representation where the operator A^* is diagonal (and A is tridiagonal) we have

$$\begin{aligned} A^* p_0(x_k) &= (1+q^{-1}abcd)p_0(x_k) \\ A^* p_1(x_k) &= (q^{-1}+abcd)p_1(x_k) \end{aligned} \tag{121}$$

In the (dual) representation where A^* is the AW second order difference operator yielding the Bethe-Ansatz equation for the zeros, the operator A is multiplication by x. We have

$$Ap_n(x_k) = x_k p_n(x_k) \tag{122}$$

which means that each of the highest weight irreducible modules $V_1(x_k)$ is an eigenstate of the operator A (but not of A^*)

$$AV_1(x_k) = x_k V_1(x_k) \tag{123}$$

On the tensor product of two irreducible modules $V_1(x_i) \otimes V_1(x_k)$ the operator A will act by means of the coproduct

$$\Delta(A) = A_{i_1} \otimes I + I \otimes A_{k_2} + A_{i_1} \otimes A_{k_2} \tag{124}$$

Iterating the coproduct we obtain the action of the operator A on the tensor product. (We denote the n-fold iteration by $\Delta^{(1)} = \Delta, \Delta^{(n)} = (\Delta \otimes I^{(n-2)})\Delta^{(n-1)}$ with $I^{(n-2)} = I \otimes ... \otimes I(n-2$ times)). To make the formulae more transparent we denote the first two terms in (124) by $\Delta_P(A)$. We have

$$\Delta_P^{(n)}(A) = \sum_{k=1}^{n} I^{(k-1)} \otimes A_{i_k} \otimes I^{(n-k)} \tag{125}$$

The complete set of eigenvalues of A on the tensor product will be given the action of the

$$\Delta^{(n)}A = \Delta_P^{(n)}A + A \otimes \Delta_P^{(n-1)}A + \Delta_P^{(n-1)}A \otimes A + ... + A^{(k)} \otimes \Delta_P^{(n-k)}A^{(k)} + \Delta_P^{(n-k)}A^{(k)} \otimes A^{(k)} + A^{(n)} \tag{126}$$

where $k = 1, ..., n-1$ and $A^{(l)} \equiv A \otimes A \otimes ... \otimes A$ (l times), for $l = k$ or $l = n$.

Proposition IV: The operators

$$A^{*(L)} = A^* \otimes I^{(L-1)} \tag{127}$$

and the operator $A^{(L)} = \Delta^{(L)}A$ as given in (126) for $n = L$ satisfy

$$[A^{*(L)}, [A^{*(L)}, [A^{*(L)}, A^{(L)}]_q]_{q^{-1}}] = \rho^*[A^{*(L)}, A^{(L)}] \tag{128}$$

The proposition can be verified by direct computation. We observe the peculiarity of the Dolan-Grady algebra for the Ising model, namely in the deformed case only one of the relations is satisfied in a given representation. The other one is constructed by using duality properties.

Proposition V: The operators

$$A^{(L)} = I^{(L-1)} \otimes A \tag{129}$$

and the operator

$$A^{(L)} = \Delta_P^{(L)} A + A \otimes \Delta_P^{(n-1)} A + \Delta_P^{(l-1)} A \otimes A, ..., A^{(k)} \otimes \Delta_P^{(L-k)} A^{(k)} + \Delta_P^{(L-k)} A^{(k)} \otimes A^{(k)} + A^{(L)} \tag{130}$$

where $k = 1, ..., L-1$ and $A^{(l)} \equiv A \otimes A \otimes ... \otimes A$ (l times), for $l = k$ or $l = L$ satisfy

$$[A^{(L)}, [A^{(L)}, [A^{(L)}, A^{*(L)}]_q]_{q^{-1}}] = \rho [A^{(L)}, A^{*(L)}] \tag{131}$$

To obtain a compete set of 2^n eigenvectors with 2^n eigenvalues for any finite n, $0 \leq n \leq L$, we associate with each lattice site i a basis vector $p_0(x_i)$ if a site is empty (occupation number $s_i = 0$) or $p_1(x_i)$ if there is a particle on the site (occupation number $s_i = 1$). Let $\psi(x_1, x_2, ..., x_L)$ denote the state of the ASEP on the lattice of L sites, depending on the set $s_{i_1}, s_{i_2}, ..., s_{i_L}$ and belonging to the $U_q(\hat{s}u(2))$ irreducible tensor product representation

$$\psi(x_1, x_2, ..., x_L) = V_1(x_1) \otimes ... \otimes V_1(x_L) \tag{132}$$

with a highest weight vector generating the $2j = L$ subrepresentation.

By definition the highest weight vector of the tensor product obeys $E^+\Omega = 0$, with $\Omega = p_0(x_1)p_0(x_2)...p_0(x_L)$. The discrete set of AW polynomials satisfy the three-term recurrence relation (89) with $p_0(x) = 1$ for $x = x_k$. Hence the highest weight vector Ω is a constant vector $\Omega = 1$ and is an eigenvector of the operator A^* with the eigenvalue determined by the condition $\mathcal{D}(1) = 1 + abcdq^{-1}$

$$A^*\Omega = (1 + abcdq^{-1})\Omega \tag{133}$$

This property will be related in a proper way to the ground state of the system. Namely, a corresponding shift of A^* will produce a unique ground state with eigenvalue zero.

Let now ψ_0 denote the lowest weight vector of the $U_q(\hat{s}u(2))$ tensor product representation, $\psi_0 = p_1(x_1)p_1(x_2)...p_1(x_L)$. A state $\psi(x_1, x_2, ..., x_L)$ corresponding to any configuration on the lattice will be generated from ψ_0 by the action of the operators

$$E^+_{n_1} E^+_{n_2} ... E^+_{n_r} \tag{134}$$

and we have

$$\psi(x_1, x_2, ..., x_L) \equiv \langle E_r| = \sum_{1 \leq n_1 < ... < n_r \leq L} a(n_1, n_2, ...n_r) \langle \psi_0 | E^+_{n_1} E^+_{n_2} ... E^+_{n_r} \tag{135}$$

where coefficients $a(n_1, n_2, ...n_r)$ depend on x_i. In order to determine the coefficients $a(n_1, n_2, ...n_r)$ in the conventional Bethe Ansatz one defines a wavenumber counting function and takes into account all $r!$ permutations of the numbers $(1, 2, ..., r)$.

By construction the state $\psi(x_i)$ becomes an eigenvector of the operator A to be interpreted as the Hamiltonian in the auxiliary space of the physical system. It acts on it by means of the coproduct which takes into account all the admissible permutations of the partition $n_1, n_2, ...n_r$ in (135). Namely, the action of the iterated coproduct according to (125) gives the eigenvalues $\sum_{k=1}^{L} x_k$, in the one occupation number *zero* $s_i = 0$ (one spin down) sector, the second operator term in (126) gives the values $\sum_{i<j} x_i x_j$ in the two occupation numbers *zero* (two spin down) sector and so on, which yields all the eigenvalues whose number is

$$\sum_{n=1}^{L} \frac{L!}{n!(L-n)!} = 2^L - 1 \tag{136}$$

We thus obtain the $2^L - 1$ distinct eigenvalues of the operator A according to the action

$$A\psi(x_1, x_2, ..., x_L) = (\Delta_P^{(n)} A + A \otimes \Delta_P^{(n-1)} A + \Delta_P^{(n-1)} A \otimes A + ... + A^{(n)})\psi(x_1, x_2, ..., x_L) \tag{137}$$

from which the eigenvalue equation with the corresponding eigenvalues for the state $\psi(x_1, x_2, ..., x_L)$ follows

$$A\psi(x_1, x_2, ..., x_L) = (\sum_{i=1}^{L} x_i + \sum_{i<j} x_i x_j + ... + x_1 x_2 ... x_L)\psi(x_1, x_2, ..., x_L) \tag{138}$$

With the interpretation of the operator A as the Hamiltonian eq.(138) yields the energy eigenvalues.

As we already pointed out in this algebraic scheme there is no need of counting function since the right number of states and distinct eigenvalues is encoded in the polynomial representation of the AW algebra. The scheme works for any L, even for $L = 0$. In the latter case there are no zeros since p_0 is the constant term and hence $p_{-1} = 0$ which is the initial condition for the AW polynomials.

The considered algebraic Bethe Ansatz based on the unique solution of the Bethe equations (99) yields for any $n = L$ an exactly solvable two-boundary value spectral problem with the identification of L with the spin value $2j$ of the finite dimensional highest weight evaluation representation of $U_q(\hat{s}u(2))$. There are two limit cases. The first one is

$$n \to \infty, \qquad L \quad finite \tag{139}$$

This limit is obtained by treating the product ab dependent on a, b and $1 - abq^n \neq 1$ is recovered, so that $b_n \neq 0$ when the infinite-dimensional representation of the AW algebra is restored corresponding to a finite-dimensional representation of $U_{q^{1/2}}(\hat{s}u(2))$. The thermodynamic limit for finite lattice systems with added boundary terms is conventionally obtained by letting

$$L \to \infty \tag{140}$$

In our scheme we start from the very beginning with models in the infinite volume/infinite chain with quantum affine $U_{q^{1/2}}(\hat{s}l(2))(U_{q^{1/2}}(\hat{s}u(2)))$ symmetry which is manifest. Boundary conditions break the infinite volume symmetry. However with suitably imposed boundary conditions a remnant of this symmetry survives and is encoded in the nonlocal conserved charges, elements of the coideal AW subalgebra of $U_{q^{1/2}}(\hat{s}l(2))$, defined through the homomorphism to the quantized affine $U_{q^{1/2}}(\hat{s}u(N))$ at the critical level $c = -h$.

We can now use this prescription to obtain the complete set of eigenvectors with distinct eigenvalues for the transition matrix of the open ASEP. We recall that the left boundary operator and right boundary operators, being shifted AW (as well as TD) algebra generators $D^L = A^* + \alpha - \gamma$ and $D^R = A + \beta - \delta$, have a diagonal and a tridiagonal infinite-dimensional representation, respectively, with basis the AW polynomials p_n, $n = 0, 1, \dots$. The parameters a, b, c, d of the AW polynomials are uniquely related to the four boundary rates, namely $a = k_+(\alpha, \gamma), b = k_+(\beta, \delta), c = k_-(\alpha, \gamma), d = k_-(\beta, \delta)$, where

$$k_\pm(u, v) = u - v - (1-q) \pm \sqrt{(u - v - (1-q))^2 + 4uv} \tag{141}$$

To obtain a finite-dimensional representation of the TD algebra we use the solution of the Bethe-Ansatz for a set of zeros of an AW polynomial of degree L for the particular choice of parameters in terms of the boundary probability rates. We denote these zeros by $\hat{x}_i, i = 1, ..., L$. Result: We identify the transition matrix Γ_M of the open ASEP in the auxiliary space of symmetric Laurent polynomials p_n, $0 \le n \le L-1$ with the representation of the right boundary operator $A + \beta - \delta$ and the left boundary operator $A^* + \alpha - \gamma$ in the dual representation. For a chain of L sites there is a representation of dimension 2^L for any finite L. In this representation the transfer matrix Γ_M (the Hamiltonian H respectively) has a unique eigenstate $(\Omega, 0, 0, ...0)$ of eigenvalue zero which is the eigenstate of the left boundary operator, to be identified with the ASEP stationary state and $2^L - 1$ eigenstates of the right boundary operator with real eigenvalues given by

$$E = \beta - \delta + \left((1-q) \sum_{i=1}^{L} \hat{x}_i + (1-q)^2 \sum_{i<j} \hat{x}_i \hat{x}_j + ... + (1-q)^L \hat{x}_1 \hat{x}_2 ... \hat{x}_L \right) \tag{142}$$

where $\hat{x}_i^{-1} = \hat{y}_i + \hat{y}_i^{-1}$ and $\hat{y}_i$ satisfy the Bethe-Ansatz equation

$$\frac{(\hat{y}_i - k_+(\alpha,\gamma))(\hat{y}_i - k_+(\beta,\delta))(\hat{y}_i - k_-(\alpha,\gamma))(\hat{y}_i - k_-(\beta,\delta))}{(k_+(\alpha,\gamma)\hat{y}_k - 1)(k_+(\beta,\delta)\hat{y}_k - 1)(k_-(\alpha,\gamma)\hat{y}_k - 1)(k_-(\beta,\delta)\hat{y}_k - 1)} = \prod_{l=1, l\neq k}^{L-1} \frac{(qy_k - y_l)(qy_k y_l - 1)}{(y_k - qy_l)(y_k y_l - q)} \tag{143}$$

with $k_\pm(u, v)$ given by (141).

There is a dual representation in an auxiliary space of symmetric Laurent polynomials p_n, $0 \le n \le L-1$ of dimension 2^L for any finite L. In this basis the transfer matrix Γ_M (the Hamiltonian H respectively) has a unique eigenstate $(p_0, 0, 0, ...0)^t$ of eigenvalue zero which is the eigenstate of the right boundary operator and $2^L - 1$ real eigenvalues, the eigenvalues of the left boundary operator given by

$$E = \alpha - \gamma + \left((1-q) \sum_{i=1}^{L} (\hat{x}_i + (1-q)^2 \sum_{i<j} \hat{x}_i \hat{x}_j + ... + (1-q)^L \hat{x}_1 \hat{x}_2 ... \hat{x}_L \right) \tag{144}$$

where the $\hat{x}_i$ satisfy the Bethe-Ansatz equation (143). This set of eigenvalues (144) is obtained from (142) by a shift and should not be considered as a different one.

We note that we can shift respectively by $\alpha + \gamma + 2\delta$ the right boundary operator $A + \beta - \delta$, or respectively the left boundary operator $A^* + \alpha - \gamma$. Then we have the term $\alpha + \beta + \gamma + \delta$ in (142) which will correspond to the result in [37], obtained for even L and for the energy eigenvalues in the one spin sector, in a different basis, with $(\hat{y}_i + \hat{y}_i^{-1}) \rightarrow \frac{1}{1-Q^2}(-Q(z_i + z_i^{-1}) + Q^2 + 1)$, $Q^2 = q$ and z_i satisfying a corresponding Bethe equation.

We stress once again the difference in the way one obtains the finite-dimensional representation of the TD algebra needed for the Bethe Ansatz. We have used the general scheme where no relation among the model parameters appear so that we can apply it to models of nonequilibrium physics. For the XXZ chain the condition for the finite-dimensional representation of the TD algebra follows directly form the three-term recurrence relation and coincides with the previously found Bethe-Ansatz condition [39, 40].

7 Conclusion

We have developed an algebraic Bethe ansatz based on Bethe equations for the Askey-Wilson polynomials with a unique solution for any $n = L$ which yields a complete set of 2^L eigenvectors with distinct eigenvalues and a unique ground state of the ASEP transition matrix.

Even though we have illustrated the algebraization of the difference eigenvalue equation for the Askey-Wilson polynomials on lattice systems, the procedure is rather general and should work for any system with boundary Askey-Wilson symmetry and/or tridiagonal symmetry. In our opinion, the developed Bethe Ansatz scheme will produce a diagonalization of the proper Hamiltonian in the auxiliary space for the sine-Gordon model and quantum affine Toda field theory as well. It might be also interesting to explore models with the second order Askey-Wilson operator in the continuum limit with an emphasis on the symmetry properties and Hamiltonian closure.

It is worth mentioning that the relation of the ASEP (or the equivalent quantum spin chain) boundary algebra to Bethe Ansatz integrability is promising form the point of view of Bethe Ansatz perspective in string theory. One is interested in closed strings with periodic boundary conditions. However, it is simpler to find the scattering matrix on the infinite line using asymptotic states and bootstrap. Then the spectrum is determined by asymptotic Bethe equations [52, 53] and they are approximate for a system of a finite size. The study of the Askey-Wilson algebra of a system on a ring with periodic boundary conditions which is interesting on its own, might be also useful for application to strings of finite length.

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