Mapping of Wave Systems to Nonlinear Schrödinger Equations

by

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B.Sc., University of Toronto
(1973)

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

November 1979

(i.e. February, 1980)

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Submitted to the Department of Meteorology on November 5, 1979 in partial fulfillment of the requirements for the Degree of Doctor of Philosophy

ABSTRACT

When appropriate conditions are present, nonlinear systems of dynamics, such as water waves, exhibit coherently structured waveforms. An example is the soliton. Under other conditions, these systems may behave quite unpredictably, with great randomness. Connections between these regimes were sought.

The isentropic dynamics of ideal gas was considered. Lagrangian coordinates were used. It was argued that the integral of the appropriate Hamilton-Jacobi equation for the time dependent system over an ensemble set of material elements be extremal. A constraint was imposed on variations of this integral. The Lagrangian velocity $\frac{\partial x}{\partial t}$, and the density ρ, were mapped to new dependent variables ψ and ψ^*. A constraint was imposed on the Jacobian relating present time coordinates to time $t=0$ coordinates. Resultant Euler-Lagrange governing equations are a generalization of the one-dimensional nonlinear Schrödinger equation. A method for solving these equations was not found.

Shallow water waves were considered. Much of what was introduced for the ideal gas dynamics example was repeated and extended. Resultant governing equations are quite similar to the Davey-Stewartson equations.

Inverse scattering techniques were applied to these latter equations. Soliton solutions were derived. The simplest single solitons were found to have sinusoidal lateral variations.

The Eulerian velocity field, convection, and free surface height corresponding to the simplest single solitons were derived. These were found to be waves of constant form.

Consideration was given to the stability of both the ideal gas dynamics example and the shallow water waves example. Results are analogous to those of Davey and Stewartson.

A general approach was not constructed for rotating systems. No new rules for soliton interactions were obtained.

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1. Introduction

Under certain circumstances, nonlinear phenomena such as sea surface waves, seem dominated by a few scales of motion and a few specific angles and seem not characterized by either a continuum of scales of motion or a continuum of angles. These phenomena seem to have a certain coherency, order and symmetry inherent to their structure. Chaos is not evident. The best analytical example of coherency is perhaps the soliton.

Admittedly, these same nonlinear systems can be chaotically random in many of their other motions at other times. Governing equations are the same for both regimes.

A similarly dichotomous situation exists between quantum and classical mechanics, respectively. Jammer (1966)[1] gives a history of the development of the former from the latter. At its beginning, the Schrödinger theory assumed the presence of a wave eigenvalue problem. A substitution, $S = K \log \psi$, was made in Hamilton's equation for the time independent system,

$$H\left(q, \frac{\partial S}{\partial q}\right) - E = 0 \qquad 1.1$$

where S is the action, E, the energy, K, a real constant, and q, the coordinates of the system in phase space. It was postulated that ψ be a finite, single-valued, twice differentiable function, such that the integral of the functional,

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) - E = 0$$

over all configuration space, be extremal. The resultant wave equation is the time independent Schrödinger equation.

The potential energy for the hydrogen atom is $U = -e^2/r$

The wave equation for ψ is

$$\nabla^2\psi + \frac{2m}{K^2}\left\{E + \frac{e^2}{r}\right\}\psi = 0 \qquad , \qquad H\psi = E\psi$$

where m and e are constants and where r is the radial spherical polar coordinate. A symmetry transformation with respect to an operator $H = -\frac{K^2}{2m}\nabla^2 + U$, is a linear coordinate transformation such that H is of the same form in the new coordinates as in the old. Examples of real symmetry transformations are,

1. translations, $\underline{x}' = \underline{x} + \underline{a}$
2. rotations, $\underline{x}' = r\underline{x}$, $|r| = 1$
3. inversions, $\underline{x}' = -\underline{x}$
4. reflections, $\underline{x}' = r\underline{x}$ $|r| = -1$

The time dependent Schrödinger equation was obtained by writing $\Psi(q,t) = \psi(q)e^{-iEt/K}$ and postulating,

$$-\frac{K^2}{2m}\nabla^2\Psi + U\Psi = iK\Psi_t \qquad 1.2$$

where U may now be time dependent.

The passage of Equation 1.2 back to classical mechanics is noteworthy. Perhaps this was the dominating concern allowing its' postulate. The substitution $\Psi = ae^{iS/K}$ is made, where a is assumed real. This follows Landau and Lifschitz (1958)[2]. Equation

1.2 becomes,

$$a\frac{\partial S}{\partial t} - iK\frac{\partial a}{\partial t} + \frac{a}{2m}(\nabla S)^2 - \frac{iK}{2m}a\nabla^2 S - \frac{iK}{m}\nabla S.\nabla a - \frac{K^2}{2m}\nabla^2 a + Ua = 0$$

Real and imaginary parts are equated,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + U - \frac{K^2\nabla^2 a}{2ma} = 0$$

$$\frac{\partial a}{\partial t} + \frac{a\nabla^2 S}{2m} + \frac{\nabla S.\nabla a}{m} = 0$$

Of the former, the limit $K \longrightarrow 0$ implies,

$$\frac{\partial S}{\partial t} + \frac{\nabla S.\nabla S}{2m} + U = 0 \qquad 1.3$$

which is the classical Hamilton-Jacobi equation for a particle with action S. The latter may be multiplied by $2a$ and written as,

$$\frac{\partial a^2}{\partial t} + \nabla.\frac{a^2\nabla S}{m} = 0 \qquad 1.4$$

The classical velocity of a particle is $\frac{\nabla S}{m}$. The probability density of finding the particle at some point in space is $\Psi\Psi^* = a^2$. Equation 1.4 appends a classical velocity $\frac{\nabla S}{m}$, to the probability density at each point in space and makes it obey a classical equation of continuity.

Chapter 2 begins with a consideration of the isentropic dynamics of ideal gas. Lagrangian coordinates are adopted. It is argued that the integral of the appropriate Hamilton-Jacobi equation for the time dependent system, over an ensemble set of material elements, be extremal. A mapping of dependent variables is made,

$$\rho = \psi\psi^* \qquad 2.21$$

$$\frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} = K^2 \left\{ \frac{J_{lm}}{J} \frac{\partial}{\partial \alpha_m} \log \psi \right\} \left\{ \frac{J_{ln}}{J} \frac{\partial}{\partial \alpha_n} \log \psi^* \right\} + \mathcal{F}(\underline{\alpha}, t) \qquad 2.22$$

where repeated indices are summed, $\underline{x}(\underline{\alpha}, t)$ is the position of a material element whose time $t=0$ position was $\underline{\alpha}$, K is a real constant, $\rho(\underline{\alpha}, t)$ is the density and

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \quad , \quad J_{ij} = \frac{\partial J}{\partial \frac{\partial x_i}{\partial \alpha_j}}$$

The function $\mathcal{F}(\underline{\alpha}, t)$ is related to $\underline{V} \cdot \nabla \rho$, the convection. Coordinates are transformed from $\underline{\alpha}$-space to $\underline{x}$-space. Appropriate constraints are assumed. Resultant Euler-Lagrange equations are a higher dimensional generalization of the usual one dimensional nonlinear Schrödinger equation.

Shallow water waves are considered in Chapter 3. Much of what was introduced in Chapter 2 is repeated and extended. The Euler-Lagrange equations in new dependent variables are eventually found to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2} \psi^2 \psi^* - 3\Phi_y \psi \qquad 3.23$$

$$\Phi_{xx} - \Phi_{yy} = -3(\psi\psi^*)_y \qquad 3.24$$

In Chapter 4, requisite transformations for the application of the inverse scattering transform to Equations 3.23 and 3.24 are made. Soliton solutions are presented.

Chapter 5 uses the solutions of Chapter 4 to determine the function $\mathcal{F}$, the Eulerian velocity $\underset{\sim}{v}$, and the free surface height h, for the shallow water waves example.

The stability of Equations 3.23 and 3.24 is considered in Chapter 6, following Hasimoto and Ono (1972)[3]. The interaction of instability and recurrence, discovered by Yuen and Ferguson (1978)[4], is mentioned.

Chapter 7 tries to describe how transition among KdV solitons might occur. An attempt is made to fashion a role for symmetry.

Equations 3.23 and 3.24, as well as the Euler-Lagrange equations of Chapter 2 are higher dimensional generalizations of the one dimensional nonlinear Schrödinger equation. This latter equation has an earlier use in the approximate description of modulational processes of free surface waves in deep water. Benjamin and Feir (1967)[5] prove that a uniform wavetrain of wavelength $2\pi/k$, in water of depth h is unstable only if $kh > 1.363$. An analysis of gravity waves using the method of multiple scales is done by Hasimoto and Ono (1972)[3]. They let the free surface height above the undisturbed level be,

$$g\zeta = i\varepsilon\omega A(\xi,\tau)e^{i(kx-\omega t)} - i\varepsilon\omega A^*(\xi,\tau)e^{-i(kx-\omega t)} + O(\varepsilon^2)$$

and obtain

$$i\frac{\partial A}{\partial \tau} + \lambda\frac{\partial^2 A}{\partial \xi^2} = \nu|A|^2 A \qquad 1.5$$

Concerning variables, g is the acceleration due to gravity, ε is a small expansion parameter, λ and ν are known real functions of g, k and h, and

$$\xi = \varepsilon(x - \omega'(k)t) \qquad , \quad \tau = \varepsilon^2 t \qquad , \quad \omega^2 = gk\tanh kh$$

Equation 1.5 is the one-dimensional nonlinear Schrödinger equation.

Freeman and Davey (1975)[6] consider three-dimensional inviscid nonlinear surface waves on water of finite depth. They show that small amplitude waves having an appropriately slow variation in the direction transverse to the propagation direction satisfy a two-dimensional generalization of the KdV equation. This is when $\Delta = \varepsilon_0^2/h^2k^2$ is finite, where ε_0 is an amplitude parameter.

When Δ is small, governing equations become the two-dimensional generalization of the one-dimensional nonlinear Schrödinger equation suggested by Davey and Stewartson (1974)[7],

$$2i\mathcal{Y}_\tau - \mathcal{Y}_{\xi\xi} + \mathcal{Y}_{\eta\eta} = \frac{9}{2}\mathcal{Y}|\mathcal{Y}|^2 + 3\mathcal{Y}\Phi_\xi \qquad 1.6$$

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = -3(\mathcal{Y}\mathcal{Y}^*)_\xi \qquad 1.7$$

This is to a first approximation.

A variational principle is quite central to the analysis. Serrin (1959)[8] and Seliger and Whitham (1968)[9] present many of the situations where variational principles have been useful in continuum mechanics. When the system is described in terms of Lagrangian coordinates a strong analogy to a collection of discrete particles is ever present. Specifically, the equations of motion are derivable from a Hamilton's principle that the kinetic energy minus the potential energy be stationary, given an arbitrary set of material elements.

Eulerian coordinates are more familiar and more widely used than Lagrangian coordinates. When these are used to describe the system, this analogy to a collection of particles is lost. Variational principles are quite difficult to find. There is no obvious set of rules relating the variational principle of one problem to another. A given set of equations may not follow from a variational principle, whereas as Seliger and Whitham (1968)[9] point out, an equivalent set of equations, obtained as a mapping of the original set, may. The extreme example is simply to map Eulerian coordinates to Lagrangian coordinates and invoke Hamilton's principle.

Consider an Eulerian fluid defined by

$$\rho\left\{\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right\} = -\frac{\partial p}{\partial x_i} \qquad 1.8$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad 1.9$$

$$\frac{\partial S}{\partial t} + v_j \frac{\partial S}{\partial x_j} = 0 \qquad 1.10$$

$$p = p(\rho, s) \qquad 1.11$$

where $\underline{V}$ is the velocity, ρ the density, p the pressure and S the entropy at position $\underline{x}$ in space and time t. The analogous thing to Hamilton's principle when the coordinates are Lagrangian is,

$$0 = \delta \iint_{R_0} \left\{ \tfrac{1}{2} v_i v_i - \rho E(\rho, s) \right\} d\underline{x}\, dt \tag{1.12}$$

The internal energy at $(\underline{x}, t)$ is $E(\rho, s)$ and variations are with respect to ρ, $\underline{V}$ and S. Variations with respect to $\underline{V}$ imply only $\underline{V} = 0$. Herivel (1955)[10] attained some nontriviality by introducing Lagrange multipliers ϕ and η and replacing Equation 1.12 with

$$0 = \delta \iint_{R} \left\{ \tfrac{1}{2} \rho v_i v_i - \rho E(\rho, s) + \phi \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_j) \right\} + \eta \left\{ \frac{\partial}{\partial t} (\rho S) + \frac{\partial}{\partial x_j} (\rho v_j S) \right\} \right\} d\underline{x}\, dt \tag{1.13}$$

Variations with respect to $\underline{V}$ imply

$$v_i = \frac{\partial \phi}{\partial x_i} + S \frac{\partial \eta}{\partial x_i} \tag{1.14}$$

integrating by parts appropriately. Variations in $\underline{V}$ vanish on the boundary ∂_R of R.

Equation 1.14 represents a subset of possible flows available to Equations 1.8 to 1.11. Isentropic flow can be rotational, which this can not be. The correction, by Lin (1963)[11], was to introduce another Lagrange multiplier β, and to require that although the representation is in terms of Eulerian variables, initial coordinates $\underline{\alpha}(\underline{x}, t)$ of the material elements are conserved following the flow,

$$\frac{\partial \alpha_i}{\partial t} + v_j \frac{\partial \alpha_i}{\partial x_j} = 0$$

Equation 1.13 is replaced by,

$$0 = \delta \iint_{R_0} \left\{ \tfrac{1}{2}\rho v_i v_i - \rho E(\rho, S) + \phi \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) \right\} + \eta \left\{ \frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x_j}(\rho v_j S) \right\} + \right.$$
$$\left. + \beta \left\{ \frac{\partial}{\partial t}(\rho \alpha) + \frac{\partial}{\partial x_j}(\rho v_j \alpha) \right\} \right\} d\underline{x}\, dt \qquad 1.15$$

Variations with respect to $\underline{V}$ yield,

$$V_i = \frac{\partial \phi}{\partial x_i} + S\frac{\partial \Lambda}{\partial x_i} + \alpha \frac{\partial \beta}{\partial x_i} \qquad 1.16$$

Bateman (1944)[12] and Lamb (1932, Article 167)[13] extend the results due to Clebsch and show that the equations,

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} \qquad , \quad p = p(\rho)$$

are generally solved by

$$V_i = \frac{\partial \chi}{\partial x_i} + \lambda \frac{\partial \mu}{\partial x_i} \qquad , \quad \int \frac{dp}{\rho} + \frac{1}{2} V_j^2 = -\frac{\partial \chi}{\partial t} - \lambda \frac{\partial \mu}{\partial t}$$

where $\frac{D\lambda}{Dt} = 0$, $\frac{D\mu}{Dt} = 0$. Thus all solutions to Equations 1.8 to 1.11 are included in Equation 1.16, when S is conserved.

2. Dynamics of Ideal Gas

2.1 Preface

The guiding element of this discussion is that a route is sought by which the usual governing equations for a given nonlinear system may be transformed to an alternate set of equations, consistent with the existence of coherently structured waveforms within the system. As will be seen, the alternate set of equations to the usual governing equations is a higher dimensional generalization of the one-dimensional nonlinear Schrödinger equation.

The dynamics of an ideal gas is described in terms of a Hamilton's principle, using Lagrangian coordinates, following Landau and Lifshitz (1969)[14] and Seliger and Whitham (1968)[9]. A similarity between material elements and particles is maintained and becomes the key to motivating many of the steps leading from the usual governing equations to an alternate set of equations. Constraints surrounding the formalism and their relation to the dynamics of the system are considered.

2.2 Classical formulation

Given a material element in an inertial frame with Lagrangian coordinates $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ at time $t = 0$, and position $\chi(\underline{\alpha}, t)$ at time t, the inviscid equations of motion are,

$$\rho \frac{\partial^2 \chi_i}{\partial t^2} = -\frac{\partial p}{\partial \chi_i} \qquad i = 1, 2, 3 \qquad 2.1$$

Coordinates (χ_1, χ_2, χ_3) are related to Cartesian coordinates (x, y, z) by

$$\chi_1 = x \quad , \quad \chi_2 = y \quad , \quad \chi_3 = \frac{t^2}{2} g + z$$

The acceleration due to gravity is g , the density $\rho(\underline{\alpha}, t)$, and the pressure $p(\underline{x}, t)$. In terms of the initial density $\rho_0(\underline{\alpha})$ the continuity relation is, following Lamb (1932, Article 14)[13]

$$\frac{1}{\rho} = \frac{J}{\rho_0} \quad , \quad J = \frac{\partial(\chi_1, \chi_2, \chi_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} = \epsilon_{ijk} \frac{\partial \chi_1}{\partial \alpha_i} \frac{\partial \chi_2}{\partial \alpha_j} \frac{\partial \chi_3}{\partial \alpha_k} \qquad 2.2$$

where $\epsilon_{ijk} = \frac{1}{2}(j-k)(k-i)(i-j)$ and $i, j, k = 1, 2, 3$.

From Equation 2.2, Equation 2.1 may be rewritten in terms of derivatives of $\chi(\underline{\alpha}, t)$ and $p(\underline{\alpha}, t)$ with respect to $\underline{\alpha}$ and t ,

$$\rho_0 \frac{\partial^2 \chi_i}{\partial t^2} = -J \frac{\partial \alpha_j}{\partial \chi_i} \frac{\partial p}{\partial \alpha_j} = -J_{ij} \frac{\partial p}{\partial \alpha_j} \quad , \quad J_{ij} = J \frac{\partial \alpha_j}{\partial \chi_i} = \frac{\partial J}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} \qquad 2.3$$

The entropy of material elements is assumed constant throughout the motion. The set of equations is completed by writing the conservation of entropy equation,

$$S = S_0(\underline{\alpha}) \qquad 2.4$$

the first law of thermodynamics,

$$dE = T\,dS - p\,d(1/\rho) \qquad 2.5$$

and equations among state variables,

$$E = p/(\gamma-1)\rho \quad , \quad T = p/R\rho \quad , \quad p = p_0 (\rho/\rho_0)^{\gamma} e^{\frac{(s-s_0)(\gamma-1)}{R}} \tag{2.6}$$

The internal energy is E, the temperature T, the time $t=0$ pressure p_0, and the ideal gas constants R, and γ.

In Eulerian coordinates, the first law of thermodynamics has the form

$$T\nabla S = C_p \nabla T - \nabla p/\rho$$

where C_p is the specific heat at constant pressure. Therefore, the momentum equation in Eulerian coordinates may be written as,

$$\frac{\partial \underline{u}}{\partial t} + \nabla \frac{\underline{u}^2}{2} + (\nabla \times \underline{u}) \times \underline{u} = T\nabla S - C_p \nabla T$$

and the vorticity equation,

$$\underline{\omega}_t + \nabla \times (\underline{\omega} \times \underline{u}) = \nabla T \times \nabla S$$

This is Crocco's theorem. The assumption of an isentropic process implies that if the flow was once irrotational, then it shall always be irrotational.

A variational principle, Hamilton's principle, leads to Equation 2.1 in the usual manner,

$$\delta \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} \left\{ \frac{1}{2} \rho_0 \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - \rho_0 E \right\} d\underline{\alpha}\, dt = 0 \qquad 2.7$$

The domain $R_0(\underline{\alpha})$ is arbitrary. Variations in material element trajectories $\delta \chi_i$, are arbitrary but vanish on the boundary ∂_{R_0} of R_0.

Let $\chi_i(\underline{\alpha}, t)$ be varied by $\delta \chi_i$ in Equation 2.7. Recognizing the relationships $\rho_0 = \rho_0(\underline{\alpha})$, $E = E(S_0, \rho)$ and $\rho = \rho_0 \frac{\partial(\alpha_1, \alpha_2, \alpha_3)}{\partial(\chi_1, \chi_2, \chi_3)}$ from Equations 2.2 and 2.5, it follows that,

$$0 = \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} \left\{ -\rho_0 \frac{\partial^2 \chi_i}{\partial t^2} + \frac{\partial}{\partial \alpha_j} \left(\rho_0 \frac{\partial E}{\partial \chi_{ij}} \right) \right\} \delta \chi_i \, d\underline{\alpha}\, dt \qquad , \chi_{ij} = \frac{\partial \chi_i}{\partial \alpha_j} \qquad 2.8$$

Both terms have been integrated by parts. As both the domain, R_0, and the variations in the trajectories of the material elements, $\delta \chi_i$, are arbitrary, the integrand of Equation 2.8 must vanish. From Equations 2.2, 2.3 and 2.6,

$$\frac{\partial E}{\partial \chi_{ij}} = \frac{\partial E}{\partial \rho} \frac{\partial \rho}{\partial \chi_{ij}} = \frac{P}{\rho^2} \frac{\partial \rho}{\partial \chi_{ij}} \qquad , \qquad \frac{\partial \rho}{\partial \chi_{ij}} = -\frac{\rho^2}{\rho_0} J_{ij} \qquad 2.9$$

and therefore Equation 2.8 implies,

$$\rho_0 \frac{\partial^2 \chi_i}{\partial t^2} = -\frac{\partial}{\partial \alpha_j}(p\, J_{ij}) = -J_{ij}\frac{\partial p}{\partial \alpha_j}$$

which is Equation 2.3. The identity, $\frac{\partial}{\partial \alpha_j} J_{ij} = 0$, follows from writing Equation 2.2,

$$J = \epsilon_{ijk}\frac{\partial \chi_1}{\partial \alpha_i}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k}$$

$$= \frac{\partial \chi_1}{\partial \alpha_1}\epsilon_{1jk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k} + \frac{\partial \chi_1}{\partial \alpha_2}\epsilon_{2jk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k} + \frac{\partial \chi_1}{\partial \alpha_3}\epsilon_{3jk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k}$$

and differentiating partially. For example,

$$\frac{\partial}{\partial \alpha_p}\frac{\partial J}{\partial \frac{\partial \chi_1}{\partial \alpha_p}} = \frac{\partial}{\partial \alpha_p}\epsilon_{pjk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k} = \epsilon_{pjk}\frac{\partial^2 \chi_2}{\partial \alpha_p \partial \alpha_j}\frac{\partial \chi_3}{\partial \alpha_k} + \epsilon_{pjk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial^2 \chi_3}{\partial \alpha_p \partial \alpha_k}$$

$$= \epsilon_{jpk}\frac{\partial^2 \chi_2}{\partial \alpha_j \partial \alpha_p}\frac{\partial \chi_3}{\partial \alpha_k} + \epsilon_{jpk}\frac{\partial \chi_2}{\partial \alpha_j}\frac{\partial^2 \chi_3}{\partial \alpha_k \partial \alpha_p}$$

which vanishes, because $\epsilon_{jpk} = -\epsilon_{pjk}$.

Let λ be the Lagrange multiplier to accommodate the constraint of mass conservation, Equation 2.2. The variational principle, Equation 2.7, becomes

$$\delta \int_{t_1}^{t_2}\int_{R_0(\underline{\alpha})} \left\{ \tfrac{1}{2}\rho_0 \frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} - \rho_0 E + \lambda(\rho_0 - \rho J) \right\} d\underline{\alpha}\, dt = 0 \qquad 2.10$$

A reference is Gelfand and Fomin (1963)[15]. Variations with respect to trajectories of the material elements, $\underline{\chi}(\underline{\alpha}, t)$, imply,

$$-\rho_\circ \frac{\partial^2 \chi_i}{\partial t^2} + \frac{\partial}{\partial \alpha_j}\left(\rho_\circ \frac{\partial E}{\partial \chi_{ij}}\right) - \frac{\partial}{\partial \chi_i}(\lambda \rho) J + \frac{\partial}{\partial \alpha_j}(\lambda \rho J_{ij}) = 0$$

Because $\frac{\partial}{\partial \alpha_j} J_{ij} = 0$ and $J \frac{\partial \alpha_j}{\partial \chi_i} = J_{ij}$, this leads to,

$$-\rho_\circ \frac{\partial^2 \chi_i}{\partial t^2} - J_{ij} \frac{\partial p}{\partial \alpha_j} - (\lambda \rho)_{\alpha_j} J_{ij} + (\lambda \rho)_{\alpha_j} J_{ij} = 0$$

which reduces to Equation 2.3.

For each material element, the Lagrangian is taken as $\mathcal{L} = \frac{1}{2}\rho_\circ \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - \rho_\circ E + \lambda(\rho_\circ - \rho J)$. In other words, for this system the Lagrangian density is $\mathcal{L} = \frac{1}{2}\rho_\circ \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - \rho_\circ E + \lambda(\rho_\circ - \rho J)$. The corresponding canonical momentum is

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}} = \rho_\circ \frac{\partial \chi_i}{\partial t} \qquad 2.11$$

and the Hamiltonian density

$$\mathcal{H} = \pi_i \frac{\partial \chi_i}{\partial t} - \mathcal{L} = \frac{1}{2}\rho_\circ \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} + \rho_\circ E - \lambda(\rho_\circ - \rho J) \qquad 2.12$$

which represents the total energy of a material element, within the context of an integral such as Equation 2.10.

Hamilton's equations of motion are sought. Consider an arbitrary number of material elements at time $t=0$ each having a Lagrangian of the form $\mathcal{L}(\chi_i, \frac{\partial \chi_i}{\partial t}, \frac{\partial \chi_i}{\partial \alpha_j})$. A Hamilton's principle, with respect to variations in material element trajectories, may be posed as,

$$\delta \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} \mathcal{L} \, d\underline{\alpha} \, dt = 0 \tag{2.13}$$

The corresponding Euler-Lagrange equations follow from taking variations in χ_i,

$$0 = \int_{t_1}^{t_2} \int_{R_0} \left\{ \frac{\partial \mathcal{L}}{\partial \chi_i} \delta \chi_i + \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}} \delta \frac{\partial \chi_i}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} \delta \frac{\partial \chi_i}{\partial \alpha_j} \right\} d\underline{\alpha} \, dt$$

and integrating by parts,

$$0 = \int_{t_1}^{t_2} \int_{R_0} \delta \chi_i \left\{ \frac{\partial \mathcal{L}}{\partial \chi_i} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}} - \frac{\partial}{\partial \alpha_j} \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} \right\} d\underline{\alpha} \, dt$$

As domain R_0 , and variations in material element trajectories are arbitrary, the integrand must vanish,

$$0 = \frac{\partial \mathcal{L}}{\partial \chi_i} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}} - \frac{\partial}{\partial \alpha_j} \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} \tag{2.14}$$

Alternately, the substitution $\mathcal{L} = \pi_i \frac{\partial \chi_i}{\partial t} - \mathcal{H}$ in Equation 2.13 implies

$$0 = \delta \int_{t_1}^{t_2} \int_{R_0} \left\{ \pi_i \frac{\partial \chi_i}{\partial t} - \mathcal{H} \right\} d\underline{\alpha} \, dt$$

Varying π_i and χ_i independently,

$$0 = \int_{t_1}^{t_2}\int_{R_0} \delta\pi_i \left\{ \frac{\partial \chi_i}{\partial t} - \frac{\partial H}{\partial \pi_i} \right\} d\underline{\alpha}\, dt + \int_{t_1}^{t_2}\int_{R_0} \delta\chi_i \left\{ -\frac{\partial \pi_i}{\partial t} + \frac{\partial}{\partial \alpha_j}\frac{\partial H}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} - \frac{\partial H}{\partial \chi_i} \right\} d\underline{\alpha}\, dt$$

where the first and second terms of the second integral have been integrated by parts. As domain R_0 , and variations $\delta\pi_i$ and $\delta\chi_i$ are arbitrary,

$$\frac{\partial \chi_i}{\partial t} = \frac{\partial H}{\partial \pi_i} \quad , \quad \frac{\partial \pi_i}{\partial t} = \frac{\partial}{\partial \alpha_j}\frac{\partial H}{\partial \frac{\partial \chi_i}{\partial \alpha_j}} - \frac{\partial H}{\partial \chi_i} \qquad 2.15$$

which are Hamilton's equations.

The action S , and the action density s , are introduced by writing

$$S = \int_{t_0}^{t}\int_{R_0} \mathcal{L}\, d\underline{\alpha}\, dt = \int_{R_0} s\, d\underline{\alpha} \qquad 2.16$$

where the first integral is indefinite with respect to time. Assuming $\mathcal{L} = \mathcal{L}\left(\chi_i, \frac{\partial \chi_i}{\partial t}, \frac{\partial \chi_i}{\partial \alpha_j}\right)$ the variation of S with respect to trajectories of the material elements is,

$$\delta S = \int_{t_0}^{t}\int_{R_0} \left\{ \frac{\partial \mathcal{L}}{\partial \chi_i}\delta\chi_i + \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}}\delta\frac{\partial \chi_i}{\partial t} + \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial \alpha_j}}\delta\frac{\partial \chi_i}{\partial \alpha_j} \right\} d\underline{\alpha}\, dt$$

and integrating by parts,

$$\delta \mathcal{S} = \int_{t_o}^{t}\!\!\int_{R_o} \left\{ \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial t}} \delta x_i \right] + \frac{\partial}{\partial \alpha_j}\left[\frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial \alpha_j}} \delta x_i \right] \right\} d\underline{\alpha}\, dt$$

$$- \int_{t_o}^{t}\!\!\int_{R_o} \delta x_i \left\{ -\frac{\partial \mathcal{L}}{\partial x_i} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial t}} + \frac{\partial}{\partial \alpha_j}\frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial \alpha_j}} \right\} d\underline{\alpha}\, dt \qquad = \int_{R_o} \frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial t}} \delta x_i \Big|_t d\underline{\alpha}$$

Variations, δx_i, are assumed to vanish on the boundary ∂_{R_o}. The motion is assumed compliant with Equation 2.14. Using the definition of canonical momentum, Equation 2.11, this implies,

$$\delta \mathcal{S} = \int_{R_o} \pi_i\, \delta x_i \Big|_t d\underline{\alpha}$$

or in view of the definition of action density in Equation 2.16,

$$\delta \mathcal{S} = \int_{R_o} \pi_i\, \delta x_i \Big|_t d\underline{\alpha} = \int_{R_o} \delta s\, d\underline{\alpha}$$

and therefore,

$$\delta s = \pi_i\, \delta x_i \qquad , \quad \pi_i = \frac{\partial s}{\partial x_i} \text{ at time } t \qquad , s = s(\underline{x}(\underline{\alpha},t),t) \tag{2.17}$$

Equation 2.16 may be expressed in the differential form,

$$\mathcal{L} = \frac{ds}{dt} = \frac{\partial s}{\partial t} + \frac{\partial x_i}{\partial t}\frac{\partial s}{\partial x_i}$$

which becomes, substituting $\pi_i = \frac{\partial s}{\partial x_i}$ from Equation 2.17,

$$\mathcal{L} = \frac{\partial s}{\partial t} + \frac{\partial \chi_i}{\partial t}\pi_i \qquad 2.18$$

The Hamiltonian density is defined as $\mathcal{H} = \pi_i \frac{\partial \chi_i}{\partial t} - \mathcal{L}$ in Equation 2.12. Thus,

$$0 = \frac{\partial s}{\partial t} + \mathcal{H} \qquad 2.19$$

which is the Hamilton-Jacobi equation.

Notice that when the first integral of Equation 2.16 is definite with respect to time

$$0 = \delta S = \delta \int_{t_1}^{t_2}\int_{R_0} \mathcal{L}\, d\underline{\alpha}\, dt = \delta \int_{t_1}^{t_2}\int_{R_0} \left\{ \frac{\partial s}{\partial t} + \pi_i \frac{\partial \chi_i}{\partial t} \right\} d\underline{\alpha}\, dt$$

using Equation 2.18. Therefore, substituting Equation 2.12,

$$0 = \delta \int_{t_1}^{t_2}\int_{R_0} \left\{ \frac{\partial s}{\partial t} + \mathcal{H} \right\} d\underline{\alpha}\, dt + \delta \int_{t_1}^{t_2}\int_{R_0} \mathcal{L}\, d\underline{\alpha}\, dt$$

which results that

$$0 = \delta \int_{t_1}^{t_2}\int_{R_0} \left\{ \frac{\partial s}{\partial t} + \mathcal{H} \right\} d\underline{\alpha}\, dt \qquad 2.20$$

2.3 Mapping of dependent variables

The usual governing equations describing the dynamics of ideal gas have been presented. An alternate set of governing equations, whose solutions are coherently structured waveforms, is sought, given that certain constraints are present within the system. It is precisely characteristics such as coherency and symmetry which occur strikingly in certain aspects of modern quantum theory. Arguing after a fashion that leads to a development of modern quantum theory from classical dynamics, as recounted by Jammer (1966)[1], a transformation from dependent variables ρ and $\frac{\partial \chi}{\partial t}$ to new dependent variables, ψ and ψ^*, is made,

$$\rho = \psi\psi^* \tag{2.21}$$

$$\frac{\partial \chi}{\partial t}\cdot\frac{\partial \chi}{\partial t} = \left\{ K \frac{J_{\ell m}}{J} \frac{\partial}{\partial \alpha_m} \log \psi \right\}\left\{ K \frac{J_{\ell n}}{J} \frac{\partial}{\partial \alpha_n} \log \psi^* \right\} + \mathcal{F}(\underline{\alpha}, t)$$

$$= K^2 \nabla_{\underline{x}} \log \psi \cdot \nabla_{\underline{x}} \log \psi^* + \mathcal{F}(\underline{x}, t) \tag{2.22}$$

It is assumed that $\mathcal{F}(\underline{x},t)$ is a know function, K, a real constant.

The second motivation for this step is that under proper conditions, nonlinear fluid mechanical systems may be represented by the KdV equation,

$$u_t + u u_x + u_{xxx} = 0 \tag{2.23}$$

As presented by Whitham (1974)[16] and Ablowitz (1978)[17], this equation is a member of the set of partial differential equations solvable by the inverse scattering transform. Within this formalism, the linear scattering problem associated to Equation 2.23 is

$$\psi_{xx} + (\zeta^2 + u(x,0))\psi = 0 \quad , \quad \psi = \psi(\underline{x},t) \quad , \quad \zeta^2 \text{ is real} \tag{2.24}$$

which is the time independent linear Schrödinger equation of quantum mechanics.

Recall that $\pi_i = \rho_0 \frac{\partial X_i}{\partial t}$ from Equation 2.11 and $\pi_i = \frac{\partial s}{\partial x_i}$ from Equation 2.17. Equation 2.22 may be written as,

$$\nabla_{\underline{x}} s . \nabla_{\underline{x}} s = \rho_0^2 K^2 \nabla_{\underline{x}} \log \psi . \nabla_{\underline{x}} \log \psi^* + \rho_0^2 \mathcal{F}(\underline{x},t) \tag{2.25}$$

Let $\psi = R(\underline{x},t) e^{i\Omega(\underline{x},t)}$. Therefore, $-2\Omega = i\log\psi - i\log\psi^*$ and Equation 2.21 implies that $\rho = \psi\psi^* = R^2$. The condition $K^2 \nabla\Omega . \nabla \log R = \frac{1}{2} \mathcal{F}(\underline{x},t)$ is imposed. Sufficient conditions for Equation 2.25 are,

$$\nabla s = \rho_0 K \left\{ \nabla \log R + \nabla\Omega \right\}$$

or equivalently,

$$s = \rho_0 \frac{K}{2} \left\{ (1-i) \log\psi + (1+i) \log\psi^* \right\} \tag{2.26}$$

where ρ_0 is a real constant.

Let Equation 2.12, $\mathcal{H} = \frac{1}{2}\rho_0 \frac{\partial \underline{X}}{\partial t} . \frac{\partial \underline{X}}{\partial t} + \rho_0 E - \lambda(\rho_0 - \rho J)$ be substituted into Equation 2.20,

$$0 = \delta \int_{t_1}^{t_2}\int_{R_0} \left\{ \frac{\partial s}{\partial t} + \mathcal{H} \right\} d\underline{\alpha}\, dt \tag{2.20}$$

implying

$$\delta \int_{t_1}^{t_2}\int_{R_o(\underline{\alpha})} \left\{ \frac{\partial s}{\partial t} + \frac{1}{2}\rho_o \frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} + \rho_o E - \lambda(\rho_o - \rho J)\right\} d\underline{\alpha}\, dt = 0$$

Transform variables via Equations 2.21, 2.22 and 2.26. The system is represented as,

$$0 = \delta \int_{t_1}^{t_2}\int_{R_o(\underline{\alpha})} \Big\{ \rho_o \frac{K}{2}(1-i)(\log\psi)_t + \rho_o \frac{K}{2}(1+i)(\log\psi^*)_t + \rho_o \frac{K^2}{2} \nabla \log\psi \cdot \nabla \log\psi^*$$

$$+ \frac{\rho_o}{2} \mathcal{F}(\underline{\alpha},t) + \rho_o E(\psi,\psi^*) - \lambda(\rho_o - \psi\psi^* J)\Big\} d\underline{\alpha}\, dt \qquad 2.27$$

where $E(\psi,\psi^*) = \dfrac{P_o(\psi\psi^*)^{\gamma-1}}{\rho_o^{\gamma}(\gamma-1)}$ from Equation 2.6. Let the coordinates be transformed from $\underline{\alpha}$-space to $\underline{\chi}$-space. Equation 2.27 becomes,

$$0 = \delta \int_{t_1}^{t_2}\int_{R_o(\underline{\chi})} \Big\{ \frac{K}{2}(1-i)\psi^*\psi_t + \frac{K}{2}(1+i)\psi\psi^*_t + \frac{K^2}{2}\nabla\psi\cdot\nabla\psi^* + \frac{\mathcal{F}}{2}(\underline{\chi},t)\psi\psi^*$$

$$+ \frac{P_o(\psi\psi^*)^{\gamma}}{\rho_o^{\gamma}(\gamma-1)} + \lambda\psi\psi^*\Big\} d\underline{\chi}\, dt - \delta\int_{t_1}^{t_2}\int_{R_o(\underline{\alpha})} \lambda\rho_o \, d\underline{\alpha}\, dt \qquad 2.28$$

Variations with respect to ψ imply, setting $c = P_o/\rho_o^{\gamma}$,

$$0 = \int_{t_1}^{t_2}\int_{R_o(\underline{\chi})} \Big\{ -iK\psi_t - \frac{K^2}{2}\nabla^2\psi + \frac{\delta}{\delta\psi^*}\frac{\mathcal{F}\psi\psi^*}{2} + \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi + \lambda\psi\Big\}\delta\psi^* \, d\underline{\chi}\, dt$$

$$+ \int_{t_1}^{t_2}\int_{R_o(\underline{\chi})} \Big\{ +iK\psi^*_t - \frac{K^2}{2}\nabla^2\psi^* + \frac{\delta}{\delta\psi}\frac{\mathcal{F}\psi\psi^*}{2} + \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi^* + \lambda\psi^*\Big\}\delta\psi \, d\underline{\chi}\, dt \qquad 2.29$$

where $\delta\psi^* \frac{\delta}{\delta\psi^*}\frac{\mathcal{F}\psi\psi^*}{2} + \delta\psi\frac{\delta}{\delta\psi}\mathcal{F}\frac{\psi\psi^*}{2}$ is the variation of $\dfrac{\mathcal{F}\psi\psi^*}{2}$ with respect to ψ . Let $Q = -iK\psi_t - \frac{K^2}{2}\nabla^2\psi + \frac{\delta}{\delta\psi^*}\frac{\mathcal{F}\psi\psi^*}{2} + \frac{c\gamma}{\gamma-1}(\psi\psi^*)^{\gamma-1}\psi + \lambda\psi$

When $\delta\psi$ is real, $\mathcal{R}e\,\mathcal{Q} = 0$. When $\delta\psi$ is imaginary, $\mathcal{I}m\,\mathcal{Q} = 0$. Therefore as $\delta\psi$ is arbitrary,

$$-i\kappa\psi_t - \frac{\kappa^2}{2}\nabla^2\psi + \frac{\delta}{\delta\psi^*}\,\frac{\mathcal{F}\psi\psi^*}{2} + \frac{C\gamma}{\gamma-1}(\psi\psi^*)^{\gamma-1}\psi + \lambda\psi = 0$$

In obtaining Equation 2.29, integration by parts has been used where appropriate. Endpoint conditions have not been written down because they sum to zero independently of other terms and constitute nothing other than boundary conditions. For details see Luke (1967)[18].

When it is assumed that no variations leading to contributions from $\frac{\mathcal{F}\psi\psi^*}{2}$ are allowed the resultant set of equations is closer to those equations solvable by the inverse scattering transform. Let $\mathcal{L} = -i\kappa\psi_t - \frac{\kappa^2}{2}\nabla^2\psi + \frac{C\gamma}{\gamma-1}(\psi\psi^*)^{\gamma-1}\psi + \lambda\psi$, $\psi = \chi + i\mathcal{Y}$ and $\delta\psi = \delta\chi + i\delta\mathcal{Y}$. Equation 2.29 implies,

$$0 = \int_{t_1}^{t_2}\int_{R_o(\underline{x})} \left\{ \mathcal{L}_R\,\delta\chi + \mathcal{L}_I\,\delta\mathcal{Y} \right\} d\underline{x}\, dt \qquad 2.30$$

where $\mathcal{L} = \mathcal{L}_R + i\mathcal{L}_I$. The condition that $\left\{\kappa^2\nabla \arctan\frac{\mathcal{Y}}{\chi}\right\}\cdot\nabla(\chi^2+\mathcal{Y}^2) = \mathcal{F}\psi\psi^*$ be invariant to allowed variations in ψ is linearly dependent in variables $\delta\chi$ and $\delta\mathcal{Y}$ to the integrand of Equation 2.30,

$$\mathcal{L}_R\,\delta\chi + \mathcal{L}_I\,\delta\mathcal{Y} = 0 \qquad 2.31$$

If $\delta\chi \neq 0$ implies $\delta\mathcal{Y} \neq 0$ or $\delta\mathcal{Y} \neq 0$ implies $\delta\chi \neq 0$ then $\mathcal{L}_R = \mathcal{L}_I = 0$. If $\delta\chi \neq 0$ implies $\delta\mathcal{Y} = 0$ and $\delta\mathcal{Y} \neq 0$ implies $\delta\chi = 0$ then $\mathcal{L}_R = \mathcal{L}_I = 0$. Therefore

$$\mathcal{L} = 0$$

$$-iK\psi_t - \frac{K^2}{2}\nabla^2\psi + \frac{C\gamma}{\gamma-1}(\psi\psi^*)^{\gamma-1}\psi + \lambda\psi = 0 \qquad 2.32$$

The variation of Equation 2.28 with respect to ψ yields no contribution from the last term,

$$\delta\int_{t_1}^{t_2}\int_{R_0(\underline{\alpha})} \lambda(\underline{\alpha})\rho_0 \, d\underline{\alpha}\, dt = \delta\int_{t_1}^{t_2}\int_{R_0(\underline{x})} \frac{\lambda(\underline{x})}{J}\rho_0 \, d\underline{x}\, dt \qquad 2.33$$

which implies $J = J(\lambda)$. No constraint is imposed on $\frac{\partial x_i}{\partial \alpha_j}$ by the mapping defined by Equations 2.21 and 2.22 except by way of Equation 2.2,

$$\frac{\rho_0}{\epsilon_{ijk}\frac{\partial x_1}{\partial \alpha_i}\frac{\partial x_2}{\partial \alpha_j}\frac{\partial x_3}{\partial \alpha_k}} = \psi\psi^* \quad , \quad J = \epsilon_{ijk}\frac{\partial x_1}{\partial \alpha_i}\frac{\partial x_2}{\partial \alpha_j}\frac{\partial x_3}{\partial \alpha_k} \qquad 2.2$$

The equations which follow become nearly solvable when a function Φ is introduced, where

$$\frac{\rho_0}{J(\lambda)} = \Phi_x - \int^x \Phi_{yy}\, dx \quad , \quad \lambda = \Phi_x \qquad 2.34$$

This restricts the system in that $J(\lambda)$ must obey,

$$\left\{\frac{\rho_0}{J(\lambda)}\right\}_{xx} = \lambda_{xx} - \lambda_{yy}$$

The resultant set of equations may be written as

$$ik\psi_t + \frac{K^2}{2}\psi_{xx} + \frac{K^2}{2}\psi_{yy} = \frac{c\gamma}{\gamma-1}(\psi\psi^*)^{\gamma-1}\psi + \Phi_x\psi \quad 2.35$$

$$\Phi_{xx} - \Phi_{yy} = (\psi\psi^*)_x \quad 2.36$$

Using the inverse scattering transform, Anker and Freeman (1978)[19] have examined soliton solutions to the Davey-Stewartson equations,

$$2i\psi_\tau - \psi_{\xi\xi} + \psi_{\eta\eta} = \frac{9}{2}\psi|\psi|^2 + 3\psi\Phi_\xi \quad 1.6$$

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = -3(\psi\psi^*)_\xi \quad 1.7$$

where $\Phi = \Phi(\xi,\eta,\tau)$ and $\psi = \psi(\xi,\eta,\tau)$. These resemble Equations 2.35 and 2.36 qualitatively and are a higher dimensional generalization of the one-dimensional nonlinear Schrödinger equation.

Had the discussion proceeded from Equation 2.7,

$$0 = \delta\int_{t_1}^{t_2}\int_{R_0} \left\{ \tfrac{1}{2}\rho_0 \frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} - \rho_0 E \right\} d\underline{\alpha}\, dt \quad 2.7$$

rather than Equation 2.10,

$$0 = \delta\int_{t_1}^{t_2}\int_{R_0} \left\{ \rho_0 \tfrac{1}{2} \frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} - \rho_0 E + \lambda(\rho_0 - \rho\,\mathcal{J}) \right\} d\underline{\alpha}\, dt \quad 2.10$$

the foregoing discussion would lead to,

$$-ik\psi_t - \frac{K^2}{2}\nabla^2\psi + \frac{c\gamma(\psi\psi^*)^{\gamma-1}\psi}{\gamma-1} = 0 \quad 2.37$$

in place of Equation 2.32. Generically this equation has the form,

$$0 = -i\psi_\tau + \nabla^2\psi + g(\psi\psi^*)\psi$$

The work of Ablowitz (1978)[17] implies that there is a systematic method of solution in the one-dimensional case when $g(\psi\psi^*) = \psi\psi^*$

At the center of this section is the mapping from dependent variables ρ and $\frac{\partial \underline{\chi}}{\partial t}$ to new dependent variables ψ and ψ^*,

$$\rho = \psi\psi^* \qquad 2.21$$

$$\frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} = \left\{K\frac{J_{\ell m}}{J}\frac{\partial}{\partial \alpha_m}\log\psi\right\}\left\{K\frac{J_{\ell n}}{J}\frac{\partial}{\partial \alpha_n}\log\psi^*\right\} + \mathcal{F}(\underline{\alpha},t) \qquad 2.22$$

$$\nabla_{\underline{x}} s\cdot\nabla_{\underline{x}} s = \rho_0^2 K^2 \nabla_{\underline{x}}\log\psi\cdot\nabla_{\underline{x}}\log\psi^* + \rho_0^2\mathcal{F}(\underline{\chi},t) \qquad 2.25$$

The substitutions $\psi = R(\underline{\chi},t)e^{i\Omega(\underline{\chi},t)}$ and $\mathcal{F}(\underline{\chi},t) = 2K^2\nabla\Omega\cdot\nabla\log R$ are made. Allowed variations in ψ are such that $\mathcal{F}\psi\psi^*$ does not vary. Sufficient conditions for Equations 2.21 and 2.22 are written as,

$$\nabla s = \rho_0 K\left\{\nabla\log R + \nabla\Omega\right\} \quad , \quad s = \rho_0 K\left\{\log R + \Omega\right\} \qquad 2.26$$

Connective arguments to this last step follow.

2.4 Justification of the mapping

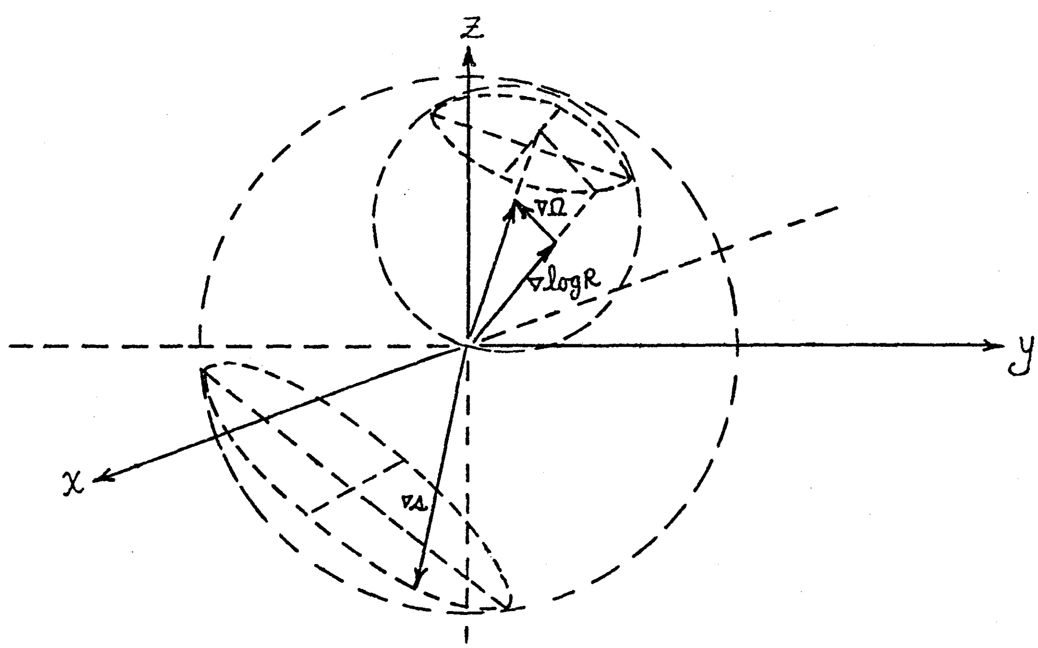
Necessary conditions for Equation 2.25, given that $\psi = R(\underline{\chi},t)e^{i\Omega(\underline{\chi},t)}$ are,

$$\nabla s = r(\underline{x},t)\, \rho_0 \kappa \left\{ \nabla \log R + \nabla \Omega \right\} \qquad 2.38$$

where $r(\underline{x},t)$ is a tensor operator of rotation. The state of the system defines ρ , $R = \sqrt{\rho}$, ∇s , and, as will become evident, $\mathcal{F}(\underline{x},t)$. The condition $\kappa^2 \nabla\Omega . \nabla \log R = \frac{1}{2}\mathcal{F}$ and Equation 2.25,

$$\nabla s . \nabla s = \rho_0^2 \kappa^2 \left\{ \nabla \log R . \nabla \log R + \nabla\Omega . \nabla\Omega \right\} + \rho_0^2\, \mathcal{F}(\underline{x},t) \qquad 2.25$$

determine the variable Ω . The operator $r(\underline{x},t)$ is defined by Equation 2.38. When $\mathcal{F}(\underline{x},t) = 0$, the general status of the system is envisaged as,



The evolution of $r(\underline{\chi},t)$ in time reflects that of the system, which in turn is specified by governing Equations 2.2 to 2.6.

The Hamilton-Jacobi equation,

$$0 = \frac{\partial \mathcal{S}}{\partial t} + \mathcal{H} \qquad 2.19$$

the definition of Hamiltonian density,

$$\mathcal{H} = \tfrac{1}{2}\rho_0 \frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} + \rho_0 E - \lambda(\rho_0 - \rho J) \qquad 2.12$$

and the relation between canonical momentum and action density,

$$\pi_i = \rho_0 \frac{\partial \chi_i}{\partial t} = \frac{\partial \mathcal{S}}{\partial \chi_i} \qquad 2.17$$

imply,

$$0 = \frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2\rho_0}\nabla \mathcal{S}.\nabla \mathcal{S} + \rho_0 E - \lambda(\rho_0 - \rho J)$$

It follows that $\mathcal{S}_t$ is independent of $r(\underline{\chi},t)$. The evolution of $\mathcal{S}$ with respect to time, given an $r(\underline{\chi},t)$ operator of arbitrary form, is the same as when $r(\underline{\chi},t) = 1$. It is consistent to choose

$$\mathcal{S}_t = \rho_0 k\left\{ (\log R)_t + \Omega_t \right\} \qquad 2.39$$

Thus dependent variables ρ and $\frac{\partial \underline{\chi}}{\partial t}$ are mapped to ψ and ψ^* resulting in Equation 2.27. The coordinates of this last relation are transformed from $\underline{\alpha}$-space to $\underline{\chi}$-space. The functional finally obtained, Equation 2.28, is taken to vanish when varied with respect to ψ .

Initially the system is defined in terms of Lagrangian coordinates. A variation in a material element trajectory, $\underline{\chi}(\underline{\alpha},t)$, corresponds to a variation in density, $\rho = \psi\psi^*$, and in velocity $\frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} = \mathcal{F}(\underline{\alpha},t) + K^2\left\{\frac{J_{\ell m}}{J}\frac{\partial}{\partial \alpha_m}\log\psi\right\}\left\{\frac{J_{\ell n}}{J}\frac{\partial}{\partial \alpha_n}\log\psi^*\right\}$, of the system. If Equation 2.10 is extremal with respect to variations in $\underline{\chi}(\underline{\alpha},t)$, Equation 2.28 should be extremal with respect to variations in ψ .

The substitution $K^2\nabla\Omega.\nabla\log R = \frac{1}{2}\mathcal{F}$ was made and it was assumed that $\mathcal{F}\psi\psi^*$ be invariant with respect to allowed variations in ψ . Alternately the mapping could be specified as,

$$\nabla s.\nabla s = \rho_0^2 K^2\left\{\nabla\log R.\nabla\log R + \nabla\Omega.\nabla\Omega\right\} + \rho_0^2\mathcal{F}(\underline{\chi},t) \qquad 2.25$$

$$s_t = \rho_0 K\left\{(\log R)_t + \Omega_t\right\} \qquad 2.39$$

where $\mathcal{F}$ is arbitrary. In one-dimension these equations imply the compatibility condition, $s_{xt} = s_{tx}$,

$$K\left\{(\log R)_{tx} + \Omega_{xt}\right\} = \frac{1}{2}\left\{K^2\left[(\log R)_x(\log R)_x + \Omega_x\Omega_x\right] + \mathcal{F}\right\}^{-\frac{1}{2}}$$

$$\cdot\left\{2\rho_0^2K^2\left[(\log R)_x(\log R)_{xt} + \Omega_x\Omega_{xt}\right] + \mathcal{F}_t\right\} \qquad 2.40$$

2.5 Equilibrium ideal gas dynamics

Suppose that the total energy of an ensemble set of material elements is conserved following the flow. That is,

$$0=\frac{d}{dt}\int_{R_0}\mathcal{H}\,d\underline{\alpha}=\int_{R_0(\underline{\alpha})}\frac{d\mathcal{H}}{dt}\,d\underline{\alpha}=\int_{R_0(\underline{\alpha})}\left\{\frac{\partial\mathcal{H}}{\partial t}+\frac{\partial\mathcal{H}}{\partial x_i}\frac{\partial x_i}{\partial t}+\frac{\partial\mathcal{H}}{\partial\frac{\partial x_i}{\partial\alpha_j}}\frac{\partial}{\partial t}\frac{\partial x_i}{\partial\alpha_j}+\frac{\partial\mathcal{H}}{\partial\pi_i}\frac{\partial\pi_i}{\partial t}\right\}d\underline{\alpha}$$

where $\mathcal{H}$ is the Hamiltonian density, and $R_0(\underline{\alpha})$, an arbitrary domain. Integrating the third term by parts and using Hamilton's equations,

$$\frac{\partial x_i}{\partial t}=\frac{\partial\mathcal{H}}{\partial\pi_i}\qquad,\qquad\frac{\partial\pi_i}{\partial t}=\frac{\partial}{\partial\alpha_j}\frac{\partial\mathcal{H}}{\partial\frac{\partial x_i}{\partial\alpha_j}}-\frac{\partial\mathcal{H}}{\partial x_i}\tag{2.15}$$

this becomes,

$$0=\int_{R_0(\underline{\alpha})}\frac{\partial\mathcal{H}}{\partial t}\,d\underline{\alpha}$$

Let the Hamiltonian density be $\mathcal{H}=\frac{1}{2}\rho_0\frac{\partial\underline{x}}{\partial t}\cdot\frac{\partial\underline{x}}{\partial t}+\rho_0 E$ as in Equation 2.7. Using the continuity relation, $\rho_0=\rho J$, of Equation 2.2 it follows that

$$\epsilon=\int_{R_0}\rho_0\left\{\frac{1}{2}\frac{\partial\underline{x}}{\partial t}\cdot\frac{\partial\underline{x}}{\partial t}+E\right\}d\underline{\alpha}=\int_{R_0(\underline{x})}\rho\left\{\frac{1}{2}\frac{\partial\underline{x}}{\partial t}\cdot\frac{\partial\underline{x}}{\partial t}+E\right\}d\underline{x}$$

where ϵ is time independent. When variables are transformed via Equations 2.21 and 2.22 this becomes,

$$\epsilon(\psi,\psi^*)=\int_{R_0(\underline{x})}\psi\psi^*\left\{\frac{K^2}{2}\nabla\log\psi\cdot\nabla\log\psi^*+\frac{\mathcal{F}}{2}+\frac{c(\psi\psi^*)^{\gamma-1}}{\gamma-1}\right\}d\underline{x}\tag{2.41}$$

where $E(\psi,\psi^*) = \frac{\mathcal{C}(\psi\psi^*)^{\gamma-1}}{\gamma-1}$ and $\mathcal{C} = P_0/\rho_0^\gamma$ from Equation 2.6. There is no need for the constraint, $K^2\nabla\Omega.\nabla \log R = \mathcal{F}/2$ or the relation $\mathcal{A}_t = \rho_0 K\left\{(\log R)_t + \Omega_t\right\}$. The function $\mathcal{F}$ is arbitrary.

A Lagrange multiplier λ_1 , is introduced to cope with the continuity constraint, $\rho_0 = \rho J$. The system is postulated to rest in the state for which ϵ is extremal, presumably minimal, with respect to variations in ψ . Equation 2.41 leads to,

$$0 = \delta\int_{R_0(\underline{x})}\left\{\frac{K^2}{2}\nabla\psi.\nabla\psi^* + \frac{\mathcal{C}(\psi\psi^*)^\gamma}{\gamma-1} + \lambda_1\psi\psi^* + \frac{\mathcal{F}}{2}\psi\psi^*\right\}d\underline{x} - \delta\int_{R_0(\underline{\alpha})}\lambda_1\rho_0\, d\underline{\alpha} \qquad 2.42$$

The conditions surrounding variations in ψ are those assumed in Sections 2.3 and 2.4. The resultant set of equations is,

$$\frac{K^2}{2}\nabla^2\psi = \frac{\mathcal{C}\gamma(\psi\psi^*)^{\gamma-1}\psi}{\gamma-1} + \Phi_x\psi \qquad 2.43$$

$$\Phi_{xx} - \Phi_{yy} = (\psi\psi^*)_x \qquad 2.44$$

Notice the similarity between this last set of equations and the corresponding set in the time dependent problem,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = \frac{\mathcal{C}\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi + \Phi_x\psi \qquad 2.35$$

$$\Phi_{xx} - \Phi_{yy} = (\psi\psi^*)_x \qquad 2.36$$

3. Shallow Water Waves

3.1 Why this problem is of interest

Coherently structured waveforms are most readily observed among surface waves on deep water. It does not seem possible to analytically handle that system at this time. Chapter 2 considered an isentropic process in an ideal gas. The internal energy depended on density to a fractional power. Mapping of dependent variables eventually led to the set of governing equations,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi + \Phi_x\psi \qquad 2.35$$

$$\Phi_{xx} - \Phi_{yy} = (\psi\psi^*)_x \qquad 2.36$$

These are qualitatively similar to the Davey-Stewartson equations,

$$2i\Psi_\tau - \Psi_{\xi\xi} + \Psi_{\eta\eta} = \frac{9}{2}\Psi|\Psi|^2 + 3\Psi\Phi_\xi \qquad 1.6$$

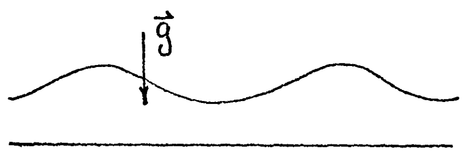
$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = -3(\Psi\Psi^*)_\xi \qquad 1.7$$

In contrast to this latter set of equations, Equations 2.35 and 2.36 seem neither to be solvable nor to be consistent with coherently structured waveforms in any obvious way.

This discussion considers shallow water waves. The internal energy does not depend on a density equivalent variable to a fractional power. Equations for mapped dependent variables are eventually matched to Equations 1.6 and 1.7 and solved by the inverse scattering transform.

3.2 Classical formulation

Given a material element with Lagrangian coordinates $(\alpha_1, \alpha_2, \alpha_3)$ at time t = 0, and position $\underline{x}(\underline{\alpha}, t)$ at time t, in an inertial frame, the governing equations for shallow water waves are,



$$\frac{\partial^2 x_i}{\partial t^2} = -g \frac{\partial h}{\partial x_i} \qquad i = 1,2 \tag{3.1}$$

$$\frac{1}{h} = \frac{1}{h_0} \frac{\partial(x_1, x_2)}{\partial(\alpha_1, \alpha_2)} = \frac{J}{h_0} \tag{3.2}$$

for momentum and mass respectively. The acceleration due to gravity is g, the free surface height h, where initially $h(\underline{x}, t=0) = h_0$ and the constant density of the system ρ. The flow is two-dimensional. The vorticity equation in Eulerian coordinates is,

$$\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} + \underline{\omega} \nabla \cdot \underline{u} = 0$$

implying that if $\underline{\omega} = 0$ initially, then it remains so. Equation 3.1 may be rewritten so that partial derivatives with respect to dependent variables x_i are expressed in terms of independent variables $(\underline{\alpha}, t)$,

$$\frac{\partial^2 x_i}{\partial t^2} = -g \frac{\partial \alpha_j}{\partial x_i} \frac{\partial h}{\partial \alpha_j} = -g \frac{h}{h_0} J_{ij} \frac{\partial h}{\partial \alpha_j} \quad , \quad J_{ij} = J \frac{\partial \alpha_j}{\partial x_i} = \frac{\partial J}{\partial \frac{\partial x_i}{\partial \alpha_j}} \tag{3.3}$$

The variational principle, Hamilton's principle for Equation 3.1 is,

$$0 = \delta \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} h_0 \left\{ \frac{1}{2} \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - gh \right\} d\underline{\alpha}\, dt \qquad 3.4$$

Domain R_0 is arbitrary, as are variations in material element trajectories $\delta \chi_i$ except for the condition that they vanish on the boundary ∂_{R_0}. The continuity constraint is accounted for by introducing a Lagrange multiplier, λ. Equation 3.4 becomes,

$$0 = \delta \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} \left\{ h_0 \left[\frac{1}{2} \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - gh \right] + \lambda (h_0 - J h) \right\} d\underline{\alpha}\, dt \qquad 3.5$$

Varying χ_i, the resultant Euler-Lagrange equation is

$$-h_0 \frac{\partial^2 \chi_i}{\partial t^2} - g h_0 h_{x_i} - (\lambda h)_{x_i} J + \frac{\partial}{\partial \alpha_j} (\lambda h J_{ij}) = 0 \qquad 3.6$$

where integration by parts has been used appropriately. Recall that $\frac{\partial}{\partial \alpha_j} J_{ij} = 0$ from the discussion following Equation 2.9. This implies that Equation 3.6 is the same as Equation 3.1.

For each material element the Lagrangian is taken as $\mathcal{L} = h_0 \left\{ \frac{1}{2} \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - gh \right\} + \lambda (h_0 - h J)$. That is, the system is represented as having Lagrangian density,

$\mathcal{L} = h_0 \left\{ \frac{1}{2} \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - gh \right\} + \lambda (h_0 - h J)$ canonical momentum,

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \frac{\partial x_i}{\partial t}} = h_o \frac{\partial x_i}{\partial t} \qquad 3.7$$

and Hamiltonian density,

$$\mathcal{H} = \pi_i \frac{\partial x_i}{\partial t} - \mathcal{L} = h_o \left\{ \frac{1}{2} \frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} + gh \right\} - \lambda (h_o - hJ) \qquad 3.8$$

The total energy density for the system is $\mathcal{H}$, given the trajectories implicit to Equation 3.5.

3.3 Mapping of dependent variables

The transformation from dependent variables h and $\frac{\partial \underline{x}}{\partial t}$ to new dependent variables ψ and ψ^* is analogous to the mapping of Equations 2.21 and 2.22 in the ideal gas example. Let

$$h = \psi \psi^* \qquad 3.9$$

$$\frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} = \left\{ K \frac{J_{\ell m}}{J} \frac{\partial}{\partial \alpha_m} \log \psi \right\} \left\{ K \frac{J_{\ell n}}{J} \frac{\partial}{\partial \alpha_n} \log \psi \right\}^* + \mathcal{F}(\underline{\alpha}, t)$$

$$= K^2 \nabla_{\underline{x}} \log \psi \cdot \nabla_{\underline{x}} \log \psi^* + \mathcal{F}(\underline{x}, t) \qquad 3.10$$

where K is a real constant and $\mathcal{F}(\underline{x}, t)$ is a known function. Recall that $\pi_i = h_o \frac{\partial x_i}{\partial t}$ from Equation 3.7 and $\pi_i = \frac{\partial \mathcal{A}}{\partial x_i}$ from Equation 2.17. Equation 3.10 may be rewritten as

$$\nabla_x s . \nabla_x s = K^2 h_o^2 \nabla_x \log \psi . \nabla_x \log \psi^* + h_o^2 \mathcal{F} \qquad 3.11$$

Let $\psi = R(x,t) e^{i\Omega(x,t)}$. Therefore $-2\Omega = i \log \psi - i \log \psi^*$ and Equation 3.9 implies $h = \psi\psi^* = R^2$. The condition $K^2 \nabla\Omega . \nabla \log R = \mathcal{F}/2$ is imposed. Sufficient conditions for Equation 3.11 are

$$\nabla s = h_o K \left\{ \nabla \log R + \nabla \Omega \right\}$$

or equivalently,

$$s = h_o \frac{K}{2} \left\{ (1-i) \log \psi + (1+i) \log \psi^* \right\} \qquad 3.12$$

where h_o is constant. This specifies initial conditions on R. Initial conditions on Ω are unspecified.

Let the Hamiltonian density,

$$H = h_o \left\{ \frac{1}{2} \frac{\partial x}{\partial t} . \frac{\partial x}{\partial t} + gh \right\} - \lambda (h_o - h J) \qquad 3.8$$

be substituted into

$$0 = \delta \int_{t_1}^{t_2} \int_{R_o(\alpha)} \left\{ \frac{\partial s}{\partial t} + H \right\} d\alpha \, dt \qquad 2.20$$

This implies,

$$0 = \delta \int_{t_1}^{t_2} \int_{R_o(\underline{\alpha})} \left\{ \frac{\partial s}{\partial t} + h_o \left[\frac{1}{2} \frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} + gh \right] - \lambda(h_o - hJ) \right\} d\underline{\alpha}\, dt$$

Transform dependent variables via Equations 3.9, 3.10 and 3.12. The system becomes represented as

$$0 = \delta \int_{t_1}^{t_2} \int_{R_o(\underline{\alpha})} \Big\{ h_o \frac{K}{2} (1-i) (\log \psi)_t + h_o \frac{K}{2} (1+i) (\log \psi^*)_t$$

$$+ h_o \frac{K^2}{2} \nabla_{\underline{x}} \log \psi \cdot \nabla_{\underline{x}} \log \psi^* + \frac{h_o}{2} \mathcal{F} + g \psi \psi^* h_o - \lambda(h_o - \psi\psi^* J) \Big\} d\underline{\alpha}\, dt \tag{3.13}$$

Let coordinates be transformed from $\underline{\alpha}$ -space to $\underline{x}$ -space. Equation 3.13 becomes,

$$0 = \delta \int_{t_1}^{t_2} \int_{R_o(\underline{x})} \Big\{ \frac{K}{2}(1-i) \psi^* \psi_t + \frac{K}{2}(1+i) \psi \psi_t^* + \frac{K^2}{2} \nabla\psi \cdot \nabla\psi^* + (\lambda + \frac{\mathcal{F}}{2}) \psi\psi^*$$

$$+ g \psi^2 \psi^{*2} \Big\} d\underline{x}\, dt \quad - \delta \int_{t_1}^{t_2} \int_{R_o(\underline{\alpha})} \lambda h_o \, d\underline{\alpha}\, dt \tag{3.14}$$

Variations with respect to ψ imply

$$0 = \int_{t_1}^{t_2} \int_{R_o(\underline{x})} \left\{ -iK\psi_t - \frac{K^2}{2} \nabla^2 \psi + \frac{\delta}{\delta \psi^*} \frac{\mathcal{F}\psi\psi^*}{2} + 2g \psi^2 \psi^* + \lambda \psi \right\} \delta\psi^* d\underline{x}\, dt$$

$$+ \int_{t_1}^{t_2} \int_{R_o(\underline{x})} \left\{ +iK\psi_t^* - \frac{K^2}{2} \nabla^2 \psi^* + \frac{\delta}{\delta \psi} \frac{\mathcal{F}\psi\psi^*}{2} + 2g \psi^{*2} \psi + \lambda \psi^* \right\} \delta\psi \, d\underline{x}\, dt \tag{3.15}$$

which leads to,

$$-i\kappa\psi_t - \frac{\kappa^2}{2}\nabla^2\psi + \frac{\delta}{\delta\psi^*}\frac{\mathcal{F}\psi\psi^*}{2} + 2g\,\psi^2\psi^* + \lambda\psi = 0$$

as $\delta\psi$ is arbitrary. In obtaining Equation 3.15 integration by parts has been used where appropriate.

A solvable resultant set of equations is sought. Suppose that no variations in ψ leading to contributions from $\mathcal{F}\psi\psi^*$ are allowed. These latter terms appeared as $\delta\psi^*\frac{\delta}{\delta\psi^*}\frac{\mathcal{F}\psi\psi^*}{2} + \delta\psi\frac{\delta}{\delta\psi}\frac{\mathcal{F}\psi\psi^*}{2}$ in Equation 3.15. Let $\mathcal{O} = -i\kappa\psi_t - \frac{\kappa^2}{2}\nabla^2\psi + 2g\psi^2\psi^* + \lambda\psi$, $\psi = \chi + i\mathcal{Y}$ and $\delta\psi = \delta\chi + i\,\delta\mathcal{Y}$. Equation 3.15 implies

$$0 = \int_{t_1}^{t_2}\int_{R_o(\underline{x})} \left\{ \mathcal{O}_R\,\delta\chi + \mathcal{O}_I\,\delta\mathcal{Y} \right\} d\underline{x}\,dt \qquad 3.16$$

where $\mathcal{O} = \mathcal{O}_R + i\,\mathcal{O}_I$. As in the ideal gas dynamics example, the condition that $\mathcal{F}\psi\psi^* = \kappa^2\left\{\nabla \arctan \mathcal{Y}/\chi\right\}\cdot\nabla(\chi^2 + \mathcal{Y}^2)$ be invariant to allowed variations in ψ is linearly dependent in variables $\delta\chi$ and $\delta\mathcal{Y}$ to the integrand of Equation 3.16,

$$\mathcal{O}_R\,\delta\chi + \mathcal{O}_I\,\delta\mathcal{Y} = 0 \qquad 3.17$$

which leads to $\mathcal{O} = 0$

$$-i\kappa\psi_t - \frac{\kappa^2}{2}\nabla^2\psi + 2g\,\psi^2\psi^* + \lambda\psi = 0 \qquad 3.18$$

The variation of Equation 3.14 yields no contribution from the last term,

$$\delta \int_{t_1}^{t_2} \int_{R_0(\underline{\alpha})} \lambda h_0 \, d\underline{\alpha} \, dt = \delta \int_{t_1}^{t_2} \int_{R_0(\underline{x})} \frac{\lambda(\underline{x}) h_0}{J} \, d\underline{x} \, dt$$

This implies $J = J(\lambda)$. Equations 3.9 and 3.10 define the mapping from old dependent variables to new dependent variables. No constraint is placed on $\frac{\partial x_i}{\partial \alpha_j}$ except via Equation 3.2,

$$\frac{h_0}{\frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_2}{\partial \alpha_2} - \frac{\partial x_1}{\partial \alpha_2}\frac{\partial x_2}{\partial \alpha_1}} = \psi\psi^* \qquad 3.2$$

Let $\frac{\partial x_i}{\partial \alpha_j}$ be chosen so that,

$$\lambda = P \Phi_y \quad , \quad \psi\psi^* = Q\left\{ \Phi_y - \int^x \Phi_{xx} \, dy \right\} \qquad 3.19$$

where P and Q are real constants.

Assuming that this is valid, the resultant set of equations is,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = 2g\psi^2\psi^* + P\Phi_y\psi \qquad 3.20$$

$$\Phi_{yy} - \Phi_{xx} = \frac{(\psi\psi^*)_y}{Q} \qquad 3.21$$

Variables are scaled via

$$\psi = \left(\frac{9}{4g}\right)^{1/3}\psi' \quad , \quad (x,y) = \frac{K}{\sqrt{2}}\left(\frac{9}{4g}\right)^{1/6}(x',y') \quad , \quad t = \frac{K}{2}\left(\frac{9}{4g}\right)^{1/3} t'$$

$$P = -\frac{2\sqrt{2g}}{K} \quad , \quad Q = \frac{1}{K\sqrt{2g}} \quad , \quad \Phi = \frac{K^2}{2}\left(\frac{9}{4g}\right)^{1/3}\Phi' \qquad 3.22$$

which leads to,

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2}\psi^2\psi^* - 3\Phi_y\psi \qquad 3.23$$

$$-\Phi_{yy} + \Phi_{xx} = -3(\psi\psi^*)_y \qquad 3.24$$

in place of Equations 3.20 and 3.21. Primes have been dropped.

Had the discussion begun at Equation 3.4

$$0 = \delta\int_{t_1}^{t_2}\int_{R_o(\underline{\alpha})} h_o\left\{\frac{1}{2}\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} - gh\right\} d\underline{\alpha}\,dt \qquad 3.4$$

rather than Equation 3.5,

$$0 = \delta\int_{t_1}^{t_2}\int_{R_o(\underline{\alpha})} \left\{\frac{1}{2}h_o\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} - h_o g h + \lambda(h_o - hJ)\right\} d\underline{\alpha}\,dt \qquad 3.5$$

the resultant Euler-Lagrange equation would be

$$-iK\psi_t - \frac{K^2}{2}\nabla^2\psi + 2g\psi^2\psi^* = 0 \qquad 3.25$$

in place of Equation 3.18. This is a two-dimensional generalization of the one-dimensional nonlinear Schrödinger equation. Ablowitz (1978)[17] states that no method of solution is known. The corresponding one-dimensional case is

$$i\kappa\psi_t + \frac{\kappa^2}{2}\psi_{xx} - 2g\psi^2\psi^* = 0$$

to which no soliton solutions are known

The mapping from dependent variables h and $\frac{\partial \underline{x}}{\partial t}$ to new dependent variables ψ and ψ^* was introduced via,

$$h = \psi\psi^* \qquad 3.9$$

$$\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} = \kappa^2 \nabla_{\underline{x}} \log\psi \cdot \nabla_{\underline{x}} \log\psi^* + \mathcal{F}(\underline{x},t) \qquad 3.10$$

$$\nabla_{\underline{x}} s \cdot \nabla_{\underline{x}} s = \kappa^2 h_0^2 \nabla_{\underline{x}} \log\psi \cdot \nabla_{\underline{x}} \log\psi^* + h_0^2 \mathcal{F}(\underline{x},t) \qquad 3.11$$

The substitutions $\psi = R(\underline{x},t)\, e^{i\Omega(\underline{x},t)}$ and $\mathcal{F}(\underline{x},t) = 2\kappa^2 \nabla\Omega \cdot \nabla \log R$ are made. Allowed variations in ψ are such that $\mathcal{F}\psi\psi^*$ does not vary. Sufficient conditions for Equations 3.9 to 3.11 are written as,

$$\nabla s = \kappa h_0 \nabla \left\{ \log R + \Omega \right\} \quad , \quad s = \kappa h_0 \left\{ \log R + \Omega \right\} \qquad 3.12$$

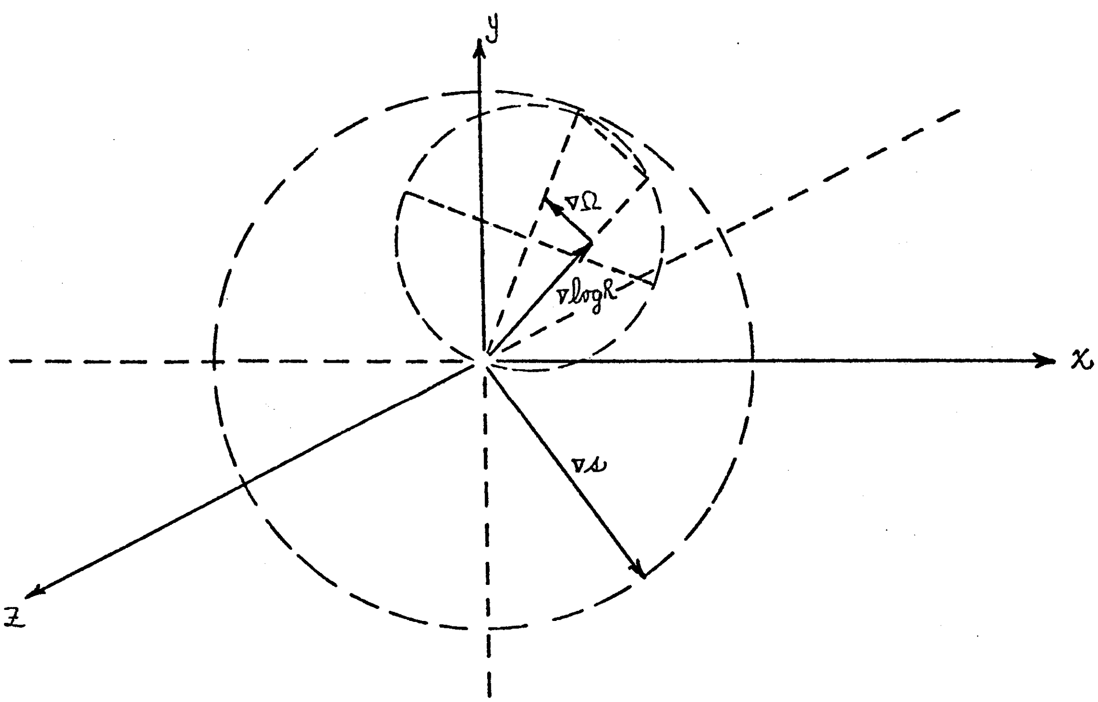
Conditions surrounding these relations are now considered.

3.4 Justification of the mapping

Necessary conditions for Equation 3.11 are,

$$\nabla s = r(x,t)\, \kappa h_0 \left\{ \nabla \log R + \nabla \Omega \right\} \qquad 3.26$$

where $r(x,t)$ is a tensor operator of rotation. The state of the system determines h , $R = \sqrt{h}$, ∇s , and, as will be seen, $\mathcal{F}$. The condition $\kappa^2 \nabla\Omega . \nabla \log R = \mathcal{F}/2$ and $\nabla_x s . \nabla_x s = \kappa^2 h_0^2 \left\{ \nabla \log R . \nabla \log R + \nabla\Omega . \nabla\Omega + \frac{\mathcal{F}}{\kappa^2} \right\}$, Equation 3.11, determine Ω . Equation 3.26 defines the operator, $r(x,t)$. When $\mathcal{F} = 0$, the general status of the system is envisaged as,



The evolution of the system, as specified by Equations 3.2 and 3.3, dictates the evolution of $r(\underline{x},t)$ in time.

The Hamilton-Jacobi equation,

$$0 = \frac{\partial s}{\partial t} + \mathcal{H} \qquad 2.19$$

the definition of the Hamiltonian density,

$$\mathcal{H} = h_0 \left\{ \frac{1}{2} \frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} + gh \right\} - \lambda (h_0 - h J) \qquad 3.8$$

and the relation between canonical momentum and action density, via Equation 3.7,

$$\pi_i = h_0 \frac{\partial x_i}{\partial t} = \frac{\partial s}{\partial x_i} \qquad 2.17$$

imply

$$0 = \frac{\partial s}{\partial t} + \frac{1}{2h_0} \nabla s \cdot \nabla s + g h h_0 - \lambda (h_0 - h J)$$

Therefore, s_t is $r(\underline{x},t)$ – independent. Given an $r(\underline{x},t)$ operator of arbitrary form the evolution of s in time is the same as if $r(\underline{x},t) = 1$. It is consistent to write,

$$s_t = \kappa h_0 \left\{ (\log R)_t + \Omega_t \right\} \qquad 3.27$$

Dependent variables h and $\frac{\partial \underline{x}}{\partial t}$ have been mapped to new dependent variables ψ and ψ^* yielding Equation 3.13. Coordinates are transformed to $\underline{\chi}$ -space from $\underline{\alpha}$ -space. The resultant is assumed extremal with respect to variations in ψ .

The system was defined in terms of Lagrangian coordinates initially. A variation in a material element trajectory, $\underline{\chi}(\underline{\alpha},t)$ corresponds to a variation in height of the free surface, $h = \psi\psi^*$, and speed $\frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} = K^2 \nabla_{\chi} \log\psi \cdot \nabla_{\chi} \log\psi^* + \mathcal{F}(\underline{\chi},t)$. Given that Equation 3.5 is extremal with respect to variations in $\underline{\chi}(\underline{\alpha},t)$, Equation 3.14 should be extremal with respect to variations in ψ .

In Chapter 1, the passageway leading from quantum mechanics to classical mechanics was described. It is of interest to consider the analogous transition regarding Equation 3.18. Let $\psi = a e^{iA/Kh_0}$ where A is the action density and a is real. Equation 3.18 implies,

$$iKa_t - \frac{a}{h_0} A_t + \frac{K^2}{2}\nabla^2 a + iK\nabla a \cdot \frac{\nabla A}{h_0} + \frac{i}{2} aK \frac{\nabla^2 A}{h_0} - \frac{a}{2}\frac{\nabla A}{h_0}\cdot\frac{\nabla A}{h_0} = 2ga^3 + a\lambda$$

Real and imaginary terms may be equated,

$$\frac{A_t}{h_0} - \frac{K^2}{2}\frac{\nabla^2 a}{a} + \frac{\nabla A \cdot \nabla A}{2h_0^2} + 2ga^2 + \lambda = 0$$

$$a_t + \nabla a \cdot \frac{\nabla A}{h_0} + \frac{a}{2}\frac{\nabla^2 A}{h_0} = 0$$

The limit $K \to 0$ implies, of the former,

$$A_t + \frac{\nabla A \cdot \nabla A}{2h_0} + 2ga^2 h_0 + \lambda h_0 = 0 \tag{3.28}$$

which is a classical Hamilton-Jacobi relation for a fluid with action density s , and time $t=0$ height h_0 . The latter may be multiplied by $2a$ and written as,

$$(a^2)_t + \nabla . \frac{a^2 \nabla s}{h_0} = 0 \tag{3.29}$$

The classical velocity is $\frac{\nabla s}{h_0}$ throughout the system. The free surface height is $h = \psi\psi^* = a^2$ at any point in space. Equation 3.29 attaches a classical velocity $\frac{\nabla s}{h_0}$ to the free surface height at each point in space and compels it to obey a classical continuity relation.

Suppose that ψ is normalized, $1 = \int_{R_0} \psi\psi^* d\underline{x}$ and the coordinates $\underline{a}$ at time $t=0$ for some material element are unknown. It is not inconsistent to regard ψ as the probability amplitude that the material element occupies coordinates $(\underline{x},t)$.

Notice that the result of multiplying Equation 3.18 by ψ^* minus the complex conjugate of Equation 3.18 multiplied by ψ is,

$$i\kappa(\psi\psi^*)_t + \frac{\kappa^2}{2}\left\{ \psi^*\nabla^2\psi - \psi\nabla^2\psi^* \right\} = 0$$

The identity $h = \psi\psi^*$ implies,

$$h_t + \frac{\kappa}{2}\nabla . i\left\{ \psi\nabla\psi^* - \psi^*\nabla\psi \right\} = 0$$

This is the continuity relation in Eulerian variables when the substitution,

$$h\underline{v} = \frac{K}{2} i \left\{ \psi \nabla \psi^* - \psi^* \nabla \psi \right\} \tag{3.30}$$

or equivalently,

$$\underline{v} = K \nabla \Omega \tag{3.31}$$

is made. The Eulerian velocity field is $\underline{v}$. The analogous situation applies to

$$iK\psi_t + \frac{K^2}{2} \nabla^2 \psi = \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1} \psi + \lambda\psi \tag{2.32}$$

concerning the ideal gas dynamics example, and to Schrödinger's linear time dependent equation,

$$iK\Psi_t = -\frac{K^2}{2m} \nabla^2 \Psi + U\Psi \tag{1.2}$$

in quantum mechanics. See Merzbacher (1970)[20], page 37.

Equation 3.31 implies that $\underline{v}$ is irrotational regardless of the introduction of λ . The condition $K^2 \nabla \Omega . \nabla \log R = \frac{1}{2} \mathcal{F}$ implies,

$$\mathcal{F}\psi\psi^* = K \underline{v} . \nabla h \tag{3.32}$$

in the shallow water waves example, and

$$\mathcal{F}\psi\psi^{*} = K\underline{v}.\nabla\rho \qquad 3.33$$

in the ideal gas dynamics example. Allowed variations in ψ in Equations 2.30 and 3.16 are such that $\mathcal{F}\psi\psi^{*}$ does not vary.

3.5 Equilibrium shallow water waves

The total energy of an arbritary number of material elements is assumed constant following the flow. The Hamiltonian density corresponding to Equation 3.4,

$$H = h_0\left\{ \frac{1}{2}\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} + gh\right\}$$

and the continuity relation, $h_0 = hJ$ imply,

$$\epsilon = \int_{R_0} h_0\left\{ \frac{1}{2}\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} + gh\right\} d\underline{\alpha} = \int_{R_0(\underline{x})} h\left\{ \frac{1}{2}\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} + gh\right\} d\underline{x} \qquad 3.34$$

where ϵ is time independent. Variables are transformed by

$$h = \psi\psi^{*} \qquad 3.9$$

$$\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} = K^2\,\nabla_{\underline{x}}\log\psi\,.\,\nabla_{\underline{x}}\log\psi^{*} + \mathcal{F}(\underline{x},t) \qquad 3.10$$

Equation 3.34 becomes

$$\epsilon(\psi,\psi^*) = \int_{R_o(x)} \psi\psi^* \left\{ \frac{K^2}{2} \nabla \log\psi . \nabla \log\psi^* + \frac{\mathcal{F}}{2} + g\psi\psi^* \right\} d\underline{x}$$

Equation 3.12, $\nabla s = h_o K \left\{ \nabla \log R + \nabla\Omega \right\}$ Equation 3.27,

$s_t = K h_o \left\{ (\log R)_t + \Omega_t \right\}$ and the condition $K^2 \nabla\Omega . \nabla \log R = \mathcal{F}/2$

do not enter the consideration.

The system is postulated to rest in the state for which ϵ is extremal, presumably minimal, with respect to variations in ψ. A Lagrange multiplier, λ_1, is introduced to cope with the continuity relation, $h_o = hJ$. Thus,

$$0 = \delta \int_{R_o(x)} \left\{ \frac{K^2}{2} \nabla\psi . \nabla\psi^* + g\psi^2\psi^{*2} + \lambda_1 \psi\psi^* + \frac{\mathcal{F}}{2}\psi\psi^* \right\} d\underline{x} - \delta \int \lambda_1 h_o \, d\underline{\alpha} \tag{3.35}$$

where conditions surrounding allowed variations in ψ are those assumed in sections 3.3 and 3.4. The resultant set of equations is,

$$\nabla^2\psi = \frac{9}{2}\psi^2\psi^* g - 3\Phi_y\psi \tag{3.36}$$

$$-\Phi_{yy} + \Phi_{xx} = -3(\psi\psi^*)_y \tag{3.37}$$

which differ from the corresponding relations of the time dependent problem in a single term,

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2}\psi^2\psi^* - 3\Phi_y\psi \tag{3.23}$$

$$-\Phi_{yy} + \Phi_{xx} = -3(\psi\psi^*)_y \qquad 3.24$$

The one-dimensional nonlinear Schrödinger equation,

$$-i\psi_t = \frac{1}{2}\psi_{yy} + \frac{9}{4}\psi^2\psi^* \qquad 3.38$$

was reported to have solutions of the form

$$\psi = A(y)e^{i\omega t}$$

by Hasimoto and Ono (1972)[3]. These are the so-called equilibrium solutions, balancing the effects of nonlinearity and dispersion. They found that $A(y) = A_0 \, dn\left\{A_0 \frac{3}{2} y, s\right\}$ is a Jacobian elliptical function and that $A(y) = \frac{\sqrt{8\omega}}{3} \operatorname{sech}(\sqrt{2\omega}\, y)$ when $s = 1$. The solution is a solitary modulational wave.

4. The Presence of Solitons

4.1 Preliminary transformation

Shallow water waves were considered in Chapter 3. A mapping of dependent variables was made which in due time led from the usual governing Equations 3.2 and 3.3 to an alternate set of equations,

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2}\psi^2\psi^* - 3\Phi_y\psi \qquad 3.23$$

$$\Phi_{xx} - \Phi_{yy} = -3(\psi\psi^*)_y \qquad 3.24$$

which are noticeably similar to the Davey-Stewartson equations,

$$2i\mathcal{S}_\tau - \mathcal{S}_{\xi\xi} + \mathcal{S}_{\eta\eta} = \frac{9}{2}\mathcal{S}|\mathcal{S}|^2 + 3\mathcal{S}\Phi_\xi \qquad 1.6$$

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = -3|\mathcal{S}|^2_\xi \qquad 1.7$$

Anker and Freeman (1978)[19] make a transformation on this latter set of equations, consistent with a search for Stokes-type waves,

$$\mathcal{S} = A_0\mathcal{S}_1 e^{ip\tau} \quad , \quad \underline{Q} = Q_0 + Q_1 \quad , \quad \Phi_\xi = \underline{Q} - 3(\mathcal{S}\mathcal{S}^*)$$

where p, A_0 and Q_0 are constants. Using the inverse scattering transform on the outcome,

$$2i\,\mathcal{Y}_{1,\tau} - \mathcal{Y}_{1,\xi\xi} + \mathcal{Y}_{1,\eta\eta} = -\frac{9}{2}A_0^2\,\mathcal{Y}_1\left\{|\mathcal{Y}_1|^2 - 1\right\} + 3\mathcal{Y}_1 Q_1$$

$$Q_{1,\xi\xi} + Q_{1,\eta\eta} = 3A_0^2\,|\mathcal{Y}_1|^2{}_{\eta\eta}$$

they deduce solition solutions.

A transformation of the same sort

$$\psi = A\varphi e^{imt} \quad , \quad q_1 = q_0 + q \quad , \quad \Phi_y = q_1 + 3|\psi|^2 \qquad 4.1$$

where A, m and q_0 are constants, is made on Equations 3.23 and 3.24. The result is

$$2i\varphi_t + \varphi_{xx} + \varphi_{yy} = -\frac{9}{2}A^2\varphi\left\{\varphi\varphi^* - 1\right\} - 3\varphi q \qquad 4.2$$

$$-q_{yy} + q_{xx} = -3A^2|\varphi|^2_{xx} \qquad 4.3$$

upon which, the inverse scattering transform is applied. Solitary waves with sinusoidal spanwise modulations are obtained.

4.2 The inverse scattering transform

Most of this section and the next repeat relevant parts of the papers of Zakharov and Shabat (1974)[21] and Anker and Freeman[19]. The essential concern is the slight modifications necessary to apply the inverse scattering transform to Equations 4.2 and 4.3.

Suppose that $\hat{F}$ is a linear integral operator acting on a vector valued function $\psi(z, w, t)$, where $\psi = \{\psi_1, \psi_2, \cdots \psi_N\}$. That is,

$$\hat{F}\psi = \int_{-\infty}^{\infty} F(z,\eta)\,\psi(\eta)\,d\eta \qquad 4.4$$

where $F(z, \eta, w, t)$ is a $N \times N$ matrix. Let $\hat{F}$ be represented as

$$1 + \hat{F} = (1 + \hat{K}_+)^{-1}(1 + \hat{K}_-) \qquad 4.5$$

where $\hat{K}_+$ and $\hat{K}_-$ are Volterra operators in the sense,

$$\hat{K}_+\psi = \int_{z}^{\infty} K_+(z,\eta)\,\psi(\eta)\,d\eta \quad , \quad \hat{K}_-\psi = \int_{-\infty}^{z} K_-(z,\eta)\,\psi(\eta)\,d\eta$$

The contours of integration of these integrals are partitions of the contour of integration used in Equation 4.4. The alternate form for Equation 4.5 is,

$$1 + \hat{K}_- = 1 + \hat{K}_+ + \hat{F} + \hat{K}_+\hat{F}$$

and therefore for any $\psi = \psi(z, w, t)$,

$$(1 + \hat{K}_-)\psi = \psi(z) + \int_{-\infty}^{z} d\eta\, K_-(z,\eta)\,\psi(\eta) = (1 + \hat{K}_+ + \hat{F} + \hat{K}_+\hat{F})\psi$$

This may be expressed as,

$$\psi(z) + \int_{-\infty}^{z} d\eta\, K_-(z,\eta)\psi(\eta) = \psi(z) + \int_{z}^{\infty} d\eta\, K_+(z,\eta)\psi(\eta) + \int_{-\infty}^{\infty} d\eta\, F(z,\eta)\psi(\eta)$$
$$+ \int_{z}^{\infty} d\eta\, K_+(z,\eta) \int_{-\infty}^{\infty} ds\, F(\eta,s)\psi(s)$$

The $\int_{-\infty}^{\infty} d\eta$ and $\int_{-\infty}^{\infty} ds$ integrals may be broken, to imply

$$\int_{-\infty}^{z} K_-(z,\eta)\psi(\eta)d\eta = \int_{z}^{\infty} d\eta\, K_+(z,\eta)\psi(\eta) + \int_{-\infty}^{z} d\eta\, F(z,\eta)\psi(\eta) + \int_{z}^{\infty} d\eta\, F(z,\eta)\psi(\eta)$$
$$+ \int_{z}^{\infty} d\eta\, K_+(z,\eta)\int_{-\infty}^{z} ds\, F(\eta,s)\psi(s) + \int_{z}^{\infty} d\eta\, K_+(z,\eta)\int_{z}^{\infty} ds\, F(\eta,s)\psi(s)$$

Because ψ is arbitrary and may be specified as nonzero in only a single neighbourhood about some point within the contour of integration, for example $(\eta_0 - \epsilon_0, \eta_0 + \epsilon_0)$, it follows that

$$K_-(z,\eta) = F(z,\eta) + \int_{z}^{\infty} ds\, K_+(z,s)\, F(s,\eta) \tag{4.6}$$

$$0 = K_+(z,\eta) + F(z,\eta) + \int_{z}^{\infty} ds\, K_+(z,s)\, F(s,\eta) \tag{4.7}$$

The latter of these is the Gelfand-Levitan equation.

Zakharov and Shabat (1974)[21] state that a sufficient condition for this discussion when the independent variables are real is

$$\sup_{x > x_0} \int_{x_0}^{\infty} |F(x,z)|\, dz < \infty \qquad , \quad x_0 > -\infty$$

and that if Equation 4.7 is solvable then K_+ and K_- also satisfy this condition. The analogous statement applies when independent variables are complex. It will be seen that for at least some circumstances, K_+ has isolated singularities not on the real axis.

Suppose there is some operator $\hat{\mathcal{M}}$ which commutes with $\hat{F}$; $0=[\hat{\mathcal{M}},\hat{F}] = \hat{\mathcal{M}}\hat{F}-\hat{F}\hat{\mathcal{M}}$. Therefore, for some operator $\hat{\hat{\mathcal{M}}}$,

$$\hat{\hat{\mathcal{M}}}(1+\hat{K}_-) - (1+\hat{K}_-)\hat{\mathcal{M}} = \hat{\hat{\mathcal{M}}}(1+\hat{K}_+)(1+\hat{F}) - (1+\hat{K}_+)(1+\hat{F})\hat{\mathcal{M}}$$

$$= \left\{\hat{\hat{\mathcal{M}}}(1+\hat{K}_+) - (1+\hat{K}_+)\hat{\mathcal{M}}\right\}(1+\hat{F}) + (1+\hat{K}_+)[\hat{\mathcal{M}},\hat{F}]$$

$$= \left\{\hat{\hat{\mathcal{M}}}(1+\hat{K}_+) - (1+\hat{K}_+)\hat{\mathcal{M}}\right\}(1+\hat{F})$$

which implies

$$\hat{\hat{\mathcal{M}}}(1+\hat{K}_{\pm}) - (1+\hat{K}_{\pm})\hat{\mathcal{M}} = 0 \qquad 4.8$$

In general let

$$\hat{\mathcal{M}} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L}^{o}_{n} \qquad , \qquad \hat{L}^{o}_{n} = \ell\frac{\partial^n}{\partial z^n} \qquad 4.9$$

where α and β are constants and ℓ is a matrix of constants. Therefore $\hat{\hat{\mathcal{M}}}$ has the form,

$$\hat{\tilde{M}} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L}_n \qquad , \quad \hat{L}_n = \ell\frac{\partial^n}{\partial z^n} + \sum_{k=0}^{n-1} u_k(z)\frac{\partial^{n-k-1}}{\partial z^{n-k-1}} \tag{4.10}$$

The proof is by applying the operators of Equation 4.8 to ψ and equating differential parts. The rationale is as before. Because ψ is arbitrary, it may be specified as non-zero in only a single neighbourhood about some point within the contour of integration, for example $(z, z+\epsilon_0)$. The differential parts are order 1 in comparison.

When $n=1$ for example, $\hat{M} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \ell\frac{\partial}{\partial z}$ and,

$$\begin{aligned}(1+\hat{K}_+)\hat{M}\psi &= \alpha\psi_t + \beta\psi_w + \ell\psi_z + \int_z^\infty d\gamma\, K_+(z,\gamma)\left\{\alpha\psi_t + \beta\psi_w + \ell\psi_\gamma\right\} \\ &= \alpha\psi_t + \beta\psi_w + \ell\psi_z + K_+(z,\gamma)\,\ell\psi(\gamma)\Big|_{\gamma=z}^{\gamma=\infty} \\ &\quad + \int_z^\infty d\gamma\left\{K_+(z,\gamma)\left[\alpha\psi_t + \beta\psi_w - K_{+\gamma}(z,\gamma)\ell\psi\right]\right\}\end{aligned} \tag{4.11}$$

integrating by parts. The second term of Equation 4.8 is,

$$\begin{aligned}\hat{\tilde{M}}(1+\hat{K}_+)\psi &= \alpha\psi_t + \beta\psi_w + \ell\psi_z + u_0(z)\psi + \ell K_+(z,\gamma)\psi(\gamma)\Big|_{\gamma=z} \\ &\quad + \int_z^\infty d\gamma\left\{\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \ell\frac{\partial}{\partial z} + u_0(z)\right\}K_+(z,\gamma)\psi(\gamma)\end{aligned} \tag{4.12}$$

Contributions at $z=\infty$ are neglected. Differential parts of Equations 4.11 and 4.12 are equated, implying

$$u_0 = \ell K_+(z,z) - K_+(z,z)\ell = [\ell, K_+(z,z)]$$

The linearity of Equation 4.8 in $\hat{M}$ and $\hat{\tilde{M}}$ implies the generalization that if

$$\hat{M} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L}^{\circ} \qquad , \qquad \hat{L}^{\circ} = \Sigma_n \ell_n \frac{\partial^n}{\partial z^n} \tag{4.13}$$

then

$$\hat{\tilde{M}} = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L} \qquad , \qquad \hat{L} = \Sigma_n \hat{L}_n \tag{4.14}$$

Substitution of these into Equation 4.8 and many partial integrations imply

$$(1+\hat{K}_+)\hat{M}\psi = \alpha\psi_t + \beta\psi_w + \sum_{n=1} \ell_n \frac{\partial^n}{\partial z^n}\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\left\{\alpha\psi_t + \beta\psi_w + \sum_n \ell_n \frac{\partial^n}{\partial \gamma^n}\psi\right\}$$

$$= \alpha\psi_t + \beta\psi_w + \sum_{n=1} \ell_n \frac{\partial^n}{\partial z^n}\psi + \sum_{n=1}\sum_{m=0}^{n-1}(-)^m \frac{\partial^m}{\partial \gamma^m} K_+(z,\gamma)\ell_n \frac{\partial^{n-1-m}}{\partial \gamma^{n-1-m}}\psi(\gamma)\Big|_{\gamma=z}^{\gamma=\infty}$$

$$+ \sum_{n=1}(-)^n \int_z^\infty d\gamma \left\{\frac{\partial^n}{\partial \gamma^n} K_+(z,\gamma)\right\}\ell_n\psi(\gamma) + \int_z^\infty d\gamma\, K_+(z,\gamma)\left\{\alpha\psi_t + \beta\psi_w\right\}$$

and,

$$\hat{\hat{M}}(1+\hat{K}_+)\psi = \left\{\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \sum_n \hat{L}_n\right\}\left\{\psi(z) + \int_z^\infty d\eta\, K_+(z,\eta)\psi(\eta)\right\}$$

$$= \alpha\psi_t + \beta\psi_w + \sum_n \hat{L}_n\psi + \int_z^\infty d\eta \left\{\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L}\right\} K_+(z,\eta)\psi(\eta)$$

$$+ \int_z^\infty d\eta\, K_+(z,\eta)\left\{\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w}\right\}\psi(\eta) +$$

$+$ some differential terms

Integrands may be equated,

$$\sum_{n=1} (-)^n \frac{\partial^n}{\partial \eta^n} K_+(z,\eta)\, l_n = \left\{\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial w} + \hat{L}\right\} K_+(z,\eta) \qquad 4.15$$

The extension of Equation 4.8 is to let $\hat{F}$ be an arbitrary linear integral operator, as before, such that

$$[\hat{M}_1, \hat{F}] = 0 \quad , \quad \hat{M}_1 = \alpha\frac{\partial}{\partial t} + \hat{L}^\circ_1 \quad , \quad \hat{L}^\circ_1 = \sum_{n=1}^{N_1} l_n^{(1)} \frac{\partial^n}{\partial z^n} \qquad 4.16$$

$$[\hat{M}_2, \hat{F}] = 0 \quad , \quad \hat{M}_2 = \beta\frac{\partial}{\partial w} + \hat{L}^\circ_2 \quad , \quad \hat{L}^\circ_2 = \sum_{n=1}^{N_2} l_n^{(2)} \frac{\partial^n}{\partial z^n} \qquad 4.17$$

$$[\hat{L}^\circ_1, \hat{L}^\circ_2] = 0 \qquad 4.18$$

where Equation 4.18 is the compatibility relation to Equations 4.16 and 4.17. Thence,

$$\hat{\hat{M}}_1(1+\hat{K}_+) = (1+\hat{K}_+)\hat{M}_1 \quad , \quad \hat{\hat{M}}_2(1+\hat{K}_+) = (1+\hat{K}_+)\hat{M}_2 \qquad 4.19$$

$$\hat{\tilde{\mathcal{M}}}_1 = \alpha\frac{\partial}{\partial t} + \hat{L}_1 \qquad , \qquad \hat{\tilde{M}}_2 = \beta\frac{\partial}{\partial w} + \hat{L}_2 \qquad 4.20$$

which means,

$$\left(\alpha\frac{\partial}{\partial t} + \hat{L}_1\right)\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\} = \left(1 + \int_z^\infty d\gamma\, K_+(z,\gamma)\right)\left\{\alpha\frac{\partial}{\partial t} + \hat{L}^\circ_1\right\}\psi \qquad 4.21$$

$$\left(\beta\frac{\partial}{\partial w} + \hat{L}_2\right)\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\} = \left(1 + \int_z^\infty d\gamma\, K_+(z,\gamma)\right)\left\{\beta\frac{\partial}{\partial w} + \hat{L}^\circ_2\right\}\psi \qquad 4.22$$

The difference of $\beta\frac{\partial}{\partial w}$ applied to Equation 4.21 and $\alpha\frac{\partial}{\partial t}$. applied to Equation 4.22 is

$$\left\{\beta\frac{\partial}{\partial w}\hat{L}_1 - \alpha\frac{\partial}{\partial t}\hat{L}_2\right\}\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\} + \hat{L}_1\beta\frac{\partial}{\partial w}\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\}$$

$$-\hat{L}_2\alpha\frac{\partial}{\partial t}\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\}$$

$$= \left\{\beta\frac{\partial}{\partial w}\hat{L}_1 - \alpha\frac{\partial}{\partial t}\hat{L}_2\right\}\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\} + \hat{L}_1\left\{-\hat{L}_2\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\}\right.$$

$$\left. + (1+\hat{K}_+)\hat{\mathcal{M}}_2\psi\right\} - \hat{L}_2\left\{-\hat{L}_1\left\{\psi + \int_z^\infty d\gamma\, K_+(z,\gamma)\psi(\gamma)\right\} + (1+\hat{K}_+)\hat{\mathcal{M}}_1\psi\right\}$$

$$= \beta\frac{\partial}{\partial w}(1+\hat{K}_+)\hat{\mathcal{M}}_1\psi - \alpha\frac{\partial}{\partial t}(1+\hat{K}_+)\hat{\mathcal{M}}_2\psi$$

where Equations 4.21 and 4.22 have been resubstituted into themselves.

A consequence of Equations 4.19 and 4.20 is

$$\left\{\beta\frac{\partial}{\partial w}\hat{L}_1-\alpha\frac{\partial}{\partial t}\hat{L}_2\right\}(1+\hat{K}_+)\psi=-[\hat{L}_2,\hat{L}_1](1+\hat{K}_+)\psi+(1+\hat{K}_+)\hat{M}_2\hat{M}_1\psi$$
$$-(1+\hat{K}_+)\hat{M}_1\hat{M}_2\psi$$

As $[M_1, M_2]=0$ from Equations 4.16 to 4.18, it follows that

$$\left\{\beta\frac{\partial}{\partial w}\hat{L}_1-\alpha\frac{\partial}{\partial t}\hat{L}_2+[\hat{L}_2,\hat{L}_1]\right\}(1+\hat{K}_+)=0 \qquad 4.23$$

This is a set of nonlinear evolution equations solvable by the inverse scattering formalism.

Suppose that operators $\hat{M}_1$ and $\hat{M}_2$ define the associated linear problem as in Equations 4.16 to 4.18. Let $\hat{F}$ satisfy this set of equations. The Gelfand-Levitan Equation 4.7,

$$0=K_+(z,y)+F(z,y)+\int_z^\infty ds\,K_+(z,s)F(s,y) \qquad 4.7$$

defines the corresponding kernel, $K_+(z,s,w,t)$ This generates an exact solution to the nonlinear evolution Equation 4.23.

4.3 Soliton solutions to the shallow water wave example

The theory of Section 4.2 is now applied to Equations 4.2 and 4.3. Let

$$\hat{L}_1=\begin{bmatrix}\partial^2/\partial z^2 & 0\\ 0 & \partial^2/\partial z^2\end{bmatrix}+2\begin{bmatrix}\Sigma_{11z} & \Sigma_{12z}\\ \Sigma_{21z} & \Sigma_{22z}\end{bmatrix} \qquad 4.24$$

$$\hat{L}_2 = \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix} + (l_1 - l_2)\begin{bmatrix} 0 & \xi_{12} - 1 \\ -\xi_{21} + 1 & 0 \end{bmatrix} \qquad 4.25$$

and assume $|\xi_{ij}| \longrightarrow 1$ as $z \rightarrow \pm\infty$ so that the linear problem is specified by

$$\hat{L}_1^0 = \begin{bmatrix} \partial^2/\partial z^2 & 0 \\ 0 & \partial^2/\partial z^2 \end{bmatrix} , \quad \hat{L}_2^0 = \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix} \qquad 4.26$$

Substitute Equations 4.24 to 4.26 into equations 4.21 and 4.22

$$\left\{\alpha\frac{\partial}{\partial t} + \begin{bmatrix} \partial^2/\partial z^2 & 0 \\ 0 & \partial^2/\partial z^2 \end{bmatrix} + 2\begin{bmatrix} \Sigma_{11_z} & \Sigma_{12_z} \\ \Sigma_{21_z} & \Sigma_{22_z} \end{bmatrix}\right\}\left\{\psi + \int_z^\infty dy\, K_+(z,y)\psi(y)\right\} =$$

$$= \alpha\psi_t + \begin{bmatrix} \partial^2/\partial z^2 & 0 \\ 0 & \partial^2/\partial z^2 \end{bmatrix}\psi + \int_z^\infty dy\, K_+(z,y)\left\{\alpha\frac{\partial}{\partial t} + \begin{bmatrix} \partial^2/\partial y^2 & 0 \\ 0 & \partial^2/\partial y^2 \end{bmatrix}\right\}\psi \qquad 4.27$$

$$\left\{\beta\frac{\partial}{\partial w} + \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix} + (l_1 - l_2)\begin{bmatrix} 0 & \xi_{12} - 1 \\ -\xi_{21} + 1 & 0 \end{bmatrix}\right\}\left\{\psi + \int_z^\infty dy\, K_+(z,y)\psi(y)\right\} =$$

$$= \beta\psi_w + \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix}\psi + \int_z^\infty dy\, K_+(z,y)\left\{\beta\frac{\partial}{\partial w} + \begin{bmatrix} l_1\partial/\partial y & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial y \end{bmatrix}\right\}\psi \qquad 4.28$$

The last integral of these equations may be integrated by parts. Differential parts may be equated,

$$\Sigma(z) = K_+(z,z) \quad , \quad \xi_{12} - 1 = K_{12}(z,z) \quad , \quad \xi_{21} - 1 = K_{21}(z,z) \tag{4.29}$$

Integrand parts of Equation 4.27 may be equated,

$$\alpha \frac{\partial}{\partial t} K_{ij} + \frac{\partial^2}{\partial z^2} K_{ij} - \frac{\partial^2}{\partial \eta^2} K_{ij} + 2\, \Sigma_{i\ell_z} K_{\ell j} = 0 \tag{4.30}$$

Similarly from Equation 4.28,

$$\ell_1 \left\{ \frac{\partial}{\partial z} K_{11} + \frac{\partial}{\partial \eta} K_{11} \right\} + \beta \frac{\partial K_{11}}{\partial w} + (\ell_1 - \ell_2) K_{21} \xi_{12} + (\ell_1 - \ell_2) K_{12} = 0$$

$$\ell_1 \frac{\partial}{\partial z} K_{12} + \ell_2 \frac{\partial}{\partial \eta} K_{12} + \beta \frac{\partial}{\partial w} K_{12} + (\ell_1 - \ell_2) \xi_{12} K_{22} - K_{11} (\ell_1 - \ell_2) = 0$$

$$\ell_2 \frac{\partial}{\partial z} K_{21} + \ell_1 \frac{\partial}{\partial \eta} K_{21} + \beta \frac{\partial}{\partial w} K_{21} - (\ell_1 - \ell_2) \xi_{21} K_{11} - K_{22} (\ell_2 - \ell_1) = 0$$

$$\ell_2 \left\{ \frac{\partial}{\partial z} K_{22} + \frac{\partial}{\partial \eta} K_{22} \right\} + \beta \frac{\partial}{\partial w} K_{22} + (\ell_2 - \ell_1) K_{12} \xi_{21} + (\ell_2 - \ell_1) K_{21} = 0$$

Evaluated on $z = \eta$ this system of equations is

$$\alpha K_{12_t} - \frac{\ell_1 + \ell_2}{\ell_1 - \ell_2} K_{12_{zz}} - \frac{2\beta}{\ell_1 - \ell_2} K_{12_{wz}} + 2 \xi_{12} \left\{ \Sigma_{11_z} - \Sigma_{22_z} \right\} = 0$$

$$\ell_1 K_{11_z} + \beta K_{11_w} + (\ell_1 - \ell_2) K_{21} \xi_{12} + (\ell_1 - \ell_2) K_{12} = 0$$

and analogously with 1 and 2 interchanged. Let $\ell_1 = -\ell_2$.
These may be cast as

$$\ell_1 \Sigma_{11_z} + \beta \Sigma_{11_w} + (\ell_1 - \ell_2)(\xi_{12}\xi_{21} - 1) = 0 \qquad 4.31$$

$$\ell_2 \Sigma_{22_z} + \beta \Sigma_{22_w} + (\ell_2 - \ell_1)(\xi_{12}\xi_{21} - 1) = 0 \qquad 4.32$$

$$\alpha \xi_{12_t} + \frac{2\beta}{\ell_2 - \ell_1} \xi_{12_{zw}} + 2\xi_{12}\beta \frac{\partial}{\partial w}\left\{ \frac{\Sigma_{22}}{\ell_2} - \frac{\Sigma_{11}}{\ell_1} \right\} = 0 \qquad 4.33$$

$$\alpha \xi_{21_t} + \frac{2\beta}{\ell_1 - \ell_2} \xi_{21_{zw}} + 2\xi_{21}\beta \frac{\partial}{\partial w}\left\{ \frac{\Sigma_{11}}{\ell_1} - \frac{\Sigma_{22}}{\ell_2} \right\} = 0 \qquad 4.34$$

Let $\ell_1 = i\gamma$ where γ is real. A set of sufficient conditions that Equations 4.31 and 4.32 are consistent, are $\Sigma_{11}^* = \Sigma_{22}$ and $\xi_{12}^* = \xi_{21}$. The same is true for Equations 4.33 and 4.34.

A complete set of equations is

$$i\gamma \Sigma_{11_z} + \beta \Sigma_{11_w} + 2i\gamma \left\{ \xi_{12}\xi_{12}^* - 1 \right\} = 0 \qquad 4.35$$

$$\alpha \xi_{12_t} - \frac{\beta}{i\gamma} \xi_{12_{zw}} + \frac{2\xi_{12}}{i\gamma} \beta \frac{\partial}{\partial w}\left\{ -\Sigma_{11} - \Sigma_{11}^* \right\} = 0 \qquad 4.36$$

Let $\Sigma_{11} = \Phi + i\Psi$ where Φ and Ψ are real. Equating real and imaginary parts of Equation 4.35,

$$\gamma\Phi_z + \beta\Psi_w + 2\gamma\left\{\xi_{12}\xi_{12}^{*}-1\right\} = 0 \qquad , \qquad -\gamma\Psi_z + \beta\Phi_w = 0$$

The latter is satisfied when $\Psi = \beta\phi_w$ and $\Phi = \gamma\phi_z$. The former is then,

$$\gamma^2\phi_{zz} + \beta^2\phi_{ww} + 2\gamma\left\{\xi_{12}\xi_{12}^{*} - 1\right\} = 0 \tag{4.37}$$

and from 4.36 one obtains

$$\alpha\,\xi_{12_t} - \frac{\beta}{i\gamma}\,\xi_{12_{zw}} - 4\,\frac{\xi_{12}}{i}\,\beta\phi_{zw} = 0 \tag{4.38}$$

Transforming coordinates,

$$x = \frac{-z + \gamma w/\beta}{\sqrt{|2\gamma|}} \qquad , \qquad y = \frac{iz + i\gamma w/\beta}{\sqrt{|2\gamma|}} \tag{4.39}$$

Equations 4.37 and 4.38 become respectively

$$\phi_{xx} - \phi_{yy} + 2S(\gamma)\left\{\xi_{12}\xi_{12}^{*} - 1\right\} = 0$$

$$\alpha\,\xi_{12_t} - \frac{i}{|2\gamma|}\left\{\xi_{12_{xx}} + \xi_{12_{yy}}\right\} - 2i\,\xi_{12}S(\gamma)\left\{\phi_{xx} + \phi_{yy}\right\} = 0$$

where $S(\gamma) =$ sign of γ , and x and y are real. Substitute $\xi_{12} = \varphi$, $\phi_{yy} = 2S(\gamma)\left\{\xi_{12}\xi_{12}^{*} - 1\right\} - \frac{2q}{3A^2}$ Whence, it follows that,

$$q_{xx} - q_{yy} = -3A^2(\varphi\varphi^*)_{xx} \tag{4.40}$$

$$2i\varphi_t + \varphi_{xx} + \varphi_{yy} = -3\varphi q - \frac{9}{2}A^2\varphi\{\varphi\varphi^* - 1\} \tag{4.41}$$

where $\gamma = -\frac{9A^2}{16}$ and $\alpha|\gamma| = 1$. This set of equations is of the same form as Equations 4.2 and 4.3.

The associated linear problem is defined by,

$$\hat{L}_1^0 = \begin{bmatrix} \partial^2/\partial z^2 & 0 \\ 0 & \partial^2/\partial z^2 \end{bmatrix} \quad , \quad \hat{L}_2^0 = \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix} \tag{4.26}$$

$$[\hat{M}_1, \hat{F}] = 0 \quad , \quad \hat{M}_1 = \alpha\frac{\partial}{\partial t} + \hat{L}_1^0 \tag{4.16}$$

$$[\hat{M}_2, \hat{F}] = 0 \quad , \quad \hat{M}_2 = \beta\frac{\partial}{\partial w} + \hat{L}_2^0 \tag{4.17}$$

Thus,

$$\left\{\alpha\frac{\partial}{\partial t} + \begin{bmatrix} \partial^2/\partial z^2 & 0 \\ 0 & \partial^2/\partial z^2 \end{bmatrix}\right\}\int_{-\infty}^{\infty} dy\, F(z,y)\psi(y) = \int_{-\infty}^{\infty} dy\, F(z,y)\left\{\alpha\frac{\partial}{\partial t} + \begin{bmatrix} \partial^2/\partial y^2 & 0 \\ 0 & \partial^2/\partial y^2 \end{bmatrix}\right\}\psi(y)$$

$$\left\{\beta\frac{\partial}{\partial w} + \begin{bmatrix} l_1\partial/\partial z & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial z \end{bmatrix}\right\}\int_{-\infty}^{\infty} dy\, F(z,y)\psi(y) = \int_{-\infty}^{\infty} dy\, F(z,y)\left\{\beta\frac{\partial}{\partial w} + \begin{bmatrix} l_1\partial/\partial y & l_1 - l_2 \\ -l_1 + l_2 & l_2\partial/\partial y \end{bmatrix}\right\}\psi(y)$$

which is equivalent to

$$\alpha \frac{\partial}{\partial t} F_{ij} + \frac{\partial^2}{\partial z^2} F_{ij} - \frac{\partial^2}{\partial y^2} F_{ij} = 0 \qquad , \qquad F_{ij} = F_{ij}(z, y, w, t) \tag{4.42}$$

$$\beta \frac{\partial}{\partial w} F_{11} + l_1 \frac{\partial}{\partial z} F_{11} + (l_1 - l_2) F_{21} = -\frac{\partial}{\partial y} F_{11} l_1 + F_{12}(-l_1 + l_2) \tag{4.43}$$

$$\beta \frac{\partial}{\partial w} F_{12} + l_1 \frac{\partial}{\partial z} F_{12} + (l_1 - l_2) F_{22} = F_{11}(l_1 - l_2) - \frac{\partial}{\partial y} F_{12} l_2 \tag{4.44}$$

$$\beta \frac{\partial}{\partial w} F_{22} + l_2 \frac{\partial}{\partial z} F_{22} + (l_2 - l_1) F_{12} = -\frac{\partial}{\partial y} F_{22} l_2 + F_{21}(-l_2 - l_1) \tag{4.45}$$

$$\beta \frac{\partial}{\partial w} F_{21} + l_2 \frac{\partial}{\partial z} F_{21} + (l_2 - l_1) F_{11} = F_{22}(l_2 - l_1) - \frac{\partial}{\partial y} F_{21} l_1 \tag{4.46}$$

A solution of the form $F_{ij} = A_{ij}(t)\, e^{qz + mw + ny}$ is sought, where q, m and n are real constants. Thus,

$$A_{11}\left\{ \beta m + l_1 q + l_1 n \right\} + (l_1 - l_2)\left\{ A_{21} + A_{12} \right\} = 0 \tag{4.47}$$

$$A_{12}\left\{ \beta m + l_1 q + l_2 n \right\} + (l_1 - l_2)\left\{ A_{22} - A_{11} \right\} = 0 \tag{4.48}$$

$$A_{22}\left\{ \beta m + l_2 q + l_2 n \right\} + (l_2 - l_1)\left\{ A_{12} + A_{21} \right\} = 0 \tag{4.49}$$

$$A_{21}\left\{ \beta m + l_2 q + l_1 n \right\} + (l_2 - l_1)\left\{ A_{11} - A_{22} \right\} = 0 \tag{4.50}$$

Let $\ell_1 = -\ell_2 = i\gamma$ where γ is real, as before. Compatibility conditions between Equations 4.47 and 4.49 are $A_{12} = A_{21}^*$ and $A_{11} = A_{22}^*$. The same is true for Equations 4.48 and 4.50. A complete set of equations is, including Equation 4.42,

$$\alpha \frac{\partial}{\partial t} A_{ij} + (q^2 - n^2) A_{ij} = 0 \tag{4.51}$$

$$A_{11}\left\{\beta m + i\gamma(n+q)\right\} + 2i\gamma\left\{A_{12} + A_{12}^*\right\} = 0 \tag{4.52}$$

$$A_{12}\left\{\beta m + i\gamma(q-n)\right\} + 2i\gamma\left\{A_{11}^* - A_{11}\right\} = 0 \tag{4.53}$$

Substitute $q = 2\sin\theta$, $n = 2\sin\phi$, and $\beta m = -2\gamma\left\{\cos\theta + \cos\phi\right\}$. Real and imaginary parts of Equations 4.52 and 4.53 may be equated to zero. Thus,

$$A_{11} = i a e^{i/2(\theta+\phi)} \quad , \quad A_{12} = a e^{i/2(\theta-\phi)} \tag{4.54}$$

and from Equation 4.51,

$$a = a_0 e^{-(q^2-n^2)t/\alpha} = a_0\, e^{2(\cos 2\theta - \cos 2\phi)t\, 9A^2/16} \tag{4.55}$$

The Gelfand-Levitan equation,

$$0 = K_{ij}(z,y) + F_{ij}(z,y) + \int_z^\infty ds\, K_{i\ell}(z,s)\, F_{\ell j}(s,y) \tag{4.7}$$

must now be solved for $K_{ij}(z,y)$. Assume $K_{ij} = K_{ij}'(z,w,t)\, e^{n y}$.

As $F_{ij} = A_{ij}(t)e^{qz+mw+n\gamma}$ it follows that,

$$0 = K'_{ij} + A_{ij}\,e^{qz+mw} + \int_z^{\infty} ds\, K'_{i\ell}(z,w,t)\,e^{ns} A_{\ell j}(t)\,e^{qs+mw}$$

Contributions at $s=\infty$ are neglected. Therefore,

$$-A_{11}e^{qz+mw} = K'_{11}\left\{1 - A_{11}\frac{e^{(n+q)z+mw}}{n+q}\right\} - K'_{12}A^*_{12}\frac{e^{(n+q)z+mw}}{n+q}$$

$$-A_{12}e^{qz+mw} = -K'_{11}A_{12}\frac{e^{(n+q)z+mw}}{n+q} + K'_{12}\left\{1 - A^*_{11}\frac{e^{(n+q)z+mw}}{n+q}\right\}$$

This set of equations may be inverted, assuming the determinant does not vanish,

$$K'_{12} = -\frac{A_{12}\,e^{qz+mw}}{1+\frac{1}{2}a\sec\frac{1}{2}(\theta-\phi)\,e^{(n+q)z+mw}}$$

which implies from Equation 4.29,

$$\xi_{12} = 1 + K_{12}(z,z) = 1 - \frac{a_0 e^{-(q^2-n^2)t/\alpha + (q+n)z + mw}\; e^{i/2(\theta-\phi)}}{1+\frac{1}{2}a_0\sec\frac{1}{2}(\theta-\phi)\,e^{-(q^2-n^2)t/\alpha+(n+q)z+mw}} \qquad 4.56$$

This is part of the solution to the set of equations,

$$\gamma^2\phi_{zz} + \beta^2\phi_{ww} + 2\gamma\left\{\xi_{12}\xi^*_{12} - 1\right\} = 0 \qquad 4.37$$

$$\alpha\,\xi_{12t} - \frac{\beta}{i\gamma}\xi_{12zw} - 4\frac{\xi_{12}}{i}\beta\phi_{zw} = 0 \qquad 4.38$$

Equation 4.56 becomes a solution to Equations 4.2 and 4.3 when coordinates are transformed,

$$x = \frac{-z + \gamma w/\beta}{\sqrt{|2\gamma|}} \quad , \quad y = \frac{iz + i\gamma w/\beta}{\sqrt{|2\gamma|}} \tag{4.39}$$

where $\gamma = -\frac{9A^2}{16}$ and $\alpha|\gamma| = 1$ and the identification $\xi_{12} = \psi$ is made. The resultant is

$$\psi = \frac{1 - e^{-\mu} e^{2i\Delta}}{1 + e^{-\mu}} \tag{4.57}$$

where

$$\mu = -\frac{9A^2}{2}\cos\Delta\left\{t\cos 2(\Theta + \pi/4)\sin\Delta - \frac{2x}{3A}\sin(\Theta + \pi/4) + \frac{2iy}{3A}\cos(\Theta + \frac{\pi}{4})\right\} + \delta_0$$

$$\Theta = \tfrac{1}{2}(\theta + \phi)$$

$$\Delta = \tfrac{1}{2}(\theta - \phi)$$

$$\delta_0 = -\log\left\{\tfrac{1}{2} a_0 \sec\Delta\right\} \tag{4.58}$$

and using the substitutions $q = 2\sin\theta$, $n = 2\sin\phi$, $\beta m = -2\gamma\{\cos\theta + \cos\phi\}$ and $\gamma = -9A^2/16$.

4.4 Multi-soliton solutions with a single propagation direction

The generalization to multi-soliton solutions begins by letting

$$F_{ij} = \sum_{k=1}^{n} A_k^{ij}\, e^{q_k z + m_k w + n_k \gamma} \qquad 4.59$$

where q_k , m_k and n_k are real constants. Thus Equations 4.42 to 4.46 imply,

$$\alpha \frac{\partial}{\partial t} A_k^{ij} + (q_k^2 - n_k^2) A_k^{ij} = 0 \qquad 4.60$$

$$\beta A_k^{11} m_k + l_1 A_k^{11} q_k + (l_1 - l_2) A_k^{21} = -l_1 A_k^{11} n_k + (-l_1 + l_2) A_k^{12}$$

$$\beta A_k^{12} m_k + l_1 A_k^{12} q_k + (l_1 - l_2) A_k^{22} = (l_1 - l_2) A_k^{11} - l_2 A_k^{12} n_k$$

$$\beta A_k^{22} m_k + l_2 A_k^{22} q_k + (l_2 - l_1) A_k^{12} = -l_2 A_k^{22} n_k + (-l_2 + l_1) A_k^{21}$$

$$\beta A_k^{21} m_k + l_2 A_k^{21} q_k + (l_2 - l_1) A_k^{11} = (l_2 - l_1) A_k^{22} - l_1 A_k^{21} n_k$$

4.61

respectively. Repeated indices are not summed in this set of equations. Let $l_1 = -l_2 = i\gamma$, where γ is real. Compatibility conditions within the set of Equations 4.61 are $A_k^{11*} = A_k^{22}$ and $A_k^{12*} = A_k^{21}$. The set of Equations 4.61 may be cast as,

$$A_k^{11}\left\{\beta m_k + i\gamma\left[q_k + n_k\right]\right\} + 2i\gamma\left\{A_k^{12*} + A_k^{12}\right\} = 0 \qquad 4.62$$

$$A_k^{12}\left\{\beta m_k + i\gamma\left[q_k - n_k\right]\right\} + 2i\gamma\left\{A_k^{11*} - A_k^{11}\right\} = 0 \qquad 4.63$$

Let $q_k = 2\sin\theta_k$, $n_k = 2\sin\phi_k$ and $\beta m_k = -2\gamma\left\{\cos\theta_k + \cos\phi_k\right\}$

Real and imaginary parts of Equations 4.62 and 4.63 may be equated, implying

$$A_k^{11} = i a_k e^{i/2(\theta_k + \phi_k)} \quad , \quad A_k^{12} = a_k e^{i/2(\theta_k - \phi_k)}$$

and from Equation 4.60,

$$a_k = a_{o_k} e^{-(q_k - n_k)t/\alpha}$$

The Gelfand-Levitan equation,

$$0 = K_{ij}(z,\gamma) + F_{ij}(z,\gamma) + \int_z^\infty ds \sum_{\ell=1}^{2} K_{i\ell}(z,s) F_{\ell j}(s,\gamma)$$

must be solved for $K_{ij}(z,\gamma)$. Assume $K_{ij} = \sum_{p=1}^{n} K_p^{ij}(z,w,t)e^{n_p\gamma}$.

Thus, as $F_{ij} = \sum_{k=1}^{n} A_k^{ij} e^{q_k z + m_k w + n_k \gamma}$

$$0 = K_p^{ij} + A_p^{ij} e^{q_p z + m_p w} + \int_z^\infty ds \sum_{b=1}^{2} \sum_{k=1}^{n} K_k^{ib} A_p^{bj} e^{(q_k + n_k)s + m_p w}$$

which integrates to, neglecting contributions at $s = \infty$,

$$0 = K_p^{ij} + A_p^{ij} e^{n_p z + m_p w} - \sum_{b=1}^{2} \sum_{k=1}^{n} K_k^{ib} A_p^{bj} \frac{e^{(q_p+n_k)z + m_p w}}{q_p + n_k} \tag{4.64}$$

Equation 4.64 is equivalent to,

$$\begin{bmatrix} \left\{ \delta_{pk} - A_p^{11} \dfrac{e^{(q_p+n_k)z+m_p w}}{q_p+n_k} \right\} & \left\{ -A_p^{12*} \dfrac{e^{(q_p+n_k)z+m_p w}}{q_p+n_k} \right\} \\ \left\{ -A_p^{12} \dfrac{e^{(q_p+n_k)z+m_p w}}{q_p+n_k} \right\} & \left\{ \delta_{pk} - A_p^{11*} \dfrac{e^{(q_p+n_k)z+m_p w}}{q_p+n_k} \right\} \end{bmatrix} \begin{bmatrix} K^{11} \\ K^{12} \end{bmatrix} = \begin{bmatrix} -A^{11} e^{qz+mw} \\ -A^{12} e^{qz+mw} \end{bmatrix} \tag{4.65}$$

where each of $\{\ \}$ are $n \times n$ matrices, K^{11}, K^{12}, $A^{11} e^{qz+mw}$ and $A^{12} e^{qz+mw}$ are n-vectors.

A solution to Equations 4.37 and 4.38

$$\gamma^2 \phi_{zz} + \beta^2 \phi_{ww} + 2\gamma \left\{ \xi_{12} \xi_{12}^* - 1 \right\} = 0 \tag{4.37}$$

$$\alpha \xi_{12_t} - \frac{\beta}{i\gamma} \xi_{12_{zw}} - \frac{4}{i} \xi_{12} \beta \phi_{zw} = 0 \tag{4.38}$$

is, using Equation 4.29, $\xi_{12} = 1 + K_{12}(z, z)$

$$\xi_{12} = 1 + \sum_{k=1}^{n} K_k^{12} e^{n_k z} = 1 + \frac{1}{\Delta} \sum_{k=n+1}^{2n} d_k e^{n_k z} \tag{4.66}$$

d_k is the determinant Δ, with the k-th column replaced by the vector on the right side of Equation 4.65, by Cramer's rule.

4.5 Propagation and modulation at various angles

Single soliton solutions of the form $F_{ij} = A_{ij}(t)\, e^{qz + mw + n\gamma}$ where q, m and n are real, were sought in Section 4.3 to the linear set of equations

$$\alpha \frac{\partial}{\partial t} F_{ij} + \frac{\partial^2}{\partial z^2} F_{ij} - \frac{\partial^2}{\partial \gamma^2} F_{ij} = 0 \qquad , \; F_{ij} = F_{ij}(z, \gamma, w, t) \tag{4.42}$$

$$\beta \frac{\partial}{\partial w} F_{11} + l_1 \frac{\partial}{\partial z} F_{11} + (l_1 - l_2) F_{21} = -\frac{\partial}{\partial \gamma} F_{11} l_1 + F_{12}(-l_1 + l_2) \tag{4.43}$$

$$\beta \frac{\partial}{\partial w} F_{12} + l_1 \frac{\partial}{\partial z} F_{12} + (l_1 - l_2) F_{22} = F_{11}(l_1 - l_2) - \frac{\partial}{\partial \gamma} F_{12} l_2 \tag{4.44}$$

$$\beta \frac{\partial}{\partial w} F_{22} + l_2 \frac{\partial}{\partial z} F_{22} + (l_2 - l_1) F_{12} = -\frac{\partial}{\partial \gamma} F_{22} l_2 + F_{21}(-l_2 + l_1) \tag{4.45}$$

$$\beta \frac{\partial}{\partial w} F_{21} + l_2 \frac{\partial}{\partial z} F_{21} + (l_2 - l_1) F_{11} = F_{22}(l_2 - l_1) - \frac{\partial}{\partial \gamma} F_{21} l_1 \tag{4.46}$$

These equations eventually define the linear problem associated to the nonlinear evolution equations of the shallow water waves example,

$$2i\varphi_t + \varphi_{xx} + \varphi_{yy} = -\frac{9}{2} A^2 \varphi \{ \varphi\varphi^* - 1 \} - 3\varphi q \tag{4.2}$$

$$q_{xx} - q_{yy} = -3 A^2 |\varphi|^2_{xx} \tag{4.3}$$

Had q, m and n been assumed complex, the set of equations subsequent to Equations 4.42 to 4.46 would be as in Section 4.3, respectively,

$$\alpha \frac{\partial}{\partial t} A_{ij} + (q^2 - n^2) A_{ij} = 0 \qquad 4.51$$

$$A_{11} \{ \beta m + l_1 q + l_1 n \} + (l_1 - l_2) \{ A_{21} + A_{12} \} = 0 \qquad 4.47$$

$$A_{12} \{ \beta m + l_1 q + l_2 n \} + (l_1 - l_2) \{ A_{22} - A_{11} \} = 0 \qquad 4.48$$

$$A_{22} \{ \beta m + l_2 q + l_2 n \} + (l_2 - l_1) \{ A_{12} + A_{21} \} = 0 \qquad 4.49$$

$$A_{21} \{ \beta m + l_2 q + l_1 n \} + (l_2 - l_1) \{ A_{11} - A_{22} \} = 0 \qquad 4.50$$

It is necessary that $l_1 = -l_2 = i\gamma$ where γ is real, if this discussion is to be consistent with that concerning the nonlinear problem, Equations 4.2 and 4.3 in Section 4.3. The compatibility conditions, $A_{11}^* = A_{22}$ and $A_{12}^* = A_{21}$, appropriate when q, m and n were real are no longer evident. Equation 4.51 implies

$$A_{ij} = A_{ij_0} e^{-(q^2 - n^2)t/\alpha} \qquad 4.67$$

Letting

$$A_{11_0} = a_0 \qquad 4.68$$

Equations 4.47 to 4.50 imply

$$A_{12_0} = \frac{4i\gamma\beta m a_0}{(\beta m + i\gamma(q-n))(\beta m - i\gamma(q+n))} \qquad 4.69$$

$$A_{21_0} = \frac{4i\gamma\beta m a_0}{(\beta m - i\gamma(q-n))(\beta m - i\gamma(q+n))} \quad , \quad A_{22_0} = \frac{-a_0(\beta m + i\gamma(q+n))}{\beta m - i\gamma(q+n)} \qquad 4.70$$

The Gelfand-Levitan equation,

$$0 = K_{ij}(z,\eta) + F_{ij}(z,\eta) + \int_z^\infty ds\, K_{i\ell}(z,s)\, F_{\ell j}(s,\eta) \qquad 4.7$$

must be solved for $K_{ij}(z,\eta)$. Assume $K_{ij} = K'_{ij}(z,w,t)\,e^{n\eta}$ and recall that $F_{ij} = A_{ij}(t)\,e^{qz+mw+n\eta}$. Whence,

$$0 = K'_{ij}(z,w,t) + A_{ij}(t)\,e^{qz+mw} + \int_z^\infty ds\, K'_{i\ell}(z,w,t)\,A_{\ell j}\,e^{(n+q)s+mw}$$

Contributions at $s=\infty$ are neglected. Thus,

$$\begin{bmatrix} 1-A_{11}\frac{e^{(n+q)z+mw}}{n+q} & -A_{21}\frac{e^{(n+q)z+mw}}{n+q} & 0 & 0 \\ -A_{12}\frac{e^{(n+q)z+mw}}{n+q} & 1-A_{22}\frac{e^{(n+q)z+mw}}{n+q} & 0 & 0 \\ 0 & 0 & 1-A_{11}\frac{e^{(n+q)z+mw}}{n+q} & -A_{21}\frac{e^{(n+q)z+mw}}{n+q} \\ 0 & 0 & -A_{12}\frac{e^{(n+q)z+mw}}{n+q} & 1-A_{22}\frac{e^{(n+q)z+mw}}{n+q} \end{bmatrix} \begin{bmatrix} K'_{11} \\ K'_{12} \\ K'_{21} \\ K'_{22} \end{bmatrix} = \begin{bmatrix} -A_{11}e^{qz+mw} \\ -A_{12}e^{qz+mw} \\ -A_{21}e^{qz+mw} \\ -A_{22}e^{qz+mw} \end{bmatrix}$$

which implies,

$$K'_{12} = \frac{-A_{12}\, e^{qz+mw}}{1-(A_{11}+A_{22})\dfrac{e^{(n+q)z+mw}}{n+q}+(A_{11}A_{22}-A_{12}A_{21})\dfrac{e^{2(n+q)z+mw}}{(n+q)^2}} \qquad 4.71$$

Substitutions for A_{ij} are made from Equations 4.67 to 4.70, and for z and w from Equation 4.39

$$w = -\beta(x-iy) \qquad , \qquad z = -\tfrac{1}{2}(x+iy) \qquad 4.39$$

where, as in Section 5.1, $A = 2\sqrt{2}/3$, $\gamma = -1/2$ and $1/\alpha = 1/2$. Unspecified parameters are q, n, m, β and a_0. Equation 4.29 requires $\xi_{12} = 1 + K_{12}(z,z)$. The relation $\varphi = \xi_{12}$ results in solutions to Equations 4.2 and 4.3 of the form,

$$\varphi = 1 - \frac{A_{12_0}\, e^{-\mu}(n+q)}{1 - A_{11_0} e^{-\mu} - A_{22_0} e^{-\mu} + (A_{11_0}A_{22_0} - A_{12_0}A_{21_0})e^{-2\mu}} \qquad 4.72$$

where

$$\mu = \tfrac{1}{2}(q^2-n^2)t + \left(\frac{q+n}{2} + m\beta\right)x + \left(\frac{q+n}{2} - m\beta\right)iy + \log(n+q)$$

$$A_{11_0} = a_0$$

$$A_{12_0} = \frac{-2i\beta m a_0}{\left(\beta m - \frac{i}{2}(q-n)\right)\left(\beta m + \frac{i}{2}(q+n)\right)}$$

$$A_{21_0} = \frac{-2i\beta m a_0}{(\beta m + \frac{i}{2}(q-n))(\beta m + \frac{i}{2}(q+n))}$$

$$A_{22_0} = \frac{-a_0(\beta m - i/2(q+n))}{\beta m + i/2(q+n)} \qquad 4.73$$

Constants q, m and n may be complex. These dictate the direction of propagation and the modulational traits of the soliton.

The multi-soliton situation may be considered analogously.

5. Eulerian Variables

5.1 Single soliton solutions of the shallow water waves example

The mapping from dependent variables h and $\frac{\partial \underline{\underline{x}}}{\partial t}$ to dependent variables ψ and ψ^* was

$$h = \psi\psi^* \qquad 3.9$$

$$\frac{\partial \underline{\underline{x}}}{\partial t} \cdot \frac{\partial \underline{\underline{x}}}{\partial t} = K^2 \nabla_{\underline{x}} \log \psi \cdot \nabla_{\underline{x}} \log \psi^* + \mathcal{F}(\underline{x},t) \qquad 3.10$$

Section 3.4 saw the identification of the Eulerian velocity field as

$$\underline{V} = K \nabla \Omega \qquad 3.31$$

where $\psi = R(\underline{x},t)\, e^{i\Omega(\underline{x},t)}$. The function $\mathcal{F}(\underline{x},t)$ was seen to satisfy,

$$\mathcal{F}(\underline{x},t)\, h = K \underline{V} . \nabla h \qquad 3.32$$

where $\underline{V}.\nabla h$ is the convection. When the scalings of Equation 3.22, in particular,

$$\psi = \left(\frac{9}{4g}\right)^{1/3} \psi' \quad , \quad (x,y) = \frac{K}{\sqrt{2}}\left(\frac{9}{4g}\right)^{1/6} (x',y') \qquad 3.22$$

are imposed, it follows that,

$$h = \left(\frac{9}{4g}\right)^{2/3} \psi'\psi'^{*} = \left(\frac{9}{4g}\right)^{2/3} h' \qquad 5.1$$

$$\underline{v} = \sqrt{2}\left(\frac{4g}{9}\right)^{1/6} \nabla'\Omega = \sqrt{2}\left(\frac{4g}{9}\right)^{1/6} \underline{v}' \qquad 5.2$$

$$h\mathcal{F} = 2\left(\frac{9}{4g}\right)^{1/3} \underline{v}'.\nabla'h' = 2\left(\frac{9}{4g}\right)^{1/3} \mathcal{F}'h' \qquad 5.3$$

Let primes on independent variables be dropped. A further transformation is made in Chapter 4,

$$\psi' = A\varphi e^{imt} \qquad 4.1$$

Single soliton solutions are found to be,

$$\varphi = \frac{1 - e^{-\mu} e^{2i\Delta}}{1 + e^{-\mu}} \qquad 4.57$$

where,

$$\mu = -\frac{9A^2}{2}\cos\Delta\left\{t\cos 2\left(\Theta+\frac{\pi}{4}\right)\sin\Delta - \frac{2x}{3A}\sin\left(\Theta+\frac{\pi}{4}\right) + \frac{2iy}{3A}\cos\left(\Theta+\frac{\pi}{4}\right)\right\} + \delta_0$$

$$\Theta = \frac{1}{2}(\theta + \phi)$$

$$\Delta = \frac{1}{2}(\theta - \phi)$$

$$\delta_0 = -\log\left\{\frac{1}{2}a_0\sec\Delta\right\} \qquad 4.58$$

For simplicity, let $\Delta = \frac{\pi}{4}$, $\Theta + \frac{\pi}{4} = \frac{\pi}{6}$, $a_0 = \sqrt{2}$ and $A = 2\sqrt{2}/3$. Therefore

$$\psi = \frac{1 - i e^{t-x+\sqrt{3}iy}}{1 + e^{t-x+\sqrt{3}iy}} \qquad 5.4$$

whence, as $\psi = R' e^{i\Omega} e^{-imt} A^{-1}$ from Equation 4.1,

$$\frac{\nabla\psi}{\psi} = \frac{\nabla R'}{R'} + i\nabla\Omega \qquad 5.5$$

$$\frac{\nabla R'}{R'} = \frac{1}{D}\Big\{\hat{x} e^{t-x}\left[1 - e^{2t-2x}\right]\sqrt{2}\cos(\sqrt{3}y + \tfrac{\pi}{4}) + \hat{y}\sqrt{6}e^{t-x}\left[1 + e^{2t-2x}\right]\sin(\sqrt{3}y + \tfrac{\pi}{4}) + \\ + \hat{y}\sqrt{3}\,2e^{2t-2x}\Big\} \qquad 5.6$$

$$\nabla\Omega = \frac{1}{D}\Big\{2\hat{x}e^{2t-2x} + \hat{x}\sqrt{2}e^{t-x}\left[1 + e^{2t-2x}\right]\sin(\sqrt{3}y + \tfrac{\pi}{4}) + \\ + \hat{y}\sqrt{6}e^{t-x}\left[-1 + e^{2t-2x}\right]\cos(\sqrt{3}y + \pi/4)\Big\} \qquad 5.7$$

where

$$D = 1 + e^{t-x}2\sqrt{2}\sin(\sqrt{3}y + \tfrac{\pi}{4}) + e^{2t-2x}\,2(1 + \sin 2\sqrt{3}y) + \\ + e^{3t-3x}\,2\sqrt{2}\sin(\sqrt{3}y + \tfrac{\pi}{4}) + e^{4t-4x} \qquad 5.8$$

Variables in Equations 5.4 to 5.8 are scaled. Equation 5.2 defines

$$\underline{v}' = \nabla\Omega$$

Equations 5.3 and 3.9,

$$\mathcal{F}' = 2\underline{v}'\cdot\nabla \log R' = 2\nabla\Omega\cdot\nabla R'/R'$$

$$= 8\frac{e^{2t-2x}}{D^2}\left\{-1+e^{2t-2x}\right\}\cos(\sqrt{3}y+\tfrac{\pi}{4})\left\{\left[1+e^{2t-2x}\right]\sin(\sqrt{3}y+\tfrac{\pi}{4})+\sqrt{2}e^{2t-2x}\right\}$$

Furthermore, from Equations 5.1 and 4.1,

$$h' = \psi'\psi'^{*} = A^2\psi\psi^{*} = \frac{8}{9}\cdot\frac{1+2e^{t-x}\sin\sqrt{3}y+e^{2t-2x}}{1+2e^{t-x}\cos\sqrt{3}y+e^{2t-2x}}$$

These variables, $\underline{v}'$, the Eulerian velocity, $\mathcal{F}'h'/K$, the convection, and h' , the free surface height, are waves of constant form propagating in the x direction with modulations along the y axis. The general case of Equations 4.57 and 4.58 is of this form.

5.2 A meaning for $\mathcal{F}=\infty$

The equations of motion for the shallow water wave examples were found to be,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = 2g\psi^2\psi^* + \lambda\psi \qquad 3.18$$

$$h_0 - \psi\psi^* J = 0 \qquad 3.2$$

Section 3.4 presented a transition in analogy to the path leading from quantum mechanics to classical mechanics. The assumed form of the solution to Equations 3.18 and 3.2 was

$$\psi = a e^{i s / k h_0} \qquad 5.9$$

where s is the action density and a is real. It follows from this that

$$\mathcal{F} = \frac{2K}{h_0} \frac{\nabla s}{a} . \nabla a = \frac{1}{K} \frac{4}{h_0} \left(\frac{4g}{9} \right)^{1/3} \nabla' s . \frac{\nabla' a'}{a'} \qquad 5.10$$

where the scaling of Equation 3.22 has been imposed.

Viewed as a WKB approximation, Equation 5.9 and the limit $K \to 0$ imply a region of rapid oscillation for ψ. The region is characterized by waves whose wavelength is K in order of magnitude. See Bender and Orszag (1978)[22]. It is evident from Equation 5.10 that when the approach of this section is taken, the limit $K \to 0$ implies $\mathcal{F} \to \infty$. An alternate situation is Equation 5.3 where $\mathcal{F}$ is independent of K. Suppose that $\psi = R e^{i\Omega}$ describes the dynamics of either situation. The former equation assumes $\Omega = O(1/K)$, the latter, $\Omega = O(1)$ as $K \to 0$.

5.3 Two soliton solutions

Suppose that both solitons have modulations along the y axis and propagate in the x direction. Equations 4.65 and 4.66 imply that when $n = 2$,

$$\xi_{12} = 1 + \frac{1}{D} \sum_{k=3}^{4} d_k e^{n_k z}$$

$$\xi_{12} = 1 - \frac{2}{D}\left\{ \cos\Delta_1 e^{i\Delta_1-\mu_1} + \cos\Delta_2 e^{i\Delta_2-\mu_2} + \cos\Delta_1 e^{i\Delta_1-\mu_1-\mu_2} \right\}$$

$$-\frac{2}{D}\left\{ \cos\Delta_2 e^{i\Delta_1-\mu_1-\mu_2} + \left[e^{\delta} - 1 \right]\left\{ 1 + e^{i(\Delta_1+\Delta_2)}\cos(\Theta_1-\Theta_2)\right\} e^{-\mu_1-\mu_2} \right\}$$

5.11

where

$$D = 1 + e^{-\mu_1} + e^{-\mu_2} + e^{\delta-\mu_1-\mu_2}$$

$$e^{\delta} = 1 - \frac{\cos\Delta_1\cos\Delta_2}{\cos\frac{1}{2}(\Theta_1+\Delta_1-\Theta_2+\Delta_2)\,\cos\frac{1}{2}(\Theta_2+\Delta_2-\Theta_1+\Delta_1)}$$

$$\mu_j = -2\sqrt{2}\cos\Delta_j\left\{ t\sqrt{2}\sin\Delta_j\cos 2(\Theta_j+\tfrac{\pi}{4}) - x\sin(\Theta_j+\tfrac{\pi}{4}) + iy\cos(\Theta_j+\tfrac{\pi}{4})\right\}$$

$$\Theta_j = \tfrac{1}{2}(\theta_j + \phi_j)$$

$$\Delta_j = \tfrac{1}{2}(\theta_j - \phi_j)$$

$$\lambda_{oj} = 2\cos\Delta_j$$

$$A = 2\sqrt{2}/3 \quad , \quad \gamma = -1/2 \quad , \quad \alpha = 1/2$$ 5.12

The limit $\mu_1 \longrightarrow \infty$ implies, concerning Equations 5.11,

$$\xi_{12} = \frac{1 - e^{2i\Delta_2-\mu_2}}{1 + e^{-\mu_2}}$$ 5.13

and similarly, the limit $\mu_2 \longrightarrow \infty$. This matches the single soliton solutions, Equations 4.57 and 4.58. The limit $\mu_1 \longrightarrow -\infty$ implies that Equation 5.11 becomes

$$\xi_{12} = -e^{2i\Delta_1} \cdot \frac{1+e^{\delta+2i\Delta_3-\mu_2}}{1+e^{\delta-\mu_2}} \qquad 5.14$$

where

$$e^{2i\Delta_3} = e^{-2i\Delta_1} + 2e^{i(\Delta_2-\Delta_1)}\cos(\Theta_1-\Theta_2) + \frac{1+e^{2i(\Delta_2-\Delta_1)}-2e^{i(\Delta_2-\Delta_1)}\cos(\Theta_1-\Theta_2)}{1-\dfrac{\cos\Delta_1\cos\Delta_2}{\cos(\Theta_1+\Delta_1-\Theta_2+\Delta_2)\cos\frac{1}{2}(\Theta_2+\Delta_2-\Theta_1+\Delta_1)}}$$

This is a soliton with center shift $\delta(2\sqrt{2}\cos\Delta_2)^{-1}$ and phase shift $2\Delta_1 \pm \pi$ relative to the sort represented by Equations 4.57 and 4.58. An analogous result corresponds to the limit $\mu_2 \longrightarrow -\infty$.

When parameters are chosen appropriately, Equations 5.11 and 5.12 represent two solitons which come together, interact in a very complex manner, and then separate. Center shifts and phase shifts are what they carry with them as marks of the interaction. It seems clear that $\underline{V}$, the Eulerian velocity, $\mathcal{F}h/_{k}$, the convection; and h, the free surface height, are composed of two waves of constant form, propagating in the x direction with modulations along the y axis. These waves come together, interact, and separate, bearing with them center shifts and phase shifts.

An attempt was made at finding $\mathcal{F}$, $\underline{V}$, and h for a simplifying choice of parameters in the general situation specified by Equations 5.11 and 5.12. The mathematics consortium symbol manipulation program, MAXSYMA, was used. The derivation was not ran to completion. As far as it went, there is one conclusion. The functions $\underline{V}$, $\mathcal{F}$ and h are composed of two waves as described in the paragraph above.

6. Stability and Recurrence

6.1 Stability of time independent states

The Euler-Lagrange governing equations to the shallow water waves example were found to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2}\psi^2\psi^* - 3\Phi_y\psi \tag{3.23}$$

$$\Phi_{xx} - \Phi_{yy} = -3(\psi\psi^*)_y \tag{3.24}$$

These reduce to become, in the one-dimensional χ -independent problem,

$$-i\psi_t = \frac{1}{2}\psi_{yy} + \frac{9}{4}\psi^2\psi^* \tag{6.1}$$

which is a special case of the one-dimensional nonlinear Schrödinger equation of Hasimoto and Ono (1972)[3]. This equation has nonlinear plane wave solutions of the form,

$$\psi = \psi_0 e^{i(\alpha t - \theta y)} \quad , \quad \alpha = -\frac{\theta^2}{2} + \frac{9}{4}|\psi_0|^2 \tag{6.2}$$

where ψ_0 and θ are constants.

Dependent variables were mapped in Chapter 3, via

$$h = \psi\psi^* \tag{3.9}$$

$$\frac{\partial \underline{x}}{\partial t}\cdot\frac{\partial \underline{x}}{\partial t} = K^2\nabla_x \log\psi . \nabla_x \log\psi^* + \mathcal{F}(\underline{x},t) \tag{3.10}$$

where $\mathcal{F} = 2K^2 \nabla\Omega . \nabla \log R$ and $\psi = R(\underline{x},t)e^{i\Omega(\underline{x},t)}$. Therefore, the free surface height corresponding to Equation 6.2 is $h = |\psi_0|^2$, and the squared Lagrangian speed $\frac{\partial \underline{x}}{\partial t} . \frac{\partial \underline{x}}{\partial t} = K^2\theta^2$. Variables h and $\frac{\partial \underline{x}}{\partial t} . \frac{\partial \underline{x}}{\partial t}$ are time independent. Variable $\mathcal{F}$ is zero.

Suppose that $\alpha = \alpha_0$, a constant, and $\theta = 0$. The linearization of Equation 6.1 is obtained by setting $\psi = (|\psi_0| + \hat{\epsilon}\hat{\psi})e^{i(\alpha_0 t + \hat{\epsilon}\hat{\theta})}$ where $\hat{\epsilon}$ and $\hat{\psi}$ are assumed real. Thus,

$$2\hat{\psi}_t + |\psi_0|\hat{\theta}_{yy} = 0 \tag{6.3}$$

$$2|\psi_0|\hat{\theta}_t = \hat{\psi}_{yy} + 9|\psi_0|^2\hat{\psi} \cdot \frac{3}{2} \tag{6.4}$$

This is a set of linear differential equations with constant coefficients. Let

$$\begin{Bmatrix} \hat{\psi} \\ \hat{\theta} \end{Bmatrix} = \begin{Bmatrix} \hat{\psi}_0 \\ \hat{\theta}_0 \end{Bmatrix} e^{i(ky - \omega t)} \quad + \quad \text{complex conjugate}$$

where $\hat{\psi}_0$ and $\hat{\theta}_0$ are constants. Equations 6.3 and 6.4 imply

$$-2i\omega\hat{\psi}_0 - |\psi_0|k^2\hat{\theta}_0 = 0$$

$$\left(-k^2 + \frac{27}{2}|\psi_0|^2\right)\hat{\psi}_0 + 2i\omega|\psi_0|\hat{\theta}_0 = 0$$

which have nontrivial solutions provided there is a dispersion relation of the form,

$$\omega^2 = \frac{k^2}{4}\left\{ k^2 - \frac{27}{2}|\psi_0|^2 \right\} \qquad 6.5$$

When $k < 3\sqrt{\frac{3}{2}}|\psi_0|$, ω becomes imaginary and the distrubance grows. The maximum growth rate is $\omega_{MAX} = \frac{27|\psi_0|^2 i}{8}$ occurring at $k = 3\frac{\sqrt{3}}{2}|\psi_0|$.

The resultant of assuming that Equations 3.23 and 3.24 are y - independent is

$$-i\psi_t = \frac{1}{2}\psi_{xx} - \frac{9}{4}\psi^2\psi^*$$

which is, as before, a special case of the one-dimensional nonlinear Schrödinger equation considered by Hasimoto and Ono (1972)[3]. The corresponding linearized equations are,

$$2\hat{\psi}_t + |\psi_0|\hat{\theta}_{xx} = 0$$

$$-2|\psi_0|\hat{\theta}_t + \hat{\psi}_{xx} - \frac{27}{2}|\psi_0|^2\hat{\psi} = 0$$

The substitution,

$$\begin{Bmatrix} \hat{\psi} \\ \hat{\theta} \end{Bmatrix} = \begin{Bmatrix} \hat{\psi}_0 \\ \hat{\theta}_0 \end{Bmatrix} e^{i(kx-\omega t)} + \text{complex conjugate}$$

yields the dispersion relation,

$$\omega^2 = \frac{k^2}{4}\left\{ k^2 + \frac{27}{2}|\psi_0|^2 \right\} \qquad 6.6$$

The system is neutrally stable. Disturbances do not grow.

The Euler-Lagrange equations governing the dynamics of ideal gas were found to be,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi + \Phi_x\psi \qquad 2.35$$

$$\Phi_{xx} - \Phi_{yy} = (\psi\psi^*)_x \qquad 2.36$$

These reduce to simply

$$-i\psi_t = \frac{K}{2}\psi_{yy} - \frac{c\gamma(\psi\psi^*)^{\gamma-1}}{K(\gamma-1)}\psi \qquad 6.7$$

in the χ -independent problem. The nonlinear plane wave solutions of this equation are

$$\psi = \psi_0 e^{i(\alpha t - \theta y)} \quad , \quad \alpha = -K\frac{\theta^2}{2} - \frac{c\gamma|\psi_0|^{2\gamma-2}}{(\gamma-1)K} \qquad 6.8$$

where θ and ψ_0 are constants.

Dependent variables were mapped in Chapter 2, via

$$\rho = \psi\psi^* \qquad 2.21$$

$$\frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} = K^2\nabla_{\underline{x}}\log\psi\cdot\nabla_{\underline{x}}\log\psi^* + \mathcal{F}(\underline{x},t) \qquad 2.22$$

where $\mathcal{F} = 2K^2\nabla\Omega\cdot\nabla\log R$ and $\psi = R(\underline{x},t)e^{i\Omega(\underline{x},t)}$. Therefore the density corresponding to Equation 6.8 is $\rho = |\psi_0|^2$, the squared Lagrangian speed, $\frac{\partial \underline{\chi}}{\partial t}\cdot\frac{\partial \underline{\chi}}{\partial t} = K^2\theta^2$. Dependent variables ρ and

$\frac{\partial \chi}{\partial t} \cdot \frac{\partial \chi}{\partial t}$ are time independent. Variable $\mathcal{F}$ is zero.

The analysis applied to the shallow water waves equations is repeated analogously. The linearized equations are

$$2\hat{\psi}_t + K|\psi_0|\hat{\theta}_{yy} = 0$$

$$K\hat{\psi}_{yy} - 2\frac{c\gamma}{K}(\psi_0\psi_0^*)^{\gamma-1}\hat{\psi}\frac{(2\gamma-1)}{\gamma-1} - 2|\psi_0|\hat{\theta}_t = 0$$

An assumed nontrivial substitution,

$$\begin{Bmatrix} \hat{\psi} \\ \hat{\theta} \end{Bmatrix} = \begin{Bmatrix} \hat{\psi}_0 \\ \hat{\theta}_0 \end{Bmatrix} e^{i(ky-\omega t)} + \text{complex conjugate}$$

implies the dispersion relation

$$\omega^2 = \frac{k^2}{4}\left\{K^2 k^2 + 2c\gamma|\psi_0|^{2\gamma-2}\frac{(2\gamma-1)}{\gamma-1}\right\}$$

The example is neutrally stable.

The one-dimensional y-independent version of Equations 2.35 and 2.36 is quite similar.

6.2 The interaction of Benjamin - Feir instability and recurrence

The one-dimensional nonlinear Schrödinger equation, as written in Equation 6.1 is solvable by the inverse scattering transform when initial conditions decay sufficiently rapidly as $|y| \longrightarrow \infty$. The characteristics of the solutions are less clear for other initial conditions. As presented in Section 6.1, when boundary conditions are periodic, uniform wavetrain solutions of the form $\psi = \psi_0 e^{i\alpha t}$ are unstable with respect to perturbations of wavenumber $|k| < 3\sqrt{\frac{3}{2}}\,\psi_0$. The maximum instability occurs at $k = \frac{3\sqrt{3}}{2}|\psi_0|$. Historically, this is the Benjamin-Feir (1967)[5] instability. Given a weakly nonlinear uniform deep-water wavetrain, a Stokes wavetrain, the instability is with respect to modulational perturbations, specifically, a pair of side bands around the primary component in the power spectrum.

Fermi-Pasta-Ulm recurrence is the situation in which the initial condition of an unstable nonlinear system is periodically reconstructed, or almost reconstructed, in time. Concerning the one-dimensional nonlinear Schrödinger equation, the recurrence begins with the growth of unstable modulations to the uniform solution. The growth is at exponential rate as predicted by Benjamin and Feir (1967)[5] and is followed by the eventual demodulation and return of the system to almost uniform state.

Yuen and Ferguson (1978)[4] investigated the relationship between initial instability and the long time evolution of the one-dimensional nonlinear Schrödinger equation. They found that the number of free modes actively taking part in the energy sharing experience following the onset of instability is governed by the number of unstable modes associated with the initial conditions. If the initial instability has no higher harmonics that are unstable, that are below the critical wavenumber for the occurrence of instability, then the instability is simple. There is only one mode which grows unstably at exponential rate when triggered by the forced oscillation. Otherwise, when higher harmonics of the initial instability are also unstable, then each in turn must dominate the evolution of the system. The recurrence is complex.

Subsequent evolution of the system, following initial instability consists of growth and decay of these unstable modes. Between each instance of maximum modulation, the initial conditions are at some point reconstructed, or almost reconstructed. If there were no critical upper bound on wavenumbers for the occurrence of instability, the system might be expected to thermalize, a permanent leakage of energy to high wavenumbers taking place.

In another study by Yuen and Ferguson (1978)[23], the nonlinear evolution equation for a deep-water gravity wave train subject to two-dimensional modulation, namely,

$$i\left\{A_t + \frac{\omega_0}{2k_0}A_x\right\} - \frac{\omega_0}{8k_0^2}A_{xx} + \frac{\omega_0}{4k_0^2}A_{yy} - \frac{1}{2}\omega_0 k_0^2 |A|^2 A = 0 \qquad 6.9$$

is considered. The frequency and wavenumber of the carrier wave are ω_0 and k_0, respectively. Numerical work using simple initial conditions such as sinusoidal perturbations in the x and y directions show recurrence. The existence of simple and complex recurrence is not clear, nor is the relationship between initial instability and long time evolution of the system.

Equation 6.9 is a two-dimensional nonlinear Schrödinger equation similar to that written in the set of Equations 3.23 and 3.24. Both reduce to the same one-dimensional nonlinear Schrödinger equations when spatial variation is regimented to one direction. Therefore, in the sense that stability was considered in Section 6.1, both have the same stability characteristics. It seems reasonable to expect that both may have a similar interaction of Benjamin-Feir instability and recurrence. As yet, these relationships are unknown.

7. Transition and Symmetry

7.1 Transition among solitons

What are the necessary conditions for a soliton to change significantly or split into a multiple of other solitons? What are the necessary conditions for the inverse to occur? Are there rules which govern the transition of a soliton into a modification of itself, or into a disintegration stage from which a new set of solitons arise? These phenomena are observed in experiments on waves generated by wind over deep water as executed by Mollo-Christensen and Ramamonjiarisoa (1979)[24]. It is not possible to analytically approach the system these experiments were done on at this time. Transition among solitons of the KdV equation is considered.

7.2 Inverse scattering transform solution to the KdV equation

Ablowitz, Kaup, Newell and Segur (1974)[25] discuss solutions to the KdV equation

$$U_t + UU_x + U_{xxx} = 0 \qquad 7.1$$

via the inverse scattering transform. Initial conditions require that $u(x,0)$ decay to zero sufficiently rapidly as $x \to \infty$.

The associated linear eigenvalue problem is,

$$\varphi_{xx} + (\zeta^2 + \tfrac{1}{6}\, u(x,t))\,\varphi = 0 \qquad 7.2$$

which appeared in Chapter 2, Equation 2.24. This is the one-dimensional time independent linear Schrödinger equation of quantum mechanics. For real $u(x,t)$, the spectrum of Equation 7.2 consists of a finite number of discrete imaginary values $\zeta = iK_n$, where $n = 1, \cdots, N$ and a continuum set of real ζ. The corresponding eigenfunctions have the asymptotic behavior, when $\zeta = iK_n$,

$$\varphi_n(x,t) \longrightarrow \begin{cases} C_n(t)\, e^{-K_n x} & , x \to \infty \\ D_n(t)\, e^{K_n x} & , x \to -\infty \end{cases}$$

and when $\zeta = k$, k is real,

$$\varphi(k,x,t) \longrightarrow \begin{cases} e^{-ikx} + R(k,t)\, e^{ikx} & , x \to \infty \\ T(k,t)\, e^{-ikx} & , x \to -\infty \end{cases}$$

There is a normalization constraint, $\int_{-\infty}^{\infty} |\varphi_n|^2\, dx = 1$. Gardiner, Greene, Kruskal and Muira (1967)[26] found that when $u(x,t)$ evolves as in Equation 7.1, the spectrum of Equation 7.2 is time invariant, if

$$\varphi_t = \left\{ \frac{u_x}{6} + c \right\} \varphi + \left\{ 4\zeta^2 - \frac{u}{3} \right\} \varphi_x \tag{7.3}$$

When ζ is real, $c = 4iR^3$. When ζ is imaginary, $c = 0$. The finding of $\varphi(\zeta,x,t)$ as a function of time requires a knowledge of $u(x,t)$. In the limit $x \to \pm\infty$, this is no longer a constraint. The time evolution of the scattering data is

$$(K_n)_t = 0 \quad , \quad T_t(k,t) = 0$$

$$R_t(k,t) = 8ik^3 R(k,t) \qquad , \quad [C_n^2(t)]_t = 8K_n^3 C_n^2(t)$$

Define,

$$B(p,t) = \sum_{n=1}^{N} C_n^2(t) e^{-K_n p} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t) e^{ikp}\, dk \tag{7.4}$$

and solve the Gelfand-Levitan equation,

$$K(x,y,t) + B(x+y,t) + \int_x^{\infty} dz\, K(x,z,t)\, B(x+z,t) = 0 \tag{7.5}$$

for $y > x$. Boundary conditions are $K(x,z) \to 0$ as $z \to \infty$. Thence, $u(x,t) = 12\frac{d}{dx}K(x,x,t)$. Segur (1973)[27] presents a computation of $u(x,t)$ when $N = 1$,

$$u(x,t) = 12\, K_1^2 \,\mathrm{sech}^2\{K_1(x - x_1 - 4K_1^2 t)\} \quad , \quad K_1 x_1 = \log\sqrt{\frac{C_1}{2K_1}} \tag{7.6}$$

and when $N = 2$

$$u(x,t) = \frac{48K_1^2\left\{\frac{1}{f_2} + \left(\frac{K_1-K_2}{K_1+K_2}\right)f_2\right\}^2 + 48K_2^2\left\{\frac{1}{f_1} + \left(\frac{K_2-K_1}{K_1+K_2}\right)f_1\right\}^2}{\left\{\frac{1}{f_1 f_2} + \frac{f_1}{f_2} + \frac{f_2}{f_1} + \left(\frac{K_1-K_2}{K_1+K_2}\right)^2 f_1 f_2\right\}^2} \tag{7.7}$$

where

$$f_1 = \frac{c_1}{\sqrt{2K_1}}\, e^{4K_1^3 t - K_1 x} \quad , \quad f_2 = \frac{c_2}{\sqrt{2K_2}}\, e^{4K_2^3 t - K_2 x}$$

When $K_1 > K_2 > 0$ and $|t| \to \infty$, Equation 7.7 implies,

$$u(x,t) \to 12K_1^2 \operatorname{sech}^2\left[K_1(x - x_1^{\pm} - 4K_1^2 t)\right] + 12K_2^2 \operatorname{sech}^2\left[K_2(x - x_2^{\pm} - 4K_2^2 t)\right] \qquad 7.8$$

where,

$$x_1^- = \frac{1}{2K_1}\log C_1^2 - \frac{1}{K_1}\log\left|\frac{K_1+K_2}{K_1-K_2}\right| \quad , \quad x_2^- = \frac{1}{2K_2}\log\frac{C_2^2}{2K_2}$$

$$x_1^+ = x_1^- + \frac{1}{K_1}\log\left|\frac{K_1+K_2}{K_1-K_2}\right| \quad , \quad x_2^+ = x_2^- - \frac{1}{K_2}\log\left|\frac{K_1+K_2}{K_1-K_2}\right|$$

Solitons survive interactions, retaining their identity, suffering only phase shift, as observed by Lax (1968)[28].

7.3 Transition among KdV solitons

Suppose that the spectrum of Equation 7.2 consists of a doubly degenerate eigenvalue. This corresponds to a system having a single soliton solution, as described by Equation 7.6. At some instant in time let a perturbative influence be imposed upon the system. In all prior time, as in all later time, the spectrum is time invariant, due to eigenfunction evolution Equation 7.3. However, with the imposition of the perturbative influence, Equation 7.2 has a spectrum of two discrete eigenvalues, assuming the degeneracy splits. The corresponding solution of the KdV equation consists of two solitons, as described by

Equations 7.7 and 7.8. Initially, as the perturbation is being applied to the system, the solution of the KdV equation looks quite like Equation 7.6 describing a single soliton. In due time, the two 2 solitons order themselves according to size and speed, as in Equation 7.8. The system has made a transition from 1 soliton to 2 solitons.

As it turns out, the author does not know of any $u(x,t)$ which would have degenerate eigenvalues when used in Equation 7.2.

There is an alternate mechanism to the above scenario. Suppose that initially the system has a single soliton solution

$$u(x,t) = 2K_1^2 \operatorname{sech}^2\left\{ K_1(x - x_1 - 4K_1^2 t)\right\} \qquad , K_1 x_1 = \log\sqrt{\frac{C_1}{2K_1}} \tag{7.6}$$

Suppose that at some instant a perturbative influence is imposed upon the system so that Equation 7.6 is modified to

$$u(x,t) = 2K_1^2 \operatorname{sech}\left\{ K_1(x - x_1 - 4K_1^2 t)\right\} + \Delta \left.\frac{\partial U}{\partial \Delta}\right|_{\Delta=0} \tag{7.9}$$

where,

$$U = \frac{48K_1^2\left\{\frac{1}{f_2} + \left(\frac{-\Delta}{2K_1+\Delta}\right) f_2\right\} + 48(K_1+\Delta)^2\left\{\frac{1}{f_1} + \left(\frac{\Delta}{2K_1+\Delta}\right) f_1\right\}^2}{\left\{\frac{1}{f_1 f_2} + \frac{f_1}{f_2} + \frac{f_2}{f_1} + \frac{\Delta^2}{(2K_1+\Delta)^2} f_1 f_2\right\}^2}$$

$$f_1 = \frac{c}{\sqrt{2K_1}} \exp\left\{4K_1^3 t - K_1 x\right\}$$

$$f_2 = \frac{c}{\sqrt{2K_1+2\Delta}} \exp\left\{4(K_1+\Delta)^3 t - (K_1+\Delta)x\right\} \qquad \frac{\Delta}{K_1} \ll 1$$

The example is simply the limit of Equation 7.7 when $K_2 = K_1 + \Delta$ and Δ approaches zero. For all time after the instant the perturbation is imposed, the spectrum consists of two very close discrete eigenvalues, K_1, and $K_1 + \Delta$. The evolution equation ensures that K_1, and $K_1 + \Delta$ remain invariant with respect to time. Initially, as the perturbation is being applied to the system, it may seem as though only one soliton is present. In due time, the two solitons will order themselves according to size and speed, as in Equation 7.8. The system has made a transition from 1 soliton to 2 solitons.

7.4 Symmetry and directional instability

The Euler-Lagrange governing equations for ideal gas were generally found to be, in Chapter 2,

$$iK\psi_t + \frac{K^2}{2}\nabla^2\psi = \frac{C\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1}\psi + \lambda\psi \qquad 2.32$$

$$\rho_0 - \psi\psi^* J = 0 \qquad 2.2$$

and when the system is assumed at equilibrium,

$$\frac{K^2}{2}\nabla^2\psi = \frac{C\gamma(\psi\psi^*)^{\gamma-1}}{\gamma-1} + \lambda_1\psi \qquad 2.43$$

$$\rho_0 - \psi\psi^* J = 0 \qquad 2.2$$

Equations 2.32 and 2.43 may be linearized by setting $\psi = V + \Psi$ where $\frac{\Psi}{V} \ll 1$. Thus,

$$i\kappa\Psi_t + \frac{\kappa^2}{2}\nabla^2\Psi - \frac{c\gamma}{\gamma-1}(VV^*)^{\gamma-1}\left\{\Psi + V(\gamma-1)\left[\frac{\Psi}{V}+\frac{\Psi^*}{V^*}\right]\right\} = \lambda\Psi \qquad 7.10$$

$$\frac{\kappa^2}{2}\nabla^2\Psi - \frac{c\gamma}{\gamma-1}(VV^*)^{\gamma-1}\left\{\Psi + V(\gamma-1)\left[\frac{\Psi}{V}+\frac{\Psi^*}{V^*}\right]\right\} = \lambda_1\Psi \qquad 7.11$$

The representation, $\Psi = pe^{iq}$ and $V = Pe^{iQ}$, is generally valid. Assume that $q = Q$ so that Equations 7.10 and 7.11 may be written as,

$$i\kappa\Psi_t + \frac{\kappa^2}{2}\nabla^2\Psi - \frac{c\gamma}{\gamma-1}(VV^*)^{\gamma-1}(2\gamma-1)\Psi - \lambda\Psi = 0 \qquad 7.12$$

$$+\frac{\kappa^2}{2}\nabla^2\Psi - \frac{c\gamma}{\gamma-1}(VV^*)^{\gamma-1}(2\gamma-1)\Psi - \lambda_1\Psi = 0 \qquad 7.13$$

These equations are of the same form as the three-dimensional time dependent and time independent linear Schrödinger equations of quantum mechanics respectively, Let VV^* dependent only on r, the spherical polar radial coordinate. Equations 7.12 and 7.13 then represent a system having central forces. The angular momentum associated with the state Ψ is conserved. Suppose that $(VV^*)^{\gamma-1} = -1/r$. Equation 7.13 then has an infinite set of eigenvalues and eigenfunctions. The n level eigenvalue is n^2-fold degenerate. Merzbacher (1970)[20] considers the situation in detail.

A perturbation of the system can be expected to cause loss of symmetry, splitting of degenerate eigenfunctions, and transitions among these eigenfunctions. Loss of all symmetry implies loss of all degeneracy. This in turn implies that all transitions among eigenfunctions are allowed. This is discussed in Falicov (1966)[29] and Tinkham (1964)[30].

There is a weak generalization from this. Higher symmetry implies fewer transitions which implies greater stability. Absolute chaos implies minimal symmetry which implies ultimate instability.

7.5 The Rayleigh-Schrödinger perturbation formalism

The usual methods for computing the effect of a perturbation in shifting eigenvalues and eigenfunctions and in splitting degenerate eigenfunctions of a system described by Equation 7.2 are presented. The reference is Merzbacher (1970)[20].

Write Equation 7.2 as

$$\varphi_{n_{xx}} + V\varphi_n = K_n^2\varphi_n \quad , \quad H_0\varphi_n = K_n^2\varphi_n \tag{7.14}$$

where $H_0 = \frac{\partial^2}{\partial x^2} + V$. A small perturbation f is imposed on V. The system may be represented as having $H = H_0 + \delta f$ where δ lies between 0 and 1,

$$H\phi_n = E_n\phi_n \tag{7.15}$$

Eigenfunctions and eigenvalues of H may be expanded in powers of δ,

$$\phi_n = \varphi_n + \delta\,\varphi_n^{(1)} + \delta^2\varphi_n^{(2)} + \cdots$$

$$E_n = K_n^2 + \delta K_n^{(1)} + \delta^2 K_n^{(2)} + \cdots \tag{7.16}$$

which implies, equating coefficients of equal powers of δ ,

$$H_0 \varphi_n = K_n^2 \varphi_n \tag{7.17}$$

$$H_0 \varphi_n^{(1)} + f \varphi_n = K_n^{(2)} \varphi_n^{(1)} + K_n^{(1)} \varphi_n \tag{7.18}$$

$$H_0 \varphi_n^{(2)} + f \varphi_n^{(1)} = K_n^2 \varphi_n^{(2)} + K_n^{(1)} \varphi_n^{(1)} + K_n^2 \varphi_n$$

$$\vdots$$

Equations 7.14 and 7.17 are the same. Equation 7.18 may be rewritten as,

$$(H_0 - K_n^2) \varphi_n^{(1)} = (K_n^{(1)} - f) \varphi_n \tag{7.19}$$

which is an inhomogeneous linear equation for $\varphi_n^{(1)}$, given $K_n^{(1)}$.

In general given a vector v , and a Hermitian operator A with a complete set of eigenvectors such that for some u ,

$$Au = v \tag{7.20}$$

then either there are nontrivial solutions u' such that $Au' = 0$ and A has zero eigenvalues or there exists a unique inverse operator A^{-1} such that $u = A^{-1} v$. In the former case, given w such that $Aw = v$, then any $u = w + u'$ is also a solution to Equation 7.20.

Of course v can not have any component in the subspace spanned by u' . The argument for this is the following. Operator A may be represented as the sum of projection operators $\sum_i A'_i P_i$. Thus,

$Au' = 0$ means that regardless of the spaces into which each of the projection operators P_i, project vectors, these spaces are not the same as the subspace of which u' is a member. Letting P_0 be the projection operator of this latter space, then clearly $P_0 \sum_i A'_i P_i = 0$. In other words, $P_0 v = P_0 A u = 0$. There is no component of v in the subspace spanned by u'.

A specific w may be constructed. Assuming $Aw = v$, then

$$Aw = (1 - P_0) v \tag{7.21}$$

There is always some $\underset{\sim}{K}$ such that $A\underset{\sim}{K} = 1 - P_0$. Operator $\underset{\sim}{K}$ is simply what is needed so that $\underset{\sim}{K} v - w$ belongs to the subspace spanned by u'. There are an infinite number of other operators $\underset{\sim}{K}$, obtained one from the other by adding $P_0 \mathcal{B}$, where $\mathcal{B}$ is arbitrary. To select $\underset{\sim}{K}$ uniquely, the constraint $P_0 \underset{\sim}{K} = 0$ is imposed. Symbolically, $\underset{\sim}{K}$ may be represented as,

$$\underset{\sim}{K} = \frac{1 - P_0}{A}$$

Equation 7.21 implies $Aw = A\underset{\sim}{K} v$, or

$$w = \frac{1 - P_0}{A} v \tag{7.22}$$

This is a particular solution to $Aw = v$. By construction it is orthogonal to the subspace spanned by u',

$$\mathcal{P}_0 w = 0$$

Thus, if $Au' = 0$ has nontrivial solutions, then $Au = v$ has the general solution $u = u' + \left\{\frac{1-\mathcal{P}_0}{A}\right\} v$ where $\mathcal{P}_0 v = 0$.

To approach Equation 7.19, match $H_0 - K_n^2$ to A . The presence of nontrivial solutions, namely φ_n , to the corresponding homogeneous equation is evident. These are orthonormal since H_0 is a Hermitian operator. The condition corresponding to $\mathcal{P}_0 v = 0$ is

$$\mathcal{P}_n^{(0)} (K_n^{(1)} - f) \varphi_n = 0$$

where $\mathcal{P}_n^{(0)}$ is the projection vector for the direction φ_n . This follows from Equation 7.17. Because $\mathcal{P}_n^{(0)} \varphi_n = \varphi_n$, this last equation implies

$$K_n^{(1)} \varphi_n = \mathcal{P}_n^{(0)} f \varphi_n$$

The result of this is, as $\mathcal{P}_n^{(0)}$ is Hermitian,

$$K_n^{(1)} = (\varphi_n , \mathcal{P}_n^{(0)} f \varphi_n) = (\varphi_n , f \varphi_n) = f_{nn}$$

where the left inner product with φ_n has been taken. This is the first order correction to the nth level eigenvalue.

Eigenvalues are assumed nondegenerate. To each eigenvalue there corresponds a single eigenfunction. Equation 7.22 implies that the general solution to Equation 7.19 is

$$\varphi_n^{(1)} = c_n^{(1)}\varphi_n - \left\{ \frac{1 - P_n^{(0)}}{K_n^2 - \mathcal{H}_0} \right\} (K_n^{(1)} - f)\varphi_n \qquad 7.23$$

As $C_n^{(1)}$ is arbitrary, it may be set to zero. The symbol for $\underset{\sim}{K}$ is $\left\{ \frac{1 - P_n^{(0)}}{K_n^2 - \mathcal{H}_0} \right\}$. This acts on any solution of the homogeneous equation to give zero, implying that Equation 7.23 may be simplified to,

$$\varphi_n^{(1)} = \left\{ \frac{1 - P_n^{(0)}}{K_n^2 - \mathcal{H}_0} \right\} f\varphi_n \qquad 7.24$$

The summation of projection operators is unity, $1 = \sum_k P_k^{(0)}$, where $P_k^{(0)} f\varphi_n = \varphi_k(\varphi_k, f\varphi_n) = \varphi_k f_{kn}$. Thus, Equation 7.23 is simply,

$$\varphi_n^{(1)} = \sum_{k \neq n} \frac{\varphi_k f_{kn}}{K_n^2 - K_k^2} \qquad 7.25$$

This is the first order correction to the nth level eigenfunction. To first order in δ, $\varphi_n + \delta\varphi_n^{(1)}$ is normalized to unity.

Suppose that the unperturbed eigenvalue K_n^2 is doubly degenerate, that to this eigenvalue there corresponds two linearly independent eigenfunctions. The assumed expansion of Equation 7.16 is no longer valid because the unperturbed eigenfunction to which ϕ_n collapses as δ approaches zero is unknown. The denominator of Equation 7.25 vanishes.

Impose a perturbation on the system and assume that the eigenvalue level splits,

$$E_{n1} = K_n^2 + \delta E_{n1}^{(1)} + \delta^2 E_{n1}^{(2)} + \cdots$$

$$E_{n2} = K_n^2 + \delta E_{n2}^{(1)} + \delta^2 E_{n2}^{(2)} + \cdots \tag{7.26}$$

The unperturbed degenerate eigenfunctions φ_{n1} and φ_{n2} are assumed orthonormal. The perturbed eigenvectors corresponding respectively to the eigenvalue expansions above are,

$$\phi_{n1} = c_{11}\varphi_{n1} + C_{21}\varphi_{n2} + \delta\varphi_{n1}^{(1)}$$

$$\phi_{n2} = c_{12}\varphi_{n1} + C_{22}\varphi_{n2} + \delta\varphi_{n2}^{(1)} \tag{7.27}$$

These expansions are substituted into Equation 7.17, $\mathcal{H}\phi_n = E_n\phi_n$, and powers of δ are equated. The results are,

$$\{\mathcal{H}_0 - K_n^2\}\varphi_{n1} = 0$$

$$\{\mathcal{H}_0 - K_n^2\}\varphi_{n2} = 0$$

$$\{\mathcal{H}_0 - K_n^2\}\varphi_{n1}^{(1)} = (E_{n1}^{(1)} - f)\{c_{11}\varphi_{n1} + c_{21}\varphi_{n2}\} \tag{7.28}$$

$$\{\mathcal{H}_0 - K_n^2\}\varphi_{n2}^{(1)} = (E_{n2}^{(1)} - f)\{c_{12}\varphi_{n1} + c_{22}\varphi_{n2}\} \tag{7.29}$$

The inhomogeneous terms, that is the right sides of Equations 7.28 and 7.29, have no component in the subspace spanned by solutions, φ_{n1} and φ_{n2}, of the corresponding homogeneous equations. The operator appropriate to this space is $\mathcal{P}_n^{(0)} = \mathcal{P}_{n1}^{(0)} + \mathcal{P}_{n2}^{(0)}$. Applying it to Equation 7.28 implies,

$$c_{11}\left\{ E_{n1}^{(1)}\varphi_{n1} - \mathcal{P}_n^{(0)} f \varphi_{n1} \right\} + c_{21}\left\{ E_{n1}^{(1)}\varphi_{n2} - \mathcal{P}_n^{(0)} f \varphi_{n2} \right\} = 0 \qquad 7.30$$

The operator $\mathcal{P}_n^{(0)}$ may be expanded,

$$\mathcal{P}_n^{(0)} f \varphi_{n1} = \varphi_{n1} f_{11} + \varphi_{n2} f_{21} \quad , \quad \mathcal{P}_n^{(0)} f \varphi_{n2} = \varphi_{n1} f_{12} + \varphi_{n2} f_{22}$$

Equation 7.30 is separable into the set of equations,

$$c_{11}\left\{ f_{11} - E_{n1}^{(1)} \right\} + c_{21} f_{12} = 0 \quad , \quad c_{11} f_{21} + c_{21}\left\{ f_{22} - E_{n1}^{(1)} \right\} = 0 \qquad 7.31$$

which has nontrivial solutions if the determinant vanishes. Equation 7.29 may be handled in the same way and leads to,

$$c_{12}\left\{ f_{11} - E_{n2}^{(1)} \right\} + c_{22} f_{12} = 0 \quad , \quad c_{12} f_{21} + c_{22}\left\{ f_{22} - E_{n2}^{(1)} \right\} = 0 \qquad 7.32$$

This set of equations is of the same form as Equation 7.31. The secular equation gives $E_{n1}^{(1)}$ and $E_{n2}^{(1)}$ as well as the correct relations of c_{11} to c_{21} and c_{12} to c_{22}.

8. Motivating Features and Concluding Remarks

The foregoing has sought clarification of an apparent duality in the nature of certain nonlinear systems of dynamics. Observationally, sea surface waves are often seen to be turbulent, chaotic and random. On the other hand, there are circumstances in which these same systems manifest coherently structured waveforms.

At the outset, an Euler-Lagrange equation similar in form to the linear Schrödinger equation, Equation 7.12, was sought. The operator of this relation is Hermitian, something to which symmetry arguments may be applied. It was not expected that nonlinear Schrödinger equations would arise.

It seems essential that a given nonlinear system be described in terms of Lagrangian coordinates, that similarities between material elements and classical particles be evident. The approach to systems described by Eulerian coordinates is less clear. Seliger and Whitham (1968)[9] show that often these systems may be represented by a variational principle where the Lagrangian is simply the pressure. However there is no analogy between material elements and classical particles. When the system was represented by Lagrangian coordinates, this analogy was the key to motivating the path taken.

Consider the ideal gas dynamics example. A material element was taken to have Lagrangian coordinates $(\alpha_1, \alpha_2, \alpha_3)$ at time $t=0$. The momentum equation is

$$\rho \frac{\partial^2 x_i}{\partial t^2} = -\frac{\partial p}{\partial x_i} \quad , \quad i = 1,2,3 \qquad 2.1$$

where $\chi(\underline{\alpha},t)$ is the material element trajectory, $\rho(\underline{\alpha},t)$ the density, and $p(\underline{\chi},t)$ the pressure. The continuity relation is,

$$\frac{1}{\rho} = \frac{J}{\rho_0} \quad , \quad J = \frac{\partial(\chi_1, \chi_2, \chi_3)}{\partial(\alpha_1, \alpha_2, \alpha_3)} \tag{2.2}$$

where $\rho_0(\underline{\alpha})$ is the initial density.

The dynamics of the system are described in terms of a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho_0 \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} - \rho_0 E + \lambda(\rho_0 - \rho J) \tag{2.10}$$

where E is the internal energy and λ, a Lagrange multiplier. The canonical momentum is,

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \frac{\partial \chi_i}{\partial t}} = \rho_0 \frac{\partial \chi_i}{\partial t} \tag{2.11}$$

the Hamiltonian density, following Goldstein (1950)[31]

$$\mathcal{H} = \pi_i \frac{\partial \chi_i}{\partial t} - \mathcal{L} = \frac{1}{2}\rho_0 \frac{\partial \underline{\chi}}{\partial t} \cdot \frac{\partial \underline{\chi}}{\partial t} + \rho_0 E - \lambda(\rho_0 - \rho J) \tag{2.12}$$

An action $\mathcal{S}$, and an action density $\mathcal{A}$, are introduced, where

$$\pi_i = \frac{\partial \mathcal{A}}{\partial \chi_i} \tag{2.17}$$

The Hamilton-Jacobi relation is

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H} = 0 \qquad 2.19$$

and a corresponding variational principle,

$$0 = \delta \int_{t_1}^{t_2}\int_{R_0} \left\{ \frac{\partial \mathcal{S}}{\partial t} + \mathcal{H} \right\} d\underline{\alpha}\, dt \qquad 2.20$$

A mapping of dependent variables is made,

$$\rho = \psi\psi^* \qquad 2.21$$

$$\frac{\partial \underline{x}}{\partial t} \cdot \frac{\partial \underline{x}}{\partial t} = \left\{ K \frac{J_{\ell m}}{J} \frac{\partial}{\partial \alpha_m} \log \psi \right\} \left\{ K \frac{J_{\ell n}}{J} \frac{\partial}{\partial \alpha_n} \log \psi^* \right\} + \mathcal{F}(\underline{\alpha}, t)$$

$$= K^2 \nabla_{\underline{x}} \log \psi \cdot \nabla_{\underline{x}} \log \psi^* + \mathcal{F}(\underline{x}, t) \qquad 2.22$$

where $\mathcal{F}(\underline{x},t) = 2K^2 \nabla\Omega \cdot \nabla \log R$ and K is a real constant.

It is important that the system have a continuity relation such as Equation 2.2 or an equivalent, as in the shallow water waves example. Otherwise an alternative to Equation 2.21 must be written.

A consistent choice for $\mathcal{S}_t$ is

$$\mathcal{S}_t = \rho_0 \frac{K}{2} \left\{ (1-i)(\log \psi)_t + (1+i)(\log \psi^*)_t \right\} \qquad 2.26$$

Coordinates of Equation 2.20 are transformed from $\underline{\alpha}$-space to $\underline{x}$-space. Variations are taken with respect to ψ . Resultant governing equations are,

$$i\kappa\psi_t + \frac{\kappa^2}{2}\nabla^2\psi = \frac{\partial}{\partial\psi^*}(E\psi\psi^*) + \lambda\psi \qquad 2.32$$

$$\rho_0 - J\rho = 0 \qquad 2.2$$

In varying the integral of Equation 2.20 with respect to ψ , the constraint $\frac{\delta}{\delta\psi}\mathcal{F}\psi\psi^* = 0$ is imposed, The Eulerian velocity is found to be $\underline{v} = \kappa\nabla\Omega$. Therefore, $\mathcal{F}\psi\psi^* = \kappa\underline{v}.\nabla h$ in the shallow water waves example, and $\mathcal{F}\psi\psi^* = \kappa\underline{v}.\nabla\rho$ in the dynamics of ideal gas example. In the former problem, a further transformation,

$$\lambda = P\Phi_y \quad , \quad \psi\psi^* = Q\left\{\Phi_y - \int^x \Phi_{xx}\,dy\right\} \qquad 3.19$$

leads to the resultant set of equations

$$2i\psi_t + \psi_{xx} + \psi_{yy} = \frac{9}{2}\psi^2\psi^* - 3\Phi_y\psi \qquad 3.23$$

$$\Phi_{xx} - \Phi_{yy} = -3(\psi\psi^*)_y \qquad 3.24$$

Application of inverse scattering techniques implies the simplest waveform solutions of constant form,

$$\psi = A\varphi\, e^{imt} \qquad 4.1$$

where

$$\varphi = \frac{1 - e^{-\mu} e^{2i\Delta}}{1 + e^{-\mu}} \qquad 4.57$$

and

$$\mu = -\frac{9A^2}{2}\cos\Delta\left\{ t\cos 2\left(\Theta + \frac{\pi}{4}\right)\sin\Delta - \frac{2x}{3A}\sin\left(\Theta + \frac{\pi}{4}\right) + \frac{2iy}{3A}\cos\left(\Theta + \frac{\pi}{4}\right)\right\} + \delta_0$$

$$\Theta = \tfrac{1}{2}(\theta + \phi) \quad , \quad \Delta = \tfrac{1}{2}(\theta - \phi)$$

$$\delta_0 = -\log\left\{ \tfrac{1}{2}\lambda_0 \sec\Delta \right\} \qquad 4.58$$

Equation 3.19 amounts to the constraint,

$$\left\{\frac{\rho_0}{J}\right\}_{yy} = \frac{\mathcal{Q}}{\mathcal{P}}\lambda_{yy} - \frac{\mathcal{Q}}{\mathcal{P}}\lambda_{xx} \qquad 8.1$$

Variations with respect to $\underline{\chi}$ in Equation 2.20, yield Euler-Lagrange equations in which λ does not appear. The transformation of coordinates from $\underline{\alpha}$ -space to $\underline{\chi}$ -space and the mapping of dependent variables from $\frac{\partial \underline{x}}{\partial t}$ and ρ to ψ and ψ^* changes the situation so that λ does appear. Were it not for the need to introduce λ in accounting for the continuity relation there would be no known method of solving the resultant Euler-Lagrange equations.

Suppose that the frame of reference is rotating uniformly at angular velocity $\underset{\sim}{\omega}$. It is not clear how to apply the foregoing formalism to the variables of the rotating frame. In place of Equation 2.1, the momentum equation is

$$\rho\left\{ \frac{\partial^2 \underset{\sim}{\chi}}{\partial t^2} + 2\left(\underset{\sim}{\omega} \times \frac{\partial \underset{\sim}{\chi}}{\partial t}\right) + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{\chi}) \right\} = -\nabla p - \rho g \qquad 8.2$$

A time independent, equilibrium, version of the problem may be considered. The Hamiltonian density integrated over an ensemble of material elements is assumed extremal with respect to variations in material element trajectories. That is,

$$0 = \delta\int_{R_0(\underset{\sim}{\alpha})} H\, d\underset{\sim}{\alpha} = \delta\int_{R_0(\underset{\sim}{\alpha})} \left\{ \frac{K^2}{2}\nabla\psi . \nabla\psi^* + E\psi\psi^* + (\lambda + \tfrac{\mathcal{F}}{2})\psi\psi^* \right\} d\underset{\sim}{x} - \delta\int_{R_0(\underset{\sim}{\alpha})} \lambda\rho_0\, d\underset{\sim}{\alpha} \qquad 2.42$$

where the mapping of Equations 2.21 and 2.22 has been substituted. Coordinates are transformed from $\underset{\sim}{\alpha}$ -space to $\underset{\sim}{\chi}$-space. Variations in ψ are taken subject to $\dfrac{\delta \mathcal{F}\psi\psi^*}{\delta\psi} = 0$. Resultant governing equations are

$$\frac{K^2}{2}\nabla^2\psi = \frac{\partial}{\partial\psi^*}(E\psi\psi^*) + \lambda\psi \qquad 2.43$$

$$\rho_0 - \psi\psi^* J \qquad 2.2$$

An attempt was made to discuss transition among KdV solitons. In general, given a nonlinear evolution equation solvable by the inverse scattering transform, there is an associated linear eigenvalue problem.

An understanding of the degeneracies, the symmetry, and the manner in which degeneracies may be eliminated within the linear eigenvalue problem is sought. The hope is that this will determine how transitions among solitons occur within the system. The KdV equation almost seems to work this way.

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Acknowledgements

I am sincerely thankful to Professor Edward Lorenz for his patient encouragement and support, and to Professor Erik Mollo-Christensen for his many very useful suggestions and encouragement.

I am very thankful to Professors Glenn Flierl and Louis Howard and visiting Professor James Curry of the University of Colorado for quite useful discussions.

I want to thank my teachers in meteorology, oceanography, applied mathematics and at the University of California for the outstanding ability, enthusiasm, and concern they have shown.

I am very grateful to my family and many, many friends and colleagues for their love, prayers, and encouragement, through whom I have gained hope, through whom the Lord has shown his love and majesty unto me.

As is written,

> Thy mercy, O Lord, is in the heavens;
> and thy faithfulness reacheth the clouds.
> Thy righteousness is like the great mountains
> thy judgements are a great deep.
> O Lord, thou preservest man and beast.
>
> Psalms 36: 5-6

Many thanks are due Liz Manzi for typing the manuscript and Steven Ricci for drawing the figures.

This research was supported by the National Science Foundation under grant 77 10093 ATM.

Biography

William Perrie was born on April 5, 1950 in Brussels, Ontario, Canada. He attended local public schools and graduated from the University of Toronto in 1973 with an honour's bachelor degree in physics and applied mathematics. He pursued graduate studies at the University of California at Berkeley in physics, applied mathematics, and biophysics, gaining an interest in nonlinear wave phenomena. In September, 1975 he came to M.I.T.