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Does the Integrability Condition on the Boundary Parameters of the Asymmetric Exclusion Process Contradict the Matrix Product Ansatz

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Abstract

This paper is due to the question whether with the constraint from the Bethe ansatz solution on the ASEP boundary parameters the matrix product ansazt fails to produce the stationary state. We discuss applicability of the matrix product ansatz from the point of view of the tridiagonal boundary algebra which reveals deep symmetry properties of the ASEP. We argue that the Bethe ansatz integrability constraint is the defining condition for a finite dimensional representation of the boundary algebra when the system reaches a steady state satisfying detailed balance.

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1 Introduction

The asymmetric simple exclusion process (ASEP) has become a paradigm in nonequilibrium physics [1, 2] due to its simplicity, rich behaviour and wide range of applicability. It is an exactly solvable model of an open many-particle stochastic system interacting with hard core exclusion. Introduced originally as a simplified model of one dimensional transport for phenomena like hopping conductivity [3] and kinetics of biopolymerization [4], it has found applications from traffic flow [5] to interface growth [6], shock formation [7], hydrodynamic systems obeying the noisy Burger equation, problems of sequence alignment in biology [8]. At large time the ASEP exhibits relaxation to a steady state, and even after the relaxation it has a nonvanishing current. An intriguing feature is the occurrence of boundary induced phase transitions [9] and the fact that the stationary bulk properties strongly depend on the boundary rates.

The ASEP dynamics is governed by a master equation for the probability distribution $P(s_i, t)$ of a stochastic variable $s_i = 0, 1$, at a site $i = 1, 2, ...L$ of a linear chain. In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration of the system $s_i = 0$ if a site i is empty, or $s_i = 1$ if the site i is occupied. On successive sites particles hop with probability $g_{01}dt$ to the left, and $g_{10}dt$ to the right. The event of hopping occurs if out of two adjacent sites one is a vacancy and the other is occupied by a particle. The symmetric simple exclusion process is the lattice gas model of particles hopping between nearest-neighbour sites with a constant rate $g_{01} = g_{10} = g$. The asymmetric simple exclusion process with hopping in a preferred direction is the driven diffusive lattice gas of particles moving under the action of an external field. The process is totally asymmetric if all jumps occur in one direction only, and partially asymmetric if there is a different non-zero probability of both left and right hopping. The number of particles in the bulk is conserved and this is the case of periodic boundary conditions. In the case of open systems, the lattice gas is coupled to external reservoirs of particles of fixed density and additional processes can take place at the boundaries. Namely, at the left boundary $i = 1$ a particle can be added with probability αdt and removed with probability γdt, and at the right boundary $i = L$ it can be removed with probability βdt and added with probability δdt. Without loss of generality we can choose the right probability rate $g_{10} = 1$ and the left probability rate $g_{01} = q$. The totally asymmetric process corresponds to $q = 0$ and the symmetric - to $q = 1$.

The master equation has the form

$$\frac{dP(s,t)}{dt} = \sum_{s'} \Gamma(s,s')P(s',t) \tag{1}$$

where the transition rate matrix (which in the general case of n-species process is an $n^2 \times n^2$ matrix) has the property that its columns sum up to zero due to probability conservation. The RHS of eq.(1) reads explicitly

$$\sum_{s' \neq s} \Gamma(s' \to s)P(s',t) - \sum_{s' \neq s} \Gamma(s \to s')P(s',t) \tag{2}$$

where the first term gives contributions of all possible hops (transitions) into configuration $\{s\}$ from other configurations $\{s'\}$ while the second term accounts for the transitions out of configuration $\{s\}$ into other configurations $\{s'\}$. In a steady state these gain and loss terms are equal providing thus a balance condition and causing the time derivative to vanish.

The master equation can be mapped to a Schroedinger equation in imaginary time $\frac{dP(t)}{dt} = -HP(t)$ for a quantum Hamiltonian with nearest-neighbour interaction in the bulk and single-site boundary terms. The ground state of the Hamiltonian, in general non-Hermitian, corresponds to the steady state of the stochastic dynamics where all probabilities are stationary. Through the mapping the equivalence to the integrable spin 1/2 XXZ quantum spin chain by means of the similarity transformation $\Gamma = -qU_\mu^{-1}H_{XXZ}U_\mu$(see [10] for the explicit form of U_μ); H_{XXZ} is the Hamiltonian of the $U_q(su(2))$ invariant quantum spin chain with anisotropy Δ_q and with added non diagonal boundary terms B_1 and B_L.

$$H_{XXZ} = H_{XXZ}^{QGr} + B_1 + B_L \tag{3}$$

$$H_{XXZ}^{QGr} = -1/2\sum_{i=1}^{L-1}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta_q\sigma_i^z\sigma_{i+1}^z + h(\sigma_{i+1}^z - \sigma_i^z) + \Delta_q) \tag{4}$$

with

$$\Delta_q = -\frac{1}{2}(q + q^{-1}), \quad h = \frac{1}{2}(q - q^{-1}) \tag{5}$$

The transition rates of the ASEP are related to the boundary terms in the following way (μ is a free parameter, irrelevant for the spectrum)

$$\begin{aligned} B_1 &= \frac{1}{2q}\left(\alpha + \gamma + (\alpha - \gamma)\sigma_1^z - 2\alpha\mu\sigma_1^- - 2\gamma\mu^{-1}\sigma_1^+\right) \\ B_L &= \frac{\left(\beta + \delta - (\beta - \delta)\sigma_L^z - 2\delta\mu q^{L-1}\sigma_L^- - 2\beta\mu^{-1}q^{-L+1}\sigma_L^+\right)}{2q} \end{aligned} \tag{6}$$

The mapping allowed for the derivation of exact results for the ASEP using Matrix Product Approach (MPA) and Bethe Ansatz (BA).

Bethe Ansatz solution was recently achieved [11, 12] for the XXZ chain Hamiltonian and subsequently for the ASEP [13] provided the model parameters satisfy a condition which in terms of the ASEP notations reads

$$(q^{\frac{1}{2}(L+2k)} - 1)(\alpha\beta - q^{\frac{1}{2}(L-2k-2)}\gamma\delta) = 0 \tag{7}$$

with $|k| \leq L/2$. Given k, the first factor zero in (1) assumes q to be a root of unity which is unacceptable for the ASEP. The second factor zero imposes a relation between the bulk and the boundary parameters and, as pointed out by de Gier and Essler, it is believed [10, 14, 15] that with this constraint the MPA fails to produce the stationary state. Quite remarkably, however, for generic q, one can satisfy the constraint [13] by choosing k to be $k = -L/2$, which, as commented in [14] implies additional symmetries for the ASEP.

Emphasizing the dependence of the steady state bulk behaviour on the boundary rates we consider a tridiagonal boundary algebra which reveals hidden symmetries of the ASEP. We discuss the consequences of the algebraic properties on the applicability of the MPA in relation to the BA integrability condition. We show that the choice of k for which (7) is automatically satisfied for arbitrary values of the parameters is intimately related to the boundary algebra of the ASEP within the matrix ansatz.

2 Matrix Product State Ansatz

The idea is that the steady state properties of the ASEP can be obtained exactly in terms of matrices obeying a quadratic algebra [7, 16]. For a given configuration $(s_1, s_2, ..., s_L)$ the stationary probability is defined by the expectation value $P(s) = \frac{\langle w|D_{s_1}D_{s_2}...D_{s_L}|v\rangle}{Z_L}$, where $D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied and $D_{s_i} = D_0$ if a site i is empty and $Z_L = \langle w|(D_0+D_1)^L|v\rangle$ is the normalization factor to the stationary probability distribution. The operators $D_i, i = 0, 1$ satisfy the quadratic (bulk) algebra

$$D_1D_0 - qD_0D_1 = x_1D_0 - D_1x_0, \qquad x_0 + x_1 = 0 \tag{8}$$

with boundary conditions of the form

$$\begin{aligned}(\beta D_1 - \delta D_0)|v\rangle &= x_0|v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= -\langle w|x_1.\end{aligned} \tag{9}$$

and $\langle w|v\rangle \neq 0$. We stress the one parameter dependence of the MPA due to $x_0 = -x_1 = \zeta$ with $0 < \zeta < \infty$. In most known applications it is fixed and restricted to the choice $\zeta = 1$. In our opinion it is important to keep this parameter since the relation $x_0 + x_1 = 0$ implies an abelian symmetry with a conserved quantity D_0+D_1. As readily seen from $D_0 \to D_0+x_0, D_1 \to D_1+x_1$ it follows $D_0 + D_1 \to D_0 + D_1$.

The MPA is an efficient method to evaluate exactly all the relevant physical quantities such as the mean density at a site i,

$$\langle s_i\rangle = \frac{\langle w|(D_0 + D_1)^{i-1}D_1(D_0 + D_1)^{L-i}|v\rangle}{Z_L} \tag{10}$$

the two-point correlation function

$$\langle s_is_j\rangle = \frac{\langle w|(D_0 + D_1)^{i-1}D_1(D_0 + D_1)^{j-i-1}D_1(D_0 + D_1)^{L-j}|v\rangle}{Z_L} \tag{11}$$

and higher the correlation functions, the current J through a bond between sites i and $i+1$ which has a very simple form $J = \zeta\frac{Z_{L-1}}{Z_L}$.

Exact results for the ASEP with open boundaries were obtained within the MPA through the relation of the stationary state to q-Hermite [18] and Al-Salam-Chihara polynomials [19] in the case $\gamma = \delta = 0$ and to the Askey-Wilson polynomials [20] in the general case. Finite dimensional representations [10, 17] have been considered too and they simplify calculations.

Due to a constraint on the model parameters they define an invariant subspace of the infinite matrices and give exact results only on some special curves of the phase diagram.

The efficiency of the matrix product ansatz is that the algebraic relations (8) provide solvable recurrences for the current and the correlation functions. For the ASEP such recurrence relations were already obtained in earlier works, however they were not readily generalized to other models. The MPA was readily generalized to many-species models and to dynamical MPA [21]. One conceptually important question remained open and it is now raised by the Bethe ansatz integrability constraint, namely the conditions under which the MPA is successfully applied. In particular, it important to have a clear interpretation of the second factor in the Bethe Ansatz constraint. The crucial point in shedding light on this is, in our opinion, to understand the importance of the parameter ζ behind $x_0 + x_1 = 0$ in eqs. (8) and (9). As we pointed out this parameter has been fixed to $\zeta = 1$. Then the boundary conditions become

$$\begin{aligned} (\beta D_1 - \delta D_0)|v\rangle &= |v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= \langle w| \end{aligned} \tag{12}$$

and with the simple argument given in Section IV of [10] it is very easy to see that for

$$\alpha\beta = \gamma\delta \tag{13}$$

the DEHP algebra as given in (8) (with $x_0 = -x_1 = 1$) and (10) does not have any representations (if $q \neq 1$). This conclusion is only true if $x_0 = -x_1 = \zeta$, for fixed nonzero ζ. As readily seen, if the RHS of eqs.(9) vanish, there exists a solution for the algebraic relations given by $\langle w|D_0|v\rangle \neq 0$ and $\langle w|D_1|v\rangle \neq 0$. From the point of the boundary algebra this solution corresponds to the finitedimensional representation of the latter. We will discuss it in detail in a later section.

3 Boundary Askey-Wilson Symmetry

In the general case of incoming and outgoing particles at both boundaries with all boundary parameters nonzero there are four operators $\beta D_1, -\delta D_0, -\gamma D_1, \alpha D_0$ and one needs an addition rule to form two linear independent boundary operators acting on the dual boundary vectors. Led by the equivalence of the ASEP to the XXZ quantum spin chain we argue that a solution to the boundary problem can be obtained by using the bulk $U_q(su(2))$ $(0 < q < 1)$ symmetry with

$$[N, A_\pm] = \pm A_\pm \qquad [A_+, A_-] = \frac{q^N - q^{-N}}{q^{1/2} - q^{-1/2}} \tag{14}$$

with a central element

$$Q = A_+A_- - \frac{q^{N-1/2} - q^{-N+1/2}}{(q^{1/2} - q^{-1/2})^2} \tag{15}$$

The representations, labelled by the values of the Casimir $Q(\kappa) = \frac{q^\kappa + q^{1-\kappa}}{q^{-1/2}(1-q)^2}$ for a fixed real parameter κ $(\kappa < 0)$, are finite dimensional and in a basis $|n, \kappa\rangle$ (where $|0, \kappa\rangle$ is the vacuum with $r_0 = 0$) are given by

$$
\begin{aligned}
N|n,\kappa\rangle &= (\kappa+n)|n,\kappa\rangle, \\
A_-|n,\kappa\rangle &= r_n|n-1,\kappa\rangle, \\
A_+|n,\kappa\rangle &= r_{n+1}|n+1,\kappa\rangle
\end{aligned} \tag{16}
$$

where $r_n^2 = \frac{(1-q^n)q^{1/2}(q^\kappa - q^{1-n-\kappa})}{(1-q)^2}$. The dimension $l+1$ of the spin $j = |\kappa|$-representation is defined by the condition

$$
q^\kappa - q^{-l-\kappa} = 0 \tag{17}
$$

for some $n = l$. As known the $U_q(su(2))$ can act on a lattice of L sites by means of the coproduct; for the XXZ spin chain the configuration space $(\mathbf{C}^2)^{\otimes L}$ is the tensor product of L two-dimensional representations which is a highly reducible representation of dimension 2^L. The highest weight subrepresentation is the spin $2j = L$ representation of dimension $L+1$. With the identification $l \equiv L$ eq.(17) becomes

$$
q^{L+2\kappa} - 1 = 0 \tag{18}
$$

which is the same condition on $\kappa \equiv k$ as the first factor in (7). Therefore the first factor in (7) can be interpreted as the constraint reflecting the $U_q(su(2))$ finitedimensional spin $2j = L$ representation of dimension $L+1$ with $j \equiv |k| \equiv |\kappa|$.

Given the $U_q(su(2))$ generators $A_\pm$ and N we can present the boundary operators in the form

$$
\begin{aligned}
\beta D_1 - \delta D_0 &= -\frac{x_1\beta}{\sqrt{1-q}} q^{N/2} A_+ - \frac{x_0\delta}{\sqrt{1-q}} A_- q^{N/2} \\
&- \frac{x_1\beta q^{1/2} + x_0\delta}{1-q} q^N - \frac{x_1\beta + x_0\delta}{1-q} \\
\alpha D_0 - \gamma D_1 &= +\frac{x_0\alpha}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{x_1\gamma}{\sqrt{1-q}} A_- q^{-N/2} \\
&+ \frac{x_0\alpha q^{-1/2} + x_1\gamma}{1-q} q^{-N} + \frac{x_0\alpha + x_1\gamma}{1-q}
\end{aligned} \tag{19}
$$

Separating the shift parts from the boundary operators and denoting the corresponding rest parts by A and A^* we can prove that the operators A and A^* defined by

$$
\begin{aligned}
A &= \beta D_1 - \delta D_0 + \frac{x_1\beta + x_0\delta}{1-q} \\
A^* &= \alpha D_0 - \gamma D_1 - \frac{x_0\alpha + x_1\gamma}{1-q}
\end{aligned} \tag{20}
$$

and their q-commutator $[A, A^*]_q = q^{1/2}AA^* - q^{-1/2}A^*A$ form a closed linear algebra

$$
\begin{aligned}
[[A, A^*]_q, A]_q &= -\rho A^* - \omega A - \eta \\
[A^*, [A, A^*]_q]_q &= -\rho^* A - \omega A^* - \eta^*
\end{aligned} \tag{21}
$$

where the operator-valued structure constants depend on $q, \alpha, \beta, \gamma, \delta$ and the $U_q(su(2))$ Casimir Q. In particular

$$\begin{aligned} -\rho &= x_0 x_1 \beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \\ -\rho^* &= x_0 x_1 \alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2 \end{aligned} \tag{22}$$

$$\begin{aligned} -\omega &= (x_1\beta q^{1/2}+x_0\delta)(x_1\gamma+x_0\alpha)-(x_1^2\beta\gamma+x_0^2\alpha\delta)(q^{1/2}-q^{-1/2})Q \\ \eta &= q1/2(q^{1/2}+q^{-1/2})\left(x_0x_1\beta\delta(x_1\gamma+x_0\alpha)Q-\frac{(x_1\beta q^{1/2}+x_0\delta)(x_1^2\beta\gamma+x_0^2\alpha\delta)}{q^{1/2}-q^{-1/2}}\right) \\ \eta^* &= q1/2(q^{1/2}+q^{-1/2})\left(x_0x_1\alpha\gamma(x_1\beta q^{1/2}+x_0\delta)Q+\frac{(x_0\alpha+x_1\gamma)(x_0^2\alpha\delta+x_1^2\beta\gamma)}{q^{1/2}-q^{-1/2}}\right) \end{aligned} \tag{23}$$

Relations (21) are the well known Askey-Wilson (AW) relations for the shifted boundary operators A, A^*. The algebra (21) was first considered in the works of Zhedanov [22, 23] and recently discussed in a more general framework of a tridiagonal algebra [24, 25]. This is an associative algebra with a unit generated by a (tridiagonal) pair of operators A, A^*, determined up to an affine transformation $A \to tA+c$, $A^* \to t^*A^*+c^*$ where t, t^*, c, c^* are some scalars. Taking the commutators of the first and second lines of (21) with A and A^* respectively one finds

$$\begin{aligned} [A[A[A,A^*]_q]_{q^{-1}}] &= \rho A, \\ [A^*[A^*[A^*,A]_q]_{q^{-1}}] &= \rho^* A^* \end{aligned} \tag{24}$$

which are the deformed version of the Dolan-Grady relations [26].

The AW algebra possesses some important properties that allow to obtain its ladder representations, spectra, overlap functions [22, 25, 27]. Namely, there exists a basis (of orthogonal polynomials) f_r according to which the operator A is diagonal and the operator A^* is tridiagonal. There exists a dual basis f_p in which the operator A^* is diagonal and the operator A is tridiagonal. The overlap function of the two basis $\langle s|r\rangle = \langle f_s^*|f_r\rangle$ is expressed in terms of the Askey-Wilson polynomials. Let $p_n = p_n(x; a, b, c, d|q)$ denote the nth AW polynomial [28] depending on four parameters a, b, c, d,

$$p_n = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ay, ay^{-1} \\ ab, ac, ad \end{matrix}\ |q;q\right) \tag{25}$$

with $p_0 = 1$, $x = y + y^{-1}$, $0 < q < 1$ and a three term recurrence relation

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0 \tag{26}$$

The explicit form of the matrix elements reads

$$a_n = a + a^{-1} - b_n - c_n \tag{27}$$

$$b_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \tag{28}$$

$$c_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})} \tag{29}$$

After rescaling $A \to \frac{1}{\beta}A$, $A^* \to \frac{1}{\alpha}A^*$ we have found the explicit form of the infinite dimensional representation (and the dual one) for the boundary operators. Let $\mathcal{A}$ denote the matrix satisfying (26) in the basis $(p_0, p_1, p_2, ...)$. *Result*: There exist a representation π in the space of Laurent polynomials with basis $(p_0, p_1, p_2, ...)^t$ with respect to which $\pi(D_1 - \frac{\delta}{\beta}D_0)$ is diagonal with diagonal eigenvalues

$$\lambda_n = \frac{1}{1-q}\left(bq^{-n} + dq^{n-1}\right) + \frac{1}{1-q}(1+bd) \tag{30}$$

and $\pi(D_0 - \frac{\gamma}{\alpha}D_1)$ is tridiagonal

$$\pi(D_0 - \frac{\gamma}{\alpha}D_1) = \frac{1}{1-q}b\mathcal{A}^t + \frac{1}{1-q}(1+ac) \tag{31}$$

The dual representation π^* has a basis $p_0, p_1, p_2, ...$ with respect to which $\pi^*(D_0 - \frac{\gamma}{\alpha}D_1)$ is diagonal with eigenvalues

$$\lambda_n^* = \frac{1}{1-q}\left(aq^{-n} + cq^{n}\right) + \frac{1}{1-q}(1+ac) \tag{32}$$

and $\pi^*(D_1 - \frac{\delta}{\beta}D_0)$ is tridiagonal

$$\pi^*(D_1 - \frac{\delta}{\beta}D_0) = \frac{1}{1-q}a\mathcal{A} + \frac{1}{1-q}(1+bd) \tag{33}$$

The choice

$$\langle w| = h_0^{-1/2}(p_0, 0, 0, ...), \qquad |v\rangle = h_0^{-1/2}(p_0, 0, 0, ...)^t \tag{34}$$

(where h_0 is a normalization) as eigenvectors of the diagonal matrices $\pi^*(D_0 - \frac{\gamma}{\alpha}D_1)$ and $\pi(D_1 - \frac{\delta}{\beta}D_0)$ respectively yields a solution to the boundary equations which uniquely relates a, b, c, d to $\alpha, \beta, \gamma, \delta$. Namely $a = \kappa_+^*(\alpha, \gamma)$, $b = \kappa_+(\beta, \delta)$, $c = \kappa_-^*(\alpha, \gamma)$, $d = \kappa_-(\beta, \delta)$ where

$$\kappa_\pm^{(*)}(\nu, \tau) = \frac{-(\nu - \tau - (1-q)) \pm \sqrt{(\nu - \tau - (1-q))^2 + 4\nu\tau}}{2\nu}. \tag{35}$$

It is important to emphasize that these are the functions of the parameters that define the phase diagram of the model. In previous applications they have been taken for granted, here they follow from the properties of the boundary algebra representations.

It can be further shown that the transfer matrix $D_0 + D_1$ and each of the boundary operators generate isomorphic AW algebras [29]. In the tridiagonal representation the transfer matrix $D_0 + D_1$ satisfies the three-term recurrence relation of the AW polynomials which was explored in [20] for the solution of the ASEP in the stationary state. The exact calculation of all the physical quantities, such as the current, correlation functions etc, in terms of the Askey-Wilson polynomials was achieved without any reference to the AW algebra. The ultimate relation of the exact solution in the stationary state to the AW polynomials was possible due to the AW boundary hidden symmetry of the ASEP with most general boundary conditions.

The condition for the representation to be finite dimensional is $b_n = 0$ for some $n = n_0 + n_f$, where $n_f = 1, 2, ...$ is the dimension of the representation. From the explicit form of b_n it follows that the representation is finite dimensional if any of the factors in the numerator in (28) is zero, in particular, with our normalization $abcd = \frac{\gamma\delta}{\alpha\beta}$, if the condition holds

$$\alpha\beta = q^{n_f - 1}\gamma\delta \tag{36}$$

Since we are using $U_q(su(2))$ besides the constraint (36) there is one more constraint (given by eq.(18)) which is in fact the first factor in (7). We thus obtain the condition for a finitedimensional representation of the boundary Askey-Wilson algebra

$$(q^{L+2\kappa} - 1)(\alpha\beta - q^{n_f - 1}\gamma\delta) = 0 \tag{37}$$

Comparison of eq. (36) with the second factor in (7) (with the value $k = -L/2$ as following from the first factor) suggests that the latter defines a finite dimensional representation of the MPA boundary operators of dimension $n_f = L$. As known [13, 30] the condition (7) is a reflection of the non-semisimplicity of an underlying Temperley Lieb algebra with two additional boundary generators (explored in the study of the XXZ chain with nondiagonal boundary terms). We thus obtain that the condition for finitedimensional representation of the Askey-Wilson boundary algebra constructed as the linear covariance for $U_q(su(2))$ using the $L+1$-dimensional representation of the latter coincides with the condition reflecting the non-semisimplicity of the two boundary Temperley-Lieb algebra.

Finite dimensional representations of the ASEP boundary algebra imply finite dimensional matrices D_0 and D_1. The examples considered in [17] correspond to the condition $ab = \kappa_+^* \kappa_+ = q^{1-L}$, which defines an L-dimensional representation of the boundary algebra according to the first factor in the nominator of b_n. A special case of the tridiagonal algebra with $c = d = 0$ is the open system with only incoming (outgoing) particles at the left (right) boundary when the (shifted) diagonal elements are of the form $q^{\pm n}$ and the tridiagonal operators satisfy the recurrence relation for either q-Hermite or Al-Salam-Chihara polynomials. A nontrivial special case of an open system with four boundary parameters related to the representations above with $b = 0$ and $c = 0$, was considered in [15] for the MPA solution of the weakly ASEP. The constraint $\alpha\beta = q^{L-1}\gamma\delta$ was found there to describe the detailed balance when the steady state is a Bernoulli measure at density, the same for both reservoirs.

4 Detailed Balance and Matrix Product Ansatz

Detailed balance (DB) means that the probability $P(\{s\})$ for a transition from a configuration $\{s\}$ to another configuration $\{s'\}$ is equal to the probability of the reverse transition. Thus for the event of hopping between site i and $i+1$ one has

$$P(s_1, ..., 0, 1, ..., s_L) = q^{-1} P(s_1, ..., 1, 0, ..., s_L), \tag{38}$$

for an event at the left boundary -

$$P(1,s_2,...,s_L)=\frac{\alpha}{\gamma}P(0,s_2,...,s_L) \tag{39}$$

and correspondingly at the right boundary

$$P(s_1,...,s_{L-1},1)=\frac{\delta}{\beta}P(s_1,...,s_{L-1},0) \tag{40}$$

Starting from a given configuration and using eqs. (38) and (39) one can always calculate the weights of all configurations by removing particles at the left boundary and the consistency with eq.(40) [15] requires $\alpha\beta=q^{L-1}\gamma\delta$ which is the DB condition. As can be readily verified the above detailed balance relations with P expressed as matrix elements of the type $\langle w|D_{s_1}D_{s_2}...D_{s_L}|v\rangle$ are consistent with the limit $x_0=-x_1=0$ of the MPA defining relations (8) and (9). The presence of the parameter dependent linear terms in the bulk algebra is due to the boundary processes and these quadratic-linear algebraic relations provide recursive expressions for matrix elements of the steady weights, the current, the correlation functions. The quadratic algebra induces a reordering property and an element of length L can be brought in a linear combination of elements of length $L-1$ with positive coefficients. Hence one can compute all matrix elements of length L if $\langle w|D_0^kD_1^{L-k}|v\rangle$, $k=0,1,...,L$, and all $(L-1)$- elements are known. From the boundary conditions (9) and after reordering one has

$$\beta\langle w|D_0^kD_1^{L-k}|v\rangle-q^{L-k-1}\delta\langle w|D_0^{k+1}D_1^{L-k-1}|v\rangle=x_0P_{L-1}$$

$$-q^k\gamma\langle w|D_0^kD_1^{L-k}|v\rangle+\alpha\langle w|D_0^{k+1}D_1^{L-k-1}|v\rangle=-x_1P'_{L-1}$$

where P_{L-1} and P'_{L-1} are positive linear combinations of matrix elements of length $L-1$. The conclusion is straightforward: If $\alpha\beta-q^{L-1}\gamma\delta\neq0$, then the recursions are consistent. If

$$\alpha\beta-q^{L-1}\gamma\delta=0 \tag{41}$$

then in order that a matrix element of length L exists, the zero determinant

$$x_0\alpha P_{L-1}-x_1q^{L-k-1}\delta P'_{L-1}=0 \tag{42}$$

implies

$$x_0=-x_1=0 \tag{43}$$

All matrix elements of length $L-k$, with $k=0,1,...,L-1$ are nonzero if $\alpha\beta-q^{L-k-1}\gamma\delta=0$ for some $\alpha,\beta,\gamma,\delta$. This has the consequence that the current

$$J=\zeta\frac{\langle w|(D_0+D_1)^{i-1}(D_1D_0-qD_0D_1)(D_0+D_1)^{L-i-1}|v\rangle}{Z_L} \tag{44}$$

vanishes and the probabilities satisfy the detailed balance condition. Thus with the constraint on the boundary parameters even though the boundary processes are present they become irrelevant and the nonequilibrium behaviour of the system is no longer maintained. The MPA in the limit $x_0 = x_1 = 0$ produces a stationary state whose probability weights satisfy detailed balance.

One may wonder that the boundary Askey-Wilson algebra has finite dimensional representations corresponding to the constraint $\alpha\beta - q^{L-1}\gamma\delta = 0$ for nonzero $x_0 = -x_1 = \zeta$ unlike the quadratic MPA algebra for which the constraint is consistent with $x_0 = -x_1 = 0$. The reason for this lies in the fact that the representations of the quadratic algebra are contained among the representations of the tridiagonal algebra whose representation theory is richer. (see [27] for details).

The boundary operators of the detailed balance case satisfy an Askey-Wilson algebra with $\eta = \eta^* = 0$ and $\rho = \rho^*$, $\omega \simeq Q$. Without writing the explicit form of the representation, related now to q-Hahn polynomials [31], we note that in the limit $x_0 = -x_1 = 0$ from the boundary equations it follows $b = d$ and $a = c$, with $b = \sqrt{-\frac{\beta}{\delta}}$ and $a = \sqrt{-\frac{\gamma}{\alpha}}$. Inserting it in the condition $abcd = a^2b^2 = q^{1-L}$ we obtain the DB condition. Formally $a^2b^2 = q^{1-L}$ for some parameters $a' = -a^2, b' = -b^2$, with $L = 1$, resembles the limit $q \to 1$ and we recover the detailed balance case at equal density $\rho_{a'} = \rho_{b'}$, with $\rho_{a'} = \frac{\alpha}{\alpha+\gamma}, \rho_{b'} = \frac{\beta}{\beta+\delta}$[32]. The system will be in a product measure state with uniform density and zero current when $a'b' = 1$, valid for the one dimensional representation. We can identify the one dimensional representation of the boundary operators with $\langle w|D_1|v\rangle$ and $\langle w|D_0|v\rangle$, which are non zero in the limit $x_0 = -x_1 = 0$. A non vanishing matrix element of length L can be formed as the product of single-site matrix elements $\langle w|D_1|v\rangle^{L-k}$ and $\langle w|D_0|v\rangle^k$. Given the detailed balance condition $\alpha\beta = q^{L-1}\gamma\delta$, we can always find corresponding $\alpha'\beta' = \gamma'\delta'$ which define the one dimensional representation of the boundary algebra and hence determine the steady state as a Bernoulli product measure at equal density for both reservoirs.

5 Conclusion

The MPA defining algebraic relations(8) and boundary conditions (9) provide solvable recursions for the stationary state, the current, the correlation functions of an open L-site nonequilibrium system for all values of the parameters in the range $0 \leq \alpha, \beta, \gamma, \delta, \leq 1$, $\alpha\gamma \neq 0, \quad \beta\delta \neq 0$, $0 < q < 1$ and $x_0 = -x_1 = \zeta > 0$. The recursions remain valid with final dimensional matrices of dimension L determined by (one of) the conditions $\kappa_+(\alpha, \gamma)\kappa_+(\beta, \delta) = \kappa_+(\alpha, \gamma)\kappa_-(\beta, \delta) = q^{1-L}$, which also define L dimensional representations of the boundary AW algebra. Exceptional subrange of parameters is given by the constraints $\alpha\beta = q^{L-1}\gamma\delta$ and $\zeta = 0$ when the MPA produces a stationary state satisfying detailed balance.

In the Bethe ansatz condition (7) the first factor zero with $k = -L/2$ defines a representation of dimension $L + 1$ of the bulk $U_q(su(2))$ symmetry. The second factor zero defines a finite L-dimensional representation of the boundary AW algebra. The BA condition relates a spin $|k| = L/2$ representation of the $U_q(su(2))$ symmetry to a finite L representation of the boundary algebra. The Bethe integrability condition on the boundary parameters corresponds to finite dimensional representations of the MPA matrices which constrain the parameters of the open system to a range when the system will eventually reach a steady state satisfying detailed balance. For arbitrary values of the ASEP parameters the Bethe ansatz condition is automatically satisfied by the spin $|k| = L/2$ representation of the bulk $U_q(su(2))$. The symmetry behind this property is the boundary algebra, which assures the exact solvability and allows for employing Bethe ansatz or applying matrix product ansatz with no restriction on the physics of the system.

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