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Statistical mechanics of strings with Y-junctions

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We investigate the Hagedorn transitions of string networks with Y-junctions as may occur, for example, with (p, q) cosmic superstrings. In a simplified model with three different types of string, the partition function reduces to three generalized coupled XY models. We calculate the phase diagram and show that, as the system is heated, the lightest strings first undergo the Hagedorn transition despite the junctions. There is then a second, higher, critical temperature above which infinite strings of all tensions, and junctions, exist. Conversely, on cooling to low temperatures, only the lightest strings remain, but they collapse into small loops.

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I. INTRODUCTION

The statistical mechanics of string networks has been the object of numerous studies because of the importance of strings or stringlike entities across all energy scales.

In general, either because of the large number of configurational microstates or because of the large number of excited quantum states that such a network possesses, the networks undergo transitions in which, as temperatures rise, strings proliferate. In the language of configurational states such a transition is termed a Feynman-Shockley transition, after Feynman's description of the λ-transition of ^{4}He in terms of vortex production [1]. From the viewpoint of counting excited states it is called a Hagedorn transition [2]. (Henceforth we follow the common usage of *Hagedorn transition* to apply to both cases, which are similar in structure in many ways.)

Specifically, in QCD, the sudden proliferation of color flux tubes (the original dual hadronic strings) explains quark deconfinement as temperature rises (see, for example, [3–5]). In cosmology at the grand unified theory scale, where cosmic strings arise in all reasonable supersymmetric models incorporating electroweak unification [6], the statistical mechanics of cosmic string networks has been investigated in order to understand their properties at formation and their late time scaling solutions, crucial for determining their cosmological consequences [7,8]. For fundamental strings there has been substantial work on exploring the effects of such transitions on the extremely early universe [9–13].

More recently, attention has turned again to fundamental string networks, following new developments in superstring theory. Indeed, a network of cosmic superstrings is expected to form when a brane and antibrane annihilate at the end of string-motivated brane inflation models. The network contains fundamental F-strings, Dirichlet D-strings, and (p, q)-strings which are bound states of p F-strings and q D-strings [14–17], meeting at Y-junctions (or vertices). The presence of Y-junctions, as well as the spectrum of tensions of the strings, is a key characteristic of such networks and leads to more complicated dynamics. Much work has been done to determine how (p, q)-like string networks evolve, both by analytic methods and numerical simulations, with particular regard to scaling solutions, their effect on the CMB, as well as other observable consequences [18–30].

Other than being stable against breakup, such strings differ from earlier superstrings in that, due to the warping of space-time, their tensions are not of the Planck scale but many orders of magnitude smaller. As a result any Hagedorn transitions may even arise later than the reheating of the universe, and hence be of direct relevance for astrophysics. A necessary first step in seeing whether this is the case is to determine the phase diagram for the Hagedorn transitions of a network with more than one type of string, and this is the goal of the present paper.

Our approach is to attempt to map the thermodynamics of string networks with junctions into the thermodynamics of a set of interacting dual fields, whereby the Hagedorn transitions of the strings become conventional transitions of the fields, a situation with which we are familiar. One can imagine several ways to attempt this. We adopt the simplest, generalizing the methods for describing quark deconfinement mentioned above (with its flux-tube Y-junctions) to something more like (p, q)-strings.

Hence we investigate the *equilibrium statistical mechanics* of cosmic superstring networks using methods motivated by [3–5]. However, it is important to note that there is at least one major difference between cosmic superstrings

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and QCD fluxlines: with multiple tensions (from different string types), we expect cosmic superstring networks to show multiple Hagedorn transitions.

In subsequent sections we derive and analyze the phase structure of a *three*-string model with junctions. This is a reduced model of realistic cosmic superstrings, for which $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ form a doubly infinite family. Since string tension (or energy/unit length) increases with p, q, all but low values will be suppressed at high temperature. We therefore adopt the simplest nontrivial scheme, taking the two lightest strings and their bound state (and antistrings), all which have different tensions σ_α, $\alpha = (1, 2, 3)$. For example, depending on parameters, these could be the (1,0), (0,1), and (1,1) strings. We show that as the system is heated, the lightest tension strings first undergo the Hagedorn transition, despite the presence of Y-junctions. Conversely, at low temperatures, only the lightest strings remain, before they disappear into loops. Our results are summarized in Fig. 1.

These conclusions may have important consequences for (p, q) string networks in that, if only the lightest strings remain after a nonadiabatic quench, no significant rôle would be played by the junctions whose properties have been studied so extensively. The dynamics would then be that of a single-string type with no junctions (though there may be loops containing strings of different types; as

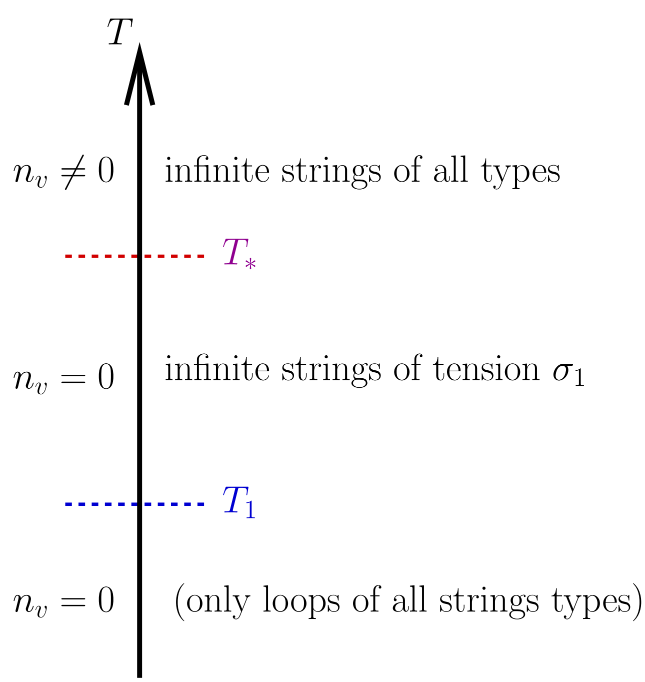

FIG. 1 (color online). Different critical temperatures for our simplified model of cosmic superstrings (with tensions $\sigma_1 < \sigma_2 < \sigma_3$) with Y-junctions. The lower Hagedorn temperature T_1 is determined by σ_1 whereas the higher Hagedorn temperature T_* is determined by all the σ_α ($\alpha = 1, 2, 3$). n_v denotes the density of vertices (or Y-junctions) joining infinite strings at temperature T.

explained below, our analysis is limited to infinite strings). This is not an idle proposition in that, although our analysis in this paper assumes adiabatic behavior, we have learned elsewhere that universality classes of equilibrium systems at their adiabatic transitions can become universality classes of nonequilibrium systems at fast quenches [31]: these points will be the content of a separate paper. Other works [18,22,25,29] based on studying the dynamics of string networks with junctions also suggest that at late times only the lightest strings may remain.

The paper is set up as follows. In Sec. II we first review some relevant aspects of string statistical mechanics in the simplest case: *one type* of string and *no* junctions. In particular, the duality between strings and fields is discussed. In Sec. III we still consider only strings of a single tension and type, but now these are allowed to meet at a junction. This section paves the way for Sec. IV in which we consider the general case of strings of three different tensions σ_α and types, meeting at junctions.

As explained in Sec. III there is significant complexity involved in adding junctions when discussing string statistical mechanics, and hence this section is central to the development of the paper. Furthermore, technically, junctions can be introduced in different ways, and as a result we are forced to discuss in detail two specific models ("bosonic" and "fermionic") to do so. While bosonic models are closer to the physical system we eventually wish to describe (and discussed in Sec. II when there are no junctions) only fermionic models can be generalized to the three-string case of Sec. IV. At the end of Sec. III we compare these two models, and conclude that they both essentially agree in their phase structure. This justifies the use of fermionic models in Sec. IV where the analysis resulting in the conclusions drawn in Fig. 1 is straightforward. Finally, we also show, following ideas from QCD, that the string system with junctions can be rewritten as a generalized spin model (XY model).

II. UNDERSTANDING THE HAGEDORN TRANSITION

In this section we discuss the nature of the Hagedorn transition for strings of a *single* type, with tension σ, and *no* Y-junctions.

As mentioned in the introduction, we proceed by using the duality between string configurations and fields to write the partition function for the string network as that of an effective field theory [32]. As a result, the Hagedorn transition can be mapped onto a transition of the effective field. Furthermore, provided the right questions are asked, one can work with the canonical rather than the microcanonical ensemble.

Consider a classical *static* picture of *noninteracting* strings in D-spatial dimensions. These are taken to lie on a hypercubic lattice of spacing a, and the energy E of the strings only depends on the total string length L through

$E = \sigma L$. Near the critical temperature, correlations are large and the details of the lattice structure should be unimportant. We also assume that the network can be thought of as a set of random walks. Now recall the *duality* between (nonoriented) Brownian paths in D spatial dimensions and a scalar field φ of mass m, as exemplified by the identity

$$\langle\varphi(\mathbf{x})\varphi(\mathbf{0})\rangle = \int_0^\infty d\tau e^{-\tau m^2}\int_{\mathbf{x}(0)=\mathbf{0}}^{\mathbf{x}(\tau)=\mathbf{x}}\mathcal{D}\mathbf{x} \times \exp\left[-\int_0^\tau d\tau' \frac{1}{4}\left(\frac{d\mathbf{x}}{d\tau'}\right)^2\right].$$

This identity can be used to construct an effective *action* (or, more accurately, a free energy) for the string partition function Z at temperature $T = \beta^{-1}$ in terms of φ as [32]

$$Z = \int \mathcal{D}\varphi \exp\left[-\int dx^D\left(\frac{a^2}{4D}(\nabla\varphi)^2 + \frac{1}{2}M^2\varphi^2\right)\right], \quad (1)$$

where the mass term is

$$M^2 = \sigma a\beta\left(1 - \frac{T}{T_H}\right).$$

The Hagedorn transition temperature, $T_H = \beta_H^{-1}$, is the solution to

$$J(\beta) \equiv e^{-\beta\sigma a} = \frac{1}{2D}. \quad (2)$$

The normalization of φ has been chosen here so that M^2 is dimensionless. (Note that one would have recovered the same temperature T_H for a gas of strings by counting single-loop configurations on the lattice [3,8]).

It is important to observe that *below* the Hagedorn transition $T < T_H$, φ is a massive free field with M^2 *positive*. For $T > T_H$, with $M^2 < 0$, it describes a *tachyon*. Here fluctuations are large and for this reason the canonical ensemble is often dropped in favor of the microcanonical ensemble [10]. However, in the conventional picture of spontaneous symmetry breaking we are familiar with the way in which tachyons describe instabilities (in field space); they are understood as corresponding to an inappropriate choice of ground state, the true ground states appearing naturally once backreaction is taken into account.

For example, the inclusion of a repulsive point-interaction modifies the free energy to [32]

$$S = \int dx^D\left(\frac{a^2}{4D}(\nabla\varphi)^2 + \frac{1}{2}M^2\varphi^2 + \lambda\varphi^4\right), \quad (3)$$

thus permitting $\langle\varphi\rangle$ to remain finite for $T > T_H$. For our (p, q) networks, the system has more complicated interactions than such a simple local repulsion. In particular, were the strings allowed to interact at Y-junctions, we would expect them to induce additional cubic $\mu\varphi^3$ terms—as we shall see in a different context below. However, the general implications are much the same.

The vanishing of the order parameter $\langle\varphi\rangle$ at $T \leq T_H$ can be understood in the following way. Examination of the partition function shows that the total string density is proportional to $\langle\varphi^2\rangle$, whereas $\langle\varphi\rangle^2$ measures the density in infinite string (i.e. string that crosses space) [32,33]. It is the vanishing of *infinite* string that characterizes the Hagedorn transition, and not the vanishing of string.

Although large loops are energetically unfavourable, some loops will always exist below the transition (in an adiabatic limit). Superficially, free energies like (3) look like those of high-temperature quantum field theories on dimensional compactification. Either from calculating the thermal propagator for excitations at the relevant ground state or by counting microstates of a loop gas we get the same result that, in the vicinity of the transition, the loop distribution is dominated by the smallest possible loops (the ultraviolet limit) [33].

III. MEAN FIELD TRANSITIONS; XY MODELS

As discussed in Sec. II, we anticipate that Y-junctions will induce cubic interaction terms in the dual field theory. However, we do not know how to introduce them in the exact framework of Sec. II, even when the junctions are between strings of the *same type and tension* σ—the setup considered in the present section.

In this section we discuss a mean field procedure which allows junctions to be incorporated, and which shows how such cubic interaction terms arise. As in Sec. II, one can then construct an analogue effective potential, $V(\varphi)$, for a field φ, whose vanishing describes the transition. Unfortunately, it is not possible to extend this construction to the full effective action, and as a result it is not possible to identify the field fluctuations that describe finite loops: our analysis is restricted to infinite string and the transitions triggered by its creation. Nonetheless, knowing that loops are there enables us to complete the picture, qualitatively. It is the mean field procedure presented in this section which will be generalized to the three-string model in Sec. IV.

Again we work in D spatial dimensions, on a periodic hyper-cubic lattice of N sites and lattice size $a = 1$. Let i label a lattice site, and $\mu = 1, \ldots, D$ the (positive) unit vectors in D-dimensional space. There is now a technical complication, related to how we allow the strings to populate the lattice. Although there is an energetic penalty in having more than one string on a link, in the first instance we do not wish to restrict the number to unity. To do so could imply an effective repulsion between strings that is a lattice artefact, and which might induce misleading terms in the effective potential for the analogue field φ. Without this restriction the models are termed "bosonic".

Models in which at most one string (of any type) can lie on a link are termed "fermionic". In practice we shall find, when we come to mimicking (p, q) strings, that only a fermionic model can accommodate junctions of three-string types.

An important result of this section is that our concern about fermionic models is largely unjustified (though we feel it is necessary, for reasons of clarity, to discuss it in detail): both bosonic and fermionic models essentially agree for the small φ values that are relevant for transitions, and for which the mean field approximation is more reliable. Further, both of these models rewrites the string system as a generalized XY model, permitting us to think of the Hagedorn transition as one of spin ordering. This suggests ways of going beyond the mean field approximation, although we shall not do so here.

A. Bosonic models

With conventional lattice notation, let $n^+_{i,\mu}$ ($n^-_{i,\mu}$) be the number $(0, 1, 2, \ldots)$ of strings (antistrings) on the link between the lattice points i and $i + \mu$.

For strings with no junctions, the Hamiltonian

$$H = \sum_{i=1}^{N} \sigma \sum_{\mu} (n^+_{i,\mu} + n^-_{i,\mu}) \tag{4}$$

gives the requisite energy $E = \sigma L$ to a network of total length L.

Now, depending on the string network we wish to model, there is more than one way to proceed. We discuss the mean field potential in each case, making links with Secs. II and IV.

1. Massless junctions

First we allow the strings to have N_v-fold *massless* junctions i.e. no extra cost in energy. (We are primarily concerned with $N_v = 3$.) Since the junctions considered are massless they do not appear in the Hamiltonian, which is still given by (4).

Rather, the existence of junctions imposes constraints on the $n^+_{i,\mu}$ ($n^-_{i,\mu}$). Junctions or antijunctions are permitted on site i provided the flux into that site is an integer multiple of N_v:

$$\alpha_i \equiv \sum_{\mu} [(n^+_{i,\mu} - n^+_{i-\mu,\mu}) - (n^-_{i,\mu} - n^-_{i-\mu,\mu})] = 0 \bmod N_v, \tag{5}$$

a constraint which can be implemented through

$$\delta_{\alpha=0 \bmod N_v} = \frac{1}{N_v} \sum_{k_i=1}^{N_v} e^{i\alpha\theta_i} \quad \text{where } \theta_i = \frac{2\pi k_i}{N_v}.$$

Using this representation in the canonical partition function

$$Z = \sum_{n^{\pm}_{i,\mu}} e^{-\beta\sigma \sum_{i,\mu}(n^+_{i,\mu} + n^-_{i,\mu})} \Big(\prod_i \delta_{\alpha_i = 0 \bmod N_v} \Big)$$

enables us to write Z as

$$Z = \prod_i \frac{1}{N_v} \sum_{k_i} \Big(\sum_{n^+_{i,\mu}} e^{-\sum_{i,\mu}[\beta\sigma n^+_{i,\mu} + i(\theta_{i+\mu} - \theta_i) n^+_{i,\mu}]} \Big) \times \Big(\sum_{n^-_{i,\mu}} e^{-\sum_{i,\mu}[\beta\sigma n^-_{i,\mu} - i(\theta_{i+\mu} - \theta_i) n^-_{i,\mu}]} \Big),$$

where the different signs in front of the lattice variables θ_i in the two terms in round brackets reflect the signs in (5). The summations can be performed, to obtain

$$Z = \Big(\frac{1}{N_v} \Big)^N \sum_{k_i} e^{-\sum_{i,\mu} \ln(1 + J(\beta)^2 - 2J(\beta)\cos(\theta_i - \theta_{i-\mu}))},$$

where $J(\beta) = e^{-\beta\sigma}$ as in (2). That is, the Hamiltonian of the network is, up to a constant,

$$\beta H = \sum_{i,\mu} \ln[1 + J(\beta)^2 - 2J(\beta)\cos(\theta_i - \theta_{i-\mu})]. \tag{6}$$

It is not possible to evaluate Z exactly. Hence we resort to the mean field approximation scheme (see for example [1]), which consists of introducing a trial Hamiltonian H_0 in which each variable of the system is decoupled from the other but depends on an external constant source φ. An obvious choice here is

$$H_0(\varphi) = -\frac{\varphi}{\beta} \sum_i \cos\theta_i. \tag{7}$$

On writing

$$H = H_0(\varphi) + [H - H_0(\varphi)],$$

then

$$Z = \sum_{\text{config}} e^{-\beta H_0(\varphi)} e^{-\beta[H - H_0(\varphi)]} = Z_0(\varphi) \langle e^{-\beta[H - H_0(\varphi)]} \rangle_{0 \geq Z_0(\varphi) e^{-\beta\langle H - H_0(\varphi)\rangle_0}},$$

where the zero subscript denotes φ-dependent averaging with regard to $H_0(\varphi)$. As a result the free energy $F = -T \ln Z$ satisfies

$$F(\varphi) \leq NV(\varphi) \equiv F_0(\varphi) + \langle H \rangle_0 - \langle H_0(\varphi) \rangle_0, \tag{8}$$

where $V(\varphi)$ is the mean field effective potential (and $F_0 = -T \ln Z_0$). Our aim is then to minimize V in order to find $\varphi_{\min}$, which determines the density of infinite string (see below). As in common practice, we assume that the upper bound of (8) is well saturated as far as transitions are concerned.

We now carry out the calculation explicitly in the case of Y-junctions for which $N_v = 3$. Then

$$Z_0(\varphi) = \left[\frac{1}{3}\left(\sum_{k=1}^{3} e^{\varphi\cos(2\pi k/3)}\right)\right]^N = \tilde{I}_0(\varphi)^N,$$

where

$$\tilde{I}_0 = \tfrac{1}{3}(e^{\varphi} + 2e^{-\varphi/2}).$$

Now use the results that

$$\langle \ln(1 + p^2 - 2p\cos\theta)\rangle = -2\sum_{m=1}^{\infty}\frac{p^m}{m}\langle\cos m\theta\rangle, \qquad (|p| < 1) \tag{9}$$

for all measures, and that

$$\langle\cos m\theta\rangle_0 = \frac{\tilde{I}_m(\varphi)}{\tilde{I}_0(\varphi)} \tag{10}$$

for the case in point, where

$$\begin{aligned}\tilde{I}_m(\varphi) &= \frac{1}{3}\sum_k e^{\varphi\cos(2\pi k/3)}\cos(2\pi mk/3)\\ &= \frac{1}{3}(e^{\varphi} + 2e^{-\varphi/2}\cos(2\pi m/3))\end{aligned}$$

is a discrete version of the Bessel function. Hence, using (8) we obtain

$$\begin{aligned}\beta V(\varphi) = &-\ln(\tilde{I}_0(\varphi)) + \varphi\left(\frac{\tilde{I}_1(\varphi)}{\tilde{I}_0(\varphi)}\right)\\ &- 2D\sum_{m=1}^{\infty}\frac{J(\beta)^m}{m}\left(\frac{\tilde{I}_m(\varphi)}{\tilde{I}_0(\varphi)}\right)^2.\end{aligned} \tag{11}$$

The periodicity (modulo 3) of the $\tilde{I}_m(\varphi)$ enables us to perform the summation explicitly, to give (up to φ-independent terms)

$$\begin{aligned}\beta V(\varphi) = &-\ln(e^{\varphi} + 2e^{-\varphi/2}) + \varphi\left(\frac{e^{\varphi} - e^{-\varphi/2}}{e^{\varphi} + 2e^{-\varphi/2}}\right)\\ &- 2DG(\beta)\left(\frac{e^{\varphi} - e^{-\varphi/2}}{e^{\varphi} + 2e^{-\varphi/2}}\right)^2,\end{aligned} \tag{12}$$

where

$$G = \frac{1}{3}\ln\left(\frac{1 + J + J^2}{(1 - J)^2}\right) = J + \frac{1}{2}J^2 + \ldots$$

for small $J(\beta)$. As commented above, in (12) and also in the remainder of this paper, we will always drop field-independent terms when writing down the mean field effective potentials.

Notice that, because the sum over m in (11) just reproduces the first term with a modified coefficient, $V(\varphi)$ of (12) can be shown to be *exactly* the mean field potential arising from the Hamiltonian

$$H_{XY}^{\rm disc} = -\frac{G(\beta)}{\beta}\sum_{i,\mu}\mathbf{s}_i\cdot\mathbf{s}_{i+\mu}, \tag{13}$$

i.e. the Hamiltonian for a system of unit spins in the plane with nearest neighbor interactions in which their relative angles are constrained to multiples of $2\pi/N_v$ (here $N_v = 3$); a discrete XY model. The mean field trial Hamiltonian H_0 in this case is $H_0(\varphi) = -\frac{\varphi}{\beta}\mathbf{n}.\sum_i\mathbf{s}_i$ for an arbitrary unit vector $\mathbf{n}$ in which the spins are decoupled; in other words, an external magnetic field proportional to φ.

In order to understand the phase structure of the model (either as a spin system or as a gas of strings with junctions), consider first the series expansion of $V(\varphi)$;

$$\beta V(\varphi) = \tfrac{1}{2}m^2\varphi^2 + \tfrac{1}{3}\mu\varphi^3 + \tfrac{1}{4}\lambda\varphi^4 + \ldots, \tag{14}$$

up to constant terms, where

$$\begin{aligned}m^2 = \tfrac{1}{2}(1 - 2DG), \qquad \mu = \tfrac{1}{4}(1 - 3DG),\\ \lambda = -\tfrac{3}{16}(1 - 2DG).\end{aligned} \tag{15}$$

Observe that the field becomes massless at the temperature for which $2DG(\beta) = 1$, which is in good agreement with the Hagedorn temperature of the free dual theory of (1) since $G(\beta) \simeq J(\beta)$ for $J = 1/2D \ll 1$. Furthermore, as anticipated, the Y-junctions have induced a cubic term in the potential. In addition they have also induced a quartic interaction, vanishing when the field becomes massless, that is repulsive when the field becomes tachyonic.

As a result of the cubic term, the potential in Eq. (12) can be shown to have a weak first-order phase transition. The critical temperature, however, cannot be obtained from (15) as it occurs at values of $\varphi \simeq 1$. Numerically, however, one finds that $2G_{\rm crit}(D = 3) \simeq 0.31$ and $2G_{\rm crit}(D = 4) \simeq 0.23$. We shall not consider the first-order transition further, since it is not reliably robust against rapid quenches which is what we ultimately have in mind.

2. Massive junctions

Alternatively one might want to model string networks with massive junctions—that is, to introduce junctions with an energy cost v. (These can model massive monopoles, which may be formed at the vertex in different symmetry breaking schemes [34]. Similarly massive monopoles or "beads" can exist in cosmic superstring networks [35].) We can then recover massless vertices by taking $v \to 0$. Furthermore this construction allows one to calculate the average density of vertices at temperature T, by simply differentiating Z with respect to v. This will be discussed in Sec. IV.

To add massive vertices, we allocate a vertex number $p_i^{\pm} = (0, 1, 2\ldots)$ to each lattice site, constrained by

$$\begin{aligned}\alpha_i \equiv &\sum_{\mu}[(n_{i,\mu}^{+} - n_{i-\mu,\mu}^{+}) - (n_{i,\mu}^{-} - n_{i-\mu,\mu}^{-})]\\ &+ 3(p_i^{+} - p_i^{-}) = 0\end{aligned} \tag{16}$$

for Y-junctions, while the Hamiltonian acquires an extra term

$$H_I = \sum_{i=1}^{N} v(p_i^+ + p_i^-). \tag{17}$$

Performing the sums over the $n_{i,\mu}^{\pm}$ and the $p_i^{\pm}$ leads to a Hamiltonian

$$\beta H = -\sum_{i,\mu} \ln[1 + J^2(\beta) - 2J(\beta)\cos(\theta_{i+\mu} - \theta_i)] - \sum_i \ln[1 + K^2(\beta) - 2K(\beta)\cos 3\theta_i], \tag{18}$$

where the θ_i are now *continuous variables*, the Lagrange multipliers that arise from imposing the constraints

$$\delta_{\alpha_i,0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta_i e^{i\alpha_i\theta_i}. \tag{19}$$

Also we have defined

$$K(\beta) = e^{-\beta v}, \tag{20}$$

analogously to J in (2). Then, carrying out the same mean field treatment as above yields

$$\beta V^{(K)}(\varphi) = -\ln(I_0(\varphi)) + \varphi\left(\frac{I_1(\varphi)}{I_0(\varphi)}\right) - 2D\sum_{m=1}^{\infty}\frac{J(\beta)^m}{m}\left(\frac{I_m(\varphi)}{I_0(\varphi)}\right)^2 - 2\sum_{m=1}^{\infty}\frac{K(\beta)^m}{m}\left(\frac{I_{3m}(\varphi)}{I_0(\varphi)}\right), \tag{21}$$

where the I_m are (continuous) Bessel functions.

For nonzero K cubic terms arise from the I_3 Bessel function, to give rise to a potential of the form (14), with coefficients

$$m^2 = \frac{1}{2}(1 - 2DJ), \qquad \mu = -\frac{K}{8}, \qquad \lambda = -\frac{3}{16}\left(1 - \frac{8DJ}{3} + \frac{DJ^2}{3}\right). \tag{22}$$

As expected, we have a tachyonic instability at $J = 1/2D$ and a cubic term in the potential.

The slightly different behavior of (22) and (15) is to be expected, since we are implementing the boundary conditions that count vertices differently in the two cases: in other words, they correspond to different implementations of the mean field approach. However, since the mean field result is, strictly, an upper bound, we could, if we wished, only retain that solution that is numerically lower. In practice, this is not necessary since there is close numerical agreement at relevant temperatures. Massless junctions correspond to taking $K = 1$ for which $\mu = -1/8$, the value arising in (15) when $2DG = 1$. Further, a numerical study of (21) shows that the transition tends to become first order as $K \to 1$, in agreement with the discussion of (15).

3. No junctions

For continuous "bosonic" string with no junctions both approaches give identical results. In the first case, we eliminate junctions by taking $N_v \to \infty$, whereby the discrete Bessel functions are replaced by their continuous counterparts. In the second, taking $v \to \infty$ ($K = 0$) just recreates the same series.

In each case, on expanding $V^{(0)}(\varphi)$ for small φ we recover the second-order transition at the Hagedorn temperature T_H of Sec. II [see Eq. (2)] when $2DJ(\beta) = 1$, and when the φ field becomes tachyonic. However, it can be seen that $V^{(0)}$ of (21) becomes unbounded below as $T \to \infty$. This is not quite the behavior of (1), for which the potential is unbounded below for all $T > T_H$, showing the limitations of the mean field approach for very large $|\varphi|$. Nonetheless, this simple example shows how the introduction of vertices induces interaction terms in the effective potential to stabilize the ground states.

B. Fermionic models

We now consider the most simple "fermionic" models. It is these which can straightforwardly be extended to the general three-string-type model of Sec. IV. We will also address the concern raised at the beginning of this section: that the "fermionic" model might add an effective repulsion between strings, which could induce misleading terms in the effective potential. We will show that this is not the case.

Thus, we now restrict the number of strings on each link to $n_{i,\mu} \in \{0, \pm 1\}$. That is, the link from site i to $i + \mu$ contains either a single-string, a single antistring, or no string at all.

1. No junctions

With no junctions, the Hamiltonian is

$$H = \sum_{i=1}^{N}\sum_{\mu=1}^{D} \sigma n_{i,\mu}^2, \tag{23}$$

subject to the constraint

$$\alpha_i \equiv \sum_{\mu}[n_{i,\mu} - n_{i-\mu,\mu}] = 0. \tag{24}$$

Performing the sums over the $n_{i,\mu}$ leads to a Hamiltonian

$$\beta H = -\sum_{i,\mu} \ln[1 + 2J(\beta)\cos(\theta_{i+\mu} - \theta_i)], \tag{25}$$

where the θ_i are again the Lagrange multipliers that arise from imposing the constraints

$$\delta_{\alpha_i,0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta_i e^{i\alpha_i\theta_i}. \tag{26}$$

Defining $\bar{J}$ by

$$J = \frac{\bar{J}}{1 + \bar{J}^2}, \tag{27}$$

whereby $J(\beta) \approx \bar{J}(\beta)$ when $J \ll 1$, a similar calculation to that above (see also Sec. IV) shows that the mean field potential is, for $\bar{J} < 1$,

$$\beta V_F^{(0)}(\varphi) = -\ln I_0(\varphi) + \varphi\left(\frac{I_1(\varphi)}{I_0(\varphi)}\right) + 2D\sum_{m=1}^{\infty}\frac{(-\bar{J}(\beta))^m}{m}\left(\frac{I_m(\varphi)}{I_0(\varphi)}\right)^2. \tag{28}$$

The φ field now becomes massless at $2D\bar{J}(\beta) = 1$, with a second-order transition. With $J \approx \bar{J}$ this is slightly displaced from that of the bosonic strings but, at the qualitative level at which we are working, can be said to agree. Note that both potentials (28) and (21) show a $\mathbb{Z}_2$ symmetry under $\varphi \rightarrow -\varphi$ that is broken *above* T_H, and restored *below* T_H, contrary to the usual pattern of symmetry breaking, but as in Sec. II.

On comparing (28) with (21) we see that they differ in that the former has alternating signs in the Bessel function expansion, whereas the latter does not. Because higher terms in the series in powers of $\bar{J}(\beta)$ become significant only at increasingly large φ, the artificial repulsion induced by the "fermionic" assumption (that is, of no more than one string per link) is a large-φ effect in the mean field approximation, and hence where the approximation is at its least reliable. However, since the transitions are determined by small φ, we can use either. This is an important result of this section.

In fact, for J small, both approximate the mean field potential of the XY model, with spin-spin Hamiltonian

$$H_{XY} = -\frac{1}{\beta}\sum_{i,\mu} 2J\cos(\theta_{i+\mu} - \theta_i) = -\frac{2J}{\beta}\sum_{i,\mu}\mathbf{s}_i \cdot \mathbf{s}_{i+\mu}. \tag{29}$$

This follows from expanding (25), for which

$$\beta V_{XY}(\varphi) = -\ln I_0(\varphi) + \varphi\left(\frac{I_1(\varphi)}{I_0(\varphi)}\right) - 2DJ(\beta)\left(\frac{I_1(\varphi)}{I_0(\varphi)}\right)^2, \tag{30}$$

showing a second-order transition at $2DJ(\beta) = 1$. Rather than just perform a series expansion in φ as in (15), more generally we see that extrema of $V_{XY}(\varphi)$ satisfy

$$\bar{\varphi} - 4DJ(\beta)u(\bar{\varphi}) = 0, \tag{31}$$

where $u(\varphi) = I_1(\varphi)/I_0(\varphi)$.

$\bar{\varphi} = 0$ is always a solution to (31). For $2DJ(\beta) > 1$ there is a further pair of solutions, $\pm\bar{\varphi}$, $\bar{\varphi} > 0$, which are the minima. We note, for future use, when we need to count extrema, that (31) behaves like the cubic equation obtained from just retaining terms up to $O(\varphi^4)$ in the expansion of the potential in the existence of three roots. The inclusion of higher terms in the series in $\bar{J}$ does not seem to affect this empirically and it is not necessary to go beyond the XY model, now and hereafter.

When the XY model is a good approximation we could, in principle, use known results about it without resorting to the mean field approximation. In practice, we know of no work on the generalized XY models appropriate to the three-string models (with or without junctions) and stay with the mean field approximation.

To give a meaning to $\bar{\varphi}$ we note that the average density of (infinite) strings is proportional to $\bar{\varphi}^2$, as anticipated, given by

$$\rho = \frac{1}{N}\left\langle\sum_{\mu} n_{i,\mu}^2\right\rangle = -J(\beta)\beta\frac{\partial V_{XY}}{\partial J} = \frac{\bar{\varphi}^2}{8DJ(\beta)}. \tag{32}$$

2. Massive junctions

We end this section by including Y-junctions in the fermionic model (still of a single-string type). Given that the occupation numbers are limited to 0, ± 1, there is no analogue of the mod 3 description for massless vertices discussed in the bosonic case [see Eq. (5)]. We therefore consider massive vertices, and furthermore take them to be noncoincident. There is thus now a single vertex number $p_i = \{0, \pm 1\}$ constrained by

$$\alpha_i \equiv \sum_{\mu}[n_{i,\mu} - n_{i-\mu,\mu}] + 3p_i = 0 \tag{33}$$

with the Hamiltonian acquiring an additional term

$$H_I = \sum_i v p_i^2. \tag{34}$$

On defining $\bar{K}$ by

$$K = \frac{\bar{K}}{1 + \bar{K}^2}, \tag{35}$$

the mean field potential is, for ($\bar{J}$, $\bar{K} < 1$),

$$\beta V_F^{(K)}(\varphi) = \beta V_F^{(0)}(\varphi) + 2\sum_{m=1}^{\infty}\frac{(-\bar{K}(\beta))^m}{m}\left(\frac{I_{3m}(\varphi)}{I_0(\varphi)}\right)^2. \tag{36}$$

[This follows from the generalization of (9), used earlier in (28) that, up to a constant,

$$\langle\ln(1 + 2K\cos\alpha)\rangle = -2\sum_{m=1}^{\infty}\frac{(-\bar{K})^m}{m}\langle\cos m\alpha\rangle \tag{37}$$

for all measures and $\bar{K} < 1$, together with the specific result

$$\langle\cos m\theta\rangle_0 \equiv \frac{\int\frac{d\theta}{2\pi}e^{\varphi\cos\theta}\cos m\theta}{\int\frac{d\theta}{2\pi}e^{\varphi\cos\theta}} = \frac{I_m(\varphi)}{I_0(\varphi)}$$

for our choice of H_0.] We note that unfortunately, for a simple cubic lattice, the requirement that $\bar{K} < 1$, necessary for convergence of the series in (37), imposes $K < 1/2$. Hence the mean field approximation is not valid for light vertices in the fermionic case [as opposed to the bosonic one in (21)]. We consider this constraint to be an artefact of the lattice fermionic approximation.

Despite that, note that the mean field potential (36) leads to an XY model in the presence of an external source [3,5] in which we retain only the first term in the power series in $\bar{K}$ in (36) [or the first term in the series in K in (21)]. As a result, there is always a second-order transition, as in the bosonic case.

Finally we also note that the density of string (32) is unchanged by the inclusion of junctions.

C. Summary of Sec. III

In summary, in this section we have seen how the inclusion of Y-junctions in a model of a single-string type can provide the backreaction necessary to prevent tachyonic instability at the Hagedorn temperature. Further, provided we restrict ourselves just to infinite string, whose density is the order parameter, we can go beyond the Hagedorn temperature, still with the canonical ensemble.

We have also discussed two models, bosonic and fermionic, and shown that the concern raised about fermionic models at the beginning of this section is unjustified: both models essentially agree for the small φ values that are relevant for transitions, and for which the mean field approximation is more reliable.

We have also shown how the value $\bar{\varphi}$ of the field at the minimum of the effective potential is related to the density of infinite strings in the system. As we discuss in Sec. IV, it is equally apparent that the density of vertices is also determined by $\bar{\varphi}$ and obtained by differentiating the partition function with respect to υ.

With this behind us, we now consider the case of three different string types with Y-junctions, as a model for (p, q) strings. We note that, oddly, the analysis of QCD confinement of [3–5], that we have called upon in this paper, was performed in the context of a single-string model, not permitting " color." Although this was not our intention, a more realistic description of QCD is given by the model that follows, in the limit of equal tensions, in which our three-string types correspond to colored flux tubes.

IV. THREE STRINGS, FERMIONIC MODEL

The basics of our model are the following.

As stated in the introduction, we model the (p, q) string network by a network of three different types of fundamental strings, labeled by $\alpha = 1, 2, 3$ as red, green, and blue, say. Generally the strings also have different tensions σ_α. The strings do not interact with each other (nor with themselves), except at a Y-junction (or vertex) which is defined to be a point at which three strings of *different* colors meet.

Following Eq. (14), our expectation is that the èffective potential will take the generic form

$$\beta V(\varphi_1, \varphi_2, \varphi_3) = \sum_\alpha \left[\frac{1}{2} m_\alpha^2 \varphi_\alpha^2 + \frac{1}{4}\lambda_\alpha \varphi_\alpha^4\right] + \mu \varphi_1 \varphi_2 \varphi_3 + \ldots, \tag{38}$$

the cubic term reflecting the junction connecting the three different string types. Potentials of the type (38), with temperature-dependent coefficients, have been studied in other contexts e.g. transformations of vortex types in superfluid ^{3}He [36].

We know that (38) is valid if Y-junctions are excluded, when $\mu = 0$. In this case, from the single-string models

$$m_\alpha^2 \propto (1 - 2DJ_\alpha(\beta)), \tag{39}$$

with $J_\alpha = e^{-\sigma_\alpha \beta}$. In the following discussion we suppose that

$$\sigma_1 \leq \sigma_2 \leq \sigma_3 \iff J_1 \geq J_2 \geq J_3. \tag{40}$$

The critical $J_\alpha^{\text{crit}} = 1/(2D)$ define three critical inverse temperatures $\beta_\alpha = T_\alpha^{-1}$ with

$$\beta_3 < \beta_2 < \beta_1 \tag{41}$$

in the vicinity of which $m_\alpha^2 \propto (1 - T/T_\alpha)$. That is, with no interactions we expect three sequential Hagedorn transitions as, on cooling, the heavier strings disappear from the picture, leaving the lightest until last before it disappears in turn, leaving just small loops.

Our aim is to understand the effect that Y-junctions have on this picture.

In practice, we are not able to recreate (38) in a bosonic model with colored Y-junctions, with arbitrary numbers of strings on each link. [The reason is that we are unable to write down a generalized form of the constraint (5) in the 3-string case.] We therefore restrict ourselves to a fermionic model, in which there is at most one string of each type on a link. As discussed in the previous section, we expect that the effective repulsion this implies can be ignored at small field values. As in the case of the single-string type, in order to be able to use mean field theory we are obliged to give the vertex a nonzero mass υ. (This may well be realistic in certain cosmic superstring models [35].)

As before, we assume that the energy of the different strings is proportional to their length $L(E_\alpha = \sigma_\alpha L)$. The different strings are described, respectively, by the variables $n_{i,\mu}^\alpha$, which all take values in $\{0, \pm 1\}$. There are also vertices, described by the variable $p_i \in \{0, \pm 1\}$, joining strings of 3 different types. The Hamiltonian of the system takes the same form as for the single-string case,

$$H = \sum_i \left[\sum_\mu \sum_\alpha \sigma_\alpha (n_{i,\mu}^\alpha)^2 + \upsilon p_i^2\right]. \tag{42}$$

We now need to impose the constraint that a junction is where three different color strings meet: this is done by

$$\gamma_i^\alpha = \sum_\mu (n_{i,\mu}^\alpha - n_{i-\mu,\mu}^\alpha) + p_i = 0, \qquad \forall \alpha. \tag{43}$$

Although summing over α would essentially recreate the constraints (33), Eq. (43) is more specific. In particular, (43) does not forbid different string types from lying on top of each other.

As in the previous section, the constraints are imposed in the standard way through Lagrange multipliers, which is equivalent to writing the Kroneker delta as

$$\delta_{\gamma_i^\alpha,0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta_i^\alpha e^{i\gamma_i^\alpha\theta_i^\alpha} \tag{44}$$

(no α summation) for each γ^α. Hence the partition function is

$$Z(\beta, v, \sigma_\alpha) = \int \prod_{i,\alpha} \frac{d\theta_i^\alpha}{2\pi} \sum_{n_{i,\mu}} e^{-\sum_{i,\mu}\sum_\alpha [\beta\sigma_\alpha (n_{i,\mu}^\alpha)^2 + in_{i,\mu}^\alpha(\theta_{i+\mu}^\alpha - \theta_i^\alpha)]} \times \sum_{p_i} e^{-\sum_i [\beta v p_i^2 + ip_i \sum_\alpha \theta_i^\alpha]} \tag{45}$$

which, on carrying out the summations gives

$$Z(\beta, v, \sigma_\alpha) = \int \prod_{i,\alpha} \frac{d\theta_i^\alpha}{2\pi} \prod_\mu [1 + 2J_\alpha \cos(\theta_{i+\mu}^\alpha - \theta_i^\alpha)] \times \left[1 + 2K\cos\left(\sum_\alpha \theta_i^\alpha\right)\right]. \tag{46}$$

This corresponds to the Hamiltonian

$$\begin{aligned}\beta H &= -\sum_{i,\mu,\alpha} \ln[1 + 2J_\alpha \cos(\theta_{i+\mu}^\alpha - \theta_i^\alpha)] - \sum_i \ln\left[1 + 2K\cos\left(\sum_\alpha \theta_i^\alpha\right)\right] \\ &\approx \sum_{i,\mu,\alpha} 2J_\alpha \cos(\theta_{i+\mu}^\alpha - \theta_i^\alpha) + \sum_i 2K\cos\left(\sum_\alpha \theta_i^\alpha\right)\end{aligned} \tag{47}$$

for small J_α and K.

The mean field treatment therefore contains three variational parameters φ_α. Following the same steps as in Sec. III, the trial partition functions which decouple different lattice sites are

$$Z_0^\alpha(\beta) = \int \prod_i \frac{d\theta_i^\alpha}{2\pi} e^{\sum_i \varphi_\alpha \cos\theta_i^\alpha} = [I_0(\varphi_\alpha)]^N, \tag{48}$$

while the mean field effective potential is

$$\begin{aligned}\beta V(\varphi_\alpha) = \sum_\alpha \Bigg[&-\ln I_0(\varphi_\alpha) + \varphi_\alpha \left(\frac{I_1(\varphi_\alpha)}{I_0(\varphi_\alpha)}\right) + 2D\sum_{m=1}^\infty \frac{(-\bar{J}_\alpha)^m}{m}\left(\frac{I_m(\varphi_\alpha)}{I_0(\varphi_\alpha)}\right)^2\Bigg] \\ &+ 2\sum_{m=1}^\infty \frac{(-\bar{K})^m}{m}\left(\frac{I_m(\varphi_1)}{I_0(\varphi_1)}\frac{I_m(\varphi_2)}{I_0(\varphi_2)}\frac{I_m(\varphi_3)}{I_0(\varphi_3)}\right),\end{aligned} \tag{49}$$

where each $\bar{J}_\alpha$ is defined as in (27), and $\bar{K}$ is given in (35).

As discussed in Sec. III, it is sufficient for our purposes to approximate $\beta V(\varphi_\alpha)$ by the first term in the series of (49),

$$\begin{aligned}\beta V_{XY}(\varphi_\alpha) = \sum_\alpha \Bigg[&-\ln I_0(\varphi_\alpha) + \varphi_\alpha \left(\frac{I_1(\varphi_\alpha)}{I_0(\varphi_\alpha)}\right) - 2DJ_\alpha\left(\frac{I_1(\varphi_\alpha)}{I_0(\varphi_\alpha)}\right)^2\Bigg] \\ &- 2K\left(\frac{I_1(\varphi_1)}{I_0(\varphi_1)}\frac{I_1(\varphi_2)}{I_0(\varphi_2)}\frac{I_1(\varphi_3)}{I_0(\varphi_3)}\right).\end{aligned} \tag{50}$$

This corresponds to making the small J, K approximation in (47) (and this approximation will be made in the remainder of this paper). That is, the model (46) is a generalized XY model, consisting of three spinlike variables defined on each lattice site i, making angles θ_i^α with respect to some fixed axis, interacting amongst themselves through the K-dependent term.

We have achieved our goal in that, if we expand $V_{XY}(\varphi_\alpha)$ of (50) [or, indeed the full $V(\varphi_\alpha)$ of (49)] in powers of φ_α we recover the generic potential (38) as the first few terms in the series.

However, we can say more. As in our earlier examples, attaching a nominal energy to each vertex allows us to calculate the density of vertices. Specifically, the density of vertices on infinite strings is

$$\begin{aligned}n_v &= \frac{1}{N}\left\langle \sum_i p_i^2 \right\rangle = -K\beta\frac{\partial V_{XY}}{\partial K} \\ &\propto \left(\frac{I_1(\bar{\varphi}_1)}{I_0(\bar{\varphi}_1)}\frac{I_1(\bar{\varphi}_2)}{I_0(\bar{\varphi}_2)}\frac{I_1(\bar{\varphi}_3)}{I_0(\bar{\varphi}_3)}\right) \propto \bar{\varphi}_1\bar{\varphi}_2\bar{\varphi}_3\end{aligned} \tag{51}$$

at the minimum $(\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ of $V_{XY}(\varphi_\alpha)$. The small loops corresponding to the field fluctuations that are invisible to our mean field analysis contain vertices not counted in (51).

As in Sec. III, we now look for the extrema of the potential in order to determine the density of infinite string and the density of vertices. As expected from Sec. III, a full numerical analysis (that we have performed) without the XY approximation does not alter our qualitative conclusions and barely changes our quantitative results.

As we noted earlier, in the main works on QCD ([3,4]) all flux strings were taken to be of a single kind, leading to a very different potential, in which $I_1(\varphi_1)I_1(\varphi_2)I_1(\varphi_3)$ is replaced by $I_3(\varphi)$ for example. In particular, as we shall see later for (49), with equal tensions there is no first-order transition when there are three-string types.

A. $K = 0$: no vertices and three independent spins

We have already anticipated the results for this simple case, but it is helpful to see them in greater detail. For $K = 0$ the XY model reduces to three independent, uncoupled, XY models with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry under $\varphi_\alpha \to -\varphi_\alpha$. The extremal points are when

$$\frac{\partial V_{XY}}{\partial \bar{\varphi}_\alpha} = 0 \Leftrightarrow \frac{\bar{\varphi}_\alpha}{4DJ_\alpha} - u(\bar{\varphi}_\alpha) = 0, \tag{52}$$

where $u(\varphi) = I_1(\varphi)/I_0(\varphi)$ as before. One possible solution is always $\bar{\varphi}_\alpha = 0$, the only real solution if $2DJ_\alpha(\beta) < 1$.

If $2DJ_\alpha(\beta) > 1$ then there are two further real solutions, denoted $\pm\bar{\varphi}_\alpha$, where we take $\bar{\varphi}_\alpha > 0$. The $3^3 = 27$ possible extrema $\varphi = (\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ then break down into a nondegenerate $\varphi = (0, 0, 0)$, three doubly degenerate solutions, exemplified by $\varphi = (\pm\bar{\varphi}_1, 0, 0)$, three fourfold degenerate solutions, exemplified by $(\pm\bar{\varphi}_1, \pm\bar{\varphi}_2, 0)$, and an eightfold degenerate solution $(\pm\bar{\varphi}_1, \pm\bar{\varphi}_2, \pm\bar{\varphi}_3)$. It is sufficient to restrict ourselves to the positive sector $\varphi_\alpha \geq 0$.

To determine which of these are maxima, which minima, and which saddle points we need to calculate the eigenvalues of the Hessian $M_{\gamma\delta} = \partial^2 V_{XY}/\partial\varphi_\gamma\partial\varphi_\delta$ at the extrema. An extremum is a minimum if all are positive, and a maximum if all are negative. Otherwise one is dealing with saddle points.

With $K = 0$, the only nonzero entries are on the diagonal with (no summation)

$$M_{\alpha\alpha} = u'(\bar{\varphi}_\alpha)[1 - 4DJ_\alpha(\beta)u'(\bar{\varphi}_\alpha)]. \tag{53}$$

For the case in hand the answer is very simple and very obvious.

(1) $\beta > \beta_1 (> \beta_2, \beta_3)$. In this range the global minimum occurs at $\vec{\varphi} = (0, 0, 0)$.

(2) $\beta_2 < \beta < \beta_1$. Now $(\bar{\varphi}_1, 0, 0)$ is the *global minimum*. [(0, 0, 0) is now a saddle point.]

(3) $\beta_3 < \beta < \beta_2$. In this range it is easy to see that $(\bar{\varphi}_1, \bar{\varphi}_2, 0)$ is the global minimum.

(4) $\beta < \beta_3$. Here it is equally straightforward to see that $(\bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ is the global minimum, (0, 0, 0) is a maximum, and all other points are saddle points.

As expected, as the temperature is increased infinite strings of the lightest tension first are nucleated at $\beta = \beta_1$; then those of the next lightest tension at $\beta = \beta_2$; and finally the heaviest strings when $\beta = \beta_3$. When one decreases the temperature from a very high one, the opposite happens.

B. $K \neq 0$: vertices and three coupled spins

Let us now consider the effect of Y-junctions in the generalized XY model of (50). For unequal σ_α the symmetry of V_{XY} is now explicitly broken from $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ to $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by

$$P_1\colon\ \varphi_1 \to \varphi_1, \qquad \varphi_2 \to -\varphi_2, \qquad \varphi_3 \to -\varphi_3$$
$$P_2\colon\ \varphi_1 \to -\varphi_1, \qquad \varphi_2 \to \varphi_2, \qquad \varphi_3 \to -\varphi_3$$
$$P_3\colon\ \varphi_1 \to -\varphi_1, \qquad \varphi_2 \to -\varphi_2, \qquad \varphi_3 \to \varphi_3.$$

If any tensions are equal the symmetry is correspondingly increased. Imposing $\partial V_{XY}/\partial\varphi_\alpha = 0$ gives (no summation)

$$u'(\bar{\varphi}_\alpha)[\bar{\varphi}_\alpha - 4DJ_\alpha(\beta)u(\bar{\varphi}_\alpha) - 2K(\beta)u(\bar{\varphi}_\beta)u(\bar{\varphi}_\gamma)] = 0, \tag{54}$$

where $\beta = (\alpha + 1) \bmod 3$, $\gamma = (\alpha + 2) \bmod 3$. There are obvious solutions to these coupled equations: (0, 0, 0) for all β; $(\bar{\varphi}_1, 0, 0)$ with $\varphi_1 = \bar{\varphi}_1$ (the standard solution provided $2DJ_1 > 1$). The important point though is that *it is not possible to have a solution with only, say $\bar{\varphi}_1 = 0$, and the other two nonzero*. One can see this from (54), where setting $\bar{\varphi}_1 = 0$ would require that one of the other two φ's must vanish.

At the extrema the Hessian has the same diagonal elements as in (53), but off-diagonal elements

$$M_{\alpha\beta} = -2K(\beta)u'(\bar{\varphi}_\alpha)u'(\bar{\varphi}_\beta)u(\bar{\varphi}_\gamma). \tag{55}$$

We now evaluate these at the different extrema identified above and discuss the consequences.

Case 1.— $\bar{\varphi}_\alpha = 0$, $\forall\alpha$.

This reduces to the free-string case above, as here the off-diagonal terms of M also vanish. We have a global minimum for $\beta > \beta_1$ as all the eigenvalues are positive. Otherwise, when $\beta_3 < \beta < \beta_1$ we have a saddle point, and for $\beta < \beta_3$ a global maximum.

Thus, as the temperature increases (or β decreases) the 1 direction will "roll" first.

Case 2.— $\bar{\varphi}_2 = \bar{\varphi}_3 = 0$ but $\bar{\varphi}_1 \neq 0$.

Now, notice that the temperatures β_1, β_2, and β_3, as defined for free strings, are in principle relevant *only* when $\bar{\varphi}_\alpha = 0$ since then the off-diagonal terms of M vanish. When nonzero $\bar{\varphi}_\alpha$ enter, we have to worry about the off-diagonal terms, and find the new eigenvalues. This in turn will introduce new critical (K-dependent) temperatures.

As before, $\bar{\varphi}_1$ is the solution of the standard equation *provided* $2D\bar{J}_1 > 1$ or $\beta < \beta_1$.

When $\beta = \beta_2$ the smallest eigenvalue is negative, showing that $(\bar{\varphi}_1, 0, 0)$ is not a local minimum. There is an intermediate temperature β_*, the solution to

$$(1 - 2DJ_2(\beta_*))(1 - 2DJ_3(\beta_*)) = K^2(\beta_*)u^2(\bar{\varphi}_1) \tag{56}$$

that denotes the transition from local minimum to saddle point. That is, strings of type 2 and 3 are nucleated at the same time.

To summarize: for $\beta > \beta_1$ there is a global minimum at $\bar{\varphi}_\alpha = 0$. For $\beta_2 < \beta_* < \beta < \beta_1$ the global minimum is at $(\bar{\varphi}_1, 0, 0)$.

Case 3.— $\bar{\varphi}_1$, $\bar{\varphi}_2$, $\bar{\varphi}_3$ all nonzero.

For $\beta < \beta_*$, type 2 and 3 strings are nucleated since one cannot have only one vanishing $\bar{\varphi}_\alpha$. Hence we expect to have nonzero $\bar{\varphi}_\alpha$ for all α. However, there is nothing at this stage to preclude the possibility of even further transitions, of first and second order.

Discussion.—We can get some help from elementary Morse theory, applied to the whole $\bar{\varphi}_\alpha$ space and not just the positive sector [36]. Empirically, for the purpose of counting extrema, Eqs. (54) also behave just like the cubic equations that would follow from taking only the leading terms (38). According to this, when we have 27 extrema, no more than 14 can be minima. The cases of all $\bar{\varphi}_\alpha = 0$ or one $\bar{\varphi}_\alpha$ nonzero may produce $7(7 = 1 + 3 \times 2)$ real extrema and therefore 20 may correspond to extrema with no $\bar{\varphi}_\alpha$ vanishing. From D_2, each is fourfold degenerate, implying that there may exist five $(5 = 20/4)$ different least symmetric extrema, of which no more than three can be local minima. This still allows for either first- or second-order Hagedorn transitions as β is reduced below β_* (or temperature increased).

Now consider the case when two string types have (approximately) the same tension, and the other is markedly different, e.g. one string is very light, and the others heavy. The cases of all $\bar{\varphi}_\alpha = 0$ or one $\bar{\varphi}_\alpha$ nonzero still may produce $7(7 = 1 + 2 + 4)$ real extrema. However, each extremum with no $\bar{\varphi}_\alpha$ vanishing is now approximately eightfold symmetric. As a result we do not expect more than two of them, of which only one can be a local minimum. This means that there cannot be any further transitions as β is reduced below β_*. Although a first-order transition cannot be precluded, empirically we have only found second-order transitions even for σ_α taking different values. The situation is summarized schematically in Fig. 2.

From the above discussion, it follows trivially that, for equal σ_α, (with 12-fold degeneracy for all φ_α nonzero) there is just one second-order transition. This is relevant to an idealized version of QCD. However, as it stands the analysis above is restricted to closed or infinite string. The addition of quarks to string ends changes the picture again. Further, since flux tubes are not fundamental in any sense, the "Hagedorn" transition in QCD has a different status, with no ambiguity about increasing the temperature beyond it.

Density of vertices.—Finally we end this section with a comment on the density of vertices in the different phases. From (51), and since $I_m(0) = 0$ for $m \geq 1$ it follows that, on differentiating $V(\varphi_\alpha)$ with respect to K,

$$n_v = 0 \tag{57}$$

when any $\bar{\varphi}_\alpha = 0$. Thus, we only have a nonzero density of vertices on infinite strings for $\beta < \beta_*$, i.e. at temperatures high enough for infinite strings of all types to be present. This is shown in Fig. 1.

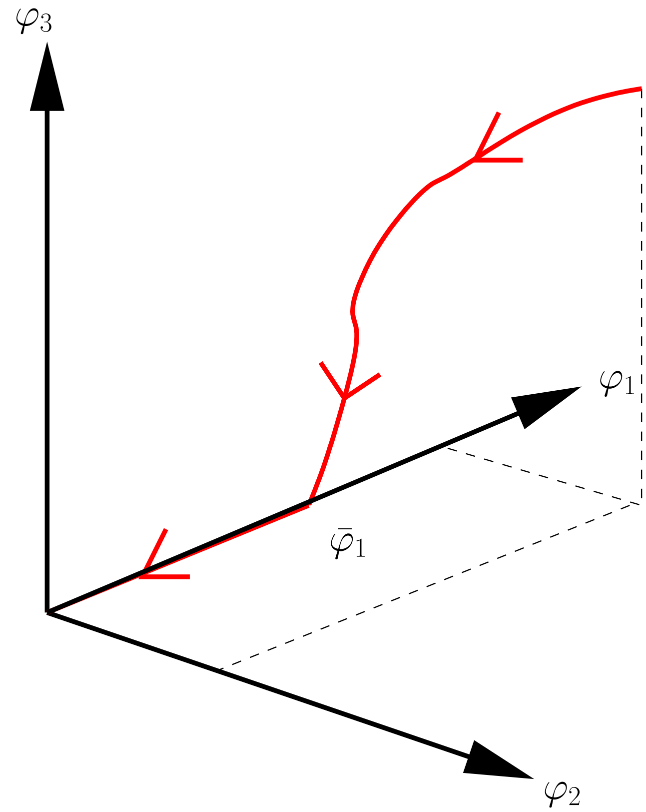

FIG. 2 (color online). Schematic representation of the trajectory of φ in field space. The arrow indicates the trajectory as a function of decreasing temperature.

V. CONCLUSIONS

The main idea of this paper has been very simple: that we can describe the thermodynamics of a network of strings of three different types (and tensions) by an effective three-field theory whose potential $V(\varphi_1, \varphi_2, \varphi_3)$ takes the form

$$\beta V = \sum_\alpha \left[\frac{1}{2} m_\alpha^2 \varphi_\alpha^2 + \frac{1}{4}\lambda_\alpha \varphi_\alpha^4\right] + \mu \varphi_1 \varphi_2 \varphi_3 + \ldots \tag{58}$$

The interaction coefficient μ reflects the presence of Y-junctions at which one string of each type meet. The coefficients are temperature dependent, with $m_\alpha^2 \propto (1 - T/T_\alpha)$ in the vicinity of its zero. If μ were zero, the T_α would be Hagedorn temperatures for the individual string types. As a result, the discrete symmetries of V are *broken* at high temperature, *restored* at low temperature, in a reversal of the usual pattern.

Our main results, summarized in Figs. 1 and 2, essentially follow from the form of (58) alone, supplemented by an understanding of the order parameters, that they characterize infinite string, and not loops. In consequence, in a network of strings of different tensions it is the lightest strings whose *infinite* strings survive last after Hagedorn transitions, and even those disappear in turn, to leave a collection of small loops. This is despite the presence of junctions between strings of different types. That is, the only rôle that the junctions play is in these small loops of string whose presence is the only memory of the initial proliferation of strings of all types.

The burden of this paper has been to provide a model in which we can see how the potential (58) is realized, almost as proof of principle. This has turned out to be a nontrivial task and the model at hand, an extension of similar models used in QCD in a much more restricted situation, has its faults. As well as picking a path through the fermionic lattice artefacts, as in the calculations for QCD strings, our strings are also assumed to be noninteracting and static. Furthermore, we are often pushed to consider the model in a limit of parameter space where approximations are not always well controlled (just as in [3–5]). Our one string bosonic model demonstrated how, for a single field, $\mu\varphi^3$ terms arise naturally. However, being unable to generalize the bosonic model to three-string types, we have also had to introduce massive vertices in the three-string model as an artefact of the lattice mean field approximation. Naturally, any specific model will give more information than just the leading terms of V of (58). In our case the model is a generalized XY model, in which transitions are seen in the language of spin ordering and which, in principle, permit better than the mean field approximation.

As suggested above, our analysis points to the final stage of the transitions as being that of a single-string type, collapsing into loops, which was the original case to be studied, primarily in the context of Nambu-Goto strings. In that case, the full statistical mechanics has been studied in detail, and can be generalized to nonstatic strings. The result, however, is the same. Indeed, rather than consider random walks in space, one can consider simultaneous independent random walks on the Kibble-Turok spheres for left- and right-moving modes respectively [12]. The microstate density at the transition is the square of that for simple random walks, but integrating over center-of-mass coordinates reduces the state density to that of (appropriately defined) single static random walks.

Another way to make this adiabatic picture dynamical is to attempt to determine the time scales of the string network transitions from the time scales of the effective field theory, using the Kibble scenario [31]. This relies on little more than causal bounds, and the analysis is under way.

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