Published by Institute of Physics Publishing for SISSA

Received: *April 11, 2008*
Accepted: *May 27, 2008*
Published: *June 10, 2008*

Optimizing the reduction of one-loop amplitudes

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Abstract: We present an optimization of the reduction algorithm of one-loop amplitudes in terms of master integrals. It is based on the exploitation of the polynomial structure of the integrand when evaluated at values of the loop-momentum fulfilling multiple cut-conditions, as emerged in the OPP-method. The reconstruction of the polynomials, needed for the complete reduction, is rendered very versatile by using a projection-technique based on the Discrete Fourier Transform. The novel implementation is applied in the context of the NLO QCD corrections to $u\bar{d} \to W^+W^-W^+$.

Keywords: NLO Computations, Standard Model.

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1. Introduction

In the last few years, we have been witnessing a boost in developing new ideas aiming to the efficient computation of one-loop amplitudes [1], as extensively reported in [2]. Besides standard techniques, where the tensor reduction is explicitly performed, new numerical and analytical developments, originally inspired by unitarity arguments [3, 4] have emerged. The common features of the so called unitarity-based methods [5 – 26], is the change of perspective they propose: instead of focusing on the actual evaluation of complete integrals, they pursue the determination of the *coefficients* of the scalar one-loop functions contributing to the unknown integrals. This possibility relies on the fact that the basis of scalar function to express any one-loop integral is known in terms of Boxes, Triangles, Bubbles and (in massive theories) Tadpoles [27]. Schematically, one can write a Master Equation for any one-loop amplitude $\mathcal{M}$ such as:

$$\mathcal{M} = \sum_i d_i \text{ Box}_i + \sum_i c_i \text{ Triangle}_i + \sum_i b_i \text{ Bubble}_i + \sum_i a_i \text{ Tadpole}_i \,, \tag{1.1}$$

where d_i, c_i, b_i and a_i are the coefficients to be determined.

Very recently in [28, 29], by exploiting the properties of spinor-integration of double-cuts of dimensionally regulated integrals [10, 12 – 16, 15, 17], general analytic formulas for the coefficients d_i, c_i, b_i were presented. These formulas can be evaluated as needed, without performing any integration, by specializing the value of input variables that are specific to the initial cut-integrand, which is assembled from tree-level amplitudes.

As an alternative to any phase-space integration, in [18, 19] there was proposed a very efficient method for the reconstruction of the so-called 4-dimensional cut-constructible term of any scattering amplitude (corresponding to the poly-logarithmic structure arising when eq. (1.1) is expanded around 4-dimensions). This method, by-now known as OPP-reduction, allows the numerical reconstruction of (the 4-dimensional limit of) the coefficients, d_i, c_i, b_i and a_i, by solving a system of algebraic equations that are obtained by: *i)* the numerical evaluation of the *integrand* at explicit values of the loop-variable, on the one side; *ii)* and the knowledge of the most general *polynomial* structure of the *integrand* itself [30], on the other one. We remark that the values of the loop momentum used for the numerical evaluation of the integrand are chosen among the set of solutions of the multiple-cut conditions, namely the solutions of the system of equations obtained by imposing the vanishing of the denominators on each 4-dimensional cut.

For the complete evaluation of scattering amplitudes, one has to consider that the 4-dimensional expansion of eq. (1.1) generates not only a poly-logarithmic term, but as well a *rational term* which cannot be detected by (massless) cuts in 4-dimensions. In [21], it has been recently shown that there are two sources of the rational terms: the first contribution, that is quite simple to calculate, originates from the generic $(n-4)$-dimensional structure of the numerator of any one-loop amplitude and it can be derived by using appropriate Feynman rules within a tree-like computation. The second contribution originates instead from the reduction of the $4-$dimensional part of the numerator in terms of the $n-$dimensional denominators appearing in the scalar integrals. This part, that is more subtle to extract, can be computed within the OPP-method in a completely automatized way, following one of the approaches discussed in [21] and numerically implemented in the public code CutTools [20].

Alternatively, the reconstruction of the rational term can be achieved as well by using techniques like direct computation [31, 32], by the bootstrapping method [33], by cuts in n-dimensions [28, 29], or by explicitly computing the amplitude at different integer value of the space-time dimensions [26]. In particular, in [25, 26] it has been proposed an extension of the OPP-reduction, implementing an integrand decomposition valid in n-dimensions, rather than in 4-dimensions, which exposed a richer, yet *polynomial*, structure of the cut-integrand.

The efficiency of the OPP-reduction has been shown in non-trivial applications, like the 6-photon amplitudes with massless and massive fermion-loop [19], the virtual QCD correction to $q\bar{q} \to ZZZ$ [2], and the complete cross section for the production of three vector bosons at LHC [34].

Within the OPP-reduction, the coefficients of the master integrals can be simply extracted by solving a system of numerical equations, rather than computing phase-space

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integrals. Cut-by-cut, in a *top-down* algorithm, from quadruple- to single-cut, one can establish a system for extracting the coefficient of each master-integral identified by the product of the cut-propagators. The general structure of the integrand [30] determines the polynomial shape of the equations forming such a system. In fact, by decomposing the loop variable in terms of a suitable basis of momenta (constructed from the external momenta and arbitrarily chosen reference momenta), the cut-conditions impose kinematical bounds on the values of the components of the loop momentum in this basis. According to the number of cuts, some component (when not all), are completely frozen, while others remain as free-variables[1]. The integrand, evaluated at a value of the loop momentum chosen among the solutions of a given cut, is a *polynomial* whose variables are the components of the loop momentum not frozen by the cut-conditions: the zero-th order coefficient of such a polynomial corresponds to the coefficient of the master integral one is interested in; the other terms of the polynomial, referred to as *spurious* terms, do not contribute to the cut (since they vanish upon integration), but they are needed later on for extracting the coefficients of lower-point master integrals.

To extract all the coefficients of this polynomial, one establishes a system, as said above, generated by evaluating numerically the integrand for values of loop momentum chosen within the solutions of the cut-conditions (parametrized by its free-components). The number of numerical evaluations must be the same as the number of the unknown coefficients to be determined.

The goal of the current paper is to exploit the polynomial structure of the integrand and the freedom in choosing the solutions of the cuts, used as numerical points for the evaluation of the integrand, to improve the system-solving algorithm. By selecting the variables of each polynomial to be proportional to the roots of unity, the extraction of the polynomial's coefficients is carried through *projections*, using the orthogonality relation among plane-waves, rather than by system inversion. The basic principle underlying this procedure is the same as for the Discrete Fourier Transform. The solutions accordingly obtained may help in getting a substantial reduction in computing-time. The effective benefit of the new implementation is mostly experienced for the extraction of the coefficients of the 3- and 2-point functions, since in the 4- and 1-point cases the polynomial structure is very simple: a degree-1 polynomial, in the case of the 4-point; an effective degree-0 polynomial, in the case of the 1-point functions.

The paper is organized as follows. In section 2, we recall the basic features of the OPP-reduction, by classifying the polynomial structures generated by the the multiple-cut integrand. In section 3, we introduce the *projections* used for extracting the coefficients out of a polynomial, which are explicitly applied, in sections 4 and 5, for the extraction of the coefficients of the 3- and 2-point functions, respectively. Finally, in section 6, we apply the optimized reduction to the computation of the Next-to-Leading-Order QCD corrections to the scattering amplitude of the process $u\bar{d} \to W^+W^-W^+$.

[1]The variables not frozen by the cut-conditions correspond to the integration variables of the phase-space integral

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2. OPP-Reduction

The starting point of the OPP reduction method [18, 19] is the general expression for the *integrand* of a generic m-point one-loop (sub-)amplitude that, using dimensional regularization, can be written as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,, \quad \bar{D}_i = (\bar{q}+p_i)^2 - m_i^2\,, \quad p_0 \neq 0\,. \tag{2.1}$$

We use a bar to denote objects living in $n = 4+\epsilon$ dimensions; therefore we have $\bar{q}^2 = q^2+\tilde{q}^2$, where $\tilde{q}^2$ is ϵ-dimensional and $(\tilde{q}\cdot q) = 0$. $N(q)$ is the 4-dimensional part of the numerator of the amplitude. If needed, the ϵ-dimensional part of the numerator should be treated separately, as explained in [35, 21]. $N(q)$ depends on the 4-dimensional denominators $D_i = (q+p_i)^2 - m_i^2$ as follows

$$\begin{aligned}
N(q) = & \sum_{i_0<i_1<i_2<i_3}^{m-1} \Big[d(i_0i_1i_2i_3) + \tilde{d}(q;i_0i_1i_2i_3)\Big] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
& + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0i_1i_2) + \tilde{c}(q;i_0i_1i_2)] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
& + \sum_{i_0<i_1}^{m-1} \Big[b(i_0i_1) + \tilde{b}(q;i_0i_1)\Big] \prod_{i\neq i_0,i_1}^{m-1} D_i \\
& + \sum_{i_0}^{m-1} [a(i_0) + \tilde{a}(q;i_0)] \prod_{i\neq i_0}^{m-1} D_i \\
& + \tilde{P}(q) \prod_{i}^{m-1} D_i\,.
\end{aligned} \tag{2.2}$$

Inserted back in eq. (2.1), this expression simply states the multi-pole nature of any m-point one-loop amplitude, that, clearly, contains a pole for any propagator in the loop, thus one has terms ranging from 1 to m poles.

The coefficients of the poles can be further split in two pieces. A piece that still depend on q (the terms $\tilde{d},\tilde{c},\tilde{b},\tilde{a}$), that vanishes upon integration, and a piece that do not depend on q (the terms d,c,b,a). Such a separation is always possible, as shown in [18], and, with this choice, the latter set of coefficients is therefore immediately interpretable as the ensemble of the coefficients of all possible 4, 3, 2, 1-point one-loop functions contributing to the amplitude. Notice that the term with no poles, namely that one proportional to $\tilde{P}(q)$ is polynomial and vanishes upon integration in dimensional regularization. Moreover, it can be shown that in the renormalizable gauge $\tilde{P}(q) = 0$, even before integration.

2.1 Top-down polynomial structures

Since the scalar 1-, 2-, 3-, 4-point functions are known, the problem of computing the one-loop amplitude is simply reduced to the algebraical problem of fitting the coefficients

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d, c, b, a by evaluating the function $N(q)$ a sufficient number of times, at different values of q, and then inverting the system.

This task can be achieved very efficiently by singling out particular choices of q such that, systematically, 4, 3, 2 or 1, among all possible denominators D_i vanish. In [18], it was shown that by proceeding *top-down* from quadruple-cuts to single-cuts, it is possible to construct a particularly simple system of equations that can be solved analytically, whose solutions yield the complete reconstruction of the unknown coefficients.

- **Quadruple-cut.** When q is solution of

$$D_0 = D_1 = D_2 = D_3 = 0\,. \tag{2.3}$$

$$N(q) = [d(0123) + \tilde{d}(q; 0123)] \prod_{i\neq 0,1,2,3} D_i(q) \equiv R(q) \prod_{i\neq 0,1,2,3} D_i(q) \tag{2.4}$$

  where $R(q)$ has a polynomial structure with 2 terms.

- **Triple-cut.** At this stage all d and $\tilde{d}$ coefficients are known. When q is solution of

$$D_0 = D_1 = D_2 = 0 \quad \text{and} \quad D_i \neq 0 \;\; \forall i \neq 0, 1, 2 \tag{2.5}$$

  eq. (2.2) reads

$$\begin{aligned} N(q) - \sum_{2<i_3} [d(012i_3) + \tilde{d}(q; 012i_3)] \prod_{i\neq 0,1,2,i_3} D_i(q) \\ \equiv R'(q) \prod_{i\neq 0,1,2} D_i(q) = [c(012) + \tilde{c}(q; 012)] \prod_{i\neq 0,1,2} D_i(q)\,, \end{aligned} \tag{2.6}$$

  where $R'(q)$ has a polynomial structure with 7 terms.

- **Double-cut.** At this stage all d, $\tilde{d}$, c and $\tilde{c}$ coefficients are known. When q is solution of

$$D_0 = D_1 = 0 \quad \text{and} \quad D_i \neq 0 \;\; \forall i \neq 0, 1 \tag{2.7}$$

  eq. (2.2) reads

$$\begin{aligned} N(q) - \sum_{1<i_2<i_3} [d(01i_2i_3) + \tilde{d}(q; 01i_2i_3)] \prod_{i\neq 0,1,i_2,i_3} D_i \\ - \sum_{1<i_2} [c(01i_2) + \tilde{c}(q; 01i_2)] \prod_{i\neq 0,1,i_2} D_i \\ \equiv R''(q) \prod_{i\neq 0,1} D_i(q) = [b(01) + \tilde{b}(q; 01)] \prod_{i\neq 0,1} D_i(q)\,, \end{aligned} \tag{2.8}$$

  where $R''(q)$ has a polynomial structure with 9 terms.

- **Single-cut.** In massless theories all 1-point functions, namely all tadpoles, vanish, also implying that, in such cases, one does not need to know all the $\tilde{b}$ coefficients.

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However, in general, also the coefficients of the tadpoles are required. Therefore we discuss how to extract them.

At this stage we assume to know all the d, $\tilde{d}$, c, $\tilde{c}$, b and $\tilde{b}$ coefficients and, when q is solution of

$$D_0 = 0 \quad \text{and} \quad D_i \neq 0 \;\; \forall i \neq 0\,, \tag{2.9}$$

eq. (2.2) reads

$$\begin{aligned} N(q) - & \sum_{0<i_1<i_2<i_3} [d(0i_1i_2i_3) + \tilde{d}(q;0i_1i_2i_3)] \prod_{i\neq 0,i_1,i_2,i_3} D_i \\ & - \sum_{0<i_1<i_2} [c(0i_1i_2) + \tilde{c}(q;0i_1i_2)] \prod_{i\neq 0,i_1,i_2} D_i \\ & - \sum_{0<i_1} [b(0i_1) + \tilde{b}(q;0i_1)] \prod_{i\neq 0,i_1} D_i \\ & \equiv R'''(q) \prod_{i\neq 0} D_i(q) = [a(0) + \tilde{a}(q;0)] \prod_{i\neq 0} D_i(q)\,. \end{aligned} \tag{2.10}$$

where $R'''(q)$ has a polynomial structure with 5 terms. We notice that the spurious coefficients $\tilde{a}_i(0)$ $(i = 1, \ldots, 4)$ are never needed, because they would be necessary only to extract what we called $\tilde{P}(q)$, that, as already observed, is irrelevant. Therefore one focuses directly on the extraction of the tadpole-coefficient $a(0)$.

We remark, that the polynomials we are going to deal with, i.e. $R(q), R'(q), R''(q)$, and $R'''(q)$, share a common structure: a *spurious* structure, depending on the loop variable, q, which does not contribute to the cut-integral; and a single q-independent term, which corresponds to the actual coefficient of the master integral identified by the cuts[2].

3. Polynomial structures and discrete fourier transform

After the general structure of eq. (2.2) is established, as we illustrated in the previous section, the calculation of the scattering amplitude reduces to the problem of extracting the coefficients of multivariable polynomials, generated at every step of the multiple-cut analysis.

Let us show how it is possible to extract efficiently the coefficients of a polynomial of degree n in the variable x, say $P_n(x)$, defined as,

$$P_n(x) = \sum_{\ell=0}^{n} c_\ell \; x^\ell\,, \tag{3.1}$$

by means of *projections*, according to the same principle underlying the Discrete Fourier Transform which works as follows.

[2]The only exception to this pattern will be the structure of the double-cut coefficients, that also includes a q-independent non vanishing term, in order to avoid numerical instabilities [19].

3.1 Discrete Fourier Transform

Consider the function F, known only numerically in N points, F_n $(n = 0, \ldots, N-1)$. Each of this values, admits a DFT, defined as,

$$F_n \equiv \sum_{k=0}^{N-1} f_k \; e^{-2\pi i \frac{k}{N} \; n} \; . \tag{3.2}$$

The coefficients f_k, can be found by using the orthogonality relation,

$$\sum_{n=0}^{N-1} e^{2\pi i \frac{k}{N} \; n} \; e^{-2\pi i \frac{k'}{N} \; n} = N \; \delta_{kk'} \; . \tag{3.3}$$

with the result

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n \; e^{2\pi i \frac{k}{N} \; n} \; . \tag{3.4}$$

3.2 Projections

Let us see how the above procedure can be implemented for extracting the coefficients c_ℓ's of the polynomial in eq. (3.1) by *projections*.

1. Generate the set of discrete values g_k $(k = 0, \ldots, n)$,

$$g_k = P_n(x_k) = \sum_{\ell=0}^{n} \; c_\ell \; \rho^\ell \; e^{-2\pi i \frac{k}{(n+1)} \ell} \, , \tag{3.5}$$

   by evaluating $P_n(x)$ at the points

$$x_k = \rho \; e^{-2\pi i \frac{k}{(n+1)}} \; . \tag{3.6}$$

2. Using the orthogonality relations for the wave planes, one can obtain the coefficient c_ℓ simply by,

$$c_\ell \; = \; \frac{\rho^{-\ell}}{n+1} \sum_{k=0}^{n} \; g_k \; e^{2\pi i \frac{k}{(n+1)} \ell} \tag{3.7}$$

This procedure is very general and can be applied as long as one needs to know the coefficients of any polynomial. In fact, it has been recently suggested in [29], for computing the coefficients of polynomials in the (-2ϵ)-dimensional mass parameter.

The projections could be extended also to the case of multi-variables polynomials, along the same line of the multi-dimensional DFT. Since we aim to minimize the computational load, we keep the number of numerical evaluations of each polynomial to be the same as the number of its coefficients. To this aim, we will see that the coefficients of a multi-variable polynomial can be equivalently found by breaking it in several one-variable polynomials, obtained from the former by freezing the values of the other variables, yielding still the use of the (one-dimensional) projections described above.

The next two sections will be devoted to the application of the projection procedure for the extraction of the coefficients of the 3- and 2-point functions, respectively out of $R'(q)$, and $R''(q)$. We won't discuss hereby the reconstruction of the coefficients of the 4- and 1-point functions, because the polynomial structures of $R(q)$ and $R'''(q)$ is very simple, and the result of the projection procedure would be the same as the one given in [18].

4. The coefficient of the 3-point functions

In this section, we show how to apply the properties of orthogonal functions for extracting the coefficient of the 3-point functions.

Let's begin from eq. (2.21) of [18],

$$R'(q) = c(012) + \sum_{j=1}^{3} \left\{ \tilde{c}_{1j}(012)[(q+p_0)\cdot \ell_3]^j + \tilde{c}_{2j}(012)[(q+p_0)\cdot \ell_4]^j \right\} \tag{4.1}$$

where $R'(q)$ appeared in eq. (2.6). By substituting the parametrization of q, solution of the triple-cut conditions given in [18], and recalled in appendix A,

$$q = -p_0 + x_1\ell_1 + x_2\ell_2 + x_3\ell_3 + x_4\ell_4 \tag{4.2}$$

one obtains

$$R'(q) = c(012) + \sum_{j=1}^{3} \left\{ \tilde{c}_{1j}(012)\ (\ell_3\cdot\ell_4)^j\ x_4^j + \tilde{c}_{2j}(012)\ (\ell_3\cdot\ell_4)^j\ x_3^j \right\}, \tag{4.3}$$

This expression can be read as a polynomial in x_3 and x_4, whose canonical form reads,

$$P(x_3,x_4) = c_{0,0} + c_{1,0}\ x_3 + c_{2,0}\ x_3^2 + c_{3,0}\ x_3^3 + c_{0,1}\ x_4 + c_{0,2}\ x_4^2 + c_{0,3}\ x_4^3 \tag{4.4}$$

with the following relations among the coefficients

$$\begin{aligned} c_{0,0} &= c(012)\,, \\ c_{j,0} &= c_{2j}(012)\ (\ell_3\cdot\ell_4)^j \\ c_{0,j} &= c_{1j}(012)\ (\ell_3\cdot\ell_4)^j \end{aligned} \tag{4.5}$$

Therefore computing the OPP-coefficients is equivalent to the computation of the 7 coefficients of eq. (4.4).

4.1 Projections

The extraction of the coefficients $c_{i,j}$ is performed, in the framework of the original OPP-method and in CutTools, by choosing a redundant set of solutions in order to avoid some fake singularities occurring in kinematical points in which $C = 0$. In practice, this is obtained by doubling the number of calls to the numerator function $N(q)$, by roughly doubling the computation time. The same problem can be more efficiently solved with the help of the proposed *projection* method. By using it, it is in fact very easy to find a set of

solutions for which the point $C = 0$ is never singular and for which $N(q)$ is called just as many times as the number of needed coefficients. This is done at the price of creating a new fake singularity at $C = 1$, but, when this situation occurrs, one can use the original solution. The described procedure explicitly shows the flexibility of the proposed *projection* method.

To extract the 7 coefficients c_{ij} by projections, we take 7 values of P, grouped in 2 sets $(4 + 3)$,

$$\begin{aligned} g_{1,k} &= P(x_{3k}, x_{4k}), \qquad x_{3k} = C\ e^{-2\pi i \frac{k}{4}}, \qquad x_{4k} = e^{2\pi i \frac{k}{4}}, \quad (k = 0, 1, 2, 3) \\ g_{2,k} &= P(x_{3k}, x_{4k}), \qquad x_{3k} = e^{-2\pi i \frac{k}{3}}, \qquad x_{4k} = C\ e^{2\pi i \frac{k}{3}}, \quad (k = 0, 1, 2) \end{aligned} \tag{4.6}$$

where C is given in appendix A.

Then, we construct the auxiliary functions

$$\mu(1, m, n) = \frac{1}{4} \sum_{k=0}^{3} g_{1,k}\ e^{2\pi i \frac{k}{4}(m-n)} \tag{4.7}$$

with: $(m, n) = (0, 0), (0, 1), (0, 2), (1, 0)$.

$$\mu(2, m, n) = \frac{1}{3} \sum_{k=0}^{2} g_{2,k}\ e^{2\pi i \frac{k}{3}(m-n)} \tag{4.8}$$

with: $(m, n) = (0, 0), (0, 1), (1, 0)$.

In terms of these auxiliary functions, the coefficients read,

$$c_{0,0} = \mu(1, 0, 0) \tag{4.9}$$

$$\begin{aligned} c_{1,0} = -\frac{1}{C^{12} - 1}(&-\mu(1, 1, 0)C^{11} - \mu(1, 0, 0)C^8 + \mu(2, 0, 0)C^8 - \mu(1, 0, 1)C^5 + \mu(2, 0, 1)C^4 \\ &-\mu(1, 0, 2)C^2 + \mu(2, 1, 0)) \end{aligned} \tag{4.10}$$

$$\begin{aligned} c_{2,0} = -\frac{1}{C^{12} - 1}(&-\mu(1, 0, 2)C^{10} + \mu(2, 1, 0)C^8 - \mu(1, 1, 0)C^7 - \mu(1, 0, 0)C^4 \\ &+\mu(2, 0, 0)C^4 - \mu(1, 0, 1)C + \mu(2, 0, 1)) \end{aligned} \tag{4.11}$$

$$\begin{aligned} c_{3,0} = -\frac{1}{C^{12} - 1}(&-\mu(1, 0, 1)C^9 + \mu(2, 0, 1)C^8 - \mu(1, 0, 2)C^6 + \mu(2, 1, 0)C^4 \\ &-\mu(1, 1, 0)C^3 - \mu(1, 0, 0) + \mu(2, 0, 0)) \end{aligned} \tag{4.12}$$

$$\begin{aligned} c_{0,1} = -\frac{1}{C^{12} - 1}(&-\mu(2, 0, 1)C^{11} + \mu(1, 0, 2)C^9 - \mu(2, 1, 0)C^7 + \mu(1, 1, 0)C^6 + \mu(1, 0, 0)C^3 \\ &-\mu(2, 0, 0)C^3 + \mu(1, 0, 1)) \end{aligned} \tag{4.13}$$

$$\begin{aligned} c_{0,2} = -\frac{1}{C^{12} - 1}(&-\mu(2, 1, 0)C^{10} + \mu(1, 1, 0)C^9 + \mu(1, 0, 0)C^6 - \mu(2, 0, 0)C^6 + \mu(1, 0, 1)C^3 \\ &-\mu(2, 0, 1)C^2 + \mu(1, 0, 2)) \end{aligned} \tag{4.14}$$

$$\begin{aligned} c_{0,3} = -\frac{1}{C^{12} - 1}(&\mu(1, 0, 0)C^9 - \mu(2, 0, 0)C^9 + \mu(1, 0, 1)C^6 - \mu(2, 0, 1)C^5 + \mu(1, 0, 2)C^3 \\ &-\mu(2, 1, 0)C + \mu(1, 1, 0)) \end{aligned} \tag{4.15}$$

Finally, to obtain the OPP-coefficients, simply use eqs. (4.5).

5. The coefficient of the 2-point functions

In this section, we show how to apply the properties of orthogonal functions for extracting the coefficient of the 2-point functions.
Let's begin from eq. (B.7) of [19]

$$\begin{aligned} R''(q) = b &+ \hat{b}_0[(q+p_0)\cdot v] + \hat{b}_{00}[(q+p_0)\cdot v]^2 \\ &+\tilde{b}_{11}[(q+p_0)\cdot \ell_7] + \tilde{b}_{21}[(q+p_0)\cdot \ell_8] \\ &+\tilde{b}_{12}[(q+p_0)\cdot \ell_7]^2 + \tilde{b}_{22}[(q+p_0)\cdot \ell_8]^2 \\ &+\tilde{b}_{01}[(q+p_0)\cdot \ell_7][(q+p_0)\cdot v] + \tilde{b}_{02}[(q+p_0)\cdot \ell_8][(q+p_0)\cdot v] \end{aligned} \tag{5.1}$$

where $R''(q)$ appeared in eq. (2.8). By substituting the parametrization of q, the solution of the double-cut conditions given in eq. (B.6) of [19], and recalled in appendix B,

$$q = -p_0 + yk_1 + y_v v + y_7 \ell_7 + x_8 \ell_8 \tag{5.2}$$

one obtains

$$\begin{aligned} N(q) = b &+ \hat{b}_0[yk_1\cdot v] + \hat{b}_{00}[yk_1\cdot v]^2 \\ &+\tilde{b}_{11}[y_8\ell_8\cdot \ell_7] + \tilde{b}_{21}[y_7\ell_7\cdot \ell_8] \\ &+\tilde{b}_{12}[y_8\ell_8\cdot \ell_7]^2 + \tilde{b}_{22}[y_7\ell_7\cdot \ell_8]^2 \\ &+\tilde{b}_{01}[y_8\ell_8\cdot \ell_7][yk_1\cdot v] + \tilde{b}_{02}[y_7\ell_7\cdot \ell_8][yk_1\cdot v] \end{aligned} \tag{5.3}$$

This expression can be read as a polynomial in y, y_7 and y_8, whose canonical form reads,

$$\begin{aligned} P(y,y_7,y_8) = a_{000} &+ a_{100}\ y + a_{200}\ y^2 \\ &+a_{010}\ y_7 + a_{020}\ y_7^2 + a_{001}\ y_8 + a_{002}\ y_8^2 \\ &+a_{110}\ y\ y_7 + a_{101}\ y\ y_8\,, \end{aligned} \tag{5.4}$$

with the following relations among the coefficients

$$\begin{aligned} a_{000} &= b \\ a_{100} &= \hat{b}_0(k_1\cdot v) \\ a_{200} &= \hat{b}_{00}(k_1\cdot v)^2 \\ a_{010} &= \tilde{b}_{21}(\ell_7\cdot \ell_8) \\ a_{020} &= \tilde{b}_{22}(\ell_7\cdot \ell_8)^2 \\ a_{001} &= \tilde{b}_{11}(\ell_7\cdot \ell_8) \\ a_{002} &= \tilde{b}_{12}(\ell_7\cdot \ell_8)^2 \\ a_{110} &= \tilde{b}_{02}(\ell_7\cdot \ell_8)(k_1\cdot v) \\ a_{101} &= \tilde{b}_{01}(\ell_7\cdot \ell_8)(k_1\cdot v) \end{aligned} \tag{5.5}$$

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5.1 Projections

Here we explicitly illustrate a solution that avoids the problem of doubling the number of calls to the numerator function $N(q)$ (used in the original implementation of the OPP-method) due to the appearence of fake singularities when $F_0 = 0$.

To extract the 9 coefficients a_{ijk} by projections, we take 9 values of P, grouped in 5 sets $(3+2+2+1+1)$,

$$\begin{aligned}
g_{00h} &= P(0, y_{7h}, y_{8h}), \quad y = 0, \quad y_{7h} = F_0\, e^{-2\pi i \frac{h}{3}}, \quad y_{8h} = e^{2\pi i \frac{h}{3}}, \quad (h = 0,1,2) \\
g_{0h0} &= P(0, y_{7h}, y_{8h}), \quad y = 0, \quad y_{7h} = e^{-2\pi i \frac{h}{2}}, \quad y_{8h} = F_0\, e^{2\pi i \frac{h}{2}} \quad (h = 0,1) \\
g_{-10h} &= P(-1, y_{7h}, y_{8h}), \quad y = -1, \quad y_{7h} = F_{-1}\, e^{-2\pi i \frac{h}{2}}, \quad y_{8h} = e^{2\pi i \frac{h}{2}} \quad (h = 0,1) \\
g_{-1} &= P(-1, 1, F_{-1}), \quad y = -1, \quad y_7 = 1, \quad y_8 = F_{-1} \\
g_1 &= P(1, F_1, 1), \quad y = 1, \quad y_7 = F_1, \quad y_8 = 1
\end{aligned} \tag{5.6}$$

where the definition of F_y is given in appendix B.

Then, we construct the auxiliary functions

$$\mu(1,0,m,n) = \frac{1}{3}\sum_{h=0}^{2} g_{00h}\, e^{2\pi i \frac{h}{3}(m-n)}, \qquad (m,n) = (0,0),(0,1),(1,0) \tag{5.7}$$

$$\mu(2,0,m,n) = \frac{1}{2}\sum_{h=0}^{1} g_{0h0}\, e^{2\pi i \frac{h}{2}(m-n)}, \qquad (m,n) = (0,0),(1,0) \tag{5.8}$$

$$\mu(1,-1,m,n) = \frac{1}{2}\sum_{h=0}^{1} g_{-10h}\, e^{2\pi i \frac{h}{2}(m-n)}, \qquad (m,n) = (0,0),(0,1) \tag{5.9}$$

$$\mu(2,-1,0,0) = g_{-1} \tag{5.10}$$

$$\mu(1,1,0,0) = g_1 \tag{5.11}$$

The coefficients can be expressed as linear combinations of these auxiliary functions. Since the expressions for generic values of F_0, F_{-1}, F_1 are rather long, we present the one obtained when $F_0 = F_{-1} = 0$, which happens when the propagators are massless, as in the case of the application later discussed. Hence, the coefficients read,

$$a_{000} = \mu(1,0,0,0) \tag{5.12}$$

$$\begin{aligned}
a_{100} = \frac{1}{2}(&\mu(1,0,0,0)F_1^2 - \mu(2,0,0,0)F_1^2 - \mu(1,-1,0,0)F_1 + \mu(1,0,0,0)F_1 \\
&+\mu(1,0,1,0)F_1 + \mu(2,-1,0,0)F_1 - \mu(2,0,0,0)F_1 - 2\mu(2,0,1,0)F_1 \\
&-\mu(1,-1,0,0) + \mu(1,-1,0,1) - 2\mu(1,0,0,1) + \mu(1,1,0,0)),
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
a_{200} = \frac{1}{2}(&\mu(1,0,0,0)F_1^2 - \mu(2,0,0,0)F_1^2 - \mu(1,-1,0,0)F_1 + \mu(1,0,0,0)F_1 \\
&+\mu(1,0,1,0)F_1 + \mu(2,-1,0,0)F_1 - \mu(2,0,0,0)F_1 - 2\mu(2,0,1,0)F_1 \\
&+\mu(1,-1,0,0) + \mu(1,-1,0,1) - 2\mu(1,0,0,0) \\
&-2\mu(1,0,0,1) - 2\mu(1,0,1,0) + \mu(1,1,0,0)),
\end{aligned} \tag{5.14}$$

$$a_{010} = \mu(2,0,1,0), \tag{5.15}$$

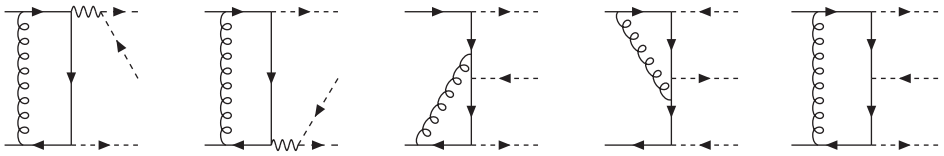

Figure 1: Box and pentagon diagrams contributing to virtual QCD corrections to $u\bar{d} \to W^+W^-W^+$. Wavy lines can be either Z or photons, dashed lines represent W^+ and W^-. Diagrams involving an exchange in the two final W^+ should also be considered.

$$a_{020} = \mu(2,0,0,0) - \mu(1,0,0,0), \tag{5.16}$$

$$a_{001} = \mu(1,0,0,1), \tag{5.17}$$

$$a_{002} = \mu(1,0,1,0), \tag{5.18}$$

$$a_{110} = \mu(1,-1,0,0) - \mu(1,0,0,0) - \mu(1,0,1,0) - \mu(2,-1,0,0) + \mu(2,0,0,0) + \mu(2,0,1,0), \tag{5.19}$$

$$a_{101} = \mu(1,0,0,1) - \mu(1,-1,0,1)\ . \tag{5.20}$$

Finally, to obtain the OPP-coefficients, simply use eqs. (5.5)

6. An example: QCD virtual corrections to $u\bar{d} \to W^+W^-W^+$ at NLO

As an example of application of the optimized algorithm, we present the results for the pentagon and box diagrams contributing to the scattering amplitudes of $u\bar{d} \to W^+W^-W^+$ at NLO in QCD. The complete cross section for the production of $W^+W^-W^+$ at LHC, as well as W^+W^-Z, W^+ZZ and ZZZ is presented in [34].

The main purpose of this application is to test the improvements on the OPP-reduction algorithm, both in terms of stability and efficiency, after the changes proposed in this paper have been included.

The complete calculation of NLO QCD virtual correction to $u\bar{d} \to W^+W^-W^+$, neglecting the contributions that depend on the Higgs boson, involves the reduction of 53 diagrams. The topologies of pentagon and box diagrams contributing to this process are depicted in figure 1. Overall, we have 2 pentagons and 12 boxes.

Most of the computing time in the calculation is spent in the evaluation of the coefficients for 3-point and 2-point scalar function arising from the reduction of the diagrams illustrated. For example, the reduction of the each pentagon implies the evaluation of ten sets of c-type coefficients and ten sets of b-type coefficients, involving each seven and nine coefficients respectively. It is therefore very important to have efficient routines to achieve this task. As a comparison, the evaluation of the d-type coefficients is only performed five times for each pentagon and involves only two coefficients.

For the purpose of this test, we fix the external momenta to the specific phase-space configuration of eq. (6.1) and the polarization vectors for the vector bosons. To perform the calculation, we use the OPP-reduction method, with and without the optimization technique illustrated before. The coefficients determined in this manner should be multiplied

Polarization	$\|A\|$
0 0 0	28.2435
0 0 −	5.13851
0 0 +	10.7870
0 − 0	1.923741
0 − +	0.718415
0 + 0	7.43599
0 + −	1.95506
− + −	0.276058
− + +	0.402302
+ + +	0.875457

Table 1: Absolute value of the amplitudes for different configurations of the polarization of the vector bosons (accounting for 5-point and 4-point Feynman diagrams only). The results are expressed in unit of e^3.

by the corresponding scalar integrals. Since, in the process that we are studying, no q-dependent massive propagator appears, we will only need massless scalar integrals. They are computed using the package `OneLOop` written by A. van Hameren [36]. Finally, we sum the contributions coming from the various pentagons and boxes: the results obtained are presented in table 1.

$$\begin{aligned} p_1 &= \{500, 0, 500, 0\} \\ p_2 &= \{500, 0, -500, 0\} \\ p_3 &= \{276.212, 97.7237, -56.2856, 238.9792\} \\ p_4 &= \{486.8926, 213.4030, 37.7214, -428.5282\} \\ p_5 &= p_1 + p_2 - p_3 - p_4 \end{aligned} \tag{6.1}$$

The results obtained with the two different implementations of the algorithm are, of course, in perfect agreement. However, with the optimized version we can roughly improve the efficiency by a factor 2. As a first test, we checked the improvements on the timing of the two new routines alone, separating them from the rest of the reduction. We experience a reduction in the computing time of about 60% for the system of the c-coefficients, and about 50% for the system of the b-coefficient.

In the overall evaluation of the amplitudes, the optimized routines are combined with other parts of the program that remain unchanged. This involves, for example, the initialization, the evaluation of scalar integrals, the evaluation of d- and a-coefficients and rational parts, and the summing over all contributions. We still retain, however, an overall improvement in the computing time of about 40%.

Concerning the stability issues, we tested the new routines for a wide set of phase-space points. We do not observe significant improvements respect to the previous implementation.

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7. Conclusions

We presented an optimization of the reduction algorithm of one-loop amplitudes in terms of master integrals. That is based on the exploitation of the polynomial structure of the integrand when evaluated at values of the loop-momentum fulfilling multiple cut-conditions, as emerged in the OPP-method. Accordingly, the integrand, evaluated at a value of the loop momentum chosen among the solutions of a given cut, is a *polynomial* whose variables are the components of the loop momentum not frozen by the cut-conditions: the zero-th order coefficient of such a polynomial corresponds to the coefficient of the master integral one is interested in; the other terms of the polynomial, though not contributing to the cut, are needed for the later determination of the coefficients of lower-point master integrals.

To extract all the polynomial coefficients, one establishes a system generated by evaluating numerically the integrand for values of loop variable chosen within the solutions of the cut-conditions (parametrized by its free-components). The freedom in choosing the solutions of the cuts has been hereby exploited to improve the system-solving algorithm. By selecting the variables of each polynomial to be proportional to the *primitive roots of unity*, the extraction of the polynomial's coefficients is carried through *projections*, using the same orthogonality relation underlying the Discrete Fourier Transform. The number of numerical evaluations is kept as low as the number of the unknown coefficients to be determined, by using one-dimensional projections also in case of polynomials in more than one-variable.

The novel implementation was applied to the reduction of the 4- and 5-point one-loop Feynman diagrams contributing the NLO QCD corrections to $u\bar{d} \to W^+W^-W^+$, where we experienced a reduction of the computational load.

The flexibility of the projection-procedure hereby presented extends its range of applicability to tackle the determination of the coefficients of polynomial structures wherever should this issue occur. Moreover, we finally remark that the parametrization of the free (integration) variables as complex unitary phases yields as well a very effective performance of Cauchy's residue theorem within the contexts of factorization- and unitarity-based methods, where the on-shellness properties are naturally captured by polar structures in complex phases.

Acknowledgments

C.G.P.'s and R.P.'s research was partially supported by the RTN European Programme MRTN-CT-2006-035505 (HEPTOOLS, Tools and Precision Calculations for Physics Discoveries at Colliders). The research of R.P. was also supported by MIUR under contract 2006020509_004 and by the MEC project FPA2006-05294.

G.O. and R.P. acknowledge the financial support of the ToK Program "ALGOTOOLS" (MTKD-CT-2004-014319).

P.M. thanks Ettore Remiddi and Thobias Motz for interesting discussions on the use of orthogonal functions, and Joe Apostolico for special support.

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A. The basis for the 3-point functions

The loop momentum solution of the triple-cut, given in eq. (4.2), is expressed in terms of auxiliary vectors defined as follows.
ℓ_1 and ℓ_2 are massless 4-vector satisfying the relations

$$k_1 = \ell_1 + \alpha_1 \ell_2 \,, \quad k_2 = \ell_2 + \alpha_2 \ell_1 \,, \tag{A.1}$$

with

$$k_i = p_i - p_0 \,. \tag{A.2}$$

Furthermore, in spinorial notation,

$$\ell_3^\mu = \langle \ell_1 | \gamma^\mu | \ell_2] \,, \quad \ell_4^\mu = \langle \ell_2 | \gamma^\mu | \ell_1] \quad \text{with} \quad (\ell_3 \cdot \ell_4) = -4(\ell_1 \cdot \ell_2) \,. \tag{A.3}$$

The solution to eq. (A.1) reads

$$\begin{aligned} \ell_1 &= \beta(k_1 - \alpha_1 k_2) \,, & \ell_2 &= \beta(k_2 - \alpha_2 k_1) \,, \\ \beta &= 1/(1 - \alpha_1 \alpha_2) \,, & \alpha_i &= \frac{k_i^2}{\gamma} \,, \\ \gamma &\equiv 2(\ell_1 \cdot \ell_2) = (k_1 \cdot k_2) \pm \sqrt{\Delta} \,, & \Delta &= (k_1 \cdot k_2)^2 - k_1^2 k_2^2 \,. \end{aligned} \tag{A.4}$$

we decompose $q^\mu + p_0^\mu$ in the basis formed by ℓ_1, ℓ_2, ℓ_3, and ℓ_4,

$$q = -p_0 + x_1 \ell_1 + x_2 \ell_2 + x_3 \ell_3 + x_4 \ell_4 \,, \tag{A.5}$$

which is solution of the triple-cut,

$$D_0 = D_1 = D_2 = 0 \,. \tag{A.6}$$

Due to the above constrains, the coefficients of the loop decomposition must fulfill the following relations,

$$x_1 = \frac{\beta}{\gamma}[d_2 - \alpha_2 d_1 - d_0(1 - \alpha_2)] \,, \tag{A.7}$$

$$x_2 = \frac{\beta}{\gamma}[d_1 - \alpha_1 d_2 - d_0(1 - \alpha_1)] \,, \tag{A.8}$$

$$x_3 \, x_4 = C \,, \tag{A.9}$$

where

$$C = \frac{1}{4}\left(x_1 x_2 - \frac{d_0}{\gamma}\right) \,, \tag{A.10}$$

$$d_i \equiv m_i^2 - k_i^2 \,. \tag{A.11}$$

B. The basis for the 2-point functions

First, we introduce a massless arbitrary 4-vector v, such that $(v \cdot k_1) \neq 0$, that we use to rewrite k_1 in terms of two massless 4-vectors (we also take $\ell^2 = 0$)

$$k_1 = \ell + \alpha\, v\,, \tag{B.1}$$

giving

$$\gamma \equiv 2\,(k_1 \cdot v) = 2\,(\ell \cdot v) \quad \text{and} \quad \alpha = \frac{k_1^2}{\gamma}\,. \tag{B.2}$$

Then, we introduce two additional independent massless 4-vectors $\ell_{7,8}$ defined as

$$\ell_7^\mu = \langle \ell | \gamma^\mu | v]\,, \quad \ell_8^\mu = \langle v | \gamma^\mu | \ell]\,, \tag{B.3}$$

for which one finds

$$(\ell_7 \cdot \ell_8) = -2\gamma\,, \tag{B.4}$$

and we decompose $q^\mu + p_0^\mu$ in the basis formed by k_1, v, ℓ_7 and ℓ_8

$$q^\mu = -p_0^\mu + y k_1^\mu + y_v v^\mu + y_7 \ell_7^\mu + y_8 \ell_8^\mu\,, \tag{B.5}$$

that fulfill the double-cut requirement

$$D_0 = D_1 = 0\,. \tag{B.6}$$

For a q written as in eq. (B.5) this implies the system

$$\begin{aligned} y_7 y_8 &= F_y \\ y_v &= \frac{d_1 - d_0 - 2y k_1^2}{\gamma}\,, \end{aligned} \tag{B.7}$$

where

$$F_y = -\frac{1}{4\gamma}\left(m_0^2 - y\,(d_1 - d_0) + y^2 k_1^2\right)\,. \tag{B.8}$$

We remark that when $m_0 = 0$, F_y vanishes for $y = -1, 0$.

References

[1] R.K. Ellis, W.T. Giele and G. Zanderighi, *The one-loop amplitude for six-gluon scattering*, *JHEP* **05** (2006) 027 [hep-ph/0602185];
C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, *Bootstrapping one-loop QCD amplitudes with general helicities*, *Phys. Rev.* **D 74** (2006) 036009 [hep-ph/0604195];
Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, *JHEP* **11** (2005) 027 [hep-ph/0507019];
T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon and C. Schubert, *An algebraic/numerical formalism for one-loop multi-leg amplitudes*, *JHEP* **10** (2005) 015 [hep-ph/0504267];

J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys.* **B 712** (2005) 59 [hep-th/0412108];
G. Bélanger et al., *Full $O(\alpha)$ electroweak corrections to double Higgs-Strahlung at the linear collider*, *Phys. Lett.* **B 576** (2003) 152 [hep-ph/0309010];
A. Denner and S. Dittmaier, *Reduction of one-loop tensor 5-point integrals*, *Nucl. Phys.* **B 658** (2003) 175 [hep-ph/0212259]; *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B 734** (2006) 62 [hep-ph/0509141];
A. Denner, S. Dittmaier, M. Roth and M.M. Weber, *Radiative corrections to Higgs-boson production in association with top-quark pairs at e^+e^- colliders*, *Nucl. Phys.* **B 680** (2004) 85 [hep-ph/0309274];
A. Denner, S. Dittmaier, M. Roth and L.H. Wieders, *Electroweak corrections to charged-current $e^+e^- \to 4$ fermion processes: technical details and further results*, *Nucl. Phys.* **B 724** (2005) 247 [hep-ph/0505042]; *Complete electroweak $O(\alpha)$ corrections to charged-current $e^+e^- \to 4$ fermion processes*, *Phys. Lett.* **B 612** (2005) 223 [hep-ph/0502063];
K. Kato et al., *Radiative corrections for Higgs study at the ILC*, PoS(HEP2005)312;
T. Binoth, G. Heinrich, T. Gehrmann and P. Mastrolia, *Six-Photon Amplitudes*, *Phys. Lett.* **B 649** (2007) 422 [hep-ph/0703311];
S. Weinzierl, *Automated calculations for multi-leg processes*, arXiv:0707.3342;
D. Maître and P. Mastrolia, *S@M, a Mathematica implementation of the spinor-helicity formalism*, arXiv:0710.5559;
Z. Nagy and D.E. Soper, *Numerical integration of one-loop Feynman diagrams for N- photon amplitudes*, *Phys. Rev.* **D 74** (2006) 093006 [hep-ph/0610028];
A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, *All-purpose numerical evaluation of one-loop multi-leg Feynman diagrams*, *Nucl. Phys.* **B 650** (2003) 162 [hep-ph/0209219];
M. Moretti, F. Piccinini and A.D. Polosa, *A fully numerical approach to one-loop amplitudes*, arXiv:0802.4171.

[2] NLO Multileg Working Group collaboration, Z. Bern et al., *The NLO multileg working group: summary report*, arXiv:0803.0494.

[3] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226].

[4] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, *Nucl. Phys.* **B 435** (1995) 59 [hep-ph/9409265].

[5] F. Cachazo, P. Svrček and E. Witten, *Gauge theory amplitudes in twistor space and holomorphic anomaly*, *JHEP* **10** (2004) 077 [hep-th/0409245].

[6] I. Bena, Z. Bern, D.A. Kosower and R. Roiban, *Loops in twistor space*, *Phys. Rev.* **D 71** (2005) 106010 [hep-th/0410054].

[7] F. Cachazo, *Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes*, hep-th/0410077.

[8] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, *Phys. Rev.* **D 71** (2005) 025012 [hep-th/0410179].

[9] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, *Nucl. Phys.* **B 725** (2005) 275 [hep-th/0412103].

[10] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, *Phys. Rev.* **D 72** (2005) 065012 [hep-ph/0503132].

JHEP06(2008)030

[11] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, *JHEP* **10** (2005) 011 [hep-th/0506068].

[12] R. Britto, B. Feng and P. Mastrolia, *The cut-constructible part of QCD amplitudes*, *Phys. Rev.* **D 73** (2006) 105004 [hep-ph/0602178].

[13] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191].

[14] P. Mastrolia, *On triple-cut of scattering amplitudes*, *Phys. Lett.* **B 644** (2007) 272 [hep-th/0611091].

[15] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089].

[16] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277].

[17] R. Britto and B. Feng, *Integral coefficients for one-loop amplitudes*, *JHEP* **02** (2008) 095 [arXiv:0711.4284].

[18] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, *Nucl. Phys.* **B 763** (2007) 147 [hep-ph/0609007].

[19] G. Ossola, C.G. Papadopoulos and R. Pittau, *Numerical evaluation of six-photon amplitudes*, *JHEP* **07** (2007) 085 [arXiv:0704.1271].

[20] G. Ossola, C.G. Papadopoulos and R. Pittau, *CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes*, *JHEP* **03** (2008) 042 [arXiv:0711.3596].

[21] G. Ossola, C.G. Papadopoulos and R. Pittau, *On the rational terms of the one-loop amplitudes*, *JHEP* **05** (2008) 004 [arXiv:0802.1876].

[22] D. Forde, *Direct extraction of one-loop integral coefficients*, *Phys. Rev.* **D 75** (2007) 125019 [arXiv:0704.1835].

[23] W.B. Kilgore, *One-loop integral coefficients from generalized unitarity*, arXiv:0711.5015.

[24] N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *Analytic structure of three-mass triangle coefficients*, *JHEP* **04** (2008) 038 [arXiv:0709.2086].

[25] R.K. Ellis, W.T. Giele and Z. Kunszt, *A numerical unitarity formalism for evaluating one-loop amplitudes*, *JHEP* **03** (2008) 003 [arXiv:0708.2398].

[26] W.T. Giele, Z. Kunszt and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049 [arXiv:0801.2237].

[27] G. 't Hooft and M.J.G. Veltman, *Scalar one loop integrals*, *Nucl. Phys.* **B 153** (1979) 365;
Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett.* **B 302** (1993) 299 [*Erratum ibid.* **318** (1993) 649] [hep-ph/9212308]; *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B 412** (1994) 751 [hep-ph/9306240];
J. Fleischer, F. Jegerlehner and O.V. Tarasov, *Algebraic reduction of one-loop Feynman graph amplitudes*, *Nucl. Phys.* **B 566** (2000) 423 [hep-ph/9907327];
T. Binoth, J.P. Guillet and G. Heinrich, *Reduction formalism for dimensionally regulated one-loop N-point integrals*, *Nucl. Phys.* **B 572** (2000) 361 [hep-ph/9911342];
G. Duplancic and B. Nizic, *Reduction method for dimensionally regulated one-loop N-point Feynman integrals*, *Eur. Phys. J.* **C 35** (2004) 105 [hep-ph/0303184];

JHEP06(2008)030

R.K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002 [arXiv:0712.1851].

[28] R. Britto, B. Feng and G. Yang, *Complete one-loop amplitudes with massless propagators*, arXiv:0803.3147.

[29] R. Britto, B. Feng and P. Mastrolia, *Closed-form decomposition of one-loop massive amplitudes*, arXiv:0803.1989.

[30] F. del Aguila and R. Pittau, *Recursive numerical calculus of one-loop tensor integrals*, *JHEP* **07** (2004) 017 [hep-ph/0404120];
R. Pittau, *Formulae for a numerical computation of one-loop tensor integrals*, hep-ph/0406105.

[31] T. Binoth, J.P. Guillet and G. Heinrich, *Algebraic evaluation of rational polynomials in one-loop amplitudes*, *JHEP* **02** (2007) 013 [hep-ph/0609054].

[32] Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. I: the general formalism*, *Nucl. Phys.* **B 758** (2006) 1 [hep-ph/0607015];
X. Su, Z. Xiao, G. Yang and C.-J. Zhu, *The rational part of QCD amplitude. II: the five-gluon*, *Nucl. Phys.* **B 758** (2006) 35 [hep-ph/0607016].

[33] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, *Phys. Rev.* **D 73** (2006) 065013 [hep-ph/0507005];
S.D. Badger, E.W.N. Glover and K. Risager, *One-loop phi-MHV amplitudes using the unitarity bootstrap*, *JHEP* **07** (2007) 066 [arXiv:0704.3914].

[34] T. Binoth, G. Ossola, C.G. Papadopoulos and R. Pittau, *NLO QCD corrections to tri-boson production*, arXiv:0804.0350.

[35] R. Pittau, *A simple method for multi-leg loop calculations*, *Comput. Phys. Commun.* **104** (1997) 23 [hep-ph/9607309]; *A simple method for multi-leg loop calculations. II: a general algorithm*, *Comput. Phys. Commun.* **111** (1998) 48 [hep-ph/9712418].

[36] A. van Hameren, J. Vollinga and S. Weinzierl, *Automated computation of one-loop integrals in massless theories*, *Eur. Phys. J.* **C 41** (2005) 361 [hep-ph/0502165].

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