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Closed-form decomposition of one-loop massive amplitudes

Ruth Britto,[1] Bo Feng,[2] and Pierpaolo Mastrolia[3]

[1]*Institute for Theoretical Physics, University of Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands*
[2]*Center of Mathematical Science, Zhejiang University, Hangzhou, China*
[3]*Theory Division, CERN, CH-1211 Geneva 23, Switzerland*

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We present formulas for the coefficients of 2-, 3-, 4-, and 5-point master integrals for one-loop massive amplitudes. The coefficients are derived from unitarity cuts in D dimensions. The input parameters can be read off from any unitarity-cut integrand, as assembled from tree-level expressions, after simple algebraic manipulations. The formulas presented here are suitable for analytical as well as numerical evaluation. Their validity is confirmed in two known cases of helicity amplitudes contributing to $gg \to gg$ and $gg \to gH$, where the masses of the Higgs and the fermion circulating in the loop are kept as free parameters.

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I. INTRODUCTION

With the approaching kick-off of the CERN Large Hadron Collider, the calculation of one-loop multileg amplitudes has been under intense consideration [1], following a program of improvement of established techniques [2] and development of new methods (see [3] for an extensive review).

The unitarity method introduced in [4,5] is designed to compute any scattering amplitude by matching its unitarity cuts onto the corresponding cuts of its expansion in a basis of master integrals [6] with rational coefficients. Each of these coefficients can be determined quantitatively from prior knowledge of the master integrals and the singularity structure of the amplitude.

As the master integrals form a basis for amplitudes, so the unitarity cuts of master integrals have uniquely identifiable analytic properties, and can be used as a basis for the cuts of any amplitude. Therefore, the coefficients of the linear combination can be extracted systematically through the phase-space integration (instead of complete loop integration).

Recently, unitarity-based methods for one-loop amplitudes have been the subject of an intense investigation, through different implementations of the cut-constraints [7–25].

The holomorphic anomaly of unitarity cuts [7,26] simplifies the phase-space integration dramatically: cut-integrals can be done analytically by evaluating residues of a complex function in spinor variables [27], reducing the problem of so-called tensor reduction to one of *algebraic* manipulation.

Accordingly, in [12,14], a systematic method was introduced to evaluate any finite four-dimensional unitarity cut, yielding compact expressions for the coefficients of the master integrals. This method was successfully applied to the final parts of the cut-constructible part of the six-gluon amplitude in QCD. The same method, based on the spinor integration of the phase space, was later extended for the evaluation of generalized cuts in D dimensions [15–18], which is essential for the complete determination of any amplitude in dimensional regularization [28–30].

In this paper, we carry out the extension to the massive case of the analytic results presented in [19], stemming from an original study of compact formulas for the coefficients of the master integrals [17]. Following the same logic as in [19], we now present general formulas for the coefficients of the master integrals which can be evaluated without performing any integration. These formulas depend on input variables (indices, momenta, and associated spinors) that are specific to the initial cut-integrand, which is assembled from tree-level amplitudes. The value of a given coefficient is thus obtained simply by pattern matching, that is by specializing the value of the input variables to be inserted in the general formulas. The implementation of the general formulas into automatic tools is straightforward, as done for the current investigation with the program `S@M` [31].

In this paper, since the formulas for the coefficients are obtained via massive double cuts in D dimension, we do not present results for the coefficients of cut-free functions like tadpoles and bubbles with massless external momentum (which can be expressed in terms of tadpoles as well). The coefficients of such functions could be fixed either by imposing the expected UV- behavior of the amplitude, as described in [29], or computed with other techniques applicable in massive calculations [20–22,24,25].

The paper is organized as follows. In Sec. II, we describe the structure of the decomposition of one-loop amplitude in terms of master integrals. In Sec. III we explain the double-cut integration with spinor variables, which leads to the formulas of the coefficients of the master integrals, presented in Sec. IV. In Secs. V and VI, we apply our formulas to two examples of one-loop scattering amplitudes, respectively $gH \to gg$ and $gg \to gg$, where the Higgs mass and the mass of the internal fermion (in both cases) are kept as free parameters. In Sec. VII, we present both analytical and numerical methods to obtain, finally,

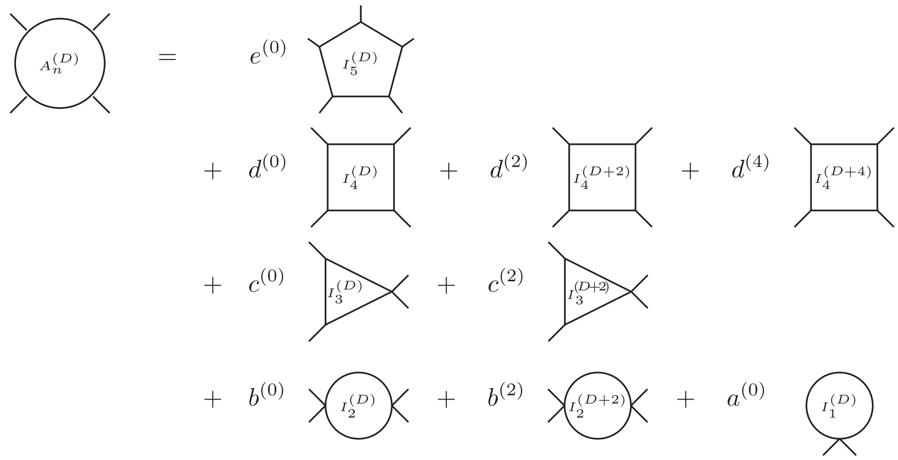

FIG. 1. Decomposition of a one-loop amplitude in D dimensions in terms of master integrals.

the explicit coefficients of the dimensionally shifted master integrals. In Appendix A, we record the translation between our basis of integrals and the ones used in the literature for the examples discussed in Secs. V and VI. In Appendix B, we present a proof of the decomposition into the dimensionally shifted basis, with rational coefficients independent of ϵ. In other words, we prove that the coefficients given by our algebraic expressions will be polynomial in our extra-dimensional variable u. As a by-product, we have produced equivalent and simpler algebraic functions for the evaluation of coefficients.

II. DECOMPOSITION IN TERMS OF MASTER INTEGRALS

We define the n-point scalar function with nonuniform masses as follows:[1]

$$I_n(M_1, M_2, m_1, \ldots, m_{n-2}) \equiv \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)((p-K)^2 - M_2^2)\prod_{j=1}^{n-2}((p-P_j)^2 - m_j^2)}. \tag{2.1}$$

Giele, Kunszt, and Melnikov [25] have given the decomposition of any one-loop amplitude in D dimensions in terms of master integrals, represented pictorially in Fig. 1.

Here, with reference to [25]: (i) we have absorbed the residual D-dependence of the coefficients in the definition of the master integrals; (ii) for ease of notation, we have given as understood the sums on the partition of the n-points of the amplitude in the number of points corresponding to each master integral. Thus, the coefficients e, d, c, b, a in Fig. 1 are independent of D.

If, on both sides of the equation in Fig. 1, we apply the standard decomposition of the $D = 4 - 2\epsilon$ dimensional loop variable L, in a four-dimensional component $\tilde{\ell}$, and its (-2ϵ)-dimensional orthogonal complement μ,

$$L = \tilde{\ell}1 + \mu, \tag{2.2}$$

then the integration measure becomes

$$\int d^{4-2\epsilon}L = \int d^{-2\epsilon}\mu \int d^4\tilde{\ell}, \tag{2.3}$$

namely the composition of a four-dimensional integration and an integration over a (-2ϵ)-dimensional masslike parameter. By taking the μ-integral to be understood, the four-dimensional integration on both sides of the equation in Fig. 1 can be represented as in Fig. 2, where $d_n(\mu^2)$, $c_n(\mu^2)$, and $b_n(\mu^2)$ are polynomials of degree n in μ^2, as discussed in Appendix B,

$$d_2(\mu^2) = d^{(0)} + d^{(2)}\mu^2 + d^{(4)}(\mu^2)^2, \tag{2.4}$$

$$c_1(\mu^2) = c^{(0)} + c^{(2)}\mu^2, \tag{2.5}$$

$$b_1(\mu^2) = b^{(0)} + b^{(2)}\mu^2, \tag{2.6}$$

whereas $\pi(\mu^2)$ is nonpolynomial in μ^2 and corresponds to the coefficients of the reduction of the pentagon to boxes, which occurs in $D = 4$.

The polynomial structure of $d_n(\mu^2)$, $c_n(\mu^2)$, and $b_n(\mu^2)$ is responsible for the dimensionally shifted integrals appearing in Fig. 1, because the μ-integration can be performed trivially by absorbing the extra powers of μ^2 into the integration measure, according to [29]:

[1]For ease of presentation, we are omitting the prefactor $i(-1)^{n+1}(4\pi)^{D/2}$ (which was included, for example, in [29]).

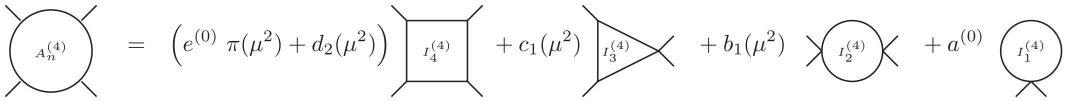

FIG. 2. Decomposition of a one-loop amplitude in 4 dimensions in terms of master integrals.

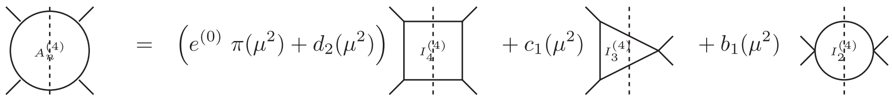

FIG. 3. Double cut of a one-loop amplitude in 4 dimensions in terms of the double-cut of master integrals.

$$\int \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}}(\mu^2)^r f(\mu^2) = -\epsilon(1-\epsilon)(2-\epsilon)\cdots(r-1-\epsilon)(4\pi)^r \times \int \frac{d^{2r-2\epsilon}\mu}{(2\pi)^{2r-2\epsilon}} f(\mu^2). \tag{2.7}$$

The presence of $\pi(\mu^2)$ in the coefficient of the four-dimensional box is a unique signature of the pentagon. We conclude that the reconstruction of the four-dimensional kernel of any one-loop amplitude, given in Fig. 2, contains all the information for the complete reconstruction of the amplitude in D dimensions, given in Fig. 1.

In the following pages, we present the general formulas of the coefficients of the box, $I_4^{(4)}$, triangle, $I_3^{(4)}$, and bubble, $I_2^{(4)}$, obtained from the double cut of the equation in Fig. 2, and represented in Fig. 3. Since the formulas for the coefficients are obtained via double cuts, we do not present the results for the coefficients of cut-free functions like tadpoles and bubbles with massless external momentum (which can be expressed in terms of tadpoles as well). Their coefficients could be fixed either by imposing the expected UV- behavior of the amplitude, as described in [29], or computed with alternative techniques [20–22,24,25].

III. THE DOUBLE-CUT PHASE-SPACE INTEGRATION

In this section, we review the D-dimensional unitarity method [15,18] as applied in cases with arbitrary masses [17]. Our goal is to describe the structure of the cut integrand, from which we will directly read off the coefficients from the formulas in the following section. The formulas will be the massive analogs of the ones in [19].

Recall the phase-space integration of a standard (double) cut in $D = 4 - 2\epsilon$ dimensions. We use the usual decomposition of the D-dimensional loop variable L, in a four-dimensional component $\tilde{\ell}$, and a transverse (-2ϵ)-dimensional remnant μ,

$$L = \tilde{\ell} + \mu. \tag{3.1}$$

The integration measure becomes

$$\int d^{4-2\epsilon}L = \int d^{-2\epsilon}\mu \int d^4\tilde{\ell} = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\int d\mu^2(\mu^2)^{-1-\epsilon}\int d^4\tilde{\ell}, \tag{3.2}$$

namely the composition of a four-dimensional integration and an integration over a (-2ϵ)-dimensional masslike parameter. In order to write the four-dimensional part in terms of spinor variables associated to massless momentum, we proceed with the following change of variables:

$$\tilde{\ell} = \ell + zK, \qquad \ell^2 = 0, \tag{3.3}$$

where ℓ is a massless momentum and K is the momentum across the cut, fixed by the kinematics. Accordingly, the four-dimensional integral measure becomes

$$\int d^4\tilde{\ell} = \int dz d^4\ell\,\delta^+(\ell^2)(2\ell\cdot K). \tag{3.4}$$

The Lorentz-invariant phase space (LIPS) of a double cut in the K^2-channel is defined by the presence of two δ-functions imposing the cut conditions:

$$\int d^{4-2\epsilon}\Phi = \int d^{4-2\epsilon}L\,\delta(L^2 - M_1^2)\delta((L-K) - M_2^2). \tag{3.5}$$

Here M_1 and M_2 are the masses of the cut lines. By using the decomposition of the loop variable in Eq. (2.2), the four-dimensional integral can be separated, so that

$$\int d^{4-2\epsilon}\Phi = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\int d\mu^2(\mu^2)^{-1-\epsilon}\int d^4\phi, \tag{3.6}$$

where the four-dimensional LIPS is

$$\int d^4\phi = \int d^4\tilde{\ell}\,\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)\delta((\tilde{\ell}-K)^2 - M_2^2 - \mu^2). \tag{3.7}$$

The change of variables in Eq. (3.3), and the z-integration (trivialized by the presence of δ's), yield the four-dimensional LIPS to appear as

$$\int d^4\phi = \int d^4\ell\,\delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K + M_1^2 - M_2^2), \tag{3.8}$$

where

$$z = \frac{(K^2+M_1^2-M_2^2) - \sqrt{\Delta[K,M_1,M_2] - 4K^2\mu^2}}{2K^2}, \tag{3.9}$$

with

$$\Delta[K,M_1,M_2] \equiv (K^2)^2 + (M_1^2)^2 + (M_2^2)^2 - 2K^2M_1^2 - 2K^2M_2^2 - 2M_1^2M_2^2. \tag{3.10}$$

We remark that the value of z in Eq. (3.9) is frozen to be the proper root ($K > 0$) of the quadratic argument of $\delta(z(1-z)K^2 + z(M_1^2 - M_2^2) - M_1^2 - \mu^2)$, coming from $\delta(\tilde{\ell}^2 - M_1^2 - \mu^2)$. For later convenience, one can redefine the μ^2-integral measure as

$$\int d\mu^2(\mu^2)^{-1-\epsilon} = \left(\frac{\Delta[K,M_1,M_2]}{4K^2}\right)^{-\epsilon}\int_0^1 du\,u^{-1-\epsilon},$$

where the relation between u and μ^2 is given by

$$u \equiv \frac{4K^2\mu^2}{\Delta[K,M_1,M_2]}, \qquad \mu^2 = \left(\frac{u\Delta[K,M_1,M_2]}{4K^2}\right). \tag{3.11}$$

We observe that the domain of u, i.e., $u \in [0,1]$, follows from the kinematical constraints, as discussed in [17].

Finally, after the above rearrangement, the D-dimensional Lorentz-invariant phase space of a double cut in the K^2-channel can be written in a suitable form,

$$\int d^{4-2\epsilon}\Phi = \chi(\epsilon,K,M_1,M_2)\int_0^1 du\,u^{-1-\epsilon}\int d^4\phi, \tag{3.12}$$

where

$$\chi(\epsilon,K,M_1,M_2) = \frac{(4\pi)^\epsilon}{\Gamma(-\epsilon)}\left(\frac{\Delta[K,M_1,M_2]}{4K^2}\right)^{-\epsilon}, \tag{3.13}$$

and where $d^4\phi$ was given in Eq. (3.8). By using the definition of u given in Eq. (3.11), we can write

$$z = \frac{\alpha - \beta\sqrt{1-u}}{2}, \tag{3.14}$$

where

$$\alpha = \frac{K^2+M_1^2-M_2^2}{K^2}, \qquad \beta = \frac{\sqrt{\Delta[K,M_1,M_2]}}{K^2}. \tag{3.15}$$

Notice that when $M_1 = M_2 = 0$ we have $\alpha = \beta = 1$, thus reproducing the massless case. A useful relation between z and u is the following:

$$(1-2z) + \frac{M_1^2 - M_2^2}{K^2} = \beta\sqrt{1-u}. \tag{3.16}$$

This relation will be used in Appendix B to prove that the coefficients given in this paper are polynomials in u, or equivalently μ^2. As discussed in the previous section, this feature is essential for the straightforward reconstruction of dimensionally shifted master integrals.

The main feature of a double-cut LIPS parametrized as in Eqs. (3.8) and (3.12), is that the kernel of the integration is represented by the four-dimensional integral. In fact, the u-integration (or equivalently, the μ^2-integration), is simply responsible for the rise of *shifted-dimension* master integrals. Thus, our interest in the extraction of the coefficients of the master integrals from a four-dimensional massive double cut, see Fig. 3, translates in focusing the discussion only on the $\int d^4\phi$.

The D-dimensional double cut of any one-loop amplitude is, in general form,

$$\int d^{4-2\epsilon}\Phi A_L^{\rm tree}\times A_R^{\rm tree} = \chi(\epsilon,K,M_1,M_2)\int_0^1 du\,u^{-1-\epsilon}\int d^4\phi A_L^{\rm tree}\times A_R^{\rm tree}, \tag{3.17}$$

where $A_L^{\rm tree}$ and $A_R^{\rm tree}$ are the two tree-level amplitudes on the left and right side of the cut. As discussed above, the kernel of the integration is represented by the four-dimensional part,

$$\int d^4\phi A_L^{\rm tree}\times A_R^{\rm tree}. \tag{3.18}$$

We proceed from the formula (3.8) by introducing spinor variables according to [26],

$$\int d^4\ell\,\delta(\ell^2) = \int\langle\ell\,d\ell\rangle[\ell\,d\ell]\int t\,dt \tag{3.19}$$

and performing the integral over t trivially, with the second delta function. The general expression of the double-cut integral will then be

$$
\begin{aligned}
\int d^4\phi A_L^{\text{tree}} \times A_R^{\text{tree}} &= \int d^4\ell\delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K + M_1^2 - M_2^2)\frac{\prod_j \langle a_j|\tilde{\ell}|b_j]}{\prod_i((\tilde{\ell}-K_i)^2 - m_i^2 - \mu^2)} \\
&= \int d^4\ell\delta^+(\ell^2)\delta((1-2z)K^2 - 2\ell\cdot K + M_1^2 - M_2^2)\frac{\prod_j \langle a_j|\ell + zK|b_j]}{\prod_i(K_i^2 + M_1^2 - m_i^2 - 2(\ell + zK)\cdot K_i)} \\
&= \int \langle \ell d\ell\rangle[\ell d\ell]\int t dt \delta((1-2z)K^2 + t\langle\ell|K|\ell] + M_1^2 - M_2^2) \\
&\quad \times \frac{\prod_j (z\langle a_j|K|b_j] + t\langle\ell|P_j|\ell])}{\prod_i(K_i^2 + M_1^2 - m_i^2 - 2zK\cdot K_i + t\langle\ell|K_i|\ell])}.
\end{aligned}
\tag{3.20}
$$

Here we have used $\tilde{\ell}^2 = M_1^2 + \mu^2$. Notice that $\langle a|\tilde{\ell}|b] = -2\tilde{\ell}\cdot P$, with $P = |a\rangle[b|$.

After using the remaining delta function to perform the integral over t, we have

$$
\int \langle\ell d\ell\rangle[\ell d\ell]\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right)\frac{(K^2)^{n+1}}{\langle\ell|K|\ell]^{n+2}} \times \frac{\prod_{j=1}^{n+k}\langle\ell|R_j|\ell]}{\prod_{i=1}^{k}\langle\ell|Q_i|\ell]},
\tag{3.21}
$$

where

$$
P_j = |a_j\rangle[b_j|,
\tag{3.22}
$$

$$
R_j = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right)P_j - \frac{z(2P_j\cdot K)}{K^2}K,
\tag{3.23}
$$

$$
Q_j = -\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right)K_j + \frac{K_j^2 + M_1^2 - m_j^2 - 2zK\cdot K_j}{K^2}K.
\tag{3.24}
$$

The vectors P_j, R_j, Q_j do not depend on the loop variables, but rather only on the external kinematics, and especially on the momentum across the cut, K. Applying (3.21) to the master integrals, we find the following results.

(1) Bubble: $k = 0$, $n + k = 0$.

$$
\int \langle\ell d\ell\rangle[\ell d\ell]\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right)\frac{(K^2)}{\langle\ell|K|\ell]^2}
\tag{3.25}
$$

(2) Triangle: $k = 1$, $n + k = 0$.

$$
\int \langle\ell d\ell\rangle[\ell d\ell]\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) \times \frac{1}{\langle\ell|K|\ell]\langle\ell|Q|\ell]}
\tag{3.26}
$$

(3) Box: $k = 2$, $n + k = 0$.

$$
\int \langle\ell d\ell\rangle[\ell d\ell]\left((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\right) \times \frac{(K^2)^{-1}}{\langle\ell|Q_1|\ell]\langle\ell|Q_2|\ell]}
\tag{3.27}
$$

These formulas are the extension to the massive case of the corresponding ones given in [19]. We notice that the presence of the masses enters only the definitions of P_j, R_j, and Q_j. Therefore the spinor integration performed in the massless case [19] is valid as well in this case.

The expression of the cut-integrand in Eq. (3.21), with its indices, n and k, and its vectors P_j, R_j, and Q_j is the key to constructing the coefficients. In the next section we present general formulas for the coefficients of the master integrals (boxes, triangles, and bubbles), which depend on exactly these input parameters. Accordingly, given a specific amplitude (or integral), one can obtain its decomposition in terms of master integrals *without any integration*. Every coefficient is obtained from the general formulas simply by substituting the input parameters characterizing the specific amplitude. These parameters are obtained by pattern-matching onto the reference form in Eq. (3.21).

IV. FORMULAS FOR THE COEFFICIENTS OF MASTER INTEGRALS

The coefficients of master integrals are obtained by the procedure described in the previous section, which is a straightforward generalization of the massless case [19]. We list the results in this section. In fact, the expressions

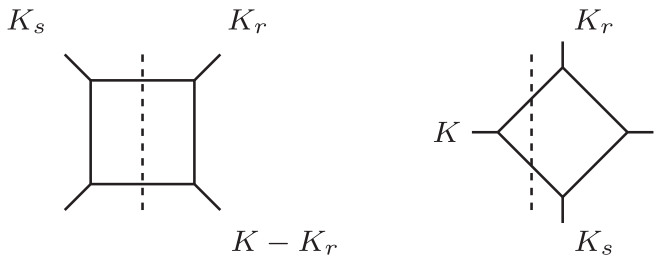

FIG. 4. Double cuts of box integrals.

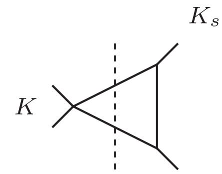

FIG. 5. Double cut of a triangle integral.

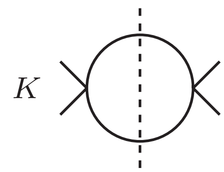

FIG. 6. Double cut of a bubble integral.

take the same form as in the massless case; the mass dependence enters directly through the definitions (3.23) and (3.24), and through these formulas into the definitions (4.2) and (4.4).

A. Box coefficient

The formula for the coefficient of either of the box functions with external kinematics as shown in Fig. 4 is

$$C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{k+n}\langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2}\prod_{t=1,t\neq i,j}^{k}\langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1} \leftrightarrow P_{sr,2}\}\right), \tag{4.1}$$

where

$$\Delta_{sr} = (2Q_s\cdot Q_r)^2 - 4Q_s^2Q_r^2, \qquad P_{sr,1} = Q_s + \left(\frac{-2Q_s\cdot Q_r + \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r, \qquad P_{sr,2} = Q_s + \left(\frac{-2Q_s\cdot Q_r - \sqrt{\Delta_{sr}}}{2Q_r^2}\right)Q_r. \tag{4.2}$$

B. Triangle coefficient

The formula for the coefficient of the triangle function with external kinematics as shown in Fig. 5 is

$$C[Q_s, K] = \frac{(K^2)^{1+n}}{2}\frac{1}{(\sqrt{\Delta_s})^{n+1}}\frac{1}{(n+1)!\langle P_{s,1}P_{s,2}\rangle^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle P_{s,1}-\tau P_{s,2}|R_jQ_s|P_{s,1}-\tau P_{s,2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}-\tau P_{s,2}|Q_tQ_s|P_{s,1}-\tau P_{s,2}\rangle} + \{P_{s,1}\leftrightarrow P_{s,2}\}\right)\bigg|_{\tau=0}, \tag{4.3}$$

where

$$\Delta_s = (2Q_s\cdot K)^2 - 4Q_s^2K^2, \qquad P_{s,1} = Q_s + \left(\frac{-2Q_s\cdot K + \sqrt{\Delta_s}}{2K^2}\right)K, \qquad P_{s,2} = Q_s + \left(\frac{-2Q_s\cdot K - \sqrt{\Delta_s}}{2K^2}\right)K. \tag{4.4}$$

Note that the triangle coefficient is present only when $n \geq -1$.

C. Bubble coefficient

The formula for the coefficient of the bubble function with the external momentum K, shown in Fig. 6, is

$$C[K] = (K^2)^{1+n}\sum_{q=0}^{n}\frac{(-1)^q}{q!}\frac{d^q}{ds^q}\left(\mathcal{B}_{n,n-q}^{(0)}(s) + \sum_{r=1}^{k}\sum_{a=q}^{n}(\mathcal{B}_{n,n-a}^{(r;a-q;1)}(s) - \mathcal{B}_{n,n-a}^{(r;a-q;2)}(s))\right)\bigg|_{s=0}, \tag{4.5}$$

where

$$\mathcal{B}_{n,t}^{(0)}(s) \equiv \frac{d^n}{d\tau^n}\left(\frac{1}{n![\eta|\tilde{\eta}K|\eta]^n}\frac{(2\eta\cdot K)^{t+1}}{(t+1)(K^2)^{t+1}}\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\eta\rangle^{n+1}\prod_{p=1}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Big|_{|\ell\rangle\to|K-\tau\tilde{\eta}|\eta]}\right)\Bigg|_{\tau=0}, \tag{4.6}$$

$$\mathcal{B}_{n,t}^{(r;b;1)}(s) \equiv \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{b+1}\langle P_{r,1}P_{r,2}\rangle^b}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,1}-\tau P_{r,2}|\eta|P_{r,1}]^{t+1}}{\langle P_{r,1}-\tau P_{r,2}|K|P_{r,1}]^{t+1}}\right.$$
$$\left.\times\frac{\langle P_{r,1}-\tau P_{r,2}|Q_r\eta|P_{r,1}-\tau P_{r,2}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,1}-\tau P_{r,2}|R_j(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}{\langle P_{r,1}-\tau P_{r,2}|\eta K|P_{r,1}-\tau P_{r,2}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,1}-\tau P_{r,2}|Q_p(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}\right)\Bigg|_{\tau=0}, \tag{4.7}$$

$$\mathcal{B}_{n,t}^{(r;b;2)}(s) \equiv \frac{(-1)^{b+1}}{b!\sqrt{\Delta_r}^{b+1}\langle P_{r,1}P_{r,2}\rangle^b}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,2}-\tau P_{r,1}|\eta|P_{r,2}]^{t+1}}{\langle P_{r,2}-\tau P_{r,1}|K|P_{r,2}]^{t+1}}\right.$$
$$\left.\times\frac{\langle P_{r,2}-\tau P_{r,1}|Q_r\eta|P_{r,2}-\tau P_{r,1}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,2}-\tau P_{r,1}|R_j(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}{\langle P_{r,2}-\tau P_{r,1}|\eta K|P_{r,2}-\tau P_{r,1}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,2}-\tau P_{r,1}|Q_p(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}\right)\Bigg|_{\tau=0}, \tag{4.8}$$

where Δ_r, $P_{r,1}$, $P_{r,2}$ are given by (4.4), and η, $\tilde{\eta}$ are arbitrary, generically chosen null vectors. Note that the bubble coefficient exists only when $n \geq 0$.

V. EXAMPLE I: s_{12}-CHANNEL CUT OF $A(1^+, 2^+, 3^+, H)$

In this section as well as the next, we check our formulas by reconstructing some helicity amplitudes contributing to $gH \to gg$ and $gg \to gg$ at next-to-leading order (NLO) in QCD, both known in the literature [29,32]. We present our calculations in detail.

Our first example is the s_{12}-channel cut of $A(1^+, 2^+, 3^+, H)$. This amplitude was first computed in [32]. Here, to facilitate comparison, we follow the setup of [30], where the amplitude was rederived using unitarity cuts. At one loop, every Feynman diagram has a massive quark circulating in the loop. The quark mass is denoted by m.

The s-channel cut of $A(1^+, 2^+, 3^+, H)$ admits a decomposition in terms of cuts of master integrals as shown in Fig. 7. Its expression, given in Eq. (4.20) of [30], reads

$$A(1^+,2^+,3^+,H)|_{s-\text{channel}} = -\frac{i}{(4\pi)^{2-\epsilon}}\frac{m^2}{v^2}\frac{st}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\left\{\frac{(t-u)}{2t(s-m_H^2)}I_3^{(4)}[4(m^2+\mu^2)-s]\right.$$
$$\left.-\frac{(s-m_H^2)}{2st}I_3^{(2)}[4(m^2+\mu^2)-s]-\frac{1}{2}I_4[4(m^2+\mu^2)-s]\right\}, \tag{5.1}$$

where $s = s_{12}$, $t = s_{23}$, $u = m_H^2 - s - t$. One can thus read the following values for the coefficients:

$$c_4^{1m} = -c_0\tfrac{1}{2}(4(m^2+\mu^2)-s), \tag{5.2}$$

$$c_{[12|3|H]} = -c_0\frac{(s-m_H^2)}{2st}(4(m^2+\mu^2)-s), \tag{5.3}$$

$$c_{[1|2|3H]} = c_0\frac{(t-u)}{2t(s-m_H^2)}(4(m^2+\mu^2)-s), \tag{5.4}$$

$$c_{[12|3H]} = 0 \tag{5.5}$$

with

$$c_0 = -\frac{i}{(4\pi)^{2-\epsilon}}\frac{m^2}{v^2}\frac{st}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}. \tag{5.6}$$

A. The reconstruction of the coefficients

We now show how to reconstruct the coefficients given above with our formulas from Sec. IV. We follow the

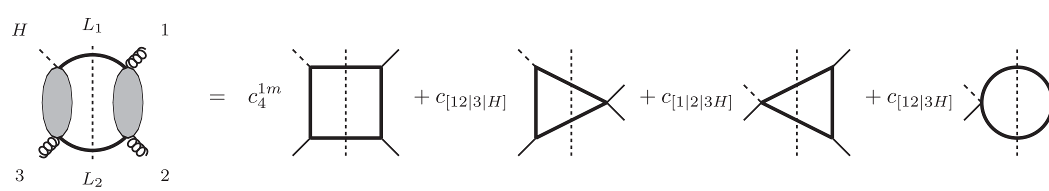

FIG. 7. Double cut in the s_{12}-channel for $A(1^+, 2^+, 3^+, H)$.

definition of the integrand given by [30]. By sewing the tree-level amplitude $A_4^{\rm tree}(-L_1, 1^+, 2^+, -L_2)$ and $A_4^{\rm tree}(-L_1, 3^+, H, -L_2)$ given in Eqs. (4.1–4.2) of [30], and using the Dirac equation for massive fermion, it is shown that the four-dimensional integrand of the s-cut, C_{12}, can be written as[2]:

$$C_{12} = c_{0,1}\frac{N_1}{D_2} + c_{0,2}\frac{N_2}{D_2D_4}, \tag{5.7}$$

where

$$c_{0,1} = -\frac{mK^2 s_{23}}{v(K^2 - m_H^2)\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}, \qquad c_{0,2} = \frac{m[1\,2]}{\sqrt{2}v\langle 1\,2\rangle}, \tag{5.8}$$

$$N_1 = m[4(m^2 + \mu^2) - s_{12}], \tag{5.9}$$

$$N_2 = 8m(m^2 + \mu^2)(\ell_1 \cdot \epsilon_3^+ + k_4 \cdot \epsilon_3^+) + \sqrt{2}m\frac{\langle 1|4|\ell_1|4|3]}{\langle 1\,3\rangle}, \tag{5.10}$$

$$D_2 = (\ell_1 - k_1)^2 - \mu^2 - m^2, \qquad D_4 = (\ell_1 + k_4)^2 - \mu^2 - m^2. \tag{5.11}$$

We need to classify the contribution of N_1 and N_2 to each coefficient. Observe that N_1 is independent of the loop momentum variable, so we consider it as a single term,

$$N_1 = m[4(m^2 + \mu^2) - s_{12}], \tag{5.12}$$

while N_2 is treated as three separate terms,

$$N_2 = N_{2,1} + N_{2,2} + N_{2,3}, \tag{5.13}$$

where

$$N_{2,1} = 8m(m^2 + \mu^2)(k_4 \cdot \epsilon_3^+) = 8m(m^2 + \mu^2)\frac{\langle 1|2|3]}{\sqrt{2}\langle 1\,3\rangle}, \tag{5.14}$$

$$N_{2,2} = 8m(m^2 + \mu^2)(\ell_1 \cdot \epsilon_3^+) = -8m(m^2 + \mu^2)\frac{\langle 1|\ell_1|3]}{\sqrt{2}\langle 1\,3\rangle}, \tag{5.15}$$

$$N_{2,3} = \sqrt{2}m\frac{\langle 1|4|\ell_1|4|3]}{\langle 1\,3\rangle}. \tag{5.16}$$

[2]Note that here we use "twistor" sign convention for the antiholomorphic spinor product, which is the opposite of the "QCD" convention followed by [30] $[xy]^{\rm Rozowsky} = -[xy]^{\rm BFM}$.

By pattern-matching onto the reference form in Eq. (3.21), each integrand can be characterized by the parameters given in the following table.

integrand	n	k	$\not{P}_1 = \vert P_1\rangle[P_1\vert$
N_1/D_2	-1	1	–
$N_{2,1}/(D_2D_4)$	-2	2	–
$N_{2,2}/(D_2D_4)$	-1	2	$\vert 1\rangle\ [3\vert$
$N_{2,3}/(D_2D_4)$	-1	2	$k_4\vert 3]\ \langle 1\vert k_4$

(5.17)

These data are the input values that we need in evaluating the formulas of the coefficients of the master integrals.

From this table we draw the following conclusions. Since N_1/D_2 has $k = 1$ and $n = -1$, it contributes only to a triangle coefficient; whereas $N_{2,i}(i = 1, 2, 3)$, having $n \leq -1$, contributes to both box and triangle coefficients. There are no bubble contributions at all. Thus we have already reproduced the absence of bubbles, Eq. (5.5), without any calculation.

To apply our formulas, we need to identify the definitions of K, K_1, and K_2. By inspection of D_2 and D_4, along with the fact of working in the s-channel cut, we choose the following consistent definitions:

$$K = k_1 + k_2; \qquad K_1 = k_1; \qquad K_2 = -k_4. \tag{5.18}$$

Since there is a single massive quark circulating in the loop, we have

$$M_1 = M_2 = m_j = m, \tag{5.19}$$

and

$$z = \frac{1 - \sqrt{1-u}\sqrt{1 - \frac{4m^2}{s_{12}}}}{2}. \tag{5.20}$$

From (5.18), we use the definition (3.24) to construct

$$Q_1 = -(1 - z)k_1 - zk_2, \tag{5.21}$$

$$Q_2 = (1 - 2z)k_4 + \left((1 - z)\frac{m_H^2}{K^2} - z\right)K. \tag{5.22}$$

Using (4.4), we also set up the following quantities useful for triangle coefficients:

$$\Delta_1 = (1 - 2z)^2(K^2)^2, \tag{5.23}$$

$$P_{1,1} = (1 - 2z)k_2, \qquad P_{1,2} = -(1 - 2z)k_1, \tag{5.24}$$

$$\Delta_2 = (1 - 2z)^2(K^2 - m_H^2)^2, \tag{5.25}$$

$$P_{2,1} = -(1 - 2z)k_3, \qquad P_{2,2} = -(1 - 2z)\frac{m_H^2}{K^2}k_3 + (1 - 2z)\left(1 - \frac{m_H^2}{K^2}\right)k_4. \tag{5.26}$$

B. The box coefficient c_4^{1m}

The box coefficient c_4^{1m} takes contributions from $N_{2,1}$, $N_{2,2}$, $N_{2,3}$:

$$c_4^{1m} = c_{0,2} 8m(m^2+\mu^2) k_4 \cdot \epsilon_3^+ C[Q_1, Q_2, K]^{(2,1)} - c_{0,2} \frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 1\, 3\rangle} C[Q_1, Q_2, K]^{(2,2)} + c_{0,2} \frac{\sqrt{2}m}{\langle 13\rangle} C[Q_1, Q_2, K]^{(2,3)}, \tag{5.27}$$

where $C[Q_r, Q_s, K]$, defined in Eq. (4.1), is

$$C[Q_r, Q_s, K] = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{k+n}\langle P_{sr,1}|R_j|P_{sr,2}]}{\langle P_{sr,1}|K|P_{sr,2}]^{n+2}\prod_{t=1,t\neq r,s}^{k}\langle P_{sr,1}|Q_t|P_{sr,2}]} + \{P_{sr,1} \leftrightarrow P_{sr,2}\}\right). \tag{5.28}$$

(i) $C[Q_1, Q_2, K]^{(2,1)}$
This term, corresponding to $n=-2$ is trivial, since $k=2$ and $N_{2,1}$ has no dependence on the loop variable,

$$C[Q_1, Q_2, K]^{(2,1)} = 1. \tag{5.29}$$

(ii) $C[Q_1, Q_2, K]^{(2,2)}$ and $C[Q_1, Q_2, K]^{(2,3)}$
$C[Q_1, Q_2, K]^{(2,2)}$ and $C[Q_1, Q_2, K]^{(2,3)}$ both correspond to $n=-1$, $k=2$. They differ only in the definition of P_1. Therefore we can compute them in parallel, and specialize later to the corresponding P_1. With $n=-1$, $k=2$, the expression is

$$\begin{aligned} C[Q_1, Q_2, K] &= \frac{(K^2)}{2}\left(\frac{\langle P_{21,1}|R_1|P_{21,2}]}{\langle P_{21,1}|K|P_{21,2}]} + \{P_{21,1} \leftrightarrow P_{21,2}\}\right) \\ &= -(1-2z)\frac{K^2}{2}\left(\frac{\langle P_{21,1}|P_1|P_{21,2}]}{\langle P_{21,1}|K|P_{21,2}]} + \frac{\langle P_{21,2}|P_1|P_{21,1}]}{\langle P_{21,2}|K|P_{21,1}]}\right) + z(-2K\cdot P_1) \\ &= -(1-2z)\frac{K^2}{2}\left(\frac{\langle P_{21,1}|P_1|P_{21,2}]\langle P_{21,2}|K|P_{21,1}] + \langle P_{21,2}|P_1|P_{21,1}]\langle P_{21,1}|K|P_{21,2}]}{\langle P_{21,1}|K|P_{21,2}]\langle P_{21,2}|K|P_{21,1}]}\right) + z(-2K\cdot P_1). \end{aligned} \tag{5.30}$$

For $|P_1\rangle = |1\rangle$, $|P_1] = |3]$, (so that, for any S, one has $2P_1 \cdot S = -\langle 1|S|3]$), one obtains

$$\begin{aligned} C[Q_1, Q_2, K]^{(2,2)} &= -(1-2z)\frac{K^2}{2}\left(-\frac{(1-2z)^4}{z(1-z)}\frac{s_{23}(s_{23}+K^2-m_H^2)\langle 1|2|3]}{K^2}\right)\left(\frac{(1-2z)^4}{z(1-z)}s_{23}(s_{23}+K^2-m_H^2)\right)^{-1} \\ &\quad + z\langle 1|2|3] = \frac{\langle 1|2|3]}{2}. \end{aligned} \tag{5.31}$$

For $|P_1\rangle = \not{k}_4|3]$, $\quad |P_1] = \not{k}_4|1\rangle$, (so that, $2P_1 \cdot S = -\langle 1|k_4 S k_4|3]$), one gets

$$\begin{aligned} C[Q_1, Q_2, K]^{(2,3)} &= -(1-2z)\frac{K^2}{2}\left(\frac{\langle P_{21,1}|P_1|P_{21,2}]\langle P_{21,2}|K|P_{21,1}] + \langle P_{21,2}|P_1|P_{21,1}]\langle P_{21,1}|K|P_{21,2}]}{\langle P_{21,1}|K|P_{21,2}]\langle P_{21,2}|K|P_{21,1}]}\right) \\ &\quad + z(-2K\cdot P_1) = -\frac{\langle 1|2|3]}{2}m_H^2. \end{aligned} \tag{5.32}$$

(iii) The result for c_4^{1m}
The total coefficient of our box is

$$\begin{aligned} c_4^{1m} &= c_{0,2} 8m(m^2+\mu^2)k_4\cdot\epsilon_3^+ - c_{0,2}\frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 1\, 3\rangle}\left(\frac{\langle 1|2|3]}{2}\right) + c_{0,2}\frac{\sqrt{2}m}{\langle 1\, 3\rangle}\left(-m_H^2\frac{\langle 1|2|3]}{2}\right) \\ &= -\frac{m^2K^2 s_{23}}{2v\langle 1\, 2\rangle\langle 2\, 3\rangle\langle 3\, 1\rangle}[4(m^2+\mu^2) - m_H^2]. \end{aligned} \tag{5.33}$$

Multiplying by $-i/(4\pi)^{2-\epsilon}$, to account for the difference in the definitions of master integrals, we confirm the result of [30].

C. The triangle coefficient $c_{[1|2|3H]}$

The coefficient $c_{[1|2|3H]}$ gets contributions from N_1, $N_{2,2}$, and $N_{2,3}$:

$$c_{[1|2|3H]} = c_{0,1}N_1 C[Q_1, K]^{(1)} - c_{0,2}\frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 13\rangle}C[Q_1,K]^{(2,2)} + c_{0,2}\frac{\sqrt{2}m}{\langle 13\rangle}C[Q_1,K]^{(2,3)}, \tag{5.34}$$

where the general triangle coefficient, given in Eq. (4.3), reads

$$C[Q_s,K] = \frac{(K^2)^{1+n}}{2}\frac{1}{(\sqrt{\Delta_s})^{n+1}}\frac{1}{(n+1)!\langle P_{s,1}P_{s,2}\rangle^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle P_{s,1}-\tau P_{s,2}|R_jQ_s|P_{s,1}-\tau P_{s,2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}-\tau P_{s,2}|Q_tQ_s|P_{s,1}-\tau P_{s,2}\rangle} + \{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Bigg|_{\tau=0}. \tag{5.35}$$

(i) $C[Q_1, K]^{(1)}$

We have already observed that the N_1 term is trivial. Here is how that shows up in our formulas.

Read $C[Q_s, K]$ for $s=1$ and $k=1$, $n=-1$, and no R_j.

The term inside the parentheses degenerates to 1.

$$C[Q_1,K]^{(1)} = \tfrac{1}{2}(1+1)|_{\tau=0} = 1. \tag{5.36}$$

(ii) $C[Q_1, K]^{(2,2)}$ and $C[Q_1, K]^{(2,3)}$

As we said already, $N_{2,2}$ and $N_{2,3}$ differ in the definition of P_1. Therefore we start by manipulating the general formula, and only at the very end we specialize each contribution using the corresponding P_1.

Since $n=-1$, $k=2$, there is no derivative at all, so we can set $\tau=0$ from the beginning:

$$C[Q_s,K] = \frac{1}{2}\left(\frac{\langle P_{s,1}|R_1Q_s|P_{s,1}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}|Q_tQ_s|P_{s,1}\rangle} + \frac{\langle P_{s,2}|R_1Q_s|P_{s,2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{s,2}|Q_tQ_s|P_{s,2}\rangle}\right). \tag{5.37}$$

The one-mass triangle $(1|2|3H)$ corresponds to the value $s=1$,

$$C[Q_1,K] = -\frac{1}{2}\left(\frac{\langle 2|P_1|1]}{\langle 2|4|1]} + \frac{\langle 1|P_1|2]}{\langle 1|4|2]}\right). \tag{5.38}$$

For $|P_1\rangle = |1\rangle$, $|P_1] = |3]$, one gets

$$C[Q_1,K]^{(2,2)} = -\frac{1}{2}\frac{\langle 1|2|3]}{s_{23}}. \tag{5.39}$$

For $|P_1\rangle = \not{k}_4|3]$, $|P_1] = \not{k}_4|1\rangle$, one obtains

$$\begin{aligned}C[Q_1,K]^{(2,3)} &= -\frac{1}{2}\left(\frac{\langle 2|4|3]\langle 1|4|1]}{\langle 2|4|1]} + \frac{\langle 1|4|3]\langle 1|4|2]}{\langle 1|4|2]}\right)\\ &= \langle 1|2|3]\left(1-\frac{m_H^2}{2s_{23}}\right).\end{aligned} \tag{5.40}$$

(iii) The result for $c_{[1|2|3H]}$

The total coefficient of triangle $(1|2|3H)$ is

$$\begin{aligned}c_{[1|2|3H]} &= -\frac{mK^2s_{23}}{(K^2-m_H^2)\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}m[4(m^2+\mu^2)-s_{12}] - \frac{m[1\,2]}{\sqrt{2}v\langle 1\,2\rangle}\frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 1\,3\rangle}\left(-\frac{\langle 1|2|3]}{2s_{23}}\right)\\ &\quad + \frac{m[12]}{\sqrt{2}v\langle 1\,2\rangle}\frac{\sqrt{2}m}{\langle 1\,3\rangle}\langle 1|2|3]\left(1-\frac{m_H^2}{2s_{23}}\right)\\ &= -\frac{m^2K^2(2s_{23}+K^2-m_H^2)}{2v(K^2-m_H^2)\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}[4(m^2+\mu^2)-m_H^2].\end{aligned} \tag{5.41}$$

Multiplying by $i/(4\pi)^{2-\epsilon}$, to account for the difference in the definitions of master integrals, we again confirm the result of [30].

D. The coefficient $c_{[12|3|H]}$

The coefficient $c_{[12|3|H]}$ gets contributions from $N_{2,2}$ and $N_{2,3}$, therefore it can be written as

$$c_{[12|3|H]} = -c_{0,2}\frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 13\rangle}C[Q_2,K]^{(2,2)} + c_{0,2}\frac{\sqrt{2}m}{\langle 1\,3\rangle}C[Q_2,K]^{(2,3)}. \quad (5.42)$$

(i) $C[Q_2,K]^{(2,2)}$ and $C[Q_2,K]^{(2,3)}$
The two-mass triangle $(12|3|H)$ corresponds to the value $s=2$, and its coefficient can be obtained from Eq. (4.3),

$$C[Q_2,K] = \frac{1}{2}\left(\frac{\langle 3|P_1K|3\rangle}{\langle 3|1|2|3\rangle} - \frac{[3|KP_1|3]}{[3|1|2|3]}\right). \quad (5.43)$$

By using $|P_1\rangle = |1\rangle$, $|P_1] = |3]$, one gets

$$C[Q_2,K]^{(2,2)} = \left(1-\frac{m_H^2}{K^2}\right)\frac{\langle 1|2|3]}{2s_{23}}. \quad (5.44)$$

By using $|P_1\rangle = \not{k}_4|3]$, $|P_1] = \not{k}_4|1\rangle$, one obtains

$$C[Q_2,K]^{(2,3)} = \frac{1}{2}\left(\frac{\langle 3|4|3]\langle 1|4K|3\rangle}{\langle 3|1|2|3\rangle} - \frac{[3|K4|3]\langle 1|4|3]}{[3|1|2|3]}\right) = \left(1-\frac{m_H^2}{K^2}\right)m_H^2\frac{\langle 1|2|3]}{2s_{23}}. \quad (5.45)$$

(ii) The result of $c_{[12|3|H]}$
The total coefficient of triangle $(12|3|H)$ is

$$c_{[12|3|H]} = -c_{0,2}\frac{8m(m^2+\mu^2)}{\sqrt{2}\langle 1\,3\rangle}\left(1-\frac{m_H^2}{K^2}\right)\frac{\langle 1|2|3]}{2s_{23}} + c_{0,2}\frac{\sqrt{2}m}{\langle 13\rangle}\left(1-\frac{m_H^2}{K^2}\right)m_H^2\frac{\langle 1|2|3]}{2s_{23}} = \frac{m^2(K^2-m_H^2)}{2v\langle 12\rangle\langle 23\rangle\langle 31\rangle}[4(m^2+\mu^2)-m_H^2]. \quad (5.46)$$

Multiplying by $i/(4\pi)^{2-\epsilon}$, to account for the difference in the definitions of master integrals, we again confirm the result of [30].

VI. EXAMPLE II: s_{23}-CHANNEL CUT OF $A(1^-, 2^-, 3^+, 4^+)$

Our second example features a nonvanishing bubble coefficient. We study the t-channel cut of the gluon amplitude $A(1^-, 2^-, 3^+, 4^+)$, with a massive quark circulating in the loop. (As usual, $t = s_{23}$.)

The t-channel cut of $A(1^-, 2^-, 3^+, 4^+)$ admits a decomposition in terms of cuts of master integrals as shown in Fig. 8, and its expression was given in Eq. (5.33) of [29]. After converting that expression into our basis of D-dimensional master integrals, as done in Appendix A, it reads

$$A_4^{\text{fermion}}(1^-,2^-,3^+,4^+)\Big|_{t-\text{cut}} = \frac{\langle 1\,2\rangle^2[3\,4]^2}{st}\left(\frac{2}{3}I_2[1] + \frac{4}{3t}I_2[m^2+\mu^2] - \frac{2}{s}I_2[m^2+\mu^2] + \frac{2t}{s}I_4[(m^2+\mu^2)^2] - tI_4[m^2+\mu^2]\right)\Big|_{t-\text{cut}}. \quad (6.1)$$

One reads the following values for the coefficients:

$$c_4^{0m} = c_0\left(\frac{2t}{s}(m^2+\mu^2) - t\right)(m^2+\mu^2), \quad (6.2)$$

$$c_{[23|4|1]} = 0, \quad (6.3)$$

$$c_{[2|3|41]} = 0, \quad (6.4)$$

$$c_{[23|41]} = c_0\left(\frac{2}{3} + \frac{4}{3t}(m^2+\mu^2) - \frac{2}{s}(m^2+\mu^2)\right) \quad (6.5)$$

with

$$c_0 = \frac{\langle 1\,2\rangle^2[3\,4]^2}{st}. \quad (6.6)$$

A. The reconstruction of the coefficients

We now apply our formulas of Sec. IV to construct the coefficients given above. We follow the definition of the integrand given by [29]. By sewing the tree-level amplitude $A_4^{\text{tree}}(-L_1, 2^-, 3^+, L_2)$ and $A_4^{\text{tree}}(-L_2, 4^+, 1^-, L_1)$

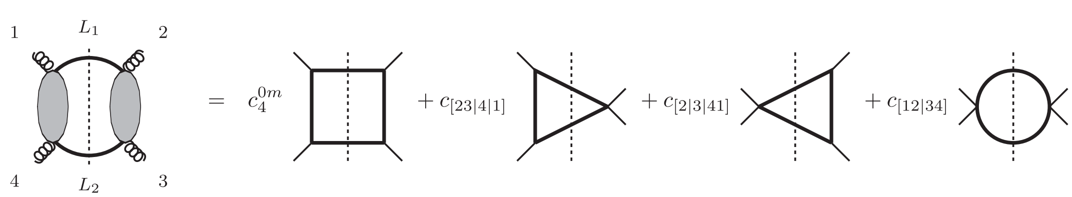

FIG. 8. Double cut in the s_{23}-channel for $A(1^-, 2^-, 3^+, 4^+)$.

given in Eq. (2.3) of [29], and using the Dirac equation for a massive fermion, it is shown that the four-dimensional integrand of the t-cut, C_{23}, can be written as[3]:

$$C_{23} = -\frac{2N_1 + N_2}{D_1 D_2}, \tag{6.7}$$

with

$$N_1 = \frac{1}{s_{23}^2}\langle 1|\ell_1|4]^2\langle 2|\ell_1|3]^2, \tag{6.8}$$

$$N_2 = -\frac{1}{s_{23}}\langle 1\,2\rangle[3\,4]\langle 1|\ell_1|4]\langle 2|\ell_1|3], \tag{6.9}$$

$$D_1 = (\ell_1 + k_1)^2 - \mu^2 - m^2, \tag{6.10}$$

$$D_2 = (\ell_1 - k_2)^2 - \mu^2 - m^2. \tag{6.11}$$

By pattern matching onto the reference form in Eq. (3.21), each integrand is characterized by the parameters given in the following table.

integrand	n	k	$\|P_1\rangle[P_1\|$	$\|P_2\rangle[P_2\|$	$\|P_3\rangle[P_3\|$	$\|P_4\rangle[P_4\|$
$N_1/(D_1D_2)$	2	2	$\|1\rangle[4\|$	$\|2\rangle[3\|$	$\|1\rangle[4\|$	$\|2\rangle[3\|$
$N_2/(D_1D_2)$	0	2	$\|1\rangle[4\|$	$\|2\rangle[3\|$	–	–

(6.12)

We define

$$K = k_2 + k_3; \qquad K_1 = -k_1; \qquad K_2 = k_2; \tag{6.13}$$

$$P_1 = P_3 = \lambda_1\tilde{\lambda}_4, \qquad P_2 = P_4 = \lambda_2\tilde{\lambda}_3. \tag{6.14}$$

Moreover, since we have a quark of mass m circulating in the loop, we take

$$M_1 = M_2 = m_j = m. \tag{6.15}$$

Then, by applying (3.9), we find

$$z(1-z) = \frac{m^2 + \mu^2}{K^2}. \tag{6.16}$$

For the N_1 term, $n = 2$. For the N_2 term, $n = 0$. Both terms give boxes, triangles and bubbles.

From the definitions (3.23) and (3.24), we have

$$Q_1 = (1-z)k_1 + zk_4, \qquad Q_2 = -(1-z)k_2 - zk_3, \tag{6.17}$$

$$\begin{aligned} R_1 = R_3 &= -(1-2z)\lambda_1\tilde{\lambda}_4, \\ R_2 = R_4 &= -(1-2z)\lambda_2\tilde{\lambda}_3. \end{aligned} \tag{6.18}$$

Further, the quantities defined in (4.2) and (4.4), become

$$\Delta_{12} = (1-2z)^4 s_{12}(K^2 + s_{12}) - (1-2z)^2 K^2 s_{12}, \tag{6.19}$$

$$\Delta_1 = (1-2z)^2(K^2)^2, \tag{6.20}$$

$$P_{1,1} = -(1-2z)k_4, \qquad P_{1,2} = (1-2z)k_1, \tag{6.21}$$

$$\Delta_2 = (1-2z)^2(K^2)^2, \tag{6.22}$$

$$P_{2,1} = (1-2z)k_3, \qquad P_{2,2} = -(1-2z)k_2. \tag{6.23}$$

B. The box coefficient c_4^{0m}

The box coefficient c_4^{0m} receives contributions from both N_1 and N_2, and can be correspondingly decomposed as

$$c_4^{0m} = -\frac{2}{(K^2)^2}C[Q_1, Q_2, K]^{(1)} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}C[Q_1, Q_2, K]^{(2)}. \tag{6.24}$$

We discuss the computation of $C[Q_1, Q_2, K]^{(1)}$ and $C[Q_1, Q_2, K]^{(2)}$ in detail, starting from the expression given in Eq. (4.1).

(i) $C[Q_1, Q_2, K]^{(1)}$
For the N_1 term, with $n = 2$, the expression is given by

$$C[Q_1, Q_2, K]^{(1)} = \frac{(K^2)^4}{2}\left(\frac{\langle P_{21,1}|R_1|P_{21,2}]^2\langle P_{21,1}|R_2|P_{21,2}]^2}{\langle P_{21,1}|K|P_{21,2}]^4} + \frac{\langle P_{21,2}|R_1|P_{21,1}]^2\langle P_{21,2}|R_2|P_{21,1}]^2}{\langle P_{21,2}|K|P_{21,1}]^4}\right). \tag{6.25}$$

For analytic simplification, the following trace identity is helpful.

$$\langle P_1|R|P_2]\langle P_2|S|P_1] = \mathrm{Tr}\left(\frac{1-\gamma_5}{2}\not{P}_1\not{R}\not{P}_2\not{S}\right) \equiv \mathrm{tr}_-(P_1RP_2S).$$

[3]Recall that here we use twistor sign convention for the antiholomorphic spinor product, which is the opposite of the QCD convention followed by [29] $[xy]^{\text{Bern-Morgan}} = -[xy]^{\text{BFM}}$.

In terms of vectors,

$$\mathrm{tr}_-(V_1V_2V_3V_4) = \tfrac{1}{2}((2V_1\cdot V_2)(2V_3\cdot V_4) + (2V_1\cdot V_4)(2V_2\cdot V_3) - (2V_1\cdot V_3)(2V_2\cdot V_4) - 4i\epsilon_{\mu\nu\sigma\rho}V_1^\mu V_2^\nu V_3^\sigma V_4^\rho).$$

The coefficient can then be expressed in terms of traces, and evaluated as follows:

$$C[Q_1, Q_2, K]^{(1)} = \frac{(K^2)^4((\mathrm{tr}_-(P_{21,2}KP_{21,1}R_1)\mathrm{tr}_-{}^2(P_{21,2}KP_{21,1}R_2))^2 + (\mathrm{tr}_-(P_{21,1}KP_{21,2}R_1)\mathrm{tr}_-{}^2(P_{21,1}KP_{21,2}R_2))^2)}{2(\mathrm{tr}_-(P_{21,1}KP_{21,2}K))^4} = \frac{(K^2)^4z^2(1-z)^2[3\,4]^2\langle 1\,2\rangle^2}{s_{12}^2}. \tag{6.26}$$

(ii) $C[Q_1, Q_2, K]^{(2)}$

For the N_2 term with $n = 0$ the expression is given by

$$C[Q_1, Q_2, K]^{(2)} = \frac{(K^2)^2}{2}\left(\frac{\langle P_{21,1}|R_1|P_{21,2}]\langle P_{21,1}|R_2|P_{21,2}]}{\langle P_{21,1}|K|P_{21,2}]^2} + \{P_{21,1}\leftrightarrow P_{21,2}\}\right). \tag{6.27}$$

Combining the two terms over a common denominator, we have

$$C[Q_1, Q_2, K]^{(2)} = \frac{(K^2)^2(\mathrm{tr}_-(P_{21,2}KP_{21,1}R_1)\mathrm{tr}_-(P_{21,2}KP_{21,1}R_2) + \mathrm{tr}_-(P_{21,1}KP_{21,2}R_1)\mathrm{tr}_-(P_{21,1}KP_{21,2}R_2))}{2(\mathrm{tr}_-(P_{21,1}KP_{21,2}K))^2} = \frac{(K^2)^2z(1-z)[3\,4]\langle 1\,2\rangle}{s_{12}}. \tag{6.28}$$

(iii) The result of c_4^{0m}

We add our two contributions together and replace z using (6.16). The total box coefficient is thus

$$c_4^{0m} = -\frac{2}{(K^2)^2}\frac{(K^2)^4z^2(1-z)^2[3\,4]^2\langle 1\,2\rangle^2}{s_{12}^2} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}\frac{(K^2)^2z(1-z)[3\,4]\langle 1\,2\rangle}{s_{12}} \tag{6.29}$$

$$= \frac{(m^2+\mu^2)[3\,4]^2\langle 1\,2\rangle^2}{s_{12}}\left(1 - \frac{2(m^2+\mu^2)}{s_{12}}\right). \tag{6.30}$$

C. The triangle coefficients $c_{[23|4|1]}$ and $c_{[2|3|41]}$

Both terms exhibit the symmetry of the amplitude, so our two triangles are not independent.

The triangle coefficients $c_{[23|4|1]}$ and $c_{[2|3|41]}$ receive contributions from both N_1 and N_2, and they can be correspondingly decomposed as

$$c_{[23|4|1]} = -\frac{2}{(K^2)^2}C[Q_1, K]^{(1)} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}C[Q_1, K]^{(2)}, \tag{6.31}$$

$$c_{[2|3|41]} = -\frac{2}{(K^2)^2}C[Q_2, K]^{(1)} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}C[Q_2, K]^{(2)}. \tag{6.32}$$

We discuss in parallel, first the contribution due to N_1 to both coefficients, namely $C[Q_1, K]^{(1)}$ and $C[Q_2, K]^{(1)}$, and later the one due to N_2, namely $C[Q_1, K]^{(2)}$ and $C[Q_2, K]^{(2)}$, where the triangle coefficient was given in Eq. (4.3).

(i) $C[Q_1, K]^{(1)}$ and $C[Q_2, K]^{(1)}$

Since the N_1 term with $n = 2$, the triangle coefficient expression is given by

$$\begin{aligned} C[Q_1, K]^{(1)} &= \frac{(K^2)^3}{2}\frac{1}{(\sqrt{\Delta_1})^3}\frac{1}{3!\langle 4\,1\rangle^3}\frac{d^3}{d\tau^3}\left(\frac{\prod_{j=1}^4\langle 4-\tau 1|R_jQ_1|4-\tau 1\rangle}{\langle 4-\tau 1|Q_2Q_1|4-\tau 1\rangle}+\frac{\prod_{j=1}^4\langle 1-\tau 4|R_jQ_1|1-\tau 4\rangle}{\langle 1-\tau 4|Q_2Q_1|1-\tau 4\rangle}\right)\Bigg|_{\tau=0} \\ &= \frac{(1-2z)}{2}\frac{1}{3!\langle 4\,1\rangle}\frac{d^3}{d\tau^3}\left(\frac{[4|Q_1|4\rangle^2\langle 4-\tau 1,2\rangle^2[3|Q_1|4-\tau 1\rangle^2}{\langle 4-\tau 1|Q_2Q_1|4-\tau 1\rangle}+\frac{\tau^4[4|Q_1|4\rangle^2\langle 1-\tau 4,2\rangle^2[3|Q_1|1-\tau 4\rangle^2}{\langle 1-\tau 4|Q_2Q_1|1-\tau 4\rangle}\right)\Bigg|_{\tau=0} \\ &= \frac{(1-2z)(1-z)^2(K^2)^2}{12\langle 4\,1\rangle}\frac{d^3}{d\tau^3}\left(\frac{\langle 4-\tau 1,2\rangle^2[3|Q_1|4-\tau 1\rangle^2}{\langle 4-\tau 1|Q_2Q_1|4-\tau 1\rangle}\right)\Bigg|_{\tau=0} \\ &= \frac{(1-z)^2(K^2)^2}{12}\frac{d^3}{d\tau^3}\left(\frac{(\langle 4\,2\rangle-\tau\langle 1\,2\rangle)^2((1-z)[3\,1]+\tau z[3\,4])^2}{-(1-z)\langle 4|3|1]+\tau s_{34}+\tau^2 z\langle 1|3|4]}\right)\Bigg|_{\tau=0}=0. \end{aligned} \tag{6.33}$$

A similar calculation shows that

$$C[Q_2, K]^{(1)} = \frac{(K^2)^3}{2}\frac{1}{(\sqrt{\Delta_2})^3}\frac{1}{3!\langle 3\,2\rangle^3}\frac{d^3}{d\tau^3}\left(\frac{\prod_{j=1}^4\langle 3-\tau 2|R_jQ_2|3-\tau 2\rangle}{\langle 3-\tau 2|Q_1Q_2|3-\tau 2\rangle}+\frac{\prod_{j=1}^4\langle 2-\tau 3|R_jQ_2|2-\tau 3\rangle}{\langle 2-\tau 3|Q_1Q_2|2-\tau 3\rangle}\right)\Bigg|_{\tau=0}=0, \tag{6.34}$$

which can also be seen by the symmetry of the amplitude and the cut.

(ii) $C[Q_1, K]^{(2)}$ and $C[Q_2, K]^{(2)}$

For the N_2 term with $n = 0$, the expression is simpler as

$$\begin{aligned} C[Q_1, K]^{(2)} &= -\frac{(1-2z)(1-z)K^2}{2}\frac{d}{d\tau}\left(\frac{\langle 4-\tau 1 2\rangle[3|Q_1|4-\tau 1\rangle}{\langle 4-\tau 1|Q_2Q_1|4-\tau 1\rangle}+\frac{\tau^2\langle 1-\tau 4\,2\rangle[3|Q_1|1-\tau 4\rangle}{\langle 1-\tau 4|Q_2Q_1|1-\tau 4\rangle}\right)\Bigg|_{\tau=0} \\ &= -\frac{(1-2z)(1-z)K^2}{2}\frac{d}{d\tau}\left(\frac{(\langle 4\,2\rangle-\tau\langle 1\,2\rangle)([3|Q_1|4\rangle-\tau[3|Q_1|1\rangle)}{\langle 4-\tau 1|Q_2Q_1|4-\tau 1\rangle}\right)\Bigg|_{\tau=0}=0. \end{aligned} \tag{6.35}$$

A similar calculation shows

$$C[Q_2, K]^{(2)} = \frac{(1-2z)}{2}\frac{d}{d\tau}\left(\frac{\langle 3-\tau 2\,1\rangle[4|Q_2|3-\tau 2\rangle[3|Q_2|3\rangle}{\langle 3-\tau 2|Q_1Q_2|3-\tau 2\rangle}+\frac{\tau^2\langle 2-\tau 3\,1\rangle[4|Q_2|2-\tau 3\rangle[3|Q_2|3\rangle}{\langle 2-\tau 3|Q_1Q_2|2-\tau 3\rangle}\right)\Bigg|_{\tau=0}=0, \tag{6.36}$$

which can also be seen by symmetry.

(iii) The results of $c_{[23|4|1]}$ and $c_{[2|3|41]}$

Every term vanishes separately, so

$$c_{[23|4|1]} = 0, \qquad c_{[2|3|41]} = 0. \tag{6.37}$$

The vanishing results for triangle coefficients is not obvious from the beginning. We suspect that there should be a more directly physical argument to see this point.

D. The bubble coefficient $c_{[23|41]}$

The bubble coefficient $c_{[23|41]}$ receives contributions from both N_1 and N_2, and can be correspondingly decomposed as

$$c_{[23|41]} = -\frac{2}{(K^2)^2}C[K]^{(1)} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}C[K]^{(2)}. \tag{6.38}$$

There is one subtlety regarding the calculation of the bubble coefficient. The formulas involve an arbitrarily chosen, generic auxiliary null vector η. If η coincides with one of the K_i, we need to use a modified formula, given in Appendix B.3.1 of [19]. In this example, we illustrate both options. First, we show the result with a generic choice of η; second, we use the formulas for the case $\eta = K_1$. Both are suitable for numerical evaluation, while the special choice of η may simplify the analytic expression. We will find that the two results agree with each other, as well as with [29].

1. Generic reference momentum η

Let us start with the formulas for generic η, given in Eq. (4.5). There are two terms we need to calculate.

(i) $C[K]^{(1)}$

For the first term, with N_1 in the numerator, and $n = 2$, the coefficient is

$$C[K]^{(1)} = (K^2)^3\sum_{q=0}^{2}\frac{(-1)^q}{q!}\frac{d^q}{ds^q}\left(\mathcal{B}_{2,2-q}^{(0)}(s)+\sum_{r=1}^{2}\sum_{a=q}^{2}(\mathcal{B}_{2,2-a}^{(r;a-q;1)}(s)-\mathcal{B}_{2,2-a}^{(r;a-q;2)}(s))\right)\Bigg|_{s=0}, \tag{6.39}$$

where

$$\mathcal{B}^{(0)}_{2,2-q}(s) = \frac{d^2}{d\tau^2}\left(\frac{1}{2[\eta|\tilde{\eta}K|\eta]^2}\frac{(2\eta\cdot K)^{3-q}}{(3-q)(K^2)^{3-q}}\frac{\prod_{j=1}^4\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\eta\rangle^3\prod_{p=1}^2\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Big|_{|\ell\rangle\to|K-\tau\tilde{\eta}|\eta]}\right)\Big|_{\tau=0}, \quad (6.40)$$

$$\mathcal{B}^{(r;a-q;1)}_{2,2-a}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_r}^{a-q+1}\langle P_{r,1}P_{r,2}\rangle^{a-q}}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{1}{(3-a)}\frac{\langle P_{r,1}-\tau P_{r,2}|\eta|P_{r,1}]^{3-a}}{\langle P_{r,1}-\tau P_{r,2}|K|P_{r,1}]^{3-a}}\right.$$
$$\left.\times\frac{\langle P_{r,1}-\tau P_{r,2}|Q_r\eta|P_{r,1}-\tau P_{r,2}\rangle^{a-q}\prod_{j=1}^4\langle P_{r,1}-\tau P_{r,2}|R_j(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}{\langle P_{r,1}-\tau P_{r,2}|\eta K|P_{r,1}-\tau P_{r,2}\rangle^3\prod_{p=1,p\neq r}^2\langle P_{r,1}-\tau P_{r,2}|Q_p(K+s\eta)|P_{r,1}-\tau P_{r,2}\rangle}\right)\Big|_{\tau=0}, \quad (6.41)$$

$$\mathcal{B}^{(r;a-q;2)}_{2,2-a}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_r}^{a-q+1}\langle P_{r,1}P_{r,2}\rangle^{a-q}}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{1}{(3-a)}\frac{\langle P_{r,2}-\tau P_{r,1}|\eta|P_{r,2}]^{3-a}}{\langle P_{r,2}-\tau P_{r,1}|K|P_{r,2}]^{3-a}}\right.$$
$$\left.\times\frac{\langle P_{r,2}-\tau P_{r,1}|Q_r\eta|P_{r,2}-\tau P_{r,1}\rangle^{a-q}\prod_{j=1}^4\langle P_{r,2}-\tau P_{r,1}|R_j(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}{\langle P_{r,2}-\tau P_{r,1}|\eta K|P_{r,2}-\tau P_{r,1}\rangle^3\prod_{p=1,p\neq r}^2\langle P_{r,2}-\tau P_{r,1}|Q_p(K+s\eta)|P_{r,2}-\tau P_{r,1}\rangle}\right)\Big|_{\tau=0}. \quad (6.42)$$

After making some substitutions, and considering the summation ranges of a and q, we get

$$\mathcal{B}^{(1;a-q;1)}_{2,2-a}(s) = 0, \quad (6.43)$$

$$\mathcal{B}^{(1;a-q;2)}_{2,2-a}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_1}^{a-q+1}\langle 4\,1\rangle^{a-q}}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{1}{(3-a)}\frac{\langle 1-\tau 4|\eta|1]^{3-a}}{\langle 1-\tau 4|K|1]^{3-a}}\right.$$
$$\left.\times\frac{\langle 1-\tau 4|Q_1\eta|1-\tau 4\rangle^{a-q}\langle -\tau 4|R_1(K+s\eta)|1-\tau 4\rangle^2\langle 1-\tau 4|R_2(K+s\eta)|1-\tau 4\rangle^2}{\langle 1-\tau 4|\eta K|1-\tau 4\rangle^3\langle 1-\tau 4|Q_2(K+s\eta)|1-\tau 4\rangle}\right)\Big|_{\tau=0}. \quad (6.44)$$

$$\mathcal{B}^{(2;a-q;1)}_{2,2-a}(s) = \frac{(-1)^{a-q+1}}{(a-q)!\sqrt{\Delta_2}^{a-q+1}\langle 3\,2\rangle^{a-q}}\frac{d^{a-q}}{d\tau^{a-q}}\left(\frac{1}{(3-a)}\frac{\langle 3-\tau 2|\eta|3]^{3-a}}{\langle 3-\tau 2|K|3]^{3-a}}\right.$$
$$\left.\times\frac{\langle 3-\tau 2|Q_2\eta|3-\tau 2\rangle^{a-q}\langle 3-\tau 2|R_1(K+s\eta)|3-\tau 2\rangle^2\langle 3|R_2(K+s\eta)|3-\tau 2\rangle^2}{\langle 3-\tau 2|\eta K|3-\tau 2\rangle^3\langle 3-\tau 2|Q_1(K+s\eta)|3-\tau 2\rangle}\right)\Big|_{\tau=0}, \quad (6.45)$$

$$\mathcal{B}^{(2;a-q;2)}_{2,2-a}(s) = 0. \quad (6.46)$$

(ii) $C[K]^{(2)}$

For the second term N_2 with $n = 0$, the expression is much simpler:

$$C[K]^{(2)} = K^2\left(\mathcal{B}^{(0)}_{0,0}(s) + \sum_{r=1}^{2}(\mathcal{B}^{(r;0;1)}_{0,0}(s) - \mathcal{B}^{(r;0;2)}_{0,0}(s))\right)\Big|_{s=0}, \quad (6.47)$$

where

$$\mathcal{B}^{(0)}_{0,0}(s=0) = \left(\frac{(2\eta\cdot K)}{K^2}\frac{\langle\ell|R_1K|\ell\rangle\langle\ell|R_2K|\ell\rangle}{\langle\ell\eta\rangle\prod_{p=1}^2\langle\ell|Q_pK|\ell\rangle}\right)\Big|_{|\ell\rangle\to|K|\eta]} = \frac{1}{K^2}\frac{[\eta|KR_1|\eta][\eta|KR_2|\eta]}{[\eta|KQ_1|\eta][\eta|KQ_2|\eta]} = -\frac{1}{K^2}\frac{[\eta\,4][\eta\,3]}{[\eta\,1][\eta\,2]}, \quad (6.48)$$

$$\mathcal{B}^{(1;0;1)}_{0,0}(s=0) = -\frac{1}{\sqrt{\Delta_1}}\frac{\langle 4|\eta|4]}{\langle 4|K|4]}\frac{\langle 4|R_1K|4\rangle\langle 4|R_2K|4\rangle}{\langle 4|\eta K|4\rangle\langle 4|Q_2K|4\rangle} = -\frac{1}{K^2}\frac{[\eta\,4]\langle 4\,2\rangle}{[\eta\,1]\langle 4\,3\rangle}, \quad (6.49)$$

$$\mathcal{B}_{0,0}^{(1;0;2)}(s=0)=0. \tag{6.50}$$

$$\mathcal{B}_{0,0}^{(2;0;1)}(s=0)=-\frac{1}{\sqrt{\Delta_2}}\frac{\langle 3|\eta|3]}{\langle 3|K|3]}\frac{\langle 3|R_1K|3\rangle\langle 3|R_2K|3\rangle}{\langle 3|\eta K|3\rangle\langle 3|Q_1K|3\rangle}=\frac{1}{K^2}\frac{[\eta\,3][4\,2]}{[\eta\,2][1\,2]}, \tag{6.51}$$

$$\mathcal{B}_{0,0}^{(2;0;2)}(s=0)=0. \tag{6.52}$$

(iii) Results:
We have used the numerical routines of S@M [31] to show that while each single term $\mathcal{B}$ entering Eq. (6.39) is η-dependent, their combination is indeed independent of the choice of η. The choice $\eta=k_3$ is found to be convenient. (Note that k_3 is not proportional to either of K_1, K_2.) Therefore we set $\eta=k_3$, and we obtain the following analytic result:

$$\begin{aligned}c_{[12|34]}&=-\frac{2}{(K^2)^2}\bigg\{\frac{\langle 1\,3\rangle[1\,3](K^2)^3}{[1\,2]^2\langle 4\,3\rangle^2}\Big(\frac{1}{2}-4z(1-z)\Big)-\frac{5(K^2)^4}{3[1\,2]^2\langle 4\,3\rangle^2}z(1-z)+\frac{(K^2)^4}{6[1\,2]^2\langle 4\,3\rangle^2}\\&\quad+\frac{K^2[1\,3]\langle 1\,3\rangle}{\langle 4\,3\rangle^2[1\,2]^2}\Big(\Big(\frac{1}{6}-\frac{2}{3}z(1-z)\Big)\langle 1\,3\rangle^2[1\,3]^2+\Big(\frac{1}{2}-3z(1-z)\Big)K^2\langle 1\,3\rangle[1\,3]\Big)\bigg\}+\frac{\langle 1\,2\rangle[3\,4]}{K^2}\frac{[4\,3]}{[12]}\\&=\frac{\langle 1\,2\rangle^2[3\,4]^2}{3s^2t^2}(-4(m^2+\mu^2)s+6(m^2+\mu^2)t-2st).\end{aligned} \tag{6.53}$$

We have used the relations $K^2=t$, $\langle 1\,3\rangle[1\,3]=-s-t$.

2. Special choice of η

Alternatively, we discuss the calculation of the bubble coefficient by using the special choice of $\eta=K_1=-k_1$ from the beginning. With this choice, we need to use formulas for the $\mathcal{B}$ which are slightly different from the ones used in the previous section. They are given in Appendix B.3.1 of [19].

Our convention for the spinors is

$$|\eta\rangle=|1\rangle,\qquad |\eta]=-|1]. \tag{6.54}$$

$$C[K]_n=(K^2)^{1+n}\sum_{q=0}^{n}\frac{(-1)^q}{q!}\frac{d^q}{ds^q}\Big(\mathcal{B}_{n,n-q}^{(0)}(s)+\sum_{r=2}^{k}\sum_{a=q}^{n}(\mathcal{B}_{n,n-a}^{(r;a-q;1)}(s)-\mathcal{B}_{n,n-a}^{(r;a-q;2)}(s))\Big)\Big|_{s=0}. \tag{6.55}$$

$$\mathcal{B}_{n,t}^{(0)}(s)=-\frac{d^{n+1}}{d\tau^{n+1}}\Big(\frac{1}{(1-2z)-sz}\frac{[\eta|\tilde{\eta}K|\eta]^{-n-1}}{(t+1)(n+1)!}\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\,\eta\rangle^{n+2}\prod_{p=2}^{k}\langle\ell|Q_p(K+s\eta)|\ell\rangle}\Big|_{|\ell\rangle\to|K-\tau\tilde{\eta}|\eta]}\Big)\Big|_{\tau\to 0}. \tag{6.56}$$

Since $k=2$, we can directly set $r=2$:

$$\mathcal{B}_{n,t}^{(2;a;1)}(s)=\frac{1}{(1-2z)-sz}\frac{(-1)^a}{(1-2z)^{a+1}(K^2)^{a+t+2}a!\langle 3\,2\rangle^a}\frac{d^a}{d\tau^a}\Big(\frac{[1\,3]^{t+1}}{(t+1)}\frac{\langle\ell|Q_2|1]^a\prod_{j=1}^{n+2}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\,1\rangle^{n+1-t-a}[1\,4]^{n+2}\langle 4\,\ell\rangle^{n+2}}\Big)\Big|_{|\ell\rangle=|3\rangle-\tau|2\rangle}, \tag{6.57}$$

$$\mathcal{B}_{n,t}^{(2;a;2)}(s)=\frac{1}{(1-2z)-sz}\frac{(-1)^a}{(1-2z)^{a+1}(K^2)^{a+t+2}a!\langle 3\,2\rangle^a}\frac{d^a}{d\tau^a}\Big(\frac{[1\,2]^{t+1}}{(t+1)}\frac{\langle\ell|Q_2|1]^a\prod_{j=1}^{n+2}\langle\ell|R_j(K+s\eta)|\ell\rangle}{\langle\ell\,1\rangle^{n+1-t-a}[1\,4]^{n+2}\langle 4\,\ell\rangle^{n+2}}\Big)\Big|_{|\ell\rangle=|2\rangle-\tau|3\rangle}. \tag{6.58}$$

Now we proceed to evaluate.

(i) $C[K]^{(2)}$
For the N_2 term with $n = 0$, the evaluation is simple. In particular, there are no derivatives in s, so we can set $s = 0$ directly.

$$C[K]^{(2)} = K^2(\mathcal{B}^{(0)}_{0,0}(s) + \mathcal{B}^{(2;0;1)}_{0,0}(s) - \mathcal{B}^{(2;0;2)}_{0,0}(s))|_{s=0}. \tag{6.59}$$

Choosing $\tilde{\eta} = 3$, we have

$$\mathcal{B}^{(0)}_{0,0}(s=0) = \frac{1}{[1\,2]\langle 3\,4\rangle}, \quad \mathcal{B}^{(2;0;1)}_{0,0}(s=0) = \frac{1}{K^2}\frac{[1\,3][4\,2]}{[1\,2]^2}, \quad \mathcal{B}^{(2;0;2)}_{0,0}(s=0) = 0 \tag{6.60}$$

so the total contribution comes to

$$C[K]^{(2)} = \frac{[4\,3]}{[1\,2]}. \tag{6.61}$$

(ii) $C[K]^{(1)}$
For the N_1 term with $n = 2$, the calculation is a bit more involved.

$$C[K]^{(1)} = (K^2)^3 \sum_{q=0}^{2} \frac{(-1)^q}{q!}\frac{d^q}{ds^q}\bigg(\mathcal{B}^{(0)}_{2,2-q}(s) + \sum_{a=q}^{2}(\mathcal{B}^{(2;a-q;1)}_{2,2-a}(s) - \mathcal{B}^{(2;a-q;2)}_{2,2-a}(s))\bigg)\bigg|_{s=0}. \tag{6.62}$$

For the various terms, we have

$$\mathcal{B}^{(0)}_{2,2-q}(s) = \frac{1}{(3-q)3!\langle 3\,4\rangle^2[1\,2]^2}\frac{d^3}{d\tau^3}\bigg(\frac{(1-2z)^4(1+s)^2}{(1-2z)-sz} \times \frac{(1-\tau)^2(K^2(1+s) - \tau(K^2 + s[3\,1]\langle 1\,3\rangle))^2}{((1+s)(1-2z)K^2 + \tau((zs+2z-s-1)K^2 - s(1-2z)[3\,1]\langle 1\,3\rangle) + \tau^2 s(1-z)[3\,1]\langle 1\,3\rangle)}\bigg)\bigg|_{\tau\to 0}, \tag{6.63}$$

$$\mathcal{B}^{(2;a-q;1)}_{2,2-a}(s) = \frac{(1+s)^2(1-2z)^{3-a+q}}{(1-2z)-sz}\frac{(-1)^{a-1}\langle 3\,2\rangle^2[1\,3]^{3-a}}{(K^2)^{4-q}(a-q)!}\frac{d^{a-q}}{d\tau^{a-q}} \times \bigg(\frac{((1-z)[1\,2] + \tau z[1\,3])^{a-q}(\langle 1\,3\rangle - \tau\langle 1\,2\rangle)^{3-q}(K^2 + s[3\,1]\langle 1\,3\rangle - s\tau[3\,1]\langle 1\,2\rangle)^2}{(3-a)[1\,4]^2(\langle 4\,3\rangle - \tau\langle 4\,2\rangle)^4}\bigg). \tag{6.64}$$

The term $\mathcal{B}^{(2;a-q;2)}_{2,2-a}(s)$ vanishes after taking the derivatives with respect to s. The reason is the following. Notice that

$$\mathcal{B}^{(2;a-q;2)}_{n,t}(s) = \frac{(1+s)^2(1-2z)^{3-a+q}}{(1-2z)-sz}\frac{(-1)^{a-q}\langle 3\,2\rangle^2[1\,2]^{t+1}}{(K^2)^{a-q+t+2}(a-q)!\langle 3\,2\rangle^{a-q}}\frac{d^{a-q}}{d\tau^{a-q}} \times \bigg(\frac{\tau^2}{(t+1)}\frac{\langle \ell|Q_2|1]^{a-q}[3|K + s\eta|\ell\rangle^2}{\langle \ell\,1\rangle^{-1-t-a+q}[1\,4]^2\langle 4\,\ell\rangle^4}\bigg)\bigg|_{|\ell\rangle=|2\rangle-\tau|3\rangle}. \tag{6.65}$$

We can see that the τ-derivative vanishes unless $a - q = 2$, in which case we get

$$\mathcal{B}^{(2;2;2)}_{n,t}(s) = \frac{(1+s)^2(1-2z)}{(1-2z)-sz}\frac{\langle 3\,2\rangle^2[1\,2]^{t+1}}{(K^2)^{4+t}\langle 3\,2\rangle^2}\bigg(\frac{1}{(t+1)}\frac{(-z)^2\langle 2|3|1]^2 s^2[3|1|2\rangle^2}{\langle 2\,1\rangle^{-3-t}[1\,4]^2\langle 4\,2\rangle^4}\bigg). \tag{6.66}$$

However, the condition $a - q = 2$ implies $a = 2$, $q = 0$. Therefore we can set $s = 0$, and the expression vanishes:

$$\mathcal{B}^{(2;2;2)}_{n,t}(s) = 0. \tag{6.67}$$

Now we collect the results of (6.63), (6.64), and (6.67). We take $\tilde{\eta} = k_3$. Define

$$C_1 \equiv \frac{[1\,3]\langle 1\,3\rangle}{(K^2)^2[1\,2]^2\langle 4\,3\rangle^2}. \tag{6.68}$$

Let us begin with the terms with $q = 0$:

$$\begin{aligned}
\mathcal{B}_{2,2}^{(0)}(s=0) &= -\frac{K^2(1-2z)^2}{3\langle 3\,4\rangle^2[2\,1]^2},\\
\mathcal{B}_{2,2}^{(2;0;1)}(s=0) &= C_1(-\tfrac{1}{3})(1-2z)^2[1\,3]^2\langle 1\,3\rangle^2,\\
\mathcal{B}_{2,1}^{(2;1;1)}(s=0) &= C_1(-\tfrac{1}{2}(1-2z)^2[1\,3]^2\langle 1\,3\rangle^2 + (\tfrac{3}{2}-\tfrac{9}{2}z+3z^2)K^2[1\,3]\langle 1\,3\rangle),\\
\mathcal{B}_{2,0}^{(2;2;1)}(s=0) &= C_1((-3+6z-3z^2)(K^2)^2 - (1-2z)^2\langle 1\,3\rangle^2[1\,3]^2 + (6-18z+12z^2)K^2[1\,3]\langle 1\,3\rangle).
\end{aligned} \tag{6.69}$$

For $q=1$:

$$\begin{aligned}
-\frac{d}{ds}\mathcal{B}_{2,1}^{(0)}(s)\Big|_{s=0} &= -\frac{(1-2z)^2}{2\langle 3\,4\rangle^2[1\,2]^2}\frac{[3\,1]\langle 1\,3\rangle-(1-2z)K^2}{(1-2z)},\\
-\frac{d}{ds}\mathcal{B}_{2,1}^{(2;0;1)}(s)\Big|_{s=0} &= C_1\Big((1-2z)^2[1\,3]^2\langle 1\,3\rangle^2 + \Big(-1+\frac{7}{2}z-3z^2\Big)K^2\langle 1\,3\rangle[1\,3]\Big),\\
-\frac{d}{ds}\mathcal{B}_{2,0}^{(2;1;1)}(s)\Big|_{s=0} &= C_1((4-10z+6z^2)(K^2)^2 + 2(1-2z)^2[1\,3]^2\langle 1\,3\rangle^2 + (-10+30z-21z^2)K^2[1\,3]\langle 1\,3\rangle),
\end{aligned} \tag{6.70}$$

For $q=2$:

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{ds^2}\mathcal{B}_{2,0}^{(0)}(s)\Big|_{s=0} &= \frac{1}{\langle 3\,4\rangle^2[1\,2]^2}z((1-2z)[3\,1]\langle 1\,3\rangle-(1-z)K^2),\\
\frac{1}{2}\frac{d^2}{ds^2}\mathcal{B}_{2,0}^{(2;0;1)}(s)\Big|_{s=0} &= C_1((-1+2z-z^2)(K^2)^2 - (1-2z)^2\langle 1\,3\rangle^2[1\,3]^2 + (4-14z+12z^2)K^2\langle 1\,3\rangle[1\,3]).
\end{aligned} \tag{6.71}$$

All together, we get the following result for the N_1 term:

$$\begin{aligned}
C[K]^{(1)} &= \frac{\langle 1\,3\rangle[1\,3](K^2)^3}{[1\,2]^2\langle 4\,3\rangle^2}\Big(\frac{1}{2}-4z(1-z)\Big) - \frac{5(K^2)^4}{3[1\,2]^2\langle 4\,3\rangle^2}z(1-z) + \frac{(K^2)^4}{6[1\,2]^2\langle 4\,3\rangle^2}\\
&\quad + \frac{K^2[1\,3]\langle 1\,3\rangle}{\langle 4\,3\rangle^2[1\,2]^2}\Big(\Big(\frac{1}{6}-\frac{2}{3}z(1-z)\Big)\langle 1\,3\rangle^2[1\,3]^2 + \Big(\frac{1}{2}-3z(1-z)\Big)K^2\langle 1\,3\rangle[1\,3]\Big).
\end{aligned} \tag{6.72}$$

(iii) The result of $c_{[12|34]}$
Final bubble coefficient:

$$\begin{aligned}
c_{[12|34]} &= -\frac{2}{(K^2)^2}\Big\{\frac{\langle 1\,3\rangle[1\,3](K^2)^3}{[1\,2]^2\langle 4\,3\rangle^2}\Big(\frac{1}{2}-4z(1-z)\Big) - \frac{5(K^2)^4}{3[1\,2]^2\langle 4\,3\rangle^2}z(1-z) + \frac{(K^2)^4}{6[1\,2]^2\langle 4\,3\rangle^2}\\
&\quad + \frac{K^2[1\,3]\langle 1\,3\rangle}{\langle 4\,3\rangle^2[1\,2]^2}\Big(\Big(\frac{1}{6}-\frac{2}{3}z(1-z)\Big)\langle 1\,3\rangle^2[1\,3]^2 + \Big(\frac{1}{2}-3z(1-z)\Big)K^2\langle 1\,3\rangle[1\,3]\Big)\Big\} + \frac{\langle 1\,2\rangle[3\,4]}{K^2}\frac{[4\,3]}{[1\,2]}\\
&= \frac{\langle 1\,2\rangle^2[3\,4]^2}{3s^2t^2}(-4(m^2+\mu^2)s + 6(m^2+\mu^2)t - 2st),
\end{aligned} \tag{6.73}$$

where we used the definitions $K^2 = t$, $\langle 1\,3\rangle[1\,3] = -s-t$.

E. Comparison with the literature

The t-channel cut of $A(1^-,2^-,3^+,4^+)$ admits a decomposition in terms of cuts of master integrals as shown in Fig. 8. Its expression was given in Eq. (5.33) of [29], and reads

$$A_4^{\text{fermion}}(1^-,2^-,3^+,4^+)|_{t-\text{cut}} = -2A_4^{\text{scalar}}(1^-,2^-,3^+,4^+)|_{t-\text{cut}} - \frac{1}{(4\pi)^{2-\epsilon}}A_4^{\text{tree}}(tJ_4 - I_2(t))|_{t-\text{cut}} \tag{6.74}$$

with

$$A_4^{\rm scalar}(1^-,2^-,3^+,4^+)\Big|_{t-{\rm cut}} = \frac{1}{(4\pi)^{2-\epsilon}}A_4^{\rm tree}\left(\frac{1}{t}I_2^{(1,3),D=6-2\epsilon}+\frac{1}{s}J_2^{(1,3)}-\frac{t}{s}K_4\right)\Big|_{t-{\rm cut}} \tag{6.75}$$

and

$$A_4^{\rm tree} = i\frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle} = -i\frac{\langle 1\,2\rangle^2[3\,4]^2}{K^2 s_{12}}, \tag{6.76}$$

where we neglected the cut-free term, I_1 and $I_2(0)$. In standard notation we have $s=s_{12}$, $t=s_{23}$, $u=-s-t=s_{13}$.

Now we translate the expression of [29] into our canonical basis, using the identities of Appendix A.

$$A_4^{\rm fermion}(1^-,2^-,3^+,4^+)\Big|_{t-{\rm cut}} = -2A_4^{\rm scalar}(1^-,2^-,3^+,4^+)\Big|_{t-{\rm cut}} - \frac{1}{(4\pi)^{2-\epsilon}}A_4^{\rm tree}(tJ_4 - I_2(t))\Big|_{t-{\rm cut}} \tag{6.77}$$

$$= -\frac{1}{(4\pi)^{2-\epsilon}}A_4^{\rm tree}\left(\frac{2}{t}I_2^{(1,3),D=6-2\epsilon}+\frac{2}{s}J_2^{(1,3)}-\frac{2t}{s}K_4+tJ_4-I_2(t)\right)\Big|_{t-{\rm cut}} \tag{6.78}$$

$$= -iA_4^{\rm tree}\bigg(\frac{2}{t}\left(\frac{t}{6}I_2^{\rm BFM}[1]+\frac{1}{3}I_1-\frac{2}{3}I_2^{\rm BFM}[m^2+\mu^2]\right)+\frac{2}{s}I_2^{\rm BFM}[m^2+\mu^2]-\frac{2t}{s}I_4^{\rm BFM}[(m^2+\mu^2)^2]$$
$$+tI_4^{\rm BFM}[m^2+\mu^2]-I_2^{\rm BFM}[1]\bigg)\Big|_{t-{\rm cut}} \tag{6.79}$$

$$= \frac{\langle 1\,2\rangle^2[3\,4]^2}{st}\left(\frac{2}{3}I_2^{\rm BFM}[1]+\frac{4}{3t}I_2^{\rm BFM}[m^2+\mu^2]-\frac{2}{s}I_2^{\rm BFM}[m^2+\mu^2]+\frac{2t}{s}I_4^{\rm BFM}[(m^2+\mu^2)^2]-tI_4^{\rm BFM}[m^2+\mu^2]\right)\Big|_{t-{\rm cut}}. \tag{6.80}$$

We have reproduced every one of these coefficients, up to an overall minus sign in the amplitude.

VII. FROM POLYNOMIALS IN *u* TO FINAL COEFFICIENTS

As proven in Appendix B, the coefficients of 2-, 3-, and 4-point functions in four dimensions are polynomials in u (or equivalently μ^2), of known degree d: for boxes, $d=[(n+2)/2]$; for triangles, $d=[(n+1)/2]$; for bubbles, $d=[n/2]$; where $[x]$ denotes the greatest integer less than or equal to x. Using this fact, we can generally represent any coefficient of the master integral as

$$P_d(u) = \sum_{r=0}^{d} c_r u^r. \tag{7.1}$$

The coefficients c_r are in one-to-one correspondence to the coefficients of the *shifted-dimension* master integrals (see Sec. II).

To compute the c_r analytically, one can proceed with the standard differentiations with respect to u, at $u=0$:

$$c_r = \frac{1}{k!}\frac{d^r}{du^r}P_d(u)\Big|_{u=0}. \tag{7.2}$$

When the differentiations are time consuming, or the analytic expression is not needed, one can switch to the following numerical procedure, and extract the c_r algebraically, by *projections*.

(1) Generate the values $P_{d,k}$, $(k=0,\ldots,d-1)$,

$$P_{d,k} = P_d(u_k), \tag{7.3}$$

by evaluating $P_d(u)$ at particular points:

$$u_k = e^{-2\pi ik/d}. \tag{7.4}$$

(2) Using the orthogonality relations for plane waves, one can obtain the coefficient c_r simply by the following formula:

$$c_r = \frac{1}{d}\sum_{k=0}^{d-1} P_{d,k}e^{2\pi irk/d}. \tag{7.5}$$

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APPENDIX A: CHANGE OF BASIS

To compare our results to the literature, we need to convert the master integrals used in [29,30] to our canonical $(4-2\epsilon)$-dimensional basis, (2.1). For clarity, we now denote the basis used in this paper by $I_n^{\rm BFM}$, while the other integrals in this appendix are defined according to [29,30]. The first point is then that

$$I_n = i(-1)^n(4\pi)^{2-\epsilon}I_n^{\rm BFM}. \tag{A1}$$

We use the identities from Appendix A.4 of [29] to perform the conversion.

$$J_4 = I_4[m^2+\mu^2] = i(4\pi)^{2-\epsilon}I_4^{\rm BFM}[m^2+\mu^2], \tag{A2}$$

$$I_2(t) = i(4\pi)^{2-\epsilon}I_2^{\rm BFM}[1], \tag{A3}$$

$$\begin{aligned} I_2^{(1,3),D=6-2\epsilon} &= \frac{t}{6}I_2(t) + \frac{1}{3}I_1 - \frac{2}{3}J_2^{(1,3)} \\ &= i(4\pi)^{2-\epsilon}\Big(\frac{t}{6}I_2^{\rm BFM}[1] + \frac{1}{3}I_1 \\ &\quad - \frac{2}{3}I_2^{\rm BFM}[m^2+\mu^2]\Big), \end{aligned} \tag{A4}$$

$$J_2^{(1,3)} = J_2^{(1,3)}[m^2+\mu^2] = i(4\pi)^{2-\epsilon}I_2^{\rm BFM}[m^2+\mu^2], \tag{A5}$$

$$K_4 = I_4[(m^2+\mu^2)^2] = i(4\pi)^{2-\epsilon}I_4^{\rm BFM}[(m^2+\mu^2)^2]. \tag{A6}$$

APPENDIX B: THE *u*-DEPENDENCE OF THE COEFFICIENTS

Here we analyze the u-dependence of the integral coefficients given by our formulas. First, we prove that they are polynomials in u. Then, we present some alternate formulas where this polynomial dependence is more explicit. This material is a straightforward generalization of the analysis in the massless case [33], so we omit many of the details here.

To begin, we rewrite our vectors R_j, Q_j from (3.23) and (3.24) in the following way:

$$\begin{aligned} R_j &= -\Big((1-2z) + \frac{M_1^2 - M_2^2}{K^2}\Big)p_j + \beta_j K, \\ p_j &\equiv \Big(P_j - \frac{P_j\cdot K}{K^2}K\Big), \end{aligned} \tag{B1}$$

$$\beta_j \equiv -\frac{(P_j\cdot K)}{K^2}\Big(1 + \frac{M_1^2-M_2^2}{K^2}\Big), \tag{B2}$$

$$Q_j = -\Big((1-2z) + \frac{M_1^2-M_2^2}{K^2}\Big)q_j + \alpha_j K, \tag{B3}$$

$$q_j \equiv \Big(K_j - \frac{K_j\cdot K}{K^2}K\Big), \tag{B4}$$

$$\alpha_j \equiv -\frac{(K_j\cdot K)}{K^2}\Big(1+\frac{M_1^2-M_2^2}{K^2}\Big) + \frac{K_j^2 + M_1^2 - m_j^2}{K^2}. \tag{B5}$$

Notice that

$$p_j\cdot K = 0, \qquad q_j \cdot K = 0. \tag{B6}$$

Using (3.16), we have

$$\begin{aligned} R_j(u) &= -\beta(\sqrt{1-u})p_j + \beta_j K, \\ Q_j(u) &= -\beta(\sqrt{1-u})q_j + \alpha_j K. \end{aligned} \tag{B7}$$

This is the same expression as in the massless case, except for the factor β. The point is that now all u-dependence is in the factor $\sqrt{1-u}$, just as in the massless case. In fact, we should now consider the factor $\beta\sqrt{1-u}$ as our basic quantity. The proof that the integral coefficients are polynomials in u was performed by considering the (demonstrably finite) series expansion in $\sqrt{1-u}$, and showing that the odd powers drop out. Therefore, the same arguments now carry over to the series expansion in $\beta\sqrt{1-u}$.

1. Triangle coefficients

Let us begin with triangle coefficients. The null vectors $P_{s,i}$ exhibit a simple dependence on u. Specifically,

$$P_{s,i}(u) = -\beta(\sqrt{1-u})P_{q_s,i},$$

where

$$P_{q_s,i} \equiv q_s \pm \Big(\sqrt{\frac{-q_s^2}{K^2}}\Big)K, \tag{B8}$$

which is manifestly independent of u. In defining the spinor components of $P_{s,i}$, we can place the u-dependent factor inside the antiholomorphic spinor, i.e.,

$$|P_{s,i}\rangle = |P_{q_s,i}\rangle, \qquad |P_{s_i}] = -\beta(\sqrt{1-u})|P_{q_s,i}]. \tag{B9}$$

Then, for the triangle coefficients, we have

$$\begin{aligned} C[Q_s,K] = {}& \frac{(K^2)^{1+n}}{2}\frac{1}{(-\beta\sqrt{1-u})^{n+1}(\sqrt{-4q_s^2K^2})^{n+1}}\frac{1}{(n+1)!\langle P_{q_s,1}P_{q_s,2}\rangle^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}} \\ &\times\Big(\frac{\prod_{j=1}^{k+n}\langle P_{q_s,1}-\tau P_{q_s,2}|R_j(u)Q_s(u)|P_{q_s,1}-\tau P_{q_s,2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{q_s,1}-\tau P_{q_s,2}|Q_t(u)Q_s(u)|P_{q_s,1}-\tau P_{q_s,2}\rangle} + \{P_{q_s,1}\leftrightarrow P_{q_s,2}\}\Big)\Big|_{\tau=0}. \end{aligned} \tag{B10}$$

Further, we make use of some identities,

$$\langle\ell|Q_t(u)Q_s(u)|\ell\rangle = \left\langle\ell|(Q_t(u) - \frac{\alpha_t}{\alpha_s}Q_s(u))Q_s(u)|\ell\right\rangle = -\beta\sqrt{1-u}\left\langle\ell|\left(q_t - \frac{\alpha_t}{\alpha_s}q_s\right)Q_s(u)|\ell\right\rangle\langle\ell|R_t(u)Q_s(u)|\ell\rangle$$
$$= -\beta\sqrt{1-u}\left\langle\ell|\left(p_t - \frac{\beta_t}{\alpha_s}q_s\right)Q_s(u)|\ell\right\rangle$$

along with the definitions

$$\tilde{q}_t \equiv \left(q_t - \frac{\alpha_t}{\alpha_s}q_s\right), \qquad \tilde{p}_t \equiv \left(p_t - \frac{\beta_t}{\alpha_s}q_s\right). \tag{B11}$$

Our final form for the triangle coefficient is

$$C[Q_s, K] = \frac{1}{2}\left(\sqrt{\frac{K^2}{-4q_s^2}}\right)^{n+1}\frac{1}{(n+1)!\langle P_{s,1}P_{s,2}\rangle^{n+1}}\frac{d^{n+1}}{d\tau^{n+1}}\left(\frac{\prod_{j=1}^{k+n}\langle P_{s,1}-\tau P_{s,2}|\tilde{p}_jQ_s(u)|P_{s,1}-\tau P_{s,2}\rangle}{\prod_{t=1,t\neq s}^{k}\langle P_{s,1}-\tau P_{s,2}|\tilde{q}_tQ_s(u)|P_{s,1}-\tau P_{s,2}\rangle} + \{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Bigg|_{\tau=0}. \tag{B12}$$

Here, the u-dependence is concentrated entirely within the vector $Q_s(u)$, since we have made sure to choose the spinor components wisely in (B9), so that the holomorphic spinors are u-independent.

2. Bubble coefficients

We follow the same procedure as with triangles, and make use of the same definitions (B8) and (B9). The u-dependence can be concentrated within the vectors $R_j(u)$ and $Q_r(u)$, $Q_p(u)$, along with the explicit factor of $\sqrt{1-u}$, in the following formulas:

$$C[K] = (K^2)^{1+n}\sum_{q=0}^{n}\frac{1}{q!}\frac{d^q}{ds^q}\left(\mathcal{B}_{n,n-q}^{(0)}(s) + \sum_{r=1}^{k}\sum_{a=q}^{n}(\mathcal{B}_{n,n-a}^{(r;a-q;1)}(s) - \mathcal{B}_{n,n-a}^{(r;a-q;2)}(s))\right)\Bigg|_{s=0}, \tag{B13}$$

where

$$\mathcal{B}_{n,t}^{(0)}(s) \equiv \frac{d^n}{d\tau^n}\left(\frac{1}{n![\eta|\tilde{\eta}K|\eta]^n}\frac{(2\eta\cdot K)^{t+1}}{(t+1)(K^2)^{t+1}}\frac{\prod_{j=1}^{n+k}\langle\ell|R_j(u)(K-s\eta)|\ell\rangle}{\langle\ell\eta\rangle^{n+1}\prod_{p=1}^{k}\langle\ell|Q_p(u)(K-s\eta)|\ell\rangle}\Bigg|_{|\ell\rangle\to|K-\tau\tilde{\eta}|\eta]}\right)\Bigg|_{\tau=0}, \tag{B14}$$

$$\mathcal{B}_{n,t}^{(r;b;1)}(s) \equiv \frac{(-1)^{b+1}}{b!(-\beta\sqrt{1-u})^{b+1}\sqrt{-4q_r^2K^2}\langle P_{r,1}P_{r,2}\rangle^b b+1}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,1}-\tau P_{r,2}|\eta|P_{q_r,1}]^{t+1}}{\langle P_{r,1}-\tau P_{r,2}|K|P_{q_r,1}]^{t+1}}\right.$$
$$\left.\times\frac{\langle P_{r,1}-\tau P_{r,2}|Q_r(u)\eta|P_{r,1}-\tau P_{r,2}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,1}-\tau P_{r,2}|R_j(u)(K-s\eta)|P_{r,1}-\tau P_{r,2}\rangle}{\langle P_{r,1}-\tau P_{r,2}|\eta K|P_{r,1}-\tau P_{r,2}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,1}-\tau P_{r,2}|Q_p(u)(K-s\eta)|P_{r,1}-\tau P_{r,2}\rangle}\right)\Bigg|_{\tau=0}, \tag{B15}$$

$$\mathcal{B}_{n,t}^{(r;b;2)}(s) \equiv \frac{(-1)^{b+1}}{b!(-\beta\sqrt{1-u})^{b+1}\sqrt{-4q_r^2K^2}^{b+1}\langle P_{r,1}P_{r,2}\rangle^b}\frac{d^b}{d\tau^b}\left(\frac{1}{(t+1)}\frac{\langle P_{r,2}-\tau P_{r,1}|\eta|P_{q_r,2}]^{t+1}}{\langle P_{r,2}-\tau P_{r,1}|K|P_{q_r,2}]^{t+1}}\right.$$
$$\left.\times\frac{\langle P_{r,2}-\tau P_{r,1}|Q_r(u)\eta|P_{r,2}-\tau P_{r,1}\rangle^b\prod_{j=1}^{n+k}\langle P_{r,2}-\tau P_{r,1}|R_j(u)(K-s\eta)|P_{r,2}-\tau P_{r,1}\rangle}{\langle P_{r,2}-\tau P_{r,1}|\eta K|P_{r,2}-\tau P_{r,1}\rangle^{n+1}\prod_{p=1,p\neq r}^{k}\langle P_{r,2}-\tau P_{r,1}|Q_p(u)(K-s\eta)|P_{r,2}-\tau P_{r,1}\rangle}\right)\Bigg|_{\tau=0}. \tag{B16}$$

3. Box and pentagon coefficients

Although the formula (4.1) for box and pentagon coefficients looks simple, the u-dependence now gets complicated. We consider the separate cases $k=2$, $k=3$, and $k\geq 4$.

a. *The case* $k = 2$

In this case, there is only one box, and no pentagons. The box coefficient is given by

$$\frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{j=1}^{n+2}\langle P_{(Q_j(u),Q_i(u));1}(u)|R_j(u)|P_{(Q_j(u),Q_i(u));2}(u)]}{\langle P_{(Q_j(u),Q_i(u));1}(u)|K|P_{(Q_j(u),Q_i(u));2}(u)]^{n+2}} + \{P_{(Q_j(u),Q_i(u));1}(u)\leftrightarrow P_{(Q_j(u),Q_i(u));2}(u)\}\right). \tag{B17}$$

Given the vectors Q_i, Q_j, K that select a particular box, it is useful to construct a vector $q_0^{(q_i,q_j,K)}$ that is orthogonal to all three, and independent of u:

$$(q_0)_\mu^{(q_i,q_j,K)} \equiv \frac{1}{K^2}\epsilon_{\mu\nu\rho\xi}q_i^\nu q_j^\rho K^\xi \tag{B18}$$

$$= \frac{1}{K^2}\epsilon_{\mu\nu\rho\xi}K_i^\nu K_j^\rho K^\xi. \tag{B19}$$

As in the massless case, the u-dependence can be concentrated in a single factor, $\alpha^{(q_i,q_j)}(u)$. If all input quantities are set to their values with $u = 0$, except for adjusting the definition of $R_s(u)$ as follows:

$$R_s(u) \to \tilde{R}_s(u) \equiv \frac{p_s \cdot q_0^{(q_i,q_j,K)}}{(q_0^{(q_i,q_j,K)})^2}(\alpha^{(q_i,q_j)}(u) - 1) \times(-\beta q_0^{(q_i,q_j,K)}) + R_s(u=0), \tag{B20}$$

$$\alpha^{(q_i,q_j)}(u) \equiv \frac{\sqrt{\beta^2(1-u) + \frac{4K^2[\alpha_i\alpha_j(2q_i\cdot q_j) - \alpha_i^2 q_j^2 - \alpha_j^2 q_i^2]}{(2q_i\cdot q_j)^2 - 4q_i^2 q_j^2}}}{\sqrt{\beta^2 + \frac{4K^2[\alpha_i\alpha_j(2q_i\cdot q_j) - \alpha_i^2 q_j^2 - \alpha_j^2 q_i^2]}{(2q_i\cdot q_j)^2 - 4q_i^2 q_j^2}}}, \tag{B21}$$

then the value of the box coefficient remains the same.

In summary, the box coefficient for $k = 2$ is given by

$$C[K_i, K_j]_{k=2} = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{s=1}^{n+2}\langle P_{ji;1}|\tilde{R}_s(u)|P_{ji;2}]}{\langle P_{ji;1}|K|P_{ji;2}]^{n+2}} + \{P_{ji;1} \leftrightarrow P_{ji;2}\}\right), \tag{B22}$$

where

$$\tilde{R}_s(u) = \frac{p_s \cdot q_0^{(q_i,q_j,K)}}{(q_0^{(q_i,q_j,K)})^2}(\alpha^{(q_i,q_j)}(u) - 1)(-\beta q_0^{(q_i,q_j,K)}) + R_s(u=0) \tag{B23}$$

and

$$P_{ji;a} = P_{(Q_j,Q_i);a}(u=0). \tag{B24}$$

In evaluating (B23), it is useful to observe the following:

$$(p_s \cdot q_0^{(q_i,q_j,K)}) = \frac{\epsilon(p_s, q_i, q_j, K)}{K^2} = \frac{\epsilon(P_s, K_i, K_j, K)}{K^2}. \tag{B25}$$

This formula (B22) looks the same as in the massless case; the difference is the appearance of β in (B23), both explicitly and through the definition (B21) of $\alpha^{(q_i,q_j)}$.

b. The case $k = 3$

Here there is a pentagon, as well as three boxes. The differences from the massless case are all based in the definitions of $R_j(u)$, $Q_j(u)$: there is always a factor of β accompanying $\sqrt{1-u}$, and mass parameters enter into the definitions (B2) and (B5) of β_j, α_j.

When we make these adjustments, we find that the pentagon coefficient takes the same form as in the massless case,

$$C[Q_i, Q_j, Q_t] = (K^2)^{3+n}\prod_{s=1}^{n+3}\beta_s^{(q_i,q_j,q_t;p_s)} \tag{B26}$$

but the definition of $\beta_s^{(q_i,q_j,q_t;p_s)}$ now includes mass parameters:

$$\beta_s^{(q_i,q_j,q_t;p_s)} \equiv \beta_s^{(K_i,K_j,K_t;P_s)} = -\frac{(K_i^2 + M_1^2 - m_i^2)\epsilon(P_s, K_j, K, K_t) + (K_j^2 + M_1^2 - m_j^2)\epsilon(K_i, P_s, K, K_t)}{K^2\epsilon(K_i, K_j, K, K_t)} - \frac{(K^2 + M_1^2 - M_2^2)\epsilon(K_i, K_j, P_s, K_t) + (K_t^2 + M_1^2 - m_t^2)\epsilon(K_i, K_j, K, P_s)}{K^2\epsilon(K_i, K_j, K, K_t)}. \tag{B27}$$

The expression (B27) is symmetric in K_i, K_j, K_t, K(recall that M_2 is the mass associated with K in this context).

The box coefficients are given by

$$C[Q_i, Q_j]_{k=3} = \frac{(K^2)^{2+n}}{2}\left(\frac{\prod_{s=1}^{n+3}\langle P_{ji;1}|\tilde{R}_s(u)|P_{ji;2}]}{\langle P_{ji;1}|K|P_{ji;2}]^{n+2}\langle P_{ji;1}|\tilde{Q}_t(u)|P_{ji;2}]} - <\prod_{s=1}^{n+3}\beta_s^{(q_i,q_j,q_t;p_s)}\frac{\langle P_{ji;1}|K|P_{ji;2}]}{\langle P_{ji;1}|\tilde{Q}_t(u)|P_{ji;2}]} + \{P_{ji;1} \leftrightarrow P_{ji;2}\}\right). \tag{B28}$$

The derivation of (B28) involved the result from the case $k = 2$. All mass dependence is already included in the definitions (B21) and (B23), along with the similarly defined vector $Q_t(u)$:

$$\tilde{R}_s(u) = \frac{p_s \cdot q_0^{(q_i,q_j,K)}}{(q_0^{(q_i,q_j,K)})^2}(\alpha^{(q_i,q_j)}(u) - 1)(-\beta q_0^{(q_i,q_j,K)}) + R_s(u = 0), \tag{B29}$$

$$\tilde{Q}_t(u) = \frac{q_t \cdot q_0^{(q_i,q_j,K)}}{(q_0^{(q_i,q_j,K)})^2}(\alpha^{(q_i,q_j)}(u) - 1)(-\beta q_0^{(q_i,q_j,K)}) + Q_t(u = 0). \tag{B30}$$

c. The case $k \geq 4$

In the derivation of the formulas, we introduce the following functions:

$$\gamma_s^{(K_i,K_j,K_s,K_t)} = \frac{(K_i^2 + M_1^2 - m_i^2)\epsilon(K, K_j, K_s, K_t) + (K_j^2 + M_1^2 - m_j^2)\epsilon(K_i, K, K_s, K_t)}{K^2\epsilon(K_i, K_j, K, K_t)} + \frac{(K_s^2 + M_1^2 - m_s^2)\epsilon(K_i, K_j, K, K_t) + (K_t^2 + M_1^2 - m_t^2)\epsilon(K_i, K_j, K_s, K)}{K^2\epsilon(K_i, K_j, K, K_t)} - \frac{\epsilon(K_i, K_j, K_s, K_t)}{\epsilon(K_i, K_j, K, K_t)}. \tag{B31}$$

The numerator of $\gamma_s^{(K_i,K_j,K_s,K_t)}$ is symmetric in K_i, K_j, K_s, K_t; the denominator breaks this symmetry by singling out K_s.

We find the following results:

The pentagon coefficients are given by

$$C[Q_i, Q_j, Q_t] = (K^2)^{3+n}\frac{\prod_{s=1}^{n+k}\beta_s^{(q_i,q_j,q_t;p_s)}}{\prod_{w=1,w\neq i,j,t}^{k}\gamma_w^{(K_i,K_j,K_w,K_t)}}. \tag{B32}$$

The box coefficients are given by

$$C[Q_i, Q_j]_{k\geq 4} = \frac{(K^2)^{2+n}}{2}\left\{\frac{\prod_{s=1}^{k+n}\langle P_{ji;1}|\tilde{R}_s(u)|P_{ji;2}]}{\langle P_{ji;1}|K|P_{ji;2}]^{n+2}\prod_{t=1,t\neq i,j}^{k}\langle P_{ji;1}|\tilde{Q}_t(u)|P_{ji;2}]} - \sum_{t=1,t\neq i,j}^{k}\frac{\prod_{s=1}^{n+k}\beta_s^{(q_i,q_j,q_t;p_s)}}{\prod_{w=1,w\neq i,j,t}^{k}\gamma_w^{(K_i,K_j;K_w,K_t)}}\frac{\langle P_{ji;1}|K|P_{ji;2}]}{\langle P_{ji;1}|\tilde{Q}_t(u)|P_{ji;2}]}\right\} + \{P_{ji;1} \leftrightarrow P_{ji;2}\}. \tag{B33}$$

Again, all the u-dependence is concentrated in $\tilde{R}(u)$ and $\tilde{Q}(u)$. The definitions of $P_{ji;a}$, $\tilde{R}_s(u)$, $\tilde{Q}_t(u)$, $\beta_s^{(q_i,q_j,q_t;p_s)}$, and $\gamma_w^{(K_i,K_j;K_w,K_t)}$ are given in (B24), (B29), (B30), (B27), and (B31), respectively.

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