N=4 mechanics, WDVV equations and roots

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ABSTRACT: $\mathcal{N}{=}4$ superconformal multi-particle quantum mechanics on the real line is governed by two prepotentials, U and F, which obey a system of partial differential equations linear in U and generalizing the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation for F. Putting $U{\equiv}0$ yields a class of models (with zero central charge) which are encoded by the finite Coxeter root systems. We extend these WDVV solutions F in two ways: the A_n system is deformed n-parametrically to the edge set of a general orthocentric n-simplex, and the BCF-type systems form one-parameter families. A classification strategy is proposed. A nonzero central charge requires turning on U in a given F background, which we show is outside the reach of the standard root-system ansatz for indecomposable systems of more than three particles. In the three-body case, however, this ansatz can be generalized to establish a series of nontrivial models based on the dihedral groups $I_2(p)$, which are permutation symmetric if 3 divides p. We explicitly present their full prepotentials.

Contents

1 Introduction

It has been known for a long time that integrable quantum systems are intimately related to Lie algebras (see, for instance, [1]). Therefore, it is natural to expect their appearance also in supersymmetric extensions of integrable multi-particle quantum mechanics models. In this paper, we revisit such systems with $\mathcal{N}{=}4$ *superconformal* symmetry in one space dimension and, within a canonical ansatz, investigate them for the superconformal algebra $su(1,1|2)$ with central charge C. Despite physical interest in these models [2], their explicit construction has remained an open problem until now.

$\mathcal{N}{=}4$ superconformal many-body quantum systems on the real line are very rigid. Their existence is governed by a system of nonlinear partial differential equations for two prepotentials, U and F, for which few solutions are known when $C{\neq}0$ [3–6]. The determination of F is decoupled from U and requires solving 'only' the well-known (generalized) Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [7, 8], which arises in topological and Seiberg-Witten field theory. The WDVV solutions known so far are all based — again — on the root systems of simple Lie algebras [9, 10].[1]

If $C{=}0$ any WDVV solution F, together with $U{\equiv}0$, will provide a valid multi-particle quantum model. For nonzero central charge, however, one is to solve a second partial

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[1] As a slight generalization, all Coxeter reflection groups appear.

differential equation for U in the presence of F. To this so-called 'flatness conditon' only particular solutions for at most four particles are in the literature [3, 6].

All considerations up to now have employed a natural ansatz for F and U in terms of a set $\{\alpha\}$ of covectors. We find, however, that for systems of less than four particles this ansatz must be generalized in order to capture all solutions. In these cases, the WDVV equation is trivially satisfied, and we can (and do) construct new three-body models for any dihedral $I_2(p)$ root system, starting with a Calogero-type A_2 model. For more than three particles, where the WDVV equation is effective, we show that even our generalized ansatz is insufficient to produce irreducible $U{\neq}0$ solutions in the root-system context. A model is reducible if, after removing the center-of-mass degree of freedom, it can be decomposed into decoupled subsystems. As for the WDVV equation alone, we generalize the solutions of [9, 10] and give a geometric interpretation of certain A_n deformations [11] in terms of orthocentric simplices.

The paper is organized as follows. In section 2 we recall the formulation of conformal mechanics of $n{+}1$ identical particles on the real line in terms of $so(1,2)$ generators including the Hamiltonian. In this description, an $\mathcal{N}{=}4$ supersymmetric extension with central charge C is straightforward to construct as we demonstrate in section 3. The closure of the superconformal algebra poses constraints on the interaction, which in section 4 lead to what we call the 'structure equations' on the prepotentials U and F. The analysis of these structure equations in section 5 suggests constructing the prepotentials in terms of a system of covectors, which reduces the differential equations to nonlinear algebraic equations. Sections 6 and 7 derive families of F solutions with $U{\equiv}0$, based on certain deformations of the root systems of the finite reflection groups. Turning on U for these F backgrounds is analyzed in sections 8 and 9, with negative results for more than three particles, but with a positive classification and the full construction of the prepotentials for three particles via the dihedral groups $I_2(p)$, including five explicit examples. Section 10 concludes.

2   Conformal quantum mechanics

Let us consider a system of $n{+}1$ identical particles with unit mass, moving on the real line according to a Hamiltonian of the generic form ($I = 1, \ldots, n{+}1$)

$$H \;=\; \frac{1}{2} p_I p_I \;+\; V_B(x^1, \ldots, x^{n+1}) \,. \tag{2.1}$$

Throughout the paper a summation over repeated indices is understood. After separating the center-of-mass motion we will work with the n degrees of freedom of relative particle motion in later sections. Also, the bosonic potential V_B will get supersymmetrically extended to a potential V including V_B.

For conformally invariant models the Hamiltonian H is a part of the $so(1,2)$ conformal algebra

$$[D, H] \;=\; -\mathrm{i}H \,, \quad [H, K] \;=\; 2\mathrm{i}D \,, \quad [D, K] \;=\; \mathrm{i}K \,, \tag{2.2}$$

where D and K are the dilatation and conformal boost generators, respectively. Their realization in term of coordinates and momenta, subject to

$$[x^I, p_J] = \mathrm{i}\delta_J{}^I, \tag{2.3}$$

reads

$$D = -\frac{1}{4}(x^I p_I + p_I x^I) \quad \text{and} \quad K = \frac{1}{2}x^I x^I. \tag{2.4}$$

The first relation in (2.2) restricts the potential via

$$(x^I \partial_I + 2) V_B = 0, \tag{2.5}$$

meaning that V_B must be homogeneous of degree -2 for the model to be conformally invariant. Imposing translation and permutation invariance and allowing only two-body interactions, we arrive at the Calogero model of $n+1$ particles interacting through an inverse-square pair potential,

$$V_B = \sum_{I<J} \frac{g^2}{(x^I - x^J)^2} \quad \longrightarrow \quad H = H_0 + V_B. \tag{2.6}$$

3   N=4 superconformal extension

Let us extend the bosonic conformal mechanics of the previous section to an $\mathcal{N}=4$ superconformal one,[2] with a single central extension [13]. The bosonic sector of the $\mathcal{N}=4$ superconformal algebra $su(1,1|2)$ includes two subalgebras. Along with $so(1,2)$ considered in the previous section one also finds the $su(2)$ R-symmetry subalgebra generated by J_a with $a=1,2,3$. The fermionic sector is exhausted by the SU(2) doublet supersymmetry generators Q_α and $\bar{Q}^\alpha$ as well as their superconformal partners S_α and $\bar{S}^\alpha$, with $\alpha=1,2$, subject to the hermiticity relations

$$(Q_\alpha)^\dagger = \bar{Q}^\alpha \quad \text{and} \quad (S_\alpha)^\dagger = \bar{S}^\alpha. \tag{3.1}$$

The bosonic generators are hermitian. The non-vanishing (anti)commutation relations in our superconformal algebra read[3]

$$
\begin{aligned}
[D,H] &= -\mathrm{i}\,H, & [H,K] &= 2\mathrm{i}\,D, \\
[D,K] &= +\mathrm{i}\,K, & [J_a,J_b] &= \mathrm{i}\,\epsilon_{abc}J_c, \\
\{Q_\alpha,\bar{Q}^\beta\} &= 2H\delta_\alpha{}^\beta, & \{Q_\alpha,\bar{S}^\beta\} &= +2\mathrm{i}\,(\sigma_a)_\alpha{}^\beta J_a - 2D\delta_\alpha{}^\beta - \mathrm{i}\,C\delta_\alpha{}^\beta, \\
\{S_\alpha,\bar{S}^\beta\} &= 2K\delta_\alpha{}^\beta, & \{\bar{Q}^\alpha,S_\beta\} &= -2\mathrm{i}\,(\sigma_a)_\beta{}^\alpha J_a - 2D\delta_\beta{}^\alpha + \mathrm{i}\,C\delta_\beta{}^\alpha, \\
[D,Q_\alpha] &= -\frac{1}{2}\mathrm{i}\,Q_\alpha, & [D,S_\alpha] &= +\frac{1}{2}\mathrm{i}\,S_\alpha, \\
[K,Q_\alpha] &= +\mathrm{i}\,S_\alpha, & [H,S_\alpha] &= -\mathrm{i}\,Q_\alpha, \\
[J_a,Q_\alpha] &= -\frac{1}{2}(\sigma_a)_\alpha{}^\beta Q_\beta, & [J_a,S_\alpha] &= -\frac{1}{2}(\sigma_a)_\alpha{}^\beta S_\beta,
\end{aligned}
$$

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[2] For a one-particle model, see [12].

[3] σ_1, σ_2 and σ_3 denote the Pauli matrices.

$$\begin{aligned}
[D, \bar{Q}^\alpha] &= -\tfrac{1}{2}\mathrm{i}\,\bar{Q}^\alpha\,, & [D, \bar{S}^\alpha] &= +\tfrac{1}{2}\mathrm{i}\,\bar{S}^\alpha\,, \\
[K, \bar{Q}^\alpha] &= +\mathrm{i}\,\bar{S}^\alpha\,, & [H, \bar{S}^\alpha] &= -\mathrm{i}\,\bar{Q}^\alpha\,, \\
[J_a, \bar{Q}^\alpha] &= \tfrac{1}{2}\,\bar{Q}^\beta (\sigma_a)_\beta{}^\alpha\,, & [J_a, \bar{S}^\alpha] &= \tfrac{1}{2}\,\bar{S}^\beta (\sigma_a)_\beta{}^\alpha\,.
\end{aligned} \tag{3.2}$$

Here $\epsilon_{123} = 1$, and C stands for the central charge.

For a mechanical realization of the $su(1,1|2)$ superalgebra, one introduces fermionic degrees of freedom represented by the operators ψ_α^I and $\bar{\psi}^{I\alpha}$, with $I = 1, \ldots, n+1$ and $\alpha = 1, 2$, which are hermitian conjugates of each other and obey the anti-commutation relations[4]

$$\{\psi_\alpha^I, \psi_\beta^J\} = 0\,, \qquad \{\bar{\psi}^{I\alpha}, \bar{\psi}^{J\beta}\} = 0\,, \qquad \{\psi_\alpha^I, \bar{\psi}^{J\beta}\} = \delta_\alpha{}^\beta \delta^{IJ}\,. \tag{3.3}$$

In the extended space it is easy to construct the free fermionic generators associated with the free Hamiltonian $H_0 = \tfrac{1}{2} p_I p_I$, namely

$$Q_{0\alpha} = p_I \psi_\alpha^I\,, \qquad \bar{Q}_0^\alpha = p_I \bar{\psi}^{I\alpha} \qquad \text{and} \qquad S_{0\alpha} = x^I \psi_\alpha^I\,, \qquad \bar{S}_0^\alpha = x^I \bar{\psi}^{I\alpha}\,, \tag{3.4}$$

as well as $su(2)$ generators

$$J_{0a} = \frac{1}{2} \bar{\psi}^{I\alpha} (\sigma_a)_\alpha{}^\beta \psi_\beta^I\,. \tag{3.5}$$

Notice that these are automatically Weyl-ordered. The free dilatation and conformal boost operators maintain their bosonic form

$$D_0 = -\frac{1}{4}(x^I p_I + p_I x^I) \qquad \text{and} \qquad K_0 = \frac{1}{2} x^I x^I\,. \tag{3.6}$$

In contrast to the $\mathcal{N} \leq 2$ cases, the free generators fail to satisfy the full algebra (3.2). Even for $C=0$, the $\{Q, \bar{S}\}$ and $\{\bar{Q}, S\}$ anticommutators require corrections to the fermionic generators, which are cubic in the fermions and can be restricted to Q and $\bar{Q}$ via

$$Q_\alpha = Q_{0\alpha} - \mathrm{i}\,[S_{0\alpha}, V] \qquad \text{and} \qquad \bar{Q}^\alpha = \bar{Q}_0^\alpha - \mathrm{i}\,[\bar{S}_0^\alpha, V] \qquad \text{where} \quad H = H_0 + V \tag{3.7}$$

and $V \neq 0$. Hence, there does not exist a free mechanical representation of the algebra (3.2). It follows further that V contains terms quadratic and quartic in the fermions, thus can be written as [3, 5, 6][5]

$$V = V_B(x) - U_{IJ}(x) \langle \psi_\alpha^I \bar{\psi}^{J\alpha} \rangle + \frac{1}{4} F_{IJKL}(x) \langle \psi_\alpha^I \psi^{J\alpha} \bar{\psi}^{K\beta} \bar{\psi}_\beta^L \rangle\,, \tag{3.8}$$

with completely symmetric unknown functions U_{IJ} and F_{IJKL} homogeneous of degree -2 in $x \equiv \{x^1, \ldots, x^{n+1}\}$. Here, the symbol $\langle \ldots \rangle$ stands for symmetric (or Weyl) ordering. The ordering ambiguity present in the fermionic sector affects the bosonic potential V_B. In contrast to the $\mathcal{N}=2$ superconformal extensions [14, 15], the quartic term is needed, and so we get

$$\begin{aligned}
Q_\alpha &= \left(p_J - \mathrm{i}\, x^I U_{IJ}(x) \right) \psi_\alpha^J - \frac{\mathrm{i}}{2} x^I F_{IJKL}(x) \langle \psi_\beta^J \psi^{K\beta} \bar{\psi}_\alpha^L \rangle\,, \\
\bar{Q}^\alpha &= \left(p_J + \mathrm{i}\, x^I U_{IJ}(x) \right) \bar{\psi}^{J\alpha} - \frac{\mathrm{i}}{2} x^I F_{IJKL}(x) \langle \psi^{J\alpha} \bar{\psi}^{K\beta} \bar{\psi}_\beta^L \rangle\,.
\end{aligned} \tag{3.9}$$

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[4] Spinor indices are raised and lowered with the invariant tensor $\epsilon^{\alpha\beta}$ and its inverse $\epsilon_{\alpha\beta}$, where $\epsilon^{12} = 1$.

[5] The classical consideration in [5] implies that (3.8) is indeed the most general quartic ansatz compatible with the $\mathcal{N}=4$ superconformal algebra.

To summarize, in order to close the algebra (3.2), the D, K, J_a, S_α and $\bar{S}^\alpha$ generators remain free, while Q_α and $\bar{Q}^\alpha$ as well as H acquire corrections as above.

4 The structure equations

Inserting the form (3.4)–(3.8) into the algebra (3.2), one produces a fairly long list of constraints on the potential V. One of the consequences is that [3, 5, 6]

$$U_{IJ} = \partial_I \partial_J U \qquad \text{and} \qquad F_{IJKL} = \partial_I \partial_J \partial_K \partial_L F, \qquad (4.1)$$

which introduces two scalar prepotentials. The constraints then turn into the following system of nonlinear partial differential equations [5, 6],

$$(\partial_I \partial_K \partial_P F)(\partial_J \partial_L \partial_P F) = (\partial_J \partial_K \partial_P F)(\partial_I \partial_L \partial_P F), \quad x^I \partial_I \partial_J \partial_K F = -\delta_{JK}, \quad (4.2)$$
$$\partial_I \partial_J U - (\partial_I \partial_J \partial_K F) \partial_K U = 0, \qquad\qquad\qquad x^I \partial_I U = -C, \qquad (4.3)$$

which we refer to as the 'structure equations'.[6] Notice that these equations are quadratic in F but only linear in U. They are invariant under $SO(n{+}1)$ coordinate transformations. The first of (4.2) is a kind of zero-curvature condition for a connection $\partial^3 F$. It coincides with the (generalized) WDVV equation known from topological field theory [7, 8]. The first of (4.3) is a kind of covariant constancy for ∂U in the $\partial^3 F$ background. Since its integrability implies the WDVV equation projected onto ∂U, we call it the 'flatness condition'.

The right equations in (4.2) and (4.3) represent homogeneity conditions for U and F. They are inhomogeneous with constants δ_{JK} and C (the central charge) on the right-hand side and display an explicit coordinate dependence. Furthermore, the second equation in (4.2) can be integrated twice to obtain

$$x^I \partial_I F - 2F + \frac{1}{2} x^I x^I = 0, \qquad (4.4)$$

where we used the freedom in the definition of F to put the integration constants — a linear function on the right-hand side — to zero. It is important to realize that the inhomogeneous term in this integrated equation excludes the trivial solution $F = 0$ equivalent to a homogeneous quadratic polynomial. This effect is absent in $\mathcal{N}{=}2$ superconformal models, where the four-fermion potential term is not required and, hence, F need not appear [15]. This issue is also discussed in [3].

To simplify the analysis of the structure equations, it is convenient to separate the center-of-mass and the relative motion of the particles. This is achieved by a rotation of the coordinate frame,

$$\{ x^I \} \longrightarrow \{ x^i, X \} \qquad \text{with} \qquad i = 1, \ldots, n \qquad \text{and} \qquad X = \frac{1}{\sqrt{n+1}} \sum_{I=1}^{n+1} x^I, \ (4.5)$$

which introduces relative-motion coordinates x^i for the hyperplane orthogonal to the center-of-mass direction. The structure equations then hold for both sets of coordinates independently, with an accompanying split of the prepotentials and the central charge,

$$F = F_{\text{com}}(X) + F_{\text{rel}}(x), \qquad U = U_{\text{com}}(X) + U_{\text{rel}}(x) \qquad \text{and} \qquad C = C_{\text{com}} + C_{\text{rel}}, \ (4.6)$$

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[6] Wyllard [3] obtained equivalent equations, but employed a different fermionic ordering.

where now $x \equiv \{x^i\}$. For the center-of-mass coordinate, the solution is trivial:

$$F_{\rm com} \;=\; -\frac{1}{2}\, X^2 \ln|X| \qquad \text{and} \qquad U_{\rm com} \;=\; -C_{\rm com} \ln|X| \;. \qquad (4.7)$$

For the relative coordinates, we simply replace $I, J, \dots$ by $i, j, \dots$ and $C \to C_{\rm rel}$ in the structure equations. In the following, we shall investigate the construction of $F_{\rm rel}$ and $U_{\rm rel}$ only and therefore drop the label 'rel' from now on. However, since these coordinates often obscure a permutation invariance for identical particles, it can be useful to go back to the original x^I by embedding $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as the hyperplane orthogonal to the vector $\rho = \frac{1}{\sqrt{n+1}}(1, 1, \dots, 1)$ for achieving a manifestly permutation-symmetric description of the $(n{+}1)$-particle system. Furthermore, the center-of-mass case is still covered in our analysis by just taking $n{=}1$.

There are some dependencies among the equations (4.2) and (4.3), now reduced to the relative coordinates. The contraction of two left equations with x^i is a consequence of the two right equations, and therefore only the components orthogonal to x are independent, effectively reducing the dimension to $n{-}1$. This means that only $\frac{1}{12}n(n{-}1)^2(n{-}2)$ WDVV equations need to be solved and only $\frac{1}{2}n(n{-}1)$ flatness conditions have to be checked. For $n{=}2$ in particular, the single WDVV equation follows from the homogeneity condition in (4.2), and the three flatness conditions are all equivalent. Hence, the nonlinearity of the structure equations becomes relevant only for $n{\geq}3$.

The scalars U and F govern the $\mathcal{N}{=}4$ superconformal extension. Note, however, that F is defined modulo a quadratic polynomial while U is defined up to a constant. Together, they determine V_B as[7]

$$V_B \;=\; \frac{1}{2}\,(\partial_i U)(\partial_i U) \;+\; \frac{\hbar^2}{8}\,(\partial_i\partial_j\partial_k F)(\partial_i\partial_j\partial_k F) \;. \qquad (4.8)$$

We note that $U{\equiv}0$ still yields nontrivial quantum models, whose potential only vanishes classically. Finally, from the two right equations in (4.2) and (4.3) it follows that

$$x^i\, F_{ijkl} \;=\; -\partial_j\partial_k\partial_l F \qquad \text{and} \qquad x^i\, U_{ij} \;=\; -\partial_j U \;, \qquad (4.9)$$

which is relevant for (3.9).

5 Prepotential ansatz and consequences

Our attack on (4.2) and (4.3) begins with the homogeneity conditions

$$(x^i\partial_i - 2)F \;=\; -\frac{1}{2}\, x^i x^i \qquad \text{and} \qquad x^i\partial_i U \;=\; -C \;. \qquad (5.1)$$

The general solution to (5.1) may be written as

$$F = -\frac{1}{4}\sum_s f_s\, Q_s(x)\, \ln|Q_s(x)| + F_{\rm hom} \qquad \text{and} \qquad U = -\frac{1}{2}\sum_s g_s\, \ln|Q_s(x)| + U_{\rm hom} \quad (5.2)$$

---

[7] We have restored $\hbar$ in the potential to illustrate that the F contribution disappears classically.

with quadratic forms $Q_s(x)$, real coefficients f_s and g_s, as well as homogeneous functions F_{hom} and U_{hom} of degree two and zero, respectively. The conditions (5.1) are obeyed if

$$Q_s(x) = \sum_{i,j} q_{ij}^s \, x^i x^j \qquad \text{satisfies} \qquad \sum_s f_s \, Q_s(x) = x^i x^i \quad \text{and} \quad \sum_s g_s = C \, . \qquad (5.3)$$

Unfortunately, it is hard to analyze the WDVV equation (4.2) and the flatness condition (4.3) in this generality. Therefore, we take the simplifying ansatz that the quadratic forms are either of rank one or proportional to the identity form,[8]

$$Q_\alpha(x) \;=\; \alpha_i \alpha_j \, x^i x^j \;=:\; (\alpha{\cdot}x)^2 \qquad \text{and} \qquad Q_R(x) \;=\; x^i x^i \;=:\; R^2 \, , \qquad (5.4)$$

which defines a set $\{\alpha\}$ of p covectors

$$\alpha \;=\; (\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad \text{with values} \qquad \alpha(x) \;=\; \alpha{\cdot}x \;=\; \alpha_i x^i \, . \qquad (5.5)$$

Replacing the label 's' by the covector name 'α' or by 'R', the prepotentials (5.2) read

$$\begin{aligned}
F &\;=\; -\frac{1}{2} \sum_\alpha f_\alpha \, (\alpha{\cdot}x)^2 \, \ln|\alpha{\cdot}x| \;-\; \frac{1}{2} f_R \, R^2 \ln R \;+\; F_{\text{hom}}(x) \, , \\
U &\;=\; -\sum_\alpha g_\alpha \, \ln|\alpha{\cdot}x| \;-\; g_R \ln R \;+\; U_{\text{hom}}(x) \, .
\end{aligned} \qquad (5.6)$$

The covector part of this ansatz is well known [3, 6, 9, 10], but the 'radial' terms (labelled 'R') are new and will be important for admitting nontrivial solutions U.

The expressions above are invariant under individual sign flips $\alpha \to -\alpha$ for each covector, and so we exclude $-\alpha$ from the set. For identical particles our relative configuration space carries an n-dimensional representation of the permutation group $\mathcal{S}_{n+1}$, whose action must leave the set $\{\pm\alpha\}$ invariant. Furthermore, the f_α and g_α couplings have to be constant along each $\mathcal{S}_{n+1}$ orbit. Finally, a rescaling of $\alpha{\cdot}x$ may be absorbed into a renormalization of f_α. Therefore, only the rays $\mathbb{R}_+\alpha$ are invariant data. We cannot, however, change the sign of f_α in this manner.

Compatibility of (5.6) with the conditions (5.1) directly yields

$$\sum_\alpha f_\alpha \, \alpha_i \alpha_j \;+\; f_R \, \delta_{ij} \;=\; \delta_{ij} \qquad \text{and} \qquad \sum_\alpha g_\alpha \;+\; g_R \;=\; C \, . \qquad (5.7)$$

The second relation fixes the central charge, and the g_α are independent free couplings if not forced to zero. The first relation amounts to a decomposition of $(1-f_R)\delta_{ij}$ into (usually non-orthogonal) rank-one projectors and imposes $\frac{1}{2}n(n+1)$ relations on the coefficients $\{f_\alpha, f_R\}$ for a given set $\{\alpha\}$.

All known solutions to the WDVV equations can be cast into the form (5.6) with $F_{\text{hom}} \equiv 0$, so from now on we drop this term. From (5.6) we then derive

$$\begin{aligned}
\partial_i \partial_j \partial_k F &\;=\; -\sum_\alpha f_\alpha \, \frac{\alpha_i \alpha_j \alpha_k}{\alpha{\cdot}x} \;-\; f_R \left\{ \frac{x_i \delta_{jk} + x_j \delta_{ki} + x_k \delta_{ij}}{R^2} - 2\frac{x_i x_j x_k}{R^4} \right\} , \\
\partial_i U &\;=\; -\sum_\alpha g_\alpha \, \frac{\alpha_i}{\alpha{\cdot}x} \;-\; g_R \, \frac{x_i}{R^2} \;+\; \partial_i U_{\text{hom}} \, ,
\end{aligned} \qquad (5.8)$$

---

[8] Our configuration space $\mathbb{R}^n$ carries the Euclidean metric (δ_{ij}), hence index position is immaterial.

and so the bosonic part of the potential takes the form

$$
\begin{aligned}
V_B \ = \ & \frac{1}{2} \sum_{\alpha,\beta} \frac{\alpha \cdot \beta}{\alpha \cdot x \ \beta \cdot x} \left(g_\alpha g_\beta + \frac{\hbar^2}{4} f_\alpha f_\beta \, (\alpha \cdot \beta)^2 \right) \ - \ \sum_\alpha g_\alpha \frac{\alpha_i}{\alpha \cdot x} \, \partial_i U_{\mathrm{hom}} \\
& + \ \frac{1}{2} \frac{1}{R^2} \left(g_R(2C - g_R) + \frac{\hbar^2}{4}(3n - 2) f_R(2 - f_R) \right) \ + \ \frac{1}{2}(\partial_i U_{\mathrm{hom}})(\partial_i U_{\mathrm{hom}}) \ .
\end{aligned}
\tag{5.9}
$$

The WDVV equation in (4.2) becomes

$$
\frac{1}{2} \sum_{\alpha,\beta} f_\alpha f_\beta \frac{\alpha \cdot \beta}{\alpha \cdot x \ \beta \cdot x} (\alpha \wedge \beta)^{\otimes 2} \ + \ f_R \, (2 - f_R) \frac{T}{R^2} \ = \ 0
\tag{5.10}
$$

with $\quad (\alpha \wedge \beta)^{\otimes 2}_{ijkl} \ = \ (\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_k \beta_l - \alpha_l \beta_k) \quad$ and

$$
T_{ijkl} \ = \ \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ik}\hat{x}_j\hat{x}_l + \delta_{il}\hat{x}_j\hat{x}_k - \delta_{jl}\hat{x}_i\hat{x}_k + \delta_{jk}\hat{x}_i\hat{x}_l \quad \text{where} \quad \hat{x}_i \equiv \frac{x_i}{R} \ .
\tag{5.11}
$$

The different singular loci of the various terms in (5.10) allow one to separate them, thus

$$
\sum_{\substack{\alpha,\beta \\ (\alpha \neq \beta)}} f_\alpha f_\beta \frac{\alpha \cdot \beta}{\alpha \cdot x \ \beta \cdot x} (\alpha \wedge \beta)^{\otimes 2} \ = \ 0 \qquad \text{and} \qquad f_R \, (2 - f_R) \ = \ 0 \ .
\tag{5.12}
$$

The two admissible choices for f_R,

$$
f_R = 0 \ \stackrel{(5.7)}{\longrightarrow} \ \sum_\alpha f_\alpha \alpha \otimes \alpha = \mathbb{1} \qquad \text{or} \qquad f_R = 2 \ \stackrel{(5.7)}{\longrightarrow} \ \sum_\alpha f_\alpha \alpha \otimes \alpha = -\mathbb{1} \,,
\tag{5.13}
$$

are related by flipping the signs of all coefficients f_α, i.e. $f_\alpha \to -f_\alpha$. Note, however, that the $n{=}2$ case is special, since then $T{\equiv}0$ and (5.10) is identically satisfied, so no restrictions on f_R arise.

The flatness condition in (4.3), on the other hand, is already nontrivial at $n{=}2$ and reads

$$
\partial_i \partial_j U \ + \ \sum_\alpha f_\alpha \frac{\alpha_i \alpha_j}{\alpha \cdot x} \alpha \cdot \partial U \ + \ f_R \left\{ \frac{x_i \partial_j U + x_j \partial_i U - \delta_{ij} C}{R^2} + \frac{2 x_i x_j C}{R^4} \right\} \ = \ 0 \ .
\tag{5.14}
$$

In particular, its trace,

$$
\partial \cdot \partial U \ + \ \sum_\alpha f_\alpha \frac{\alpha \cdot \alpha}{\alpha \cdot x} \alpha \cdot \partial U \ = \ C \, f_R \, \frac{n}{R^2} \,,
\tag{5.15}
$$

and its projection onto some covector β,

$$
(\beta \cdot \partial)^2 U \ + \ \sum_\alpha f_\alpha \frac{(\alpha \cdot \beta)^2}{\alpha \cdot x} \alpha \cdot \partial U \ + \ 2 f_R \frac{\beta \cdot x}{R^2} \beta \cdot \partial U \ = \ C \, f_R \left(\frac{\beta \cdot \beta}{R^2} - \frac{2(\beta \cdot x)^2}{R^4} \right),
\tag{5.16}
$$

prove to be useful. They are potentially singular at $R{=}0$ and on the hyperplanes $\alpha \cdot x{=}0$. For example, near $\beta \cdot x{=}0$ (but away from $R{=}0$) we may approximate (5.16) by

$$
(\beta \cdot \partial)^2 U \ + \ \frac{f_\beta \, \beta \cdot \beta}{\beta \cdot x} \beta \cdot \partial U \ \approx \ 0 \qquad \stackrel{f_\beta \geq -1}{\longrightarrow} \qquad U \ \sim \ (\beta \cdot x)^{1 - f_\beta} \quad \text{for} \quad \beta \cdot x \sim 0,
\tag{5.17}
$$

which displays the leading singularity structure of U (and thus of V_B) on the $\beta \cdot x = 0$ hyperplane provided that f_β is sufficiently large.

Of course, there is always the trivial $C=0$ solution, which puts $g_R = g_\alpha = 0 \quad \forall \alpha$. As long as we keep U_{hom} to be nonzero, it is not too illuminating to insert the covector expression (5.8) into the above equations. So let us, for a moment, ponder the consequences of putting $U_{\mathrm{hom}} \equiv 0$ in (5.6). In such a case for $n>2$, (5.14) together with (5.8) implies

$$
g_\alpha (1 - \alpha \cdot \alpha f_\alpha) = 0, \quad \sum_{\substack{\alpha, \beta \\ (\alpha \neq \beta)}} g_\alpha f_\beta \frac{\alpha \cdot \beta \; \beta_i \beta_j}{\alpha \cdot x \; \beta \cdot x} = 0, \quad g_\alpha f_R = C f_R = g_R = 0, \quad (5.18)
$$

which essentially kills all radial terms and fixes $f_\alpha = \frac{1}{\alpha \cdot \alpha}$ unless $g_\alpha = 0$. Turning on all g_α would then saturate the first option in (5.13),

$$
\sum_\alpha \frac{\alpha \otimes \alpha}{\alpha \cdot \alpha} = \mathbb{1} \qquad \longrightarrow \qquad \alpha \cdot \beta = 0 \quad \forall \alpha, \beta, \qquad (5.19)
$$

because this partition of unity is an orthonormal one and the number p of covectors α must be equal to n. Clearly, such a system is reducible: If a set of covectors decomposes into mutually orthogonal subsets, (5.10) and (5.14) — at $f_R = 0 = g_R$ — hold for each subset individually. Then, the partial prepotentials just add up to the total F or U. In fact, we have already encountered such a decomposition when separating the center-of-mass degree of freedom. Here, however, it is the *relative* motion of the particles which can be factored into independent parts. Since the irreducible relative-particle systems are the building blocks for all models, the case of $p = n$ is just a collection of $n=1$ systems and does not provide an interesting solution. We learn that $U_{\mathrm{hom}} \equiv 0$ is not an option for $n>2$.

Let us finally take a look at the special case of $n=2$, i.e relative motion in a three-particle system. First, as already mentioned, the $n=2$ WDVV equation is empty; it follows from (5.7), which can be fulfilled for any set of more than one covector. Hence, f_R is unrestricted. Second, at $n=2$ the content of (5.14) is fully captured by its trace (5.15), which in this case allows nontrivial solutions even with $U_{\mathrm{hom}} \equiv 0$. Namely, inserting the second line of (5.8) with $U_{\mathrm{hom}} \equiv 0$ into (5.15) one obtains

$$
\sum_\alpha g_\alpha (1 - \alpha \cdot \alpha f_\alpha) \frac{\alpha \cdot \alpha}{(\alpha \cdot x)^2} \; - \; \sum_{\substack{\alpha, \beta \\ (\alpha \neq \beta)}} g_\alpha f_\beta \frac{\alpha \cdot \beta \; \beta \cdot \beta}{\alpha \cdot x \; \beta \cdot x} \; - \; \frac{1}{R^2} \Big(2(n-1) g_R + n (C - g_R) f_R \Big) = 0, \qquad (5.20)
$$

which splits into

$$
g_\alpha (1 - \alpha \cdot \alpha f_\alpha) = 0, \quad \sum_{\substack{\alpha, \beta \\ (\alpha \neq \beta)}} g_\alpha f_\beta \frac{\alpha \cdot \beta \; \beta \cdot \beta}{\alpha \cdot x \; \beta \cdot x} = 0, \quad g_R = \frac{f_R}{f_R - 2 + 2/n} C. \quad (5.21)
$$

If all couplings g_α are nonzero, then

$$
f_\alpha = \frac{1}{\alpha \cdot \alpha} > 0 \quad \xrightarrow{(5.7)} \quad f_R = 1 - \frac{p}{n} \qquad \text{and} \qquad g_R = \frac{p-n}{p+n-2} C \quad (5.22)
$$

besides

$$
\sum_\alpha \frac{\alpha \otimes \alpha}{\alpha \cdot \alpha} = \frac{p}{n} \mathbb{1} \qquad \text{and} \qquad \sum_{\substack{\alpha, \beta \\ (\alpha \neq \beta)}} g_\alpha \frac{\alpha \cdot \beta}{\alpha \cdot x \beta \cdot x} = 0. \quad (5.23)
$$

These equations will be analyzed in section 8. We already see that the radial terms are essential for having $p > n$. Of course, we are to put $n{=}2$ in the equations above, but we have displayed the general formulae to make explicit the conflict between (5.22) and (5.12) for $n{\geq}3$ and $p{>}n$, which essentially rules out $U_{\text{hom}}{\equiv}0$ solutions beyond $n{=}2$.

6 $U{=}0$ solutions: root systems

The obvious strategy for solving the structure equations is to first construct a prepotential F satisfying (4.2), i.e. find covectors α (and coefficients f_α) subject to (5.13) and (5.12). Without loss of generality we restrict ourselves to the first of the two cases in (5.13) and put $f_R{=}0$. The structure equations are linear in the prepotential U, and so a solution to the WDVV equation trivially extends to a full solution $(F, U{\equiv}0)$ for $C{=}0$.

In 1999, Martini and Gragert [9] discovered that, in (5.6) with $f_R{=}0{=}g_R$, taking $\{\alpha\}$ to be a (positive) root system of any simple Lie algebra yields a valid prepotential F. Shortly thereafter, it was proved [10] that certain deformations of root systems are also allowed, as well as the root systems of *any* finite reflection group, thus adding the non-crystallographic Coxeter groups to the list. In the following, we shall rederive these results and generalize them.

Let us begin with the simply-laced root systems. Here, any two positive roots α and β are either orthogonal, or else add or subtract to another positive root, then giving rise to an equilateral triangle

$$\alpha{+}\beta{+}\gamma \,=\, 0 \qquad \longrightarrow \qquad \alpha{\wedge}\beta \,=\, \beta{\wedge}\gamma \,=\, \gamma{\wedge}\alpha \qquad \text{and} \qquad \alpha{\cdot}\beta \,=\, \beta{\cdot}\gamma \,=\, \gamma{\cdot}\alpha\,. \tag{6.1}$$

The contribution of the pairs (α, β), (β, γ) and (γ, α) to (5.12) is thus proportional to

$$\frac{f_\alpha\, f_\beta}{\alpha{\cdot}x\ \beta{\cdot}x} \;+\; \frac{f_\beta\, f_\gamma}{\beta{\cdot}x\ \gamma{\cdot}x} \;+\; \frac{f_\gamma\, f_\alpha}{\gamma{\cdot}x\ \alpha{\cdot}x}\,, \tag{6.2}$$

which vanishes precisely when $f_\alpha = f_\beta = f_\gamma$. We recognize the triple $(\alpha, \beta, -\gamma)$ as the positive roots of A_2.

It is not hard to see that in (5.12) the sum over all non-orthogonal pairs (α, β) of positive ADE roots can be decomposed into partial sums over the three pairs of a triple. Two triples may share a single root but not a pair. Since all triples are connected in this way, all f_α are equal,[9] and their value is fixed by the homogeneity condition (5.13), which implies that our root system must be of rank n. To find f, recall that, for any Lie algebra and with $\alpha{\cdot}\alpha{=}2$ for the long roots, one has

$$\sum_{\alpha\in\Phi^+} \alpha\otimes\alpha \,=\, h^\vee\,\mathbb{1} \qquad \text{and} \qquad \sum_{\alpha\in\Phi^+} 2\,\frac{\alpha\otimes\alpha}{\alpha\cdot\alpha} \,=\, h\,\mathbb{1}\,, \tag{6.3}$$

Φ^+	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2	H_3	H_4	$I_2(p)$
h	$n{+}1$	$2n$	$2n$	$2n{-}2$	12	18	30	12	6	10	30	p
$h^\vee$	$n{+}1$	$2n{-}1$	$n{+}1$	$2n{-}2$	12	18	30	9	4	—	—	—

Table 1. Coxeter and dual Coxeter numbers for all Coxeter groups.
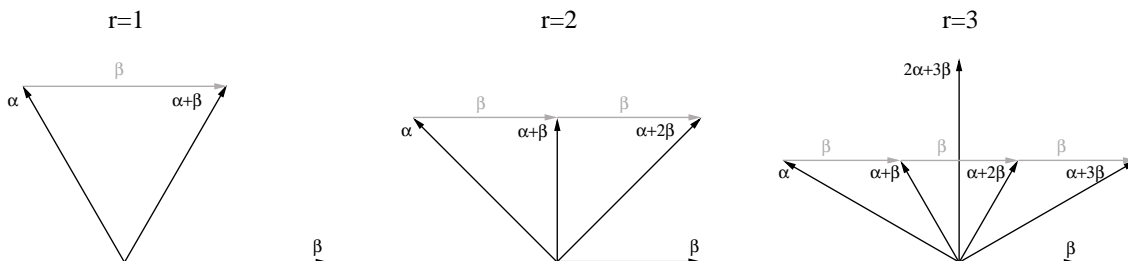
Figure 1. Short-root strings through a long root, for length^2 ratios $r = 1, 2, 3$

where Φ^+ is the set of positive roots, and h and $h^\vee$ denote the Coxeter and dual Coxeter numbers, respectively, which are listed in table 1 for the reader's convenience.. Thus, $f = 1/h^\vee$ in the ADE case, where $h = h^\vee$.

In essence, the root systems of all ADE Lie algebras provide us with prepotentials [9]

$$F_{\text{ADE}} \;=\; -\frac{1}{2h^\vee} \sum_{\alpha \in \Phi^+} (\alpha{\cdot}x)^2 \ln|\alpha{\cdot}x| \;. \tag{6.4}$$

What about the other root systems? There, we have long roots, with length$^2 = 2$, and short roots, with length$^2 = 2/r$, where $r = 2$ or 3. Any two non-orthogonal short roots add or subtract to another short root, and the same is true for the long roots. Hence, for the short/short or long/long pairs in our double sum we can again employ (6.2)=0, which identifies the f coefficients in each triple. However, we also encounter long/short pairs in (5.12). The key is to realize that the ADE triple $(\alpha, \beta, \alpha{+}\beta)$ represents the β-string of roots through α. The root string concept works for any pair of roots and in general groups together $r{+}2$ coplanar roots $(\alpha, \beta, \alpha{+}\beta, \alpha{+}2\beta, \ldots, \alpha{+}r\beta)$, with α being long, β short and $\alpha \cdot \beta = -1$.

For the long/short pairs in B_n, C_n and F_4 ($r{=}2$) the role of (6.2)=0 is then taken by a four-root identity based on the quadruple $(\alpha, \beta, \alpha{+}\beta, \alpha{+}2\beta)$. With the scalar products from table 2 and the relevant wedge products all equal modulo sign, the equal-length pairs drop out, and the quadruple yields just four long/short pairs for the sum in (5.12),

$$-\,\frac{f_\alpha\, f_\beta}{\alpha{\cdot}x\;\beta{\cdot}x} \;+\; \frac{f_\alpha\, f_{\alpha+\beta}}{\alpha{\cdot}x\;(\alpha{+}\beta){\cdot}x} \;+\; \frac{f_{\alpha+2\beta}\, f_\beta}{(\alpha{+}2\beta){\cdot}x\;\beta{\cdot}x} \;+\; \frac{f_{\alpha+2\beta}\, f_{\alpha+\beta}}{(\alpha{+}2\beta){\cdot}x\;(\alpha{+}\beta){\cdot}x} \;. \tag{6.5}$$

This expression vanishes only when it must, namely for

$$f_\alpha \;=\; f_{\alpha+2\beta} \;=:\; f_{\text{L}} \qquad \text{and} \qquad f_\beta \;=\; f_{\alpha+\beta} \;=:\; f_{\text{S}} \;. \tag{6.6}$$

---

[9] The trivial way to avoid this conclusion puts $f_\alpha{=}0$ for sufficiently many roots such that the system decomposes into mutually orthogonal parts, with their f_α values determined individually via (5.13).

·	β	$\alpha+\beta$	α	$\alpha+2\beta$
β	1	0	-1	1
$\alpha+\beta$	0	1	1	1
α	-1	1	2	0
$\alpha+2\beta$	1	1	0	2

Table 2. Scalar products for an $r=2$ root string.

Like in the ADE case, each non-orthogonal pair of roots defines a unique plane, which carries either a triple or a quadruple. Hence, the sum in (5.12) again splits into sums over the pairs of a triple or a quadruple, which yield zero individually. Since each plane shares its roots with other planes and all are connected unless the system is decomposable, all long roots come with the same coefficient $f_{\rm L}$, and all short roots with $f_{\rm S}$. The normalization in (5.13) then reads

$$f_{\rm L} \sum_{\alpha \in \Phi_{\rm L}^+} \alpha \otimes \alpha \; + \; f_{\rm S} \sum_{\alpha \in \Phi_{\rm S}^+} \alpha \otimes \alpha \; = \; \mathbb{1} \,, \tag{6.7}$$

where $\Phi_{\rm L}^+$ and $\Phi_{\rm S}^+$ stand for the positive long and short roots, respectively. Using

$$\sum_{\alpha \in \Phi_{\rm L}^+} \alpha \otimes \alpha \; = \; \frac{rh^\vee - h}{r-1}\, \mathbb{1} \qquad \text{and} \qquad \sum_{\alpha \in \Phi_{\rm S}^+} \alpha \otimes \alpha \; = \; \frac{h-h^\vee}{r-1}\, \mathbb{1} \,, \tag{6.8}$$

the solution to (6.7) is a one-parameter family,

$$
\begin{aligned}
f_{\rm L} \;&=\; \frac{1}{h^\vee} + (h-h^\vee)t \;\;=\;\; \frac{1}{h} + (h-h^\vee)t' \,, \\
f_{\rm S} \;&=\; \frac{1}{h^\vee} + (h-rh^\vee)t \;\;=\;\; r\left\{ \frac{1}{h} + \left(\frac{h}{r}-h^\vee\right)t' \right\} \,,
\end{aligned}
\tag{6.9}
$$

with $t = t' - \frac{1}{hh^\vee} \in \mathbb{R}$. Therefore, we arrive at a *family* of prepotentials

$$F \;=\; -\frac{1}{2}f_{\rm L} \sum_{\alpha \in \Phi_{\rm L}^+} (\alpha{\cdot}x)^2 \ln|\alpha{\cdot}x| \;-\; \frac{1}{2}f_{\rm S} \sum_{\alpha \in \Phi_{\rm S}^+} (\alpha{\cdot}x)^2 \ln|\alpha{\cdot}x| \,. \tag{6.10}$$

Incidentally, the formulae (6.9) and (6.10) hold for *all* root systems, including the ADE ($r=1$) and G_2 ($r=3$) cases. The only $r=3$ example, G_2, is trivial since of rank two, but let us anyway also prove the assertion for this case. The six positive roots of G_2 contribute 3 short/short, 3 long/long and 6 long/short pairs to the sum in (5.12). As argued before, the contributions of the equal-length pairs vanish by virtue of (6.2)=0, provided $f_\alpha = f_{\rm S}$ for the short roots and $f_\alpha = f_{\rm L}$ for the long ones. The mixed pairs yield

$$
\begin{aligned}
&\frac{-1}{\alpha{\cdot}x\,\beta{\cdot}x} + \frac{1}{\alpha{\cdot}x\,(\alpha+\beta){\cdot}x} + \frac{1}{(\alpha+3\beta){\cdot}x\,\beta{\cdot}x} + \frac{1}{(\alpha+3\beta){\cdot}x\,(\alpha+2\beta){\cdot}x} \\
&+\frac{1}{(2\alpha+3\beta){\cdot}x\,(\alpha+\beta){\cdot}x} + \frac{1}{(2\alpha+3\beta){\cdot}x\,(\alpha+2\beta){\cdot}x}
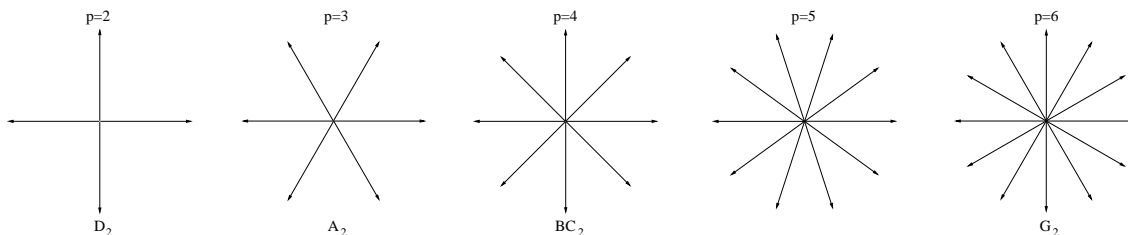\end{aligned}
\tag{6.11}
$$

Figure 2. Root systems of the dihedral groups $I_2(p)$ for $p = 2, 3, 4, 5, 6$

for a long root α and a short root β, with $\alpha{\cdot}\beta = -1$, which as simple roots generate the G_2 system. It is quickly verified that the above expression indeed vanishes, which proves our claim. Hence, for all Lie-algebra root systems and any plane Π, we have proved the identity

$$\sum_{\substack{\alpha\beta\in\Pi \\ (\alpha\neq\beta)}} \frac{\alpha{\cdot}\beta}{\alpha{\cdot}x\ \beta{\cdot}x} \;=\; 0 \qquad \text{for} \qquad (\alpha,\beta) \;\in\; \left(\Phi_{\mathrm{L}}^+, \Phi_{\mathrm{L}}^+\right) \text{ or } \left(\Phi_{\mathrm{S}}^+, \Phi_{\mathrm{S}}^+\right) \text{ or } \left(\Phi_{\mathrm{L}}^+, \Phi_{\mathrm{S}}^+\right), \qquad (6.12)$$

which is effectively equivalent to the WDVV equation. Our solution (6.9) for the f coefficients generalizes the one of [9, 10] and reduces to them at $t{=}0$. One might think that the one-parameter freedom is ficticious since f_{L} and f_{S} may be absorbed into the roots. However, this is not so because f_{L} and f_{S} may have opposite signs, which is crucial for constructing U solutions in this F background.

We have also checked the non-crystallographic Coxeter groups H_3, H_4 and $I_2(p)$ for $p{=}5$ and $p{>}6$.[10] Of these, the dihedral I_2 series

$$\{\alpha\} \;=\; \{\ \cos(k\pi/p)\,e_1 + \sin(k\pi/p)\,e_2 \mid k = 0, 1, \ldots, p{-}1\ \} \qquad (6.13)$$

trivially fulfils (5.10), as it is of rank two.

7 $U{=}0$ solutions: orthocentric simplices

In order to generalize the root-system solutions found in the previous section, in this section we take a more general look at the $n{=}3$ case. Again, the goal is to solve the WDVV equation (5.12) and the homogeneity condition (5.13) for $f_R{=}0$.

Previously we have mentioned that any set of $p{\geq}2$ covectors in $n{=}2$ dimensions solves (4.2), because the WDVV equation is empty and (5.7) only serves to restrict f_α and f_R. We now deliver a simple argument. Let us represent a covector $\alpha \in \mathbb{R}^2$ by a complex number $a \in \mathbb{C}$. Then, the traceless and the trace part of the homogeneity condition (5.7) translate to

$$\sum_a f_a\, a^2 \;=\; 0 \qquad \text{and} \qquad \sum_a f_a\, a\bar{a} \;=\; 2\,(1{-}f_R), \qquad (7.1)$$

respectively, where $\bar{a}$ is the complex conjugate of a and $f_a \equiv f_\alpha \in \mathbb{R}$. Since the length of each covector can be changed by rescaling the corresponding f, it is evident that for more

---

[10] Up to a root rescaling, $I_2(2) = A_1{\oplus}A_1$, $I_2(3) = A_2$, $I_2(4) = B_2$ or C_2, and $I_2(6) = G_2$.
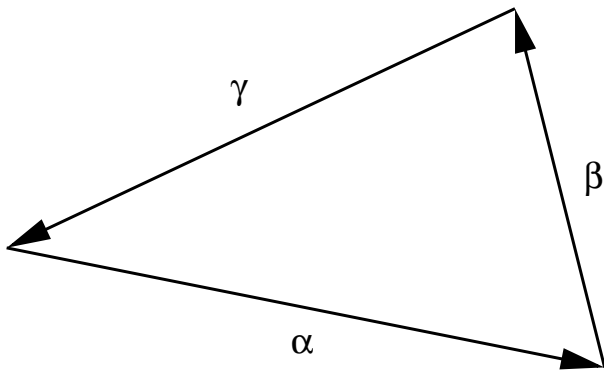
Figure 3. Triangular configuration of covectors

than one covector one can always select these coefficients in such a way that the complex numbers $f_a a^2$ form a closed polygonal chain in two dimensions, thus satisfying the first of (7.1). A common rescaling then takes care of the second equation as well, while f_R can still be dialed at will. Therefore, by taking the complex square roots of the edge vectors of any closed polygonal chain, we obtain an admissible set of covectors.

Before moving on to three dimensions, it is instructive to work out the f_α coefficients from the homogeneity condition (5.13) for $f_R{=}0$ and $n{=}2$. For the case of two covectors $\{\alpha, \beta\}$, necessarily $\alpha{\cdot}\beta = 0$. For $p{=}3$ coplanar covectors $\{\alpha, \beta, \gamma\}$, the homogeneity condition (5.13) uniquely fixes the f coefficients to

$$f_\alpha = -\frac{\beta \cdot \gamma}{\alpha{\wedge}\beta \ \gamma{\wedge}\alpha} \qquad \text{and cyclic}, \tag{7.2}$$

due to the identity

$$\beta{\wedge}\gamma \ \beta{\cdot}\gamma \ \alpha_i \alpha_j \ + \ \text{cyclic} \ = \ -\alpha{\wedge}\beta \ \beta{\wedge}\gamma \ \gamma{\wedge}\alpha \ \delta_{ij} \ . \tag{7.3}$$

The traceless part of the homogeneity condition should imply the single WDVV equation (5.12) in two dimensions. Indeed, the choice (7.2) turns the latter into

$$\alpha{\wedge}\beta \ \gamma{\cdot}x \ + \ \beta{\wedge}\gamma \ \alpha{\cdot}x \ + \ \gamma{\wedge}\alpha \ \beta{\cdot}x \ = \ 0 \tag{7.4}$$

which is identically true. Without loss of generality we may assume that $\alpha + \beta + \gamma = 0$, i.e. the three covectors form a triangle. In this case we have $\alpha{\wedge}\beta = \beta{\wedge}\gamma = \gamma{\wedge}\alpha = 2A$, where the area A of the triangle may still be scaled to $\frac{1}{2}$, and (7.2) simplifies to

$$f_\alpha = -\frac{\beta \cdot \gamma}{4 A^2} \qquad \text{and cyclic} \ . \tag{7.5}$$

In dimension $n{=}3$, the minimal set of three covectors must form an orthogonal basis, with $f_\alpha = 1/\alpha{\cdot}\alpha$. Let us skip the cases of four and five covectors and go to the situation of $p{=}6$ covectors because the homogeneity condition (5.13) then precisely determines all f coefficients. However, it is not true that six generic covectors can be scaled to form the edges
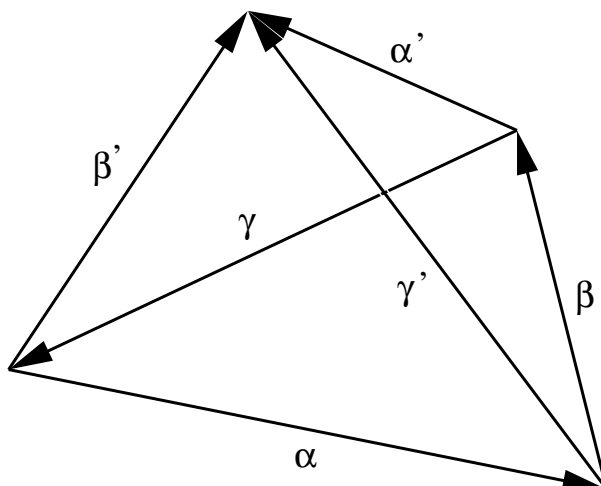
Figure 4. Tetrahedral configuration of covectors

of a polytope. The space of six rays in $\mathbb{R}^3$ modulo rigid $SO(3)$ is nine dimensional, while the space of tetrahedral shapes (modulo size) has only five dimensions. In order to generalize the $n{=}2$ solution above, let us assume that our six covectors can be scaled to form a tetrahedron, with edges $\{\alpha, \beta, \gamma, \alpha', \beta', \gamma'\}$ where α' is dual to α and so on. Any such tetrahedron is determined by giving three nonplanar covectors, say $\{\alpha, \beta, \gamma'\}$, which up to rigid rotation are fixed by six parameters, corresponding to the shape and size of the tetrahedron.

Let us try employing the triangle result (7.5) to patch together the unique solution to the homogeneity condition (5.13) for the tetrahedron. To satisfy the traceless part of the relation, we take the f coefficients around any face to be proportional to the triangular ones (7.5). Now each edge is shared by two triangular faces, so we should have

$$f_\alpha \;=\; -\lambda_{\alpha\beta\gamma}\,\beta{\cdot}\gamma \;=\; -\lambda_{\alpha\beta'\gamma'}\,\beta'{\cdot}\gamma' \tag{7.6}$$

and so forth cyclicly around the triangles $\langle\alpha\beta\gamma\rangle$ and $\langle\alpha\beta'\gamma'\rangle$, with coefficients $\lambda_{...}$ depending only on the triangle indicated. It is then tempting to put

$$f_\alpha \;=\; -\lambda\,\beta{\cdot}\gamma\,\beta'{\cdot}\gamma', \qquad f_\beta \;=\; -\lambda\,\gamma{\cdot}\alpha\,\gamma'{\cdot}\alpha', \qquad f_\gamma \;=\; -\lambda\,\alpha{\cdot}\beta\,\alpha'{\cdot}\beta' \tag{7.7}$$

and so on using the tetrahedral incidences, with λ depending only on the volume V of the tetrahedron. However, comparing the two previous sets of equations we see that this can only work if

$$\beta'{\cdot}\gamma' \;=\; \gamma'{\cdot}\alpha' \;=\; \alpha'{\cdot}\beta' \;=\; \frac{\lambda_{\alpha\beta\gamma}}{\lambda} \tag{7.8}$$

and likewise for any three convergent edges dual to some face. These eight relations are non-generic but immediately equivalent to the three conditions

$$\alpha \cdot \alpha' \;=\; 0, \qquad \beta \cdot \beta' \;=\; 0, \qquad \gamma \cdot \gamma' \;=\; 0 \tag{7.9}$$

for the pairs of dual (skew) edges of the tetrahedron. Such tetrahedra, called 'ortho-centric' [16], are characterized by the fact that all four altitudes are concurrent (in the

orthocenter) and their feet are the orthocenters of the faces. The space of orthocentric tetrahedra is of codimension two inside the space of all tetrahedra and represents a three-parameter deformation of the A_3 root system (ignoring the overall scale).

For orthocentric tetrahedra, our ansatz (7.7) is successful: Due to the identity

$$\beta{\cdot}\gamma \; \beta'{\cdot}\gamma' \; \alpha_i \alpha_j \;+\; \beta{\cdot}\gamma' \; \beta'{\cdot}\gamma \; \alpha'_i \alpha'_j \;+\; \text{cyclic} \;=\; -36\,V^2\,\delta_{ij}\,, \qquad (7.10)$$

the homogeneity condition (5.13) is obeyed for

$$f_\alpha \;=\; -\frac{\beta{\cdot}\gamma \; \beta'{\cdot}\gamma'}{36\,V^2} \qquad \text{and} \qquad f_{\alpha'} \;=\; -\frac{\beta{\cdot}\gamma' \; \beta'{\cdot}\gamma}{36\,V^2} \qquad (7.11)$$

plus their cyclic images. What about the WDVV equation in this case? The 15 pairs of edges in the double sum of (5.12) group into four triples corresponding to the tetrahedron's faces plus the three skew pairs. Using (7.11), the contribution of the $\langle\alpha\beta\gamma\rangle$ face becomes proportional to $\beta'{\cdot}\gamma' \, \gamma{\cdot}x + \text{cyclic}$, which vanishes thanks to (7.8). Repeating this argument for the other faces, we see that the concurrent edge pairs do not contribute to the double sum in (5.12), which leaves us with the three skew pairs. At this point, the orthocentricity again comes to the rescue via (7.9), and the WDVV equation is obeyed. Apparently, any reduction of the WDVV equation to some face already follows from the homogeneity condition, and the only independent projection is associated with the skew edge pairs.

Although we do not know the f coefficients for a general tetrahedron, we can employ a dimensional reduction argument to prove that the WDVV equation already enforces the orthocentricity. Consider the limit $\hat{n} \cdot x \to \infty$ for some fixed covector $\hat{n}$ of unit length. Decomposing

$$\alpha \;=\; \alpha{\cdot}\hat{n}\,\hat{n} + \alpha_\perp \qquad \longrightarrow \qquad \alpha{\cdot}x \;=\; \alpha{\cdot}\hat{n}\,\hat{n}{\cdot}x + \alpha_\perp{\cdot}x \qquad (7.12)$$

we see that any factor $\frac{1}{\alpha{\cdot}x}$ vanishes in this limit unless $\alpha{\cdot}\hat{n} = 0$. Thus, only covectors perpendicular to $\hat{n}$ survive in (5.10), reducing the system to the hyperplane orthogonal to $\hat{n}$. In addition, $\frac{1}{R} \to 0$ as well, killing all radial terms in the process.[11] In a general tetrahedron, take $\hat{n}$ to point in the direction of $\alpha{\wedge}\alpha'$. Then, the limit $\hat{n}{\cdot}x \to \infty$ retains only the covectors α and α', and the WDVV equation reduces to a single term, which vanishes only for $\alpha{\cdot}\alpha' = 0$. Equivalently, the plane spanned by α and α' contains no further covector, and two covectors in two dimensions must be orthogonal. The same argument applies to $\beta{\cdot}\beta'$ and $\gamma{\cdot}\gamma'$, completing the proof.

This scheme may be taken to any dimension n. A simplicial configuration of $\frac{1}{2}n(n{+}1)$ covectors is already determined by n independent covectors, which modulo $SO(n)$ are given by $\frac{1}{2}n(n{+}1)$ parameters. The homogeneity condition (5.13) uniquely fixes the f coefficients. Employing an iterated dimensional reduction to any plane spanned by a skew pair of edges and realizing that no other edge lies in such a plane, we see that the WDVV equation always demands such an edge pair to be orthogonal. This condition renders the n-simplex orthocentric and reduces the number of degrees of freedom to $n{+}1$ (now including the

---

[11] Note, however, that the reduced system in general does not fulfil the homogeneity conditions (5.7) since the 'lost covectors' have nonzero projections onto the hyperplane.
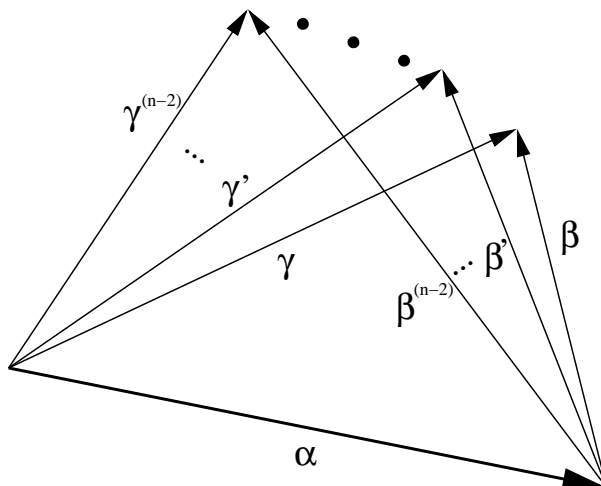
Figure 5. Faces sharing an edge of an n-simplex

overall scale given by the n-volume V). In this situation we can write down the unique solution to both the homogeneity condition and the WDVV equation,

$$f_\alpha \;=\; \frac{\beta{\cdot}\gamma\;\;\beta'{\cdot}\gamma'\;\;\beta''{\cdot}\gamma''\;\cdots\;\beta^{(n-2)}{\cdot}\gamma^{(n-2)}}{(n!\;V)^2}\,, \tag{7.13}$$

where the edge α is shared by the $n-1$ faces $\langle\alpha\beta\gamma\rangle$, $\langle\alpha\beta'\gamma'\rangle$, ..., $\langle\alpha\beta^{(n-2)}\gamma^{(n-2)}\rangle$, and we have oriented all edges as pointing away from α. This formula works because any subsimplex, in particular any tetrahedral building block, is itself orthocentric. To summarize, the WDVV solutions for simplicial covector configurations in any dimension are exhausted by an n-parameter deformation of the A_n root system. The n moduli are relative angles and do not include the $\frac{1}{2}n(n+1)$ trivial covector rescalings, which, apart from the common scale, destroy the tetrahedron. It has to be checked whether our deformation coincides with the A_n deformation found in [11] in a different setting.

As a concrete example, the reader is invited to work out the details for the generic (scaled) orthocentric 4-simplex with vertices

$$\begin{aligned} &A:(0,0,0,0) &&B:(1,0,0,0) &&C:(x,y,0,0)\\ &D:\left(x,\frac{x(1-x)}{y},z,0\right) &&E:\left(x,\frac{x(1-x)}{y},\frac{x(1-x)(y^2-x(1-x))}{y^2\,z},w\right). \end{aligned} \tag{7.14}$$

Orthocentric simplices are not the only generalization of our analysis of six covectors in three dimensions. Recalling that $A_3 = D_3$, we know that the six edges of a regular tetrahedron can be reassembled into one-half of a regular octahedron. Let us relax the regularity and look at a more general octahedron defined by six vertices $\pm v_1$, $\pm v_2$ and $\pm v_3$, which are fixed (up to rigid rotations) by six parameters, just like for the tetrahedron. For the full set of edges we need to include here also the negatives of all positive covectors,

$$\{\pm\alpha\} \;=\; \left\{\alpha_{(\pm i\pm j)}=\pm v_i\pm v_j \quad\text{for}\quad 1\le i<j\le 3\right\}. \tag{7.15}$$
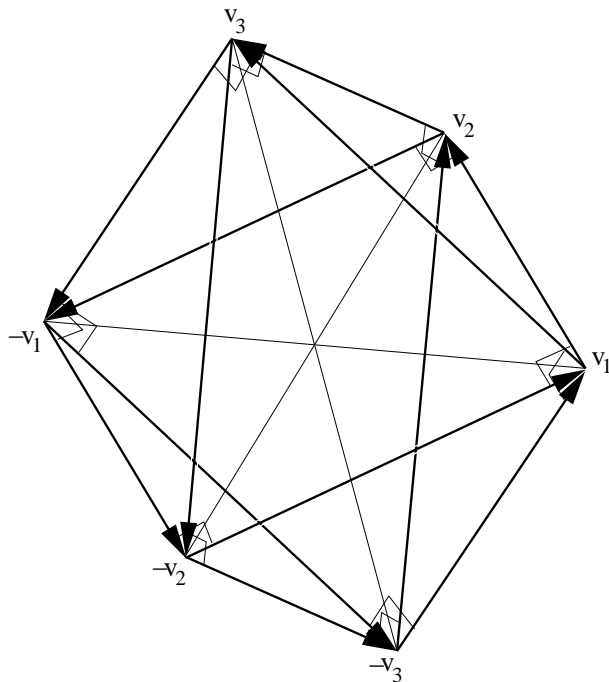
Figure 6. Octahedral configuration of covectors

With $f_{-\alpha} = f_\alpha$, the homogeneity condition uniquely fixes all f coefficients. For the WDVV equation, let us again consider the dimensional reduction to the plane spanned by any pair of covectors, and restrict to the positive ones. Like for the tetrahedron, it turns out that such a plane contains either a triangular face or just two convergent covectors $\alpha_{(i+j)}$ and $\alpha_{(i-j)}$. The reduced WDVV equation requires a right angle between the latter, which puts $v_i \cdot v_i = v_j \cdot v_j$, and so all three vertices must have the same distance from the origin. We are not aware of a particular name for such octahedra, which admit a circumsphere. In any case, these two conditions and ignoring the overall scale reduce the modular space to a three-dimensional one, which we already identified as the space of orthocentric tetrahedral shapes.

The virtue of this alternative picture is a different generalization: In addition to the simplicial polytopes (related to A_n) we obtain as well hyperoctahedral polytopes (related to D_n) for WDVV solutions in any dimension, by letting $i<j$ in (7.15) run up to n.[12] Such a configuration consists of $n(n-1)$ covectors plus their negatives, but is completely determined again by n of these, for which $\frac{1}{2}n(n+1)$ parameters are needed. Beyond $n=3$ the homogeneity condition (5.13) no longer fixes the f coefficients. The WDVV equation now demands not only that $v_i \cdot v_i = v_j \cdot v_j$ but also that $\alpha_{(\pm i \pm j)} \cdot \alpha_{(\pm k \pm l)} = 0$ for all indices mutually different. This is strong enough to enforce $v_i \cdot v_j \propto \delta_{ij}$, i.e. complete regularity for the hyperoctahedron. What remains for $n>3$ is just the D_n root system (up to scale).

Our findings suggest that covector configurations corresponding to deformations of other roots systems may solve the WDVV equations as well. For verification, we propose

---

[12] Note that our covectors (plus their negatives) form the edges of these polytopes and not their vertices.

to consider the polytopes associated with the weight systems of a given Lie algebra, since their edge sets are built from the root covectors. The idea is then to relax the angles of such polytopes and analyze the constraints from the homogeneity and WDVV equations. The n-dimensional hyper-tetrahedra and -octahedra we found emerge simply from the fundamental and vector representations of A_n and D_n, respectively. Extending this strategy to other representations and Lie algebras could lead to many more solutions.

8 $U{\neq}0$ solutions: three-particle systems with $U_{\mathrm{hom}}{\equiv}0$

Let us finally try to turn on the other prepotential, $U{\neq}0$, in the background of the F solutions already found. Unfortunately, we have no good strategy to solve (5.14) unless $U_{\mathrm{hom}}{\equiv}0$. Hence, in this section let us make the ansatz

$$U \;=\; -\sum_\alpha g_\alpha \,\ln|\alpha{\cdot}x| \;-\; g_R \ln R\,, \tag{8.1}$$

and face the conditions (5.18) (for $n{>}2$) or (5.21) (for $n{=}2$). In the background of our irreducible root-system solutions, the Weyl group identifies the f_α and g_α coefficients for all roots of the same length. Hence, besides the f_{L} and f_{S} values in (6.9) we have couplings g_{L} and g_{S} for a number p_{L} and p_{S} of long and short positive roots, respectively.[13] This simplifies the 'sum rule'

$$\sum_\alpha f_\alpha\,\alpha{\otimes}\alpha \;=\; (1{-}f_R)\,\mathbb{1} \qquad \overset{\text{trace}}{\longrightarrow} \qquad \sum_\alpha \alpha{\cdot}\alpha\, f_\alpha \;=\; n\,(1{-}f_R) \tag{8.2}$$

$$\text{to} \qquad 2 f_{\mathrm{L}}\, p_{\mathrm{L}} \;+\; \frac{2}{r} f_{\mathrm{S}}\, p_{\mathrm{S}} \;=\; n\,(1{-}f_R) \qquad \overset{g_{\mathrm{L}},g_{\mathrm{S}}\neq 0}{\longrightarrow} \qquad p \;=\; n\,(1{-}f_R)\,. \tag{8.3}$$

We first consider $n{>}2$, hence $g_R{=}0$ and $f_R{=}0$ for $C{\neq}0$. Since the total number p of positive roots exceeds n (except for $A_1^{\oplus n}$), we are forced to put either $g_{\mathrm{S}} = 0$ or $g_{\mathrm{L}} = 0$. This fixes all coefficients for $n{\geq}3$ to

$$\text{either} \quad g_{\mathrm{S}} = 0\,, \quad g_{\mathrm{L}} = g \quad \overset{(5.18)(8.3)}{\longrightarrow} \quad f_{\mathrm{S}} = \frac{r}{2}\frac{n - p_{\mathrm{L}}}{p_{\mathrm{S}}}\,, \quad f_{\mathrm{L}} = \frac{1}{2} \tag{8.4}$$

$$\text{or} \quad g_{\mathrm{S}} = g\,, \quad g_{\mathrm{L}} = 0 \quad \overset{(5.18)(8.3)}{\longrightarrow} \quad f_{\mathrm{S}} = \frac{r}{2}\,, \qquad f_{\mathrm{L}} = \frac{1}{2}\frac{n - p_{\mathrm{S}}}{p_{\mathrm{L}}}\,. \tag{8.5}$$

All simply-laced (ADEH) systems are immediately excluded because they have $f_\alpha = \frac{1}{h^\vee}$, as is seen in (6.4). In the non-simply-laced (BCFG) one-parameter family (6.9) with (6.10), however, there is always one member which obeys (8.4) or (8.5) and therefore (8.2). Furthermore, the trace of (5.18) follows from (6.12) because g_α and f_β are constant on Φ_{L}^+ and Φ_{S}^+. The same consideration simplifies the expression (5.9) for the bosonic potential at $f_R{=}0{=}g_R$ to

$$V_B \;=\; \sum_{\alpha\in\Phi^+} \frac{v_\alpha}{(\alpha{\cdot}x)^2}\,, \qquad \text{where} \qquad v_\alpha = \hbar^2\frac{f_\alpha^2}{r_\alpha^3} \quad \text{or} \quad v_\alpha = \frac{1}{r_\alpha}\left(g_\alpha^2{+}\frac{\hbar^2}{4}\right) \tag{8.6}$$

---

[13] For expliciteness, $p_{\mathrm{L}} = \frac{n}{2}\,\frac{r h^\vee - h}{r-1}$ and $p_{\mathrm{S}} = \frac{n}{2}\,\frac{r(h - h^\vee)}{r-1}$, with the sum $p = p_{\mathrm{L}} + p_{\mathrm{S}} = \frac{n}{2}h$.

for any positive root α with length$^2 = \frac{2}{r_\alpha}$, depending on whether g_α vanishes or not. It remains to check the traceless part of (5.18) for the choice (8.4) or (8.5). Unfortunately, this is never fulfilled for $n>2$, except in the reducible case of $A_1^{\oplus n}$. This failure extends to the deformed root systems, e.g. our orthocentric simplex backgrounds. This rules out $U_{\mathrm{hom}}\equiv 0$ solutions to the flatness condition for all known irreducible WDVV backgrounds at $n>2$.

Therefore, in our search for $C\neq 0$ solutions (F,U) with $U_{\mathrm{hom}}\equiv 0$, we are forced back to two dimensions, i.e. systems of not more than three particles. The plethora of $n=2$ WDVV solutions F (parametrized by polygonal chains) may be cut down by invoking physical arguments. If a solution is supposed to describe the relative motion of three identical particles, then permuting their coordinates x^I must be equivalent to permuting the covectors (up to sign). After separating the center-of-mass coordinate, the planar set $\{\pm\alpha\}$ should thus be invariant under the irreducible two-dimensional representation of $\mathcal{S}_3$. To visualize the situation, consider the $\mathbb{R}^3$ frame rotation by the orthogonal matrix

$$
O \;=\; \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \;:\qquad e_I \;\overset{O}{\longmapsto}\; \sqrt{\frac{2}{3}}\begin{pmatrix} \cos\phi_I \\ \sin\phi_I \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{with}\quad \phi_I = \frac{2\pi I}{3} - \frac{\pi}{2}\;. \quad (8.7)
$$

In the rotated frame, the 3-direction describes the center-of-mass motion, and the first two entries correspond to the relative-motion plane, on which the $\mathcal{S}_3$ representation acts by reflections and $\frac{2\pi}{3}$ rotations. Reversely, the relative-motion plane is embedded back into the $\mathbb{R}^3$ configuration space of the total motion and rotated to the x^I frame via

$$
\alpha_{\mathrm{rel}} \;=\; \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \;\hookrightarrow\; \alpha_{\mathrm{tot}} \;=\; \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \;\overset{O^T}{\longmapsto}\; \sqrt{\frac{2}{3}}\begin{pmatrix} \sin(\phi+\frac{\pi}{3}) \\ \sin(\phi-\frac{\pi}{3}) \\ -\sin\phi \end{pmatrix} \;=\; \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}\;, \quad (8.8)
$$

so that the new direction $(0,0,1)$ becomes the center-of-mass covector $\rho = \frac{1}{\sqrt{3}}(1,1,1)$. The $\mathcal{S}_3$ action is generated by $\phi \to \phi + \frac{2\pi}{3}$ and $\phi \to \pi - \phi$, which produces all permutations of the α_{tot} entries and hence permutes the $\{x^I\}$ as required. The $\mathcal{S}_3$ orbit of α_{rel} is given by the angle set

$$
\left\{\; \pm\phi\,,\; \pm\phi + \frac{2\pi}{3}\,,\; \pm\phi - \frac{2\pi}{3}\;\right\} \qquad \overset{\phi\ \mathrm{special}}{\longrightarrow} \qquad \left\{\, 0\,,\, \pm\frac{2\pi}{3}\,\right\} \quad \text{or}\quad \left\{\, \pi\,,\, \pm\frac{\pi}{3}\,\right\}\,, \qquad (8.9)
$$

where the shorter orbits occur for $\phi = 0$ or $\phi = \pi$, modulo $\frac{2\pi}{3}$. The upshot is that the two-dimensional covectors must form a reflection-symmetric arrangement of A_2 systems!

In two dimensions, we take advantage of the radial terms in the structure equations and turn on all g couplings, which yields (cf. (5.22))

$$
f_\alpha = \frac{1}{\alpha\cdot\alpha}\;\forall\alpha\,,\quad f_R = 1 - \frac{p}{2}\,,\quad g_R = \frac{p-2}{p}\,C \qquad \text{and}\qquad \sum_{\alpha<\beta}(g_\alpha + g_\beta)\,\frac{\alpha\cdot\beta}{\alpha\cdot x\ \beta\cdot x} = 0 \quad (8.10)
$$

for some ordering of covectors. The bosonic potential (5.9) specializes to

$$
V_B = \frac{1}{2}\sum_\alpha \left(g_\alpha^2 + \frac{\hbar^2}{4}\right)\frac{\alpha\cdot\alpha}{(\alpha\cdot x)^2} + \frac{p^2-4}{2}\left(\frac{C^2}{p^2} - \frac{\hbar^2}{4}\right)\frac{1}{R^2} \qquad \text{with}\quad R^2 = (x^1)^2 + (x^2)^2\,. \quad (8.11)
$$

This formula remains correct in the full three-dimensional configuration space, where one may add the center-of-mass contribution $V_B^{\mathrm{com}} = \frac{1}{2}X^{-2}(C_{\mathrm{com}}^2 + \frac{\hbar^2}{4})$. Please note, however, that R still refers to the relative-motion subspace,

$$
R^2 \quad \hookrightarrow \quad x^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x \quad \xrightarrow{O^T} \quad \frac{1}{3} x^T \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} x \quad \neq \quad \sum_I (x^I)^2 . \tag{8.12}
$$

Consider now for $\{\alpha\}$ a collection of A_2 systems, each with its own g value and oriented at a particular angle in the relative-motion plane. Because each A_2 system fulfils the flatness condition by itself, we only have to compute the 'cross terms' in (8.10). Introducing the polar angles ϕ_α, ϕ_β and ϕ_x of α, β and x, respectively, the contributions

$$
\frac{\alpha \cdot \beta}{\alpha \cdot x \ \beta \cdot x} \quad = \quad \frac{\cos(\phi_\alpha - \phi_\beta)}{\cos(\phi_x - \phi_\alpha) \ \cos(\phi_x - \phi_\beta)} \quad = \quad \frac{\tan(\phi_x - \phi_\alpha) - \tan(\phi_x - \phi_\beta)}{\tan(\phi_\beta - \phi_\alpha)} \tag{8.13}
$$

to (8.10) collapse in telescopic sums, if and only if the reflection of any covector on any other one produces again a covector, and the couplings of mirror-image covectors are identified. Therefore, the orientations of the various A_2 systems must be isotropic, i.e. their collection forms an $I_2(p)$ system with $p = 3q$. Ordering the positive roots according to their polar angles $\phi_k = k\frac{\pi}{p}$ with $k = 0, 1, \ldots, p-1$, we get

$$
g_k = g \quad \text{for } p \text{ odd} \qquad \text{or} \qquad g_{2\ell} = g \quad \text{and} \quad g_{2\ell+1} = g' \quad \text{for } p \text{ even}, \tag{8.14}
$$

so that $\sum_\alpha g_\alpha = \frac{2}{p}C = p\,g$ or $\frac{p}{2}(g+g')$, respectively. Via (8.8) we further obtain

$$
\frac{\alpha \cdot x}{\sqrt{\alpha \cdot \alpha}} \quad \rightarrow \quad \sqrt{\frac{2}{3}} \left(\sin\left(k\frac{\pi}{p}+\frac{\pi}{3}\right) \cdot x^1 \ + \ \sin\left(k\frac{\pi}{p}-\frac{\pi}{3}\right) \cdot x^2 \ - \ \sin\left(k\frac{\pi}{p}\right) \cdot x^3 \right) . \tag{8.15}
$$

To see a few simple examples, let us give explicit results for $p = 3$, 6 and 12.

A_2 model. The minimal model, $p=3$, has $f_R = -\frac{1}{2}$ and $g_R = \frac{1}{3}C$ and a single free coupling $g = \frac{2}{9}C$. The radial terms are essential. In F and U appear the coordinate combinations

$$
\frac{\alpha \cdot x}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{1}{\sqrt{2}}(x^1 - x^2), \ \frac{1}{\sqrt{2}}(x^1 - x^3), \ \frac{1}{\sqrt{2}}(x^2 - x^3) \right\} \qquad \text{and}
$$
$$
R^2 \equiv \frac{1}{3} x^T \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} x \ = \ \frac{1}{3} \left((x^1 - x^2)^2 + (x^2 - x^3)^2 + (x^3 - x^1)^2 \right) , \tag{8.16}
$$

so that the bosonic potential becomes

$$
V_B \ = \ \left(g^2 + \frac{\hbar^2}{4} \right) \left(\frac{1}{(x^1 - x^2)^2} + \frac{1}{(x^2 - x^3)^2} + \frac{1}{(x^3 - x^1)^2} \right) \ + \ \frac{5}{8} \left(9g^2 - \hbar^2 \right) \frac{1}{R^2} . \tag{8.17}
$$

G_2 model. At $p=6$, two A_2 systems (with couplings g and g') are superposed with a relative angle of $\frac{\pi}{6}$. With $f_R = -2$ and $g_R = \frac{2}{3}C$ one has $g + g' = \frac{1}{9}C$. We read off the combinations

$$
\frac{\alpha \cdot x}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{x^1 - x^2}{\sqrt{2}}, \ \frac{2x^1 - x^2 - x^3}{\sqrt{6}}, \ \frac{x^1 - x^3}{\sqrt{2}}, \ \frac{x^1 + x^2 - 2x^3}{\sqrt{6}}, \ \frac{x^2 - x^3}{\sqrt{2}}, \ \frac{-x^1 + 2x^2 - x^3}{\sqrt{6}} \right\} \quad \text{and}
$$
$$
R^2 = \frac{1}{3} \left((x^1 - x^2)^2 + \text{cyclic} \right) \ = \ \frac{1}{9} \left((2x^1 - x^2 - x^3)^2 + \text{cyclic} \right)
$$
$$
\tag{8.18}
$$

and obtain

$$V_B \;=\; \frac{g^2+\frac{\hbar^2}{4}}{(x^1-x^2)^2} \;+\; \frac{3\,(g'^2+\frac{\hbar^2}{4})}{(2x^1-x^2-x^3)^2} \;+\; \text{cyclic} \;+\; \frac{36(g+g')^2-4\hbar^2}{R^2} \;. \tag{8.19}$$

$I_2(12)$ **model.** Integrable three-particle models based on A_2 and G_2 have been discussed in the literature before. Among the infinity of novel models, we take $p{=}12$, which yields $f_R{=}{-}5$ and $g_R{=}\frac{5}{6}C$, thus $g+g' = \frac{1}{36}C$. In addition to the positive roots of the G_2 model (now all 'even' with coupling g), we have six 'odd' roots (with coupling g'),

$$\left.\frac{\alpha\cdot x}{\sqrt{\alpha\cdot\alpha}}\right|_{\text{odd}} \in \left\{ \begin{array}{c} \frac{\tau x^1-x^2-\bar\tau x^3}{\sqrt{3}} , \frac{\tau x^1-\bar\tau x^2-x^3}{\sqrt{3}} , \frac{x^1+\bar\tau x^2-\tau x^3}{\sqrt{3}} , \\[2mm] \frac{\bar\tau x^1+x^2-\tau x^3}{\sqrt{3}} , \frac{-\bar\tau x^1+\tau x^2-x^3}{\sqrt{3}} , \frac{-x^1+\tau x^2-\bar\tau x^3}{\sqrt{3}} \end{array} \right\} \tag{8.20}$$

where $\tau = \frac{1}{2}(\sqrt{3}+1)$ and $\bar\tau = \frac{1}{2}(\sqrt{3}-1)$. The bosonic potential reads

$$V_B = \frac{g^2+\frac{\hbar^2}{4}}{(x^1-x^2)^2} \;+\; \frac{3\,(g^2+\frac{\hbar^2}{4})}{(2x^1-x^2-x^3)^2} \;+\; \frac{\frac{3}{2}\,(g'^2+\frac{\hbar^2}{4})}{(\tau x^1-\bar\tau x^2-x^3)^2} \;+\; \frac{\frac{3}{2}\,(g'^2+\frac{\hbar^2}{4})}{(\tau x^1-x^2-\bar\tau x^3)^2} \;+\; \text{cyclic}$$
$$+\; \frac{630(g+g')^2-\frac{35}{2}\hbar^2}{R^2} \;. \tag{8.21}$$

As has been displayed in (8.16) and (8.18), for permutation symmetric models the radial coordinate R may be expressed via any triple Γ of roots related by $\frac{\pi}{3}$ rotations,

$$\sum_{\alpha\in\Gamma} \frac{\alpha\otimes\alpha}{\alpha\cdot\alpha} \;=\; \frac{3}{2}\,\mathbb{1} \qquad \longrightarrow \qquad R^2 \;=\; \frac{2}{3}\sum_{\alpha\in\Gamma}\frac{(\alpha\cdot x)^2}{\alpha\cdot\alpha} \;, \tag{8.22}$$

so that, for instance, the radial parts of the prepotentials (5.6) may be rewritten as

$$F_R \;=\; -\frac{1}{6}f_R\left(\sum_{\alpha\in\Gamma}\frac{(\alpha\cdot x)^2}{\alpha\cdot\alpha}\right)\ln\left(\sum_{\alpha\in\Gamma}\frac{(\alpha\cdot x)^2}{\alpha\cdot\alpha}\right) , \quad U_R \;=\; -\frac{1}{2}g_R\ln\left(\sum_{\alpha\in\Gamma}\frac{(\alpha\cdot x)^2}{\alpha\cdot\alpha}\right) \;. \tag{8.23}$$

The appearance of *sums of roots* under the logarithm is new.

We further comment that the radial terms for $I_2(3q)$ models with q even can be eliminated in two ways. First, choosing $g+g' = 0$ and the classical limit $\hbar \to 0$, one obtains a conventional model (with covector terms only), but at the expense of putting $C{=}0$. Second, taking $g' = 0$ we can relax the condition $\alpha\cdot\alpha f_\alpha{=}1$ for the odd roots and thus put $f_R{=}0{=}g_R$ in this case, which then yields $\alpha\cdot\alpha f_\alpha = \frac{4-p}{p}$ for the odd roots and fixes $g = \frac{2}{p}C$. The bosonic potential in this special situation becomes

$$V_B \;=\; \frac{1}{2}\left(g^2 + \frac{\hbar^2}{4}\right)\sum_{\alpha \text{ even}}\frac{\alpha\cdot\alpha}{(\alpha\cdot x)^2} \;+\; \frac{\hbar^2}{8}\frac{(4-p)^2}{p^2}\sum_{\alpha \text{ odd}}\frac{\alpha\cdot\alpha}{(\alpha\cdot x)^2} \;, \tag{8.24}$$

so in the classical limit only half of the roots remain. Please note that this result differs from any limit of the generic case (8.11). Of course, the role of even and odd roots may be interchanged. The results of [3] and [6] describe examples of this kind.

The other dihedral groups may also be used to construct three-particle models, which however lack the permutation symmetry. Again we give a couple of prominent examples:

$A_1 \oplus A_1$ model. This model is reducible from the outset. From $p=2$ it follows that $f_R=0=g_R$ so that $g+g' = C$. The two orthogonal positive roots are mapped via (8.15) to

$$\frac{\alpha \cdot x}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{1}{\sqrt{2}}(x^1-x^2), \frac{1}{\sqrt{6}}(x^1+x^2-2x^3) \right\}, \tag{8.25}$$

and one finds

$$V_B = \frac{g^2+\frac{\hbar^2}{4}}{(x^1-x^2)^2} + \frac{3\,(g'^2+\frac{\hbar^2}{4})}{(x^1+x^2-2x^3)^2}. \tag{8.26}$$

Adding the cyclic permutations, one seems to arrive at the G_2 model but cannot produce the (necessary) radial term in this manner.

BC_2 model. The $p=4$ case features angles of $\frac{\pi}{4}$. With $f_R=-1$, $g_R=\frac{1}{2}C$ and $g+g' = \frac{1}{4}C$, the one-forms

$$\frac{\alpha \cdot x}{\sqrt{\alpha \cdot \alpha}} \in \left\{ \frac{1}{\sqrt{2}}(x^1-x^2), \frac{1}{\sqrt{3}}(\tau x^1-\bar{\tau}x^2-x^3), \frac{1}{\sqrt{6}}(x^1+x^2-2x^3), \frac{1}{\sqrt{3}}(-\bar{\tau}x^1+\tau x^2-x^3) \right\} \tag{8.27}$$

enter in

$$V_B = \frac{g^2+\frac{\hbar^2}{4}}{(x^1-x^2)^2} + \frac{3\,(g^2+\frac{\hbar^2}{4})}{(x^1+x^2-2x^3)^2} + \frac{\frac{3}{2}\,(g'^2+\frac{\hbar^2}{4})}{(\tau x^1-\bar{\tau}x^2-x^3)^2} + \frac{\frac{3}{2}\,(g'^2+\frac{\hbar^2}{4})}{(-\bar{\tau}x^1+\tau x^2-x^3)^2} + \frac{6(g+g')^2-\frac{3}{2}\hbar^2}{R^2}. \tag{8.28}$$

Again, this looks like a truncation of the model with $p \to 3p$. Regarding the explicit form of the above expressions, those are unique only up to rotations around the center-of-mass axis $\rho = \frac{1}{\sqrt{3}}(1,1,\ldots,1)$. Our convention has been to take the first root as $e_1 \in \mathbb{R}^2$, which maps to $\frac{1}{\sqrt{2}}(e_1-e_2) \in \mathbb{R}^3$ under O^T in (8.8).

The given examples should suffice to illustrate the general pattern of dihedral $n=2$ solutions with $U_{\text{hom}} \equiv 0$: The root systems of odd or even order give rise to one- or two-parameter three-particle models, which are permutation invariant only when the order is a multiple of three. Except for the reducible case of $I_2(2)$, the radial contributions are needed; they may disappear only when one of the two couplings in the even case vanishes.

9 $U \neq 0$ solutions: three-particle systems in full

Any solution U (including the trivial $U \equiv 0$ one) for a given F background can be modified by adding to it a homogeneous function U_{hom} satisfying $x^i \partial_i U_{\text{hom}} = 0$ and (5.14) for $C=0$. As we have seen in the previous section, including this freedom is in fact mandatory for finding $n>2$ solutions in the first place. In the three-particle case ($n=2$), however, we have identified an infinite series of special solutions, for which we now investigate the corresponding extension by U_{hom}. In effect, this will add one additional coupling parameter to the models of the previous section.

To construct $U_{\rm hom}$ for a rank-two system specified by $\{\alpha, f_\alpha, f_R\}$, it suffices to solve (5.15) for $C{=}0$, so that f_R drops out. As $U_{\rm hom}$ depends only on the ratio x^2/x^1 we change to polar angles ϕ and ϕ_α via

$$\frac{x^2}{x^1} \;=\; \tan\phi \qquad \text{and} \qquad \frac{\alpha\cdot x}{\sqrt{\alpha\cdot\alpha}} \;=\; R\,\cos(\phi{-}\phi_\alpha) \tag{9.1}$$

and arrive at

$$U_{\rm hom}''(\phi) - h(\phi)\,U_{\rm hom}'(\phi) \;=\; 0 \qquad \text{with} \qquad h(\phi) \;=\; \sum_\alpha f_\alpha\,\alpha{\cdot}\alpha\,\tan(\phi{-}\phi_\alpha)\,. \tag{9.2}$$

This is easily integrated (with an integration constant λ) to

$$U_{\rm hom}'(\phi) \;=\; \lambda \prod_\alpha \bigl[\cos(\phi{-}\phi_\alpha)\bigr]^{-f_\alpha\,\alpha\cdot\alpha} \;\propto\; R^{2(1-f_R)} \prod_\alpha (\alpha{\cdot}x)^{-f_\alpha\,\alpha\cdot\alpha} \tag{9.3}$$

and blows up on the lines orthogonal to the covectors α. Generically, the singularities are $\sim (\alpha{\cdot}x)^{-1}$. Only in case some g_α vanishes, the corresponding $f_\alpha\alpha{\cdot}\alpha$ need not equate to one, thus $U_{\rm hom}'$ may have a more general singularity structure. For a dihedral configuration with nonvanishing couplings g_α we can go further since $\phi_\alpha = k\frac{\pi}{p}$ with $k = 0,\ldots,p{-}1$, which yields

$$h(\phi) = \begin{Bmatrix} p\,\tan(p\phi) & \text{for } p \text{ odd} \\ -p\,\cot(p\phi) & \text{for } p \text{ even} \end{Bmatrix} \quad\longrightarrow\quad U_{\rm hom}' = \begin{Bmatrix} \lambda\,\bigl[\cos(p\phi)\bigr]^{-1} & \text{for } p \text{ odd} \\ \lambda\,\bigl[\sin(p\phi)\bigr]^{-1} & \text{for } p \text{ even} \end{Bmatrix} \tag{9.4}$$

and thus ('$\simeq$' means 'modulo constant terms')

$$U_{\rm hom}(\phi) \;\simeq\; \frac{1}{p}\lambda\,\ln\left|\tan\left(\frac{p}{2}\phi{+}\delta\right)\right| \qquad \text{with} \qquad \delta = \begin{cases} \frac{\pi}{4} & \text{for } p \text{ odd} \\ 0 & \text{for } p \text{ even} \end{cases}. \tag{9.5}$$

This may be compared with the particular solution (8.1),

$$U_{\rm part} \;=\; -\sum_\alpha g_\alpha\,\ln|\alpha{\cdot}x| - g_R\,\ln R \;\simeq\; -\sum_\alpha g_\alpha\,\ln|\cos(\phi{-}\phi_\alpha)| - C\,\ln R\,, \tag{9.6}$$

which, in the dihedral case, can be simplified to (remember that $\sum_\alpha g_\alpha{+}g_R = C$)

$$U_{\rm part} \;\simeq\; -C\,\ln R \;-\; \begin{cases} g\,\ln|\cos(p\phi)| & \text{for } p \text{ odd} \\ g\,\ln|\cos(\frac{p}{2}\phi)| + g'\,\ln|\sin(\frac{p}{2}\phi)| & \text{for } p \text{ even}\,. \end{cases} \tag{9.7}$$

Combining $U_{\rm part} + U_{\rm hom} = U$ and lifting to the full configuration space $\mathbb{R}^3 \ni (x^I)$, we find

$$\partial_I U \;=\; -\sum_\alpha g_\alpha\,\frac{\alpha_i}{\alpha{\cdot}x} \;-\; \frac{p-2}{p}C\frac{x_i}{R} \;+\; \lambda \begin{pmatrix} x^2{-}x^3 \\ x^3{-}x^1 \\ x^1{-}x^2 \end{pmatrix} R^{p-2} \prod_\alpha (\alpha{\cdot}x)^{-1}\,. \tag{9.8}$$

For the simplest dihedral example, the A_2 system, with $(x^{ij} := x^i{-}x^j)$

$$F \;=\; -\frac{1}{4}\bigl[(x^{12})^2\ln|x^{12}| + (x^{23})^2\ln|x^{23}| + (x^{31})^2\ln|x^{31}|\bigr] \;+\; \frac{1}{4}\,R^2\ln R\,, \tag{9.9}$$

one gets

$$\partial_I U \;=\; [x^{12}x^{23}x^{31}]^{-1} \begin{pmatrix} [\lambda R - g(x^{31}-x^{12})]\,x^{23} \\ [\lambda R - g(x^{12}-x^{23})]\,x^{31} \\ [\lambda R - g(x^{23}-x^{31})]\,x^{12} \end{pmatrix} \;-\; \frac{3}{2}g\,R^{-2} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{9.10}$$

which extends the bosonic potential (8.17) to

$$\begin{aligned}
V_B \;&=\; \left(g^2 + \frac{2}{3}\lambda^2 + \frac{\hbar^2}{4}\right)\left(\frac{1}{(x^{12})^2} + \frac{1}{(x^{23})^2} + \frac{1}{(x^{31})^2}\right) \\
&\quad +\; \frac{5}{8}\left(9g^2 - \hbar^2\right)\frac{1}{R^2} \;+\; \lambda\,g\,R\,\frac{(x^{12}-x^{23})(x^{23}-x^{31})(x^{31}-x^{12})}{(x^{12}x^{23}x^{31})^2} \;.
\end{aligned} \tag{9.11}$$

10   Conclusion

In this paper we systematically constructed conformal $(n{+}1)$-particle quantum mechanics in one space dimension with $\mathcal{N}{=}4$ supersymmetry, i.e. $su(1,1|2)$ invariance, and a central charge C. To begin with, the closure of the superalgebra produced a set of 'structure equations' (4.2) and (4.3) for two scalar prepotentials U and F, which determine the potential schematically as $V = \frac{1}{2}U'U' + \frac{\hbar^2}{8}F'''F'''$ plus fermionic terms. The structure equations consist of homogeneity conditions depending on C, a (generalized) WDVV equation (for F alone) and a 'flatness condition' (for U in the F background).

Separating the center-of-mass degree of freedom reduces the configuration space from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$ for the relative motion. The ansatz (5.6) for the many-body functions U and F turned the structure equations into a decomposition of the identity (5.7) and nonlinear algebraic relations (5.10) and (5.18), for a set $\{\alpha\}$ of covectors in $\mathbb{R}^n$ and real coupling coefficients g_α (for U) and f_α (for F). The homogeneous part of U' is governed by a linear differential equation (5.14) (with $C{=}0$) of Fuchsian type. The case of three particles is special, because the WDVV equation is empty and so anything goes for F, but the flatness condition for U is still nontrivial.

To find the prepotential F it suffices to solve the WDVV equation (5.10). It is known that the roots of any finite reflection group provide a solution [9, 10], each giving rise to an interacting quantum mechanics model with $U{\equiv}0$ and thus $C{=}0$. Besides rederiving this result in a new fashion, we were able to generalize it in two ways: First, the A_n root system may be deformed to a system of edges for a general orthocentric n-simplex, yielding a nontrivial n-parameter family of WDVV solutions which might agree with one found in [11]. Second, the relative weights for the long and the short roots contributing to F are undetermined even in sign, so that the BCF-type solutions form one-parameter families.

For a nonzero central charge, in any given F background one must turn on the prepotential U by solving (5.14). Within our ansatz (5.6), this requires finding a suitable homogeneous part U_{hom} — an unsolved task. Only if in appropriate coordinates the system decomposes into subsystems not larger than rank two, then U_{hom} is not needed but can easily be found. Thus for the special case of three particles, i.e. $n{=}2$, the situation is simpler: the flatness condition (5.20) then permits the novel 'radial terms' which provided the

necessary flexibility in our ansatz (5.6). Again the covectors were forced into a root system, which as of rank two must be dihedral. We explicitly constructed the full prepotentials (including U_{hom}) for the new infinite dihedral series and displayed several examples lifted back to the original configuration space $\mathbb{R}^3 \ni (x^1, x^2, x^3)$. When the dihedral group and the central charge are fixed, the model depends on one or two tunable coupling parameters depending on the group order p being odd or even. Permutation symmetry requires p to be a multiple of 3. The previously found models [3, 6] turned out to be either decomposable or peculiar special cases of our dihedral systems, for which the 'radial terms' could by omitted. To summarize, we have classified all one-dimensional $\mathcal{N}{=}4$ superconformal quantum three-particle models based on covectors.

It remains an open problem to construct any irreducible $U{\neq}0$ solutions with more than three particles and to find all $U{\equiv}0$ solutions, i.e. the complete moduli space of the WDVV equation. To complement recent progress in mathematics on this issue [17], we would like to propose another strategy towards this goal: take any simple Lie algebra, select one of its irreducible representations and form the convex hull of its weight system. The edges of this polytope reproduce the roots, with certain multiplicities. Now consider a deformation of this polytope. Generically, the degeneracy of the edge orientations will be lifted, but the deformed collection of covectors still satisfies the incidence relation of the polytope. We suggest to test the WDVV equation on such configurations, generalizing the method successful for the fundamental A_n representation. We are confident that this is feasible and will lead to further beautiful mathematical structures.

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Note added. After a first version of this work had appeared on the arXiv, several aspects discussed here have been developed further in the three related papers [18]–[20].

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