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# $d = 4$ black hole attractors in $N = 2$ supergravity with Fayet-Iliopoulos terms

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(Received 1 February 2008; published 28 April 2008)

We generalize the description of the $d = 4$ *attractor mechanism* based on an effective black hole potential to the presence of a gauging which does not modify the derivatives of the scalars and does not involve hypermultiplets. The obtained results do not rely necessarily on supersymmetry, and they can be extended to $d > 4$, as well. Thence, we work out the example of the *stu* model of $N = 2$ supergravity in the presence of Fayet-Iliopoulos terms, for the supergravity analogues of the *magnetic* and $D0$-$D6$ black hole charge configurations, and in three different symplectic frames: the $(SO(1,1))^2$, $SO(2,2)$ covariant and $SO(8)$-truncated ones. The attractive nature of the critical points, related to the semipositive definiteness of the Hessian matrix, is also studied.

DOI: 10.1103/PhysRevD.77.085027 PACS numbers: 04.65.+e, 04.70.Dy, 11.25.Mj

## I. INTRODUCTION

The so-called *attractor mechanism* was discovered in the mid-1990's [1–5] in the context of extremal BPS (Bogomol'ny-Prasad-Sommerfeld) black holes (BHs). Recently, extremal BH *attractors* have been intensively studied [6–57], mainly due to the (re)discovery of new classes of scalar attractor configurations. Differently from the BPS ones, such configurations do not saturate the BPS bound [58] and, when considering a supergravity theory, they break all supersymmetries at the BH event horizon.

The *attractor mechanism* was first discovered in *ungauged* supergravities, with various supercharges and in various space-time dimensions. In all cases such a mechanism of purely charge-dependent stabilization of the scalars at the event horizon of an extremal BH can be *critically implemented*, i.e. the *attractor* configurations are nothing but the critical points of a suitably defined, positive definite *BH effective potential* function $V_{BH}$. The formalism based on $V_{BH}$ is intrinsically *off shell*; i.e. it holds in the whole scalar manifold, and it allows one to determine the classical Bekenstein-Hawking [59] BH entropy by going *on shell*, that is, by evaluating $V_{BH}$ at its *nondegenerate* critical points, i.e. at those critical points for which $V_{BH} \neq 0$. Furthermore, one can study the stability of the attractors by considering the sign of the eigenvalues of the (real form of the) Hessian matrix of $V_{BH}$ at its critical points. Actually, only those critical points corresponding to *all* strictly positive Hessian eigenvalues should be rigorously named *attractors*.

In this work we deal with the extension of the *effective potential formalism* in the presence of a particularly simple *gauging*. In other words, we determine the generalization of $V_{BH}$ to an *effective potential* $V_{eff}$, which takes into account also the (not necessarily positive definite) potential $V$ originated from the gauging. We show that the *attractor mechanism* can still be implemented in a critical fashion, i.e. that the stabilized configurations of the scalar fields, depending on the BH conserved electric and magnetic charges and on the gauge coupling constant $g$, can be determined as the *nondegenerate* critical points of $V_{eff}$. Concerning the $d = 4$ metric background, we consider a dyonic, extremal, static, spherically symmetric BH, which generally is asymptotically *nonflat*, due to the presence of a nonvanishing gauge potential $V$. As we will discuss further below, such a space-time background is *nonsupersymmetric*, and thus *all* attractors in such a framework will break *all* supersymmetries down.

As a particular working example, next we consider the so-called *stu* model with Fayet-Iliopoulos (FI) terms, in two manageable BH charge configurations, namely, in the supergravity analogues of the *magnetic* and $D0$-$D6$ ones. It is worth anticipating here that the obtained results (and their physical meaning) crucially depend on the choice of the symplectic basis used to describe the physical system at hand. Such a feature strongly distinguishes the gauged case from the ungauged one, the latter being insensitive to the choice of the symplectic basis. However, such a phenomenon, observed some time ago e.g. in [60], can be easily understood by noticing that, in general, the gauge potential $V$ is independent of the BH charges, and thus it is necessarily not symplectic invariant. Indeed, as we explicitly compute, the resulting explicit form of the *generalized* BH effective potential $V_{eff}$, its critical points, their positions in the scalar manifold, their stability features, and the corresponding BH entropies generally do depend on the symplectic frame being considered.

It is worth noticing here that the $SO(8)$-truncated symplectic basis (treated in Sec. III C) has the remarkable property that the FI potential $V_{FI,SO(8)}$ [given by

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Eq. (3.34) below] admits critical points at a finite distance from the origin. This is not the case for the FI potentials in the $(SO(1,1))^2$ and $SO(2,2)$ covariant bases treated in Secs. III A and III B. Indeed, $V_{\mathrm{FI},(SO(1,1))^2}$ and $V_{\mathrm{FI},SO(2,2)}$ [respectively given by Eqs. (3.9) and (3.18) below] only have a runaway behavior, admitting only minima at an infinite distance from the origin.

Finally, we address the issue of the stability of the various obtained *gauged attractors*. In light of the previous statement, not surprisingly, we find that the lifting features of the two non-BPS $Z \neq 0$ *massless* Hessian modes pertaining to the ungauged limit ($g = 0$) of the *stu* model depend on the symplectic basis being considered.

The plan of the paper is as follows. In Sec. II we exploit a general treatment of the *effective potential formalism* in the presence of a gauging which does not modify the derivatives of the scalars and does not involve hypermultiplets; such a treatment does not rely necessarily on supersymmetry.[1] Then, in Sec. III we apply the obtained general results to the case of the *stu* model in the presence of Fayet-Iliopoulos terms, in three different symplectic bases: the so-called $(SO(1,1))^2$, $SO(2,2)$ covariant and $SO(8)$-truncated ones, respectively, considered in Secs. III A, III B, and III C. Thus, in Sec. IV we compute, for the so-called magnetic and $D0$-$D6$ BH charge configurations (respectively treated in Secs. IV A and IV B), the critical points of the effective potential $V_{\mathrm{eff}}$ of the *stu* model in the symplectic frames introduced above, and then compare the resulting BH entropies, obtaining that the results will generally depend on the symplectic frame considered. Section V is devoted to the analysis of the stability of the critical points of $V_{\mathrm{eff}}$ computed in Sec. IV, in both the magnetic and $D0$-$D6$ BH charge configurations (respectively treated in Secs. V A and V B). Final comments and ideas for further developments are given in Sec. VI.

## II. GENERAL ANALYSIS

Let us start from the bosonic sector of *gauged* $N = 2$, $d = 4$ supergravity. We consider a gauging which does not modify the derivatives of the scalars and does not involve hypermultiplets. The appropriate action has the following form ($\mu = 0, 1, 2, 3$, $a = 1, \ldots, n_V$ and $\Lambda = 0, 1, \ldots, n_V$, $n_V$ being the number of vector multiplets; see e.g. [61]):

$$S = \int\Big(-\frac{1}{2}R + G_{a\bar{b}}(z,\bar{z})\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} + \mu_{\Lambda\Sigma}(z,\bar{z})\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\Sigma|\mu\nu} + \nu_{\Lambda\Sigma}(z,\bar{z})\mathcal{F}^\Lambda_{\mu\nu}{}^*\mathcal{F}^{\Sigma|\mu\nu} - V(z,\bar{z})\Big)\sqrt{-g}d^4x^\mu, \tag{2.1}$$

where $\mu_{\Lambda\Sigma} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$, $\nu_{\Lambda\Sigma} = \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$, $\mathcal{F}^\Lambda_{\mu\nu} = \frac{1}{2} \times (\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu)$, and $*$ denotes Hodge duality ($\varepsilon^{0123} = -\varepsilon_{0123} \equiv -1$),

$${}^*\mathcal{F}^{\Lambda|\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\lambda\rho}\mathcal{F}^\Lambda_{\lambda\rho}. \tag{2.2}$$

Varying the action (2.1) with respect to $g^{\mu\nu}$, $z^a$, and $A^\Lambda_\mu$, one can easily find the following equations of motion:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} - 2G_{a\bar{b}}\partial_{(\mu}z^a\partial_{\nu)}\bar{z}^{\bar{b}} + G_{a\bar{b}}\partial_\lambda z^a\partial^\lambda\bar{z}^{\bar{b}}g_{\mu\nu} = T_{\mu\nu} + Vg_{\mu\nu}; \tag{2.3}$$

$$\frac{G_{a\bar{b}}}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\bar{z}^{\bar{b}}) + \frac{\partial G_{a\bar{b}}}{\partial\bar{z}^{\bar{c}}}\partial_\lambda\bar{z}^{\bar{b}}\partial^\lambda\bar{z}^{\bar{c}} = \frac{\partial\mu_{\Lambda\Sigma}}{\partial z^a}\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\Sigma|\mu\nu} + \frac{\partial\nu_{\Lambda\Sigma}}{\partial z^a}\mathcal{F}^\Lambda_{\mu\nu}{}^*\mathcal{F}^{\Sigma|\mu\nu} - \frac{\partial V}{\partial z^a}; \tag{2.4}$$

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\nu\mathcal{G}_{\Lambda|\lambda\rho} = 0, \tag{2.5}$$

where $\mathcal{G}_{\Lambda|\mu\nu} \equiv \nu_{\Lambda\Sigma}\mathcal{F}^\Sigma_{\mu\nu} - \mu_{\Lambda\Sigma}{}^*\mathcal{F}^\Sigma_{\mu\nu}$ and

$$T_{\mu\nu} \equiv 4\mu_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\mu\lambda}\mathcal{F}^\Sigma_{\nu\rho}g^{\lambda\rho} - \mu_{\Lambda\Sigma}\mathcal{F}^\Lambda_{\lambda\rho}\mathcal{F}^{\Sigma|\lambda\rho}g_{\mu\nu}. \tag{2.6}$$

Moreover, one also obtains

$$\varepsilon^{\mu\nu\lambda\rho}\partial_\nu\mathcal{F}^\Lambda_{\lambda\rho} = 0, \tag{2.7}$$

which directly follows from the very definition of $\mathcal{F}^\Lambda_{\lambda\rho}$, and it is nothing but the Bianchi identities.

Let us now consider the dyonic, static, spherically symmetric extremal BH background. For this case, the most general metric reads

$$ds^2 = e^{2A(r)}dt^2 - e^{2B(r)}dr^2 - e^{2C(r)}r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.8}$$

It is worth pointing out that only two functions out of $A(r)$, $B(r)$, and $C(r)$ are independent; one of the three can be absorbed by a suitable redefinition of the radial coordinate $r$. As a consequence of the staticity and spherical symmetry, all scalar fields depend only on $r$, and intensities of electrical and magnetic fields are directed along the radial direction.

Let us now investigate the equation (2.5) of the electromagnetic field and the Bianchi identities (2.7). The electromagnetic field strength has only two nonzero components, namely, $\mathcal{F}^\Lambda_{01}$ and $\mathcal{F}^\Lambda_{23}$, corresponding to radially directed electrical and magnetic fields, respectively. Indeed, by solving the Bianchi identities (2.7), one gets that $\mathcal{F}^\Lambda_{01} = \mathcal{F}^\Lambda_{01}(r)$ and $\mathcal{F}^\Lambda_{23} = \mathcal{F}^\Lambda_{23}(\theta)$. The same holds for $\mathcal{G}_{\Lambda|\mu\nu}$, because by duality arguments Eq. (2.5) is nothing but the Bianchi identities for $\mathcal{G}_{\Lambda|\mu\nu}$. Substituting $\mathcal{F}^\Lambda_{01}$ and $\mathcal{F}^\Lambda_{23}$ into $\mathcal{G}_{\Lambda|01}$ and $\mathcal{G}_{\Lambda|23}$ and recalling the peculiar dependence on $r$ and $\theta$, one finally gets the following solution for the nonvanishing components of the electromagnetic field strength:

[1] In some respect, a similar treatment in $d = 5$ was recently given in [54].

$$\mathcal{F}^{\Lambda}_{01} = \frac{e^{A+B-2C}}{2r^2}(\mu)^{-1|\Lambda\Sigma}(\nu_{\Sigma\Gamma}p^{\Gamma} - q_{\Sigma}); \qquad \mathcal{F}^{\Lambda}_{23} = -\frac{1}{2}p^{\Lambda}\sin\theta, \tag{2.9}$$

where the constants $q_\Lambda$ and $p^\Lambda$, respectively, are the electrical and magnetic charges of the BH, i.e. the fluxes

$$\begin{pmatrix} p^{\Lambda} \\ q_{\Lambda} \end{pmatrix} \equiv \frac{1}{4\pi}\begin{pmatrix} \int_{S^2_\infty} \mathcal{F}^{\Lambda} \\ \int_{S^2_\infty} \mathcal{G}_{\Lambda} \end{pmatrix}. \tag{2.10}$$

$T_{\mu\nu}$ can be computed by substituting Eqs. (2.9) into Eq. (2.6). By doing so, one obtains

$$T^0_0 = T^1_1 = -T^2_2 = -T^3_3 = \frac{e^{-4C}}{r^4}V_{\rm BH}, \tag{2.11}$$

where $V_{\rm BH}$ is the standard, so-called *BH potential*, defined as (see e.g. [5])

$$V_{\rm BH} \equiv -\frac{1}{2}(p^{\Lambda}, q_{\Lambda}) \times \begin{pmatrix} \mu_{\Lambda\Sigma} + \nu_{\Lambda\Gamma}(\mu)^{-1|\Gamma\Pi}\nu_{\Pi\Sigma} & -\nu_{\Lambda\Gamma}(\mu)^{-1|\Gamma\Sigma} \\ -(\mu)^{-1|\Lambda\Gamma}\nu_{\Gamma\Sigma} & (\mu)^{-1|\Lambda\Sigma} \end{pmatrix} \times \begin{pmatrix} p^{\Sigma} \\ q_{\Sigma} \end{pmatrix}. \tag{2.12}$$

One can also easily check that

$$\frac{\partial \mu_{\Lambda\Sigma}}{\partial z^a}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma|\mu\nu} + \frac{\partial \nu_{\Lambda\Sigma}}{\partial z^a}\mathcal{F}^{\Lambda}_{\mu\nu}{}^{*}\mathcal{F}^{\Sigma|\mu\nu} = -\frac{e^{-4C}}{r^4}\frac{\partial V_{\rm BH}}{\partial z^a}. \tag{2.13}$$

Let us now write down Eqs. (2.3) and (2.4) explicitly. In order to do this, we substitute the metric (2.8) and Eqs. (2.11) and (2.13) into them, getting the following independent equations:

$$\begin{aligned}
& -e^{-2B}\left[C'(C'+2A') + \frac{2}{r}(C'+A') + \frac{1}{r^2}(1-e^{2(B-C)})\right] + e^{-2B}z^{a\prime}\bar{z}^{\bar{b}\prime}G_{a\bar{b}} = \frac{e^{-4C}}{r^4}V_{\rm BH} + V; \\
& -e^{-2B}\left[A'' + C'' + A'(A'-B') + C'(C'-B'+A') + \frac{2}{r}(A'-B'+2C')\right] - e^{-2B}z^{a\prime}\bar{z}^{\bar{b}\prime}G_{a\bar{b}} = -\frac{e^{-4C}}{r^4}V_{\rm BH} + V; \\
& e^{-2B}\left[\bar{z}^{\bar{b}\prime\prime}G_{a\bar{b}} + \bar{z}^{\bar{b}\prime}\bar{z}^{\bar{c}\prime}G_{a\bar{b},\bar{c}} + \bar{z}^{\bar{b}\prime}G_{a\bar{b}}\left(A'-B'+2C'+\frac{2}{r}\right)\right] = \frac{e^{-4C}}{r^4}\frac{\partial V_{\rm BH}}{\partial z^a} + \frac{\partial V}{\partial z^a},
\end{aligned} \tag{2.14}$$

where the prime denotes the derivative with respect to $r$.

Let us now perform the *near-horizon limit* of Eqs. (2.14). Within the assumptions (2.8) made for the 4-dimensional BH background, extremality implies that the near-horizon geometry is $AdS_2 \times S_2$. Thus, in the *near-horizon limit* the functions appearing in the metric (2.8) read

$$A = -B = \log\frac{r}{r_A}, \qquad C = \log\frac{r_H}{r}, \tag{2.15}$$

where the relation $A = -B$ has been obtained by suitably redefining the radial coordinate, and the constants $r_A$, $r_H$ are the *radii* of $AdS_2$ and $S_2$, respectively.

By assuming that the scalar fields $z^a$ are regular when approaching the BH event horizon ($\exists \lim_{r\to r_H} z^a(r) \equiv z^a_H$, $|z^a_H| < \infty$), the *near-horizon limits* of Eqs. (2.14) read

$$\frac{1}{r_H^2} = \frac{1}{r_H^4}V_{\rm BH} + V; \tag{2.16}$$

$$\frac{1}{r_A^2} = \frac{1}{r_H^4}V_{\rm BH} - V; \tag{2.17}$$

$$\frac{1}{r_H^4}\frac{\partial V_{\rm BH}}{\partial z^a} + \frac{\partial V}{\partial z^a} = 0, \tag{2.18}$$

respectively, yielding the following solutions:

$$r_H^2 = V_{\rm eff}(z_H, \bar{z}_H); \tag{2.19}$$

$$r_A^2 = \left.\frac{V_{\rm eff}}{\sqrt{1 - 4V_{\rm BH}|V|}}\right|_{z=z_H}; \tag{2.20}$$

$$\left.\frac{\partial V_{\rm eff}}{\partial z^a}\right|_{z=z_H} = 0, \tag{2.21}$$

where

$$V_{\rm eff} \equiv \frac{1 - \sqrt{1 - 4V_{\rm BH}V}}{2V} \tag{2.22}$$

and Eq. (2.21) characterizes the horizon configurations $z^a_H$ of the scalars as the critical points of $V_{\rm eff}$ in the scalar manifold. In other words, near the BH horizon the scalars $z^a$ are *attracted* towards the purely charge-dependent configurations $z^a_H(p, q)$ satisfying Eq. (2.21).

In the considered spherically symmetric framework, the Bekenstein-Hawking [59] BH entropy reads

$$S_{\rm BH} = \frac{A_H}{4} = \pi r_H^2 = V_{\rm eff}(z_H, \bar{z}_H). \tag{2.23}$$

Thus, in the presence of a nonvanishing scalar potential $V$ in the $d = 4$ Lagrangian density, the *attractor mechanism* [1–5] still works, with the usual $V_{\rm BH}$ replaced by $V_{\rm eff}$ defined by Eq. (2.22).

From its very definition (2.22), $V_{\rm eff}$ is defined in the region of the scalar manifold where $V_{\rm BH}V \le \frac{1}{4}$. Moreover, analogously to $V$, it is not necessarily positive, as instead $V_{\rm BH}$ is. It also holds that

$$\lim_{V\to 0} V_{\rm eff} = V_{\rm BH}; \tag{2.24}$$

$$\lim_{V_{\rm BH}\to 0^+} V_{\rm eff} = 0. \tag{2.25}$$

By denoting $\frac{\partial}{\partial z^a}$ as $\partial_a$, Eq. (2.22) yields

$$\partial_a V_{\rm eff} = \frac{2V^2\partial_a V_{\rm BH} - (\sqrt{1-4V_{\rm BH}V} + 2V_{\rm BH}V - 1)\partial_a V}{2V^2\sqrt{1-4V_{\rm BH}V}}, \tag{2.26}$$

which further restricts the region of definition of $V_{\rm eff}$ and $\partial_a V_{\rm eff}$ to $V_{\rm BH}V < \frac{1}{4}$. In such a region, $V_{\rm eff}$ also enjoys the following power series expansion:

$$\begin{aligned} V_{\rm eff} &= \frac{1}{2V} + \frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(V_{\rm BH})^n V^{n-1}}{2n-1} \\ &= V_{\rm BH} + (V_{\rm BH})^2 V + 2(V_{\rm BH})^3 V^2 + 5(V_{\rm BH})^4 V^3 \\ &\quad + \mathcal{O}((V_{\rm BH})^5 V^4). \end{aligned} \tag{2.27}$$

Let us finally point out that the extension of the formalism based on $V_{\rm BH}$ to the presence of a nonvanishing scalar potential $V$, obtained above, generally holds for a not necessarily supersymmetric theory described by the action (2.1), and does not rely on supersymmetry, and thus on $N$, at all. It is also clear that one can easily generalize the procedure exploited above for $d > 4$ (for the case $d = 5$, see also [54]).

However, when supersymmetry enters the game, one might ask about the supersymmetry-preserving features of the critical points of $V_{\rm eff}$. This makes sense whenever the near-horizon geometry of the considered class of BH backgrounds preserves some amount of supersymmetry, i.e. when it allows the supersymmetric variation of the (field strength of the) gravitino to vanish. In such a case, by studying the conditions of vanishing of the supersymmetric variation of the gauginos, one can establish the possible supersymmetry-preserving features of the scalar configuration corresponding to the considered critical point of $V_{\rm eff}$.

In the procedure performed above, we considered a dyonic, static, spherically symmetric, extremal BH background (2.8), whose near-horizon limit gives an $AdS_2 \times S_2$ geometry. Such factor spaces, respectively, have radii $r_A$ and $r_H$, which, as yielded by Eqs. (2.19) and (2.20), in the presence of a nonvanishing gauging and for $V_{\rm BH} \neq 0$, do *not* coincide. Thus, such a near-horizon geometry is *not* the Bertotti-Robinson (BR) one [62], corresponding to $AdS_2 \times S_2$ with $r_A = r_H$. The BR geometry, which has vanishing scalar curvature and is conformally flat, yields a maximally supersymmetric $N = 2$, $d = 4$ background (see e.g. [63,64], and references therein). In contrast, the near-horizon limit (2.15) of the metric (2.8) neither has vanishing scalar curvature nor is conformally flat; indeed its scalar curvature and nonvanishing components of the Weyl tensor can, respectively, be computed as

$$R = 2\left(\frac{1}{r_A^2} - \frac{1}{r_H^2}\right); \tag{2.28}$$

$$\begin{cases} C_{trtr} = -\frac{1}{3}\left(\frac{1}{r_A^2} - \frac{1}{r_H^2}\right); \\ C_{t\theta t\theta} = -\frac{r^2}{6r_A^4}(r_H^2 - r_A^2); \\ C_{t\varphi t\varphi} = -\frac{r^2}{6r_A^4}(r_H^2 - r_A^2)\sin^2\theta; \\ C_{r\theta r\theta} = \frac{r_A^2 - r_H^2}{6r^2}; \\ C_{r\varphi r\varphi} = \frac{r_A^2 - r_H^2}{6r^2}\sin^2\theta; \\ C_{\theta\varphi\theta\varphi} = -\frac{r_H^2}{3r_A^2}(r_H^2 - r_A^2)\sin^2\theta. \end{cases} \tag{2.29}$$

As shown in [65–67] (see also [68]), in the presence of a nonvanishing gauging, the vanishing of the $N = 2$, $d = 4$ supersymmetry variation of the gravitino in a static, spherically symmetric, extremal BH background holds only in the case of *naked singularities*. Thus, one can conclude that the near-horizon limit (2.15) of the metric (2.8) (with $r_A \neq r_H$) does *not* preserve any supersymmetry at all. Consequently, it does not make sense to address the issue of the supersymmetry-preserving features of the critical points of $V_{\rm eff}$ in such a framework. Necessarily, all scalar configurations determining critical points of $V_{\rm eff}$ will be *nonsupersymmetric*.

## III. APPLICATION TO THE *stu* MODEL WITH FAYET-ILIOPOULOS TERMS

We are now going to apply the formalism and the general results obtained above to a particular model of $N = 2$, $d = 4$ gauged supergravity, in which the scalar potential $V(z, \bar{z})$ is the FI one (see e.g. [61] and references therein),

$$V_{\rm FI}(z, \bar{z}) \equiv (U^{\Lambda\Sigma} - 3\bar{L}^{\Lambda}L^{\Sigma})\xi^x_{\Lambda}\xi^x_{\Sigma}, \tag{3.1}$$

where

$$U^{\Lambda\Sigma} \equiv G^{a\bar{b}}(D_a L^{\Lambda})\bar{D}_{\bar{b}}\bar{L}^{\Lambda} = -\tfrac{1}{2}(\mu)^{-1|\Lambda\Sigma} - \bar{L}^{\Lambda}L^{\Sigma}, \tag{3.2}$$

and $\xi^x_{\Lambda}$ are some real constants. This corresponds to the gauging of a $U(1)$, contained in the $\mathcal{R}$-symmetry $SU(2)_{\mathcal{R}}$ and given by a particular moduli-dependent alignment of the $n_V + 1$ Abelian vector multiplets. Such a gauging does not modify the derivatives of the scalar fields; moreover, in the considered framework the hypermultiplets decouple, and thus we will not be dealing with them. In this sense, we consider a gauging which is "complementary" to the one recently studied in [57].

By using some properties of the special Kähler geometry endowing the vector multiplets' scalar manifold (see e.g. [61,69] and references therein), one obtains

$$D_a V_{\rm FI} = \partial_a V_{\rm FI} = [iC_{abc}G^{b\bar{b}}G^{c\bar{c}}(\bar{D}_{\bar{b}}\bar{L}^\Lambda)\bar{D}_{\bar{c}}\bar{L}^\Lambda - 2\bar{L}^\Lambda D_a L^\Sigma]\xi^x_\Lambda \xi^x_\Sigma. \tag{3.3}$$

We will consider the FI potential in the so-called *stu* model, based on the factorized homogeneous symmetric special Kähler manifold ($\dim_{\mathbb{C}} = n_V = 3$) $(\frac{SU(1,1)}{U(1)})^3$ [23,70,71], in three different frames of symplectic coordinates. Remarkably, we will find that in the presence of gauging [in our case, the *FI gauging* of the $U(1) \subset SU(2)_{\mathcal{R}}$] the symplectic invariance is lost, and one can obtain different behavior of $V_{\rm eff}$ (and thus different physics) by changing the symplectic frame. This confirms the observation made long ago in the framework of $N = 4$, $d = 4$ gauged supergravity in [60], and it is ultimately due to the fact that, in general, differently from $V_{\rm BH}$, $V$ is not symplectic invariant.

## A. $(SO(1,1))^2$ symplectic basis

The first basis we consider is the one manifesting the full $d = 5$ isometry, i.e. $(SO(1,1))^2$, which is a particular case, $n = 2$, of the $SO(1,1) \otimes SO(1, n-1)$ ($n_V = n + 1$) covariant basis [72]. In such a frame, the holomorphic prepotential is simply $F = stu$ (with Kähler gauge fixed such that $X^0 \equiv 1$). The notation used for the symplectic coordinates is as follows ($a = 1, 2, 3$):

$$z^a = \{s \equiv x_1 - ix_2, t \equiv y_1 - iy_2, u \equiv z_1 - iz_2\}. \tag{3.4}$$

This symplectic frame is obtained by the $N = 2$ truncation of the $USp(8)$ gauged $N = 8$ supergravity [73].

Here and further below we will perform computations in the so-called *magnetic* BH charge configuration,[2] in which the nonvanishing BH charges are $q_0$, $p^1$, $p^2$, and $p^3$. This does not imply any loss of generality, due to the homogeneity of the manifold $(\frac{SU(1,1)}{U(1)})^3$ (as recently shown in [53]).

Within such a notation and BH charge configuration, the relevant geometrical quantities look as follows (see e.g. [41])

$$K = -\log(8x_2y_2z_2), \qquad G_{a\bar{b}} = \tfrac{1}{4}{\rm diag}[x_2^{-2}, y_2^{-2}, z_2^{-2}]; \tag{3.5}$$

$$\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} = -x_2y_2z_2\begin{pmatrix} 1 + \frac{x_1^2}{x_2^2} + \frac{y_1^2}{y_2^2} + \frac{z_1^2}{z_2^2} & -\frac{x_1}{x_2^2} & -\frac{y_1}{y_2^2} & -\frac{z_1^2}{z_2^2} \\ -\frac{x_1}{x_2^2} & \frac{1}{x_2^2} & 0 & 0 \\ -\frac{y_1}{y_2^2} & 0 & \frac{1}{y_2^2} & 0 \\ -\frac{z_1}{z_2^2} & 0 & 0 & \frac{1}{z_2^2} \end{pmatrix}; \tag{3.6}$$

$$\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} = \begin{pmatrix} x_1y_1z_1 & -y_1z_1 & -x_1z_1 & -x_1y_1 \\ -y_1z_1 & 0 & z_1 & y_1 \\ -x_1z_1 & z_1 & 0 & x_1 \\ -x_1y_1 & y_1 & x_1 & 0 \end{pmatrix}, \tag{3.7}$$

and $V_{\rm BH}$ reads

$$V_{\rm BH} = \frac{1}{2x_2y_2z_2}[(q_0)^2 - 2q_0(p^1y_1z_1 + p^2x_1z_1 + p^3x_1y_1) + (p^1)^2(y_1^2 + y_2^2)(z_1^2 + z_2^2) + (p^2)^2(x_1^2 + x_2^2)(z_1^2 + z_2^2) + (p^3)^2(x_1^2 + x_2^2)(y_1^2 + y_2^2) + 2p^1p^2x_1y_1(z_1^2 + z_2^2) + 2p^1p^3x_1z_1(y_1^2 + y_2^2) + 2p^2p^3y_1z_1(x_1^2 + x_2^2)]. \tag{3.8}$$

By using Eq. (3.1), $V_{\rm FI}$ is easily computed to be

$$V_{{\rm FI},(SO(1,1))^2} = -\frac{\xi_2^x\xi_3^x}{x_2} - \frac{\xi_1^x\xi_3^x}{y_2} - \frac{\xi_1^x\xi_2^x}{z_2}, \tag{3.9}$$

and $V_{{\rm eff},(SO(1,1))^2}$ can be calculated by recalling its definition (2.22).

It is worth pointing out that $V_{\rm BH}$ is symplectic invariant, and thus Eq. (3.8) holds for the *stu* model in the magnetic BH charge configuration and *in any symplectic frame*. On the other hand, $V$ (whose particular form considered here is $V_{\rm FI}$) and consequently $V_{\rm eff}$ are *not* symplectic invariant.

Thus, Eq. (3.9) and the corresponding expression of $V_{{\rm eff},(SO(1,1))^2}$ computable by using Eq. (2.22) hold for the *stu* model in the magnetic BH charge configuration *only in the* $(SO(1,1))^2$ *covariant basis*.

As one can see from Eq. (3.9), $V_{{\rm FI},(SO(1,1))^2}$ does not depend on the real parts of moduli. This implies that the criticality conditions of $V_{{\rm eff},(SO(1,1))^2}$ with respect to the real parts of moduli coincide with the analogous ones for $V_{\rm BH}$:

[2] In Secs. IV B and V B we will treat also the case of the so-called $D0$-$D6$ BH charge configuration.

$$\frac{\partial V_{\mathrm{eff},(SO(1,1))^2}}{\partial x^1}=0\Leftrightarrow\frac{\partial V_{BH}}{\partial x^1}=0;\quad \frac{\partial V_{\mathrm{eff},(SO(1,1))^2}}{\partial y^1}=0\Leftrightarrow\frac{\partial V_{BH}}{\partial y^1}=0;\quad \frac{\partial V_{\mathrm{eff},(SO(1,1))^2}}{\partial z^1}=0\Leftrightarrow\frac{\partial V_{BH}}{\partial z^1}=0. \tag{3.10}$$

It is well known that criticality conditions on the right-hand side of Eq. (3.10) imply in the BH magnetic charge configuration the "*vanishing axions*" conditions $x_1 = y_1 = z_1 = 0$ (see e.g. [41]); i.e. they yield the purely imaginary nature of the critical moduli. Thus, by using such *on-shell* conditions for the real part of the moduli, $V_{\mathrm{BH}}$ can be rewritten in the following partially *on-shell*, *reduced* form:

$$V_{\mathrm{BH,red}}=\frac{1}{2x_2y_2z_2}[(q_0)^2+(p^1)^2y_2^2z_2^2+(p^2)^2x_2^2z_2^2+(p^3)^2x_2^2y_2^2]. \tag{3.11}$$

The symplectic invariance of the reduced forms of $V_{\mathrm{BH}}$ depends on the symplectic invariance of the conditions implemented in order to go *on shell* for the variables that dropped out. Because of Eqs. (3.10), $V_{\mathrm{BH,red}}$ given by Eq. (3.11) holds for the *stu* model in the magnetic BH charge configuration and *in any symplectic frame*. By using Eqs. (3.9) and (3.11), one can also compute $V_{\mathrm{eff},(SO(1,1))^2,\mathrm{red}}$, whose expression clearly holds only in the considered $(SO(1,1))^2$ covariant basis.

### B. $SO(2, 2)$ symplectic basis

We now consider a symplectic frame exhibiting the maximum noncompact symmetry $SO(2,2)\sim(SU(1,1))^2$; it is a particular case, $n = 2$, of the so-called Calabi-Visentini [74] $SO(2,n)$ ($n_V = n+1$) symplectic frame, considered in [75], having heterotic stringy origin. The special Kähler geometry of $N = 2$, $d = 4$ supergravity in such a symplectic frame is completely specified by the following holomorphic symplectic section $\Omega$:

$$\Omega(u,s)=\begin{pmatrix}X^\Lambda\\ F_\Lambda\end{pmatrix}\equiv\begin{pmatrix}X^\Lambda(u)\\ s\eta_{\Lambda\Sigma}X^\Sigma(u)\end{pmatrix}, \tag{3.12}$$

where $\eta_{\Lambda\Sigma}=\mathrm{diag}(1,1,-1,-1)$ and $X^\Lambda(u)$ satisfies the condition $X^\Lambda(u)\eta_{\Lambda\Sigma}X^\Sigma(u)=0$. The axion-dilaton field $s\equiv x_1 - ix_2$ parametrizes the coset $\frac{SU(1,1)}{U(1)}$, whereas the two independent complex coordinates $u_1$, $u_2$ parametrize the coset $\frac{SO(2,2)}{SO(2)\times SO(2)}$.

Note that, as shown in [75], in this symplectic frame a prepotential does not exist at all. However, it is still possible to calculate all the relevant geometrical quantities, using the standard formulas of special Kähler geometry (see e.g. [69] and references therein). The Kähler potential, the vector kinetic matrix, and its real and imaginary parts (along with the inverse of the imaginary part), respectively, read

$$K=-\log(2x_2)-\log(X^\Lambda\eta_{\Lambda\Sigma}\bar{X}^\Sigma); \tag{3.13}$$

$$\mathcal{N}_{\Lambda\Sigma}=-2ix_2(\bar{\Phi}_\Lambda\Phi_\Sigma+\Phi_\Lambda\bar{\Phi}_\Sigma)+(x_1+ix_2)\eta_{\Lambda\Sigma}; \tag{3.14}$$

$$\nu_{\Lambda\Sigma}=x_1\eta_{\Lambda\Sigma}; \tag{3.15}$$

$$\mu_{\Lambda\Sigma}=x_2[\eta_{\Lambda\Sigma}-2(\bar{\Phi}_\Lambda\Phi_\Sigma+\Phi_\Lambda\bar{\Phi}_\Sigma)]; \tag{3.16}$$

$$(\mu^{-1})^{\Lambda\Sigma}=\frac{1}{x_2^2}\eta^{\Lambda\Gamma}\eta^{\Sigma\Delta}\mu_{\Gamma\Delta}=\frac{1}{x_2}[\eta^{\Lambda\Sigma}-2(\bar{\Phi}^\Lambda\Phi^\Sigma+\Phi^\Lambda\bar{\Phi}^\Sigma)], \tag{3.17}$$

where $\Phi^\Lambda\equiv X^\Lambda/\sqrt{X^\Lambda\bar{X}_\Lambda}$ ($\Phi^\Lambda\Phi_\Lambda=0$, $\Phi^\Lambda\bar{\Phi}_\Lambda=1$), and all the indexes here and below are raised and lowered by using $\eta_{\Lambda\Sigma}$ and its inverse $\eta^{\Lambda\Sigma}\equiv(\eta^{-1})^{\Lambda\Sigma}=\eta_{\Lambda\Sigma}$. By using such expressions and recalling the definition (3.1), one can calculate $V_{\mathrm{FI},SO(2,2)}$ to be

$$V_{\mathrm{FI},SO(2,2)}=-\frac{1}{2x_2}\xi^x_\Lambda\eta^{\Lambda\Sigma}\xi^x_\Sigma. \tag{3.18}$$

The relation between the $SO(2,2)$ symplectic frame specified by Eq. (3.12) and the $(SO(1,1))^2$ symplectic frame is given by the formula [75]

$$X^\Lambda_{SO(2,2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}1-tu\\ -(t+u)\\ -(1+tu)\\ t-u\end{pmatrix}. \tag{3.19}$$

Correspondingly, the BH charges in both symplectic frames are related as follows [71,75]:

$$\begin{pmatrix}p^\Lambda\\ q_\Lambda\end{pmatrix}_{SO(2,2)}=\frac{1}{\sqrt{2}}\begin{pmatrix}p^0-q_1\\ -p^2-p^3\\ -p^0-q_1\\ p^2-p^3\\ p^1+q_0\\ -q_2-q_3\\ p^1-q_0\\ q_2-q_3\end{pmatrix}_{(SO(1,1))^2}. \tag{3.20}$$

### C. $SO(8)$-truncated symplectic basis

This symplectic frame, used in [76] and recently in [26], can be obtained from the $N = 2$ truncation of the $SO(8)$ gauged $N = 8$ supergravity [77,78].

Let us start from the Lagrangian density[3] [see Eq. (2.11) of [76]]

[3]Whereas in [76] the space-time signature $(-,+,+,+)$ was used, we use $(+,-,-,-)$ instead.

$$\mathcal{L} = \sqrt{-g}\{-\tfrac{1}{2}R + \tfrac{1}{4}[(\partial_\mu \phi^{(12)})^2 + (\partial_\mu \phi^{(13)})^2 + (\partial_\mu \phi^{(14)})^2] + -(e^{-\lambda_1}(\mathcal{F}^1_{\mu\nu})^2 + e^{-\lambda_2}(\mathcal{F}^2_{\mu\nu})^2 + e^{-\lambda_3}(\mathcal{F}^3_{\mu\nu})^2 + e^{-\lambda_4}(\mathcal{F}^4_{\mu\nu})^2) - V_{\mathrm{FI},DL}\}, \quad (3.21)$$

where the $\lambda$s are defined as

$$\begin{aligned} \lambda_1 &\equiv -\phi^{(12)} - \phi^{(13)} - \phi^{(14)};\\ \lambda_2 &\equiv -\phi^{(12)} + \phi^{(13)} + \phi^{(14)};\\ \lambda_3 &\equiv \phi^{(12)} - \phi^{(13)} + \phi^{(14)};\\ \lambda_4 &\equiv \phi^{(12)} + \phi^{(13)} - \phi^{(14)}, \end{aligned} \quad (3.22)$$

and the FI potential reads

$$V_{\mathrm{FI}} \equiv -2g^2(\cosh\phi^{(12)} + \cosh\phi^{(13)} + \cosh\phi^{(14)}). \quad (3.23)$$

By switching to new variables

$$\phi^{(12)} \equiv \log(X_1X_2), \qquad \phi^{(13)} \equiv \log(X_1X_3), \qquad \phi^{(14)} \equiv \log(X_1X_4), \quad (3.24)$$

where the $X_I$ ($I = 1, 2, 3, 4$) are real and $X_1X_2X_3X_4 = 1$, it is easy to rewrite the Lagrangian (3.21) into the form [26] [the subscript $SO(8)$ stands for "$SO(8)$-truncated"]

$$\mathcal{L}_{SO(8)} = \sqrt{-g}[-\tfrac{1}{2}R + \tfrac{1}{4}X_I^{-2}\partial_\mu X_I \partial^\mu X_I - X_I^2 \mathcal{F}^I_{\mu\nu}\mathcal{F}^{I|\mu\nu} - V_{\mathrm{FI},SO(8)}], \quad (3.25)$$

where

$$V_{\mathrm{FI},SO(8)} \equiv -g^2 \sum_{I<J} X_I X_J. \quad (3.26)$$

It is worth stressing that in the symplectic frame exploited by the models above only electrical charges are nonvanishing.

The model studied in [26] and based on the Lagrangian (3.25) [which, as shown above, is nothing but a rewriting of the model considered in [76] and based on the Lagrangian (3.21)] can be related to the $stu$ model in the symplectic frame treated in Sec. III A in the following way.

First, one has to perform a symplectic transformation from the magnetic BH charge configuration to the purely electric configuration (then identifying $q_1 \equiv p^1$, $q_2 \equiv p^2$, and $q_3 \equiv p^3$):

$$\begin{pmatrix} 0\\0\\0\\0\\q_0\\q_1\\q_2\\q_3 \end{pmatrix} = \mathcal{S} \begin{pmatrix} 0\\P^1\\P^2\\P^3\\Q_0\\0\\0\\0 \end{pmatrix}, \quad (3.27)$$

where $\mathcal{S}$ is an $8 \times 8$ matrix ($\Lambda = 0, 1, 2, 3$) defined as

$$\mathcal{S} \equiv \begin{pmatrix} A^\Lambda_\Sigma & B^{\Lambda\Sigma}\\ C_{\Lambda\Sigma} & D_\Lambda^\Sigma \end{pmatrix}, \quad (3.28)$$

whose only nonvanishing components are $A^0_0 = D^0_0 = 1$ and $B^{11} = B^{22} = B^{33} = -C_{11} = -C_{22} = -C_{33} = -1$.

This yields a corresponding symplectic transformation of sections of the $(SO(1,1))^2$ symplectic frame treated in Sec. III A. By fixing the Kähler gauge such that $X^0 \equiv 1$, it reads

$$\mathcal{S}\begin{pmatrix} X^\Lambda\\ F_\Lambda \end{pmatrix}_{(SO(1,1))^2, X^0\equiv 1} = \mathcal{S}\begin{pmatrix} 1\\s\\t\\u\\-stu\\tu\\su\\st \end{pmatrix} = \begin{pmatrix} 1\\-tu\\-su\\-st\\-stu\\s\\t\\u \end{pmatrix} \equiv \begin{pmatrix} X^\Lambda\\ F_\Lambda \end{pmatrix}_{SO(8), X^0\equiv 1}. \quad (3.29)$$

One can also compute that

$$F_{SO(8),X^0\equiv 1} = -2stu = -2F_{(SO(1,1))^2, X^0\equiv 1}, \quad (3.30)$$

or, in terms of the relevant $X^\Lambda$s,[4]

$$F_{SO(8),X^0\equiv 1} = -2\sqrt{-X^0 X^1_{SO(8)} X^2_{SO(8)} X^3_{SO(8)}}\Big|_{X^0\equiv 1} = -2\frac{X^1_{(SO(1,1))^2} X^2_{(SO(1,1))^2} X^3_{(SO(1,1))^2}}{X^0}\Big|_{X^0\equiv 1}. \quad (3.31)$$

Finally, by recalling the symplectic-invariant "*vanishing axions*" conditions (3.10) and the definitions (3.4), one can introduce the following holomorphic sections:

$$\begin{pmatrix} \hat{X}^\Lambda\\ \hat{F}_\Lambda \end{pmatrix}_{SO(8),X^0\equiv 1} \equiv \begin{pmatrix} X^\Lambda\\ F_\Lambda \end{pmatrix}_{SO(8),X^0\equiv 1, x_1=y_1=z_1=0} = \begin{pmatrix} 1\\ y_2z_2\\ x_2z_2\\ x_2y_2\\ -ix_2y_2z_2\\ -ix_2\\ -iy_2\\ -iz_2 \end{pmatrix}, \quad (3.32)$$

where $x_2$, $y_2$, and $z_2$ have been defined in Sec. III A.

By using the general transformation rule for the matrix $\mathcal{N}$ (see e.g. [79]) and the explicit expressions (3.6) and (3.7) of its imaginary and real parts in the $stu$ model, one can easily compute the kinetic vector matrix corresponding

[4]Notice that $X^0_{(SO(1,1))^2} = X^0_{SO(8)} \equiv X^0$, because the symplectic transformation $\mathcal{S}$ does not act on $X^0$.

to sections (3.32), denoted by $\hat{\mathcal{N}}$. Its real part vanishes, while its imaginary reads as follows:

$$\hat{\mu}_{00} = -x_2y_2z_2, \qquad \hat{\mu}_{11} = -\frac{x_2}{y_2z_2}, \qquad \hat{\mu}_{22} = -\frac{y_2}{z_2x_2}, \qquad \hat{\mu}_{33} = -\frac{z_2}{x_2y_2}. \tag{3.33}$$

By inserting Eqs. (3.32) and (3.33) into the definition (3.1), one obtains the following expression of the FI potential:

$$V_{\mathrm{FI},SO(8)} = -\xi_0^x\left(\frac{\xi_1^x}{x_2} + \frac{\xi_2^x}{y_2} + \frac{\xi_3^x}{z_2}\right) - \xi_2^x\xi_3^x x_2 - \xi_1^x\xi_3^x y_2 - \xi_1^x\xi_2^x z_2. \tag{3.34}$$

As anticipated in the Introduction, $V_{\mathrm{FI},SO(8)}$ is the only gauge potential (among the considered ones) admitting critical points at a finite distance from the origin. By comparing Eq. (3.34) with Eqs. (3.9) and (3.18) above, it is easy to realize that this is due to the simultaneous presence of "$\frac{1}{x}$" and "$x$" terms, thus allowing for a "balance" of opposite behaviors of the potential.

Comparing the matrix $\hat{\mu}_{\Lambda\Sigma}$ given by Eq. (3.33) with the one considered in [26], namely, with ($I = 1, 2, 3, 4$)

$$\mu_{IJ} = -X_I^2\delta_{IJ}, \qquad \nu_{IJ} = 0, \tag{3.35}$$

one can relate the $(SO(1,1))^2$ symplectic coordinates with the ones used in [26]:

$$X_1^2 = \frac{x_2}{y_2z_2}, \qquad X_2^2 = \frac{y_2}{x_2z_2}, \qquad X_3^2 = \frac{z_2}{y_2x_2}, \qquad X_4^2 = x_2y_2z_2. \tag{3.36}$$

It is worth pointing out that the FI potential (3.36) of [26] corresponds to the FI potential (3.34) with the particular choice $\xi_0^x = \xi_1^x = \xi_2 = \xi_3^x = \sqrt{2g^2}$.

By recalling that $V_{\mathrm{BH,red}}$ is symplectic invariant and given by Eq. (3.11), and using Eqs. (2.22) and (3.34), one can then compute $V_{\mathrm{eff,red},SO(8)}$.

## IV. DEPENDENCE OF BLACK HOLE ENTROPY ON THE SYMPLECTIC FRAME IN THE *stu* MODEL

We will now go *on shell* for the formalism based on the effective BH potential $V_{\mathrm{eff}}$. In other words, we will compute, for the considered magnetic BH charge configuration, the critical points of $V_{\mathrm{eff}}$ of the *stu* model in the three symplectic frames introduced above, and then compare the resulting BH entropies. In the $SO(8)$-truncated symplectic basis we will match perfectly the results of [26], obtained by using the (intrinsically *on-shell*) Sen's *entropy function formalism* (see [6,8], and [48] and references therein). Furthermore, as previously mentioned, the formal and physical results will generally depend on the symplectic basis being considered. We will also consider the case of the $D0$-$D6$ BH charge configuration.

### A. Magnetic configuration

It is now convenient to introduce the following new coordinates:

$$x_2 \equiv \sqrt{\left|\frac{q_0p^1}{p^2p^3}\right|}x, \qquad y_2 \equiv \sqrt{\left|\frac{q_0p^2}{p^1p^3}\right|}y, \qquad z_2 \equiv \sqrt{\left|\frac{q_0p^3}{p^2p^1}\right|}z. \tag{4.1}$$

By doing so, Eqs. (3.9), (3.11), (3.18), and (3.34), holding in the magnetic BH charge configuration and expressing the symplectic-invariant $V_{\mathrm{BH}}$ and the not symplectic-invariant $V_{\mathrm{FI}}$ in the $(SO(1,1))^2$, $SO(2,2)$ covariant, and $SO(8)$-truncated symplectic frames, respectively, can be rewritten as follows:

$$\begin{cases} V_{\mathrm{BH,red}} = \frac{\sqrt{|q_0p^1p^2p^3|}}{2}\tilde{V}_{\mathrm{BH,red}}, \\ \tilde{V}_{\mathrm{BH,red}} \equiv \frac{1+x^2y^2+x^2z^2+y^2z^2}{xyz}; \end{cases} \tag{4.2}$$

$$\begin{cases} V_{\mathrm{FI},(SO(1,1))^2} = -\frac{1}{2\sqrt{|q_0p^1p^2p^3|}}\tilde{V}_{\mathrm{FI},(SO(1,1))^2}, \\ \tilde{V}_{\mathrm{FI},(SO(1,1))^2} \equiv \frac{\tilde{\xi}_2\tilde{\xi}_3}{x} + \frac{\tilde{\xi}_1\tilde{\xi}_3}{y} + \frac{\tilde{\xi}_1\tilde{\xi}_2}{z}, \\ \tilde{\xi}_i \equiv \sqrt{2}|p^i|\xi_i^x, \qquad i = 1, 2, 3; \end{cases} \tag{4.3}$$

$$\begin{cases} V_{\mathrm{FI},SO(2,2)} = -\frac{1}{2\sqrt{|q_0p^1p^2p^3|}}\tilde{V}_{SO(2,2)}, \\ \tilde{V}_{\mathrm{FI},SO(2,2)} \equiv \frac{\tilde{\alpha}}{x}, \\ \tilde{\alpha} \equiv |p^2p^3|\xi_\Lambda^x\eta^{\Lambda\Sigma}\xi_\Sigma^x; \end{cases} \tag{4.4}$$

$$\begin{cases} V_{\mathrm{FI},SO(8)} = -\frac{1}{2\sqrt{|q_0p^1p^2p^3|}}\tilde{V}_{\mathrm{FI},SO(8)}, \\ \tilde{V}_{\mathrm{FI},SO(8)} \equiv \tilde{\xi}_0(\frac{\tilde{\xi}_1}{x} + \frac{\tilde{\xi}_2}{y} + \frac{\tilde{\xi}_3}{z}) + \tilde{\xi}_2\tilde{\xi}_3x + \tilde{\xi}_1\tilde{\xi}_3y + \tilde{\xi}_1\tilde{\xi}_2z, \\ \tilde{\xi}_0 \equiv \sqrt{|2\frac{p^1p^2p^3}{q_0}|}\xi_0^x, \ \tilde{\xi}_1 \equiv \sqrt{|2\frac{q_0p^2p^3}{p^1}|}\xi_1^x, \ \tilde{\xi}_2 \equiv \sqrt{|2\frac{q_0p^1p^3}{p^2}|}\xi_2^x, \ \tilde{\xi}_3 \equiv \sqrt{|2\frac{q_0p^1p^2}{p^3}|}\xi_3^x. \end{cases} \tag{4.5}$$

Let us also write here the expressions of the (symplectic-invariant) superpotential and of its covariant derivatives (*matter charges*):

$$W = q_0\{1 + \mathrm{sgn}(q_0)[\mathrm{sgn}(p^1)yz + \mathrm{sgn}(p^2)xz + \mathrm{sgn}(p^3)xy]\}; \tag{4.6}$$

$$D_xW = -\frac{q_0}{x}[1 + \mathrm{sgn}(q_0p^1)yz]; \tag{4.7}$$

$$D_yW = -\frac{q_0}{y}[1 + \mathrm{sgn}(q_0p^2)xz]; \tag{4.8}$$

$$D_zW = -\frac{q_0}{z}[1 + \mathrm{sgn}(q_0p^3)xy]. \tag{4.9}$$

Now, by recalling the definition (2.22), the reduced effective potential can be written as follows:

$$\begin{cases} V_{\mathrm{eff,red}} = \sqrt{|q_0p^1p^2p^3|}\tilde{V}_{\mathrm{eff,red}}; \\ \tilde{V}_{\mathrm{eff,red}} \equiv \frac{\sqrt{1+\tilde{V}_{\mathrm{BH,red}}\tilde{V}_{\mathrm{FI}}}-1}{\tilde{V}_{\mathrm{FI}}}, \end{cases} \tag{4.10}$$

where $\tilde{V}_{\mathrm{BH,red}}$ is given by Eq. (4.2), and $\tilde{V}_{\mathrm{FI}}$, depending on the symplectic frame, is given by Eqs. (4.3), (4.4), and (4.5).

### 1. $SO(2,2)$

$\tilde{V}_{\mathrm{eff,red},SO(2,2)}$ reads

$$\tilde{V}_{\mathrm{eff,red},SO(2,2)} = x\frac{\sqrt{1 + \frac{\tilde{\alpha}}{x^2yz}(1 + x^2y^2 + x^2z^2 + y^2z^2)} - 1}{\tilde{\alpha}}. \tag{4.11}$$

Attractor equations for such a potential yield the following solutions:

$$\frac{\partial\tilde{V}_{\mathrm{eff,red},SO(2,2)}}{\partial x} = \frac{\partial\tilde{V}_{\mathrm{eff,red},SO(2,2)}}{\partial y} = \frac{\partial\tilde{V}_{\mathrm{eff,red},SO(2,2)}}{\partial z} = 0 \Leftrightarrow \begin{cases} \tilde{\alpha} > -\frac{1}{2}: x = \frac{1}{\sqrt{1+2\tilde{\alpha}}}, \quad y = z = \pm 1; \\ \tilde{\alpha} < \frac{1}{2}: x = -\frac{1}{\sqrt{1-2\tilde{\alpha}}}, \quad y = -z = \pm 1, \end{cases} \tag{4.12}$$

and the corresponding BH entropy reads

$$S_{\mathrm{BH},SO(2,2)} = \begin{cases} \tilde{\alpha} > -\frac{1}{2}: 2\pi\frac{\sqrt{|q_0p^1p^2p^3|}}{\sqrt{1+2\tilde{\alpha}}}; \\ \tilde{\alpha} < \frac{1}{2}: 2\pi\frac{\sqrt{|q_0p^1p^2p^3|}}{\sqrt{1-2\tilde{\alpha}}}. \end{cases} \tag{4.13}$$

As it can be seen by recalling Eqs. (4.6), (4.7), (4.8), and (4.9), for both ranges of $\tilde{\alpha}$ the solutions have at least one nonvanishing matter charge. The condition $Z = 0$ requires $p^2p^3 < 0$ and $p^1q_0 < 0$, whereas $Z \neq 0$ requires $p^2p^3 > 0$ and $p^1q_0 < 0$.

The ungauged limit $\tilde{\alpha} \to 0$ yields $x = \pm 1$ and $S_{\mathrm{BH},SO(2,2)} = 2\pi\sqrt{|q_0p^1p^2p^3|}$, consistently with the known results [10,35,80]. On the other hand, by recalling Eq. (3.5), the limit $\tilde{\alpha} \to \infty$ in the branch $\tilde{\alpha} > -\frac{1}{2}$ and the limit $\tilde{\alpha} \to -\infty$ in the branch $\tilde{\alpha} < \frac{1}{2}$ yield a singular geometry.

### 2. $(SO(1,1))^2$

As it can be seen from Eqs. (4.3) and (4.4), in the case $\xi_1 = 0$ one obtains that $V_{\mathrm{FI},(SO(1,1))^2} \propto V_{\mathrm{FI},SO(2,2)}$, and everything, up to some redefinitions, goes as in the $SO(2,2)$ symplectic basis.

In the following treatment we will assume the simplifying assumption (implying asymptotically AdS BHs)

$$\tilde{\xi}_1 = \tilde{\xi}_2 = \tilde{\xi}_3 = \sqrt{\frac{g^2}{3}}, \tag{4.14}$$

corresponding to the most symmetric implementation of the *FI gauging* in the considered symplectic frame. Under such an assumption, one gets

$$\tilde{V}_{\mathrm{eff,red},(SO(1,1))^2} = \frac{\sqrt{x^2y^2z^2 + \frac{g^2}{3}(xy + xz + yz)(1 + x^2y^2 + x^2z^2 + y^2z^2)} - xyz}{\frac{g^2}{3}(xy + xz + yz)}. \tag{4.15}$$

Attractor equations for such a potential yield the following solution:

$$\frac{\partial\tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial x} = \frac{\partial\tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial y} = \frac{\partial\tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial z} = 0; \Updownarrow x = y = z = \left(\frac{3+6g^2+\sqrt{9+24g^2}}{6+18g^2}\right)^{1/4}. \tag{4.16}$$

The corresponding BH entropy reads

$$S_{\mathrm{BH},(SO(1,1))^2} = \pi\sqrt{|q_0p^1p^2p^3|}\frac{[-\sqrt{2} + \sqrt{5 + 12g^2 + \sqrt{9 + 24g^2}}]}{2g^2}\left(\frac{6 + 12g^2 + 2\sqrt{9 + 24g^2}}{3 + 9g^2}\right)^{1/4}. \tag{4.17}$$

In the ungauged limit $g \to 0$ one gets $x = y = z = 1$ and the well-known expression of the BH entropy is recovered [10,35,80]:

$$\lim_{g\to 0} S_{\mathrm{BH},(SO(1,1))^2} = 2\pi\sqrt{|q_0 p^1 p^2 p^3|}. \tag{4.18}$$

For $g \to \infty$, $S_{\mathrm{BH},(SO(1,1))^2}$ vanishes and $x = y = z = 3^{-1/4}$. As it can be seen by recalling Eqs. (4.6), (4.7), (4.8), and (4.9), the solution (4.16) has all $Z$, $D_xZ$, $D_yZ$, and $D_zZ$ nonvanishing.

#### *3. SO(8)-truncated*

As it can be seen from Eqs. (4.4) and (4.5), in the case $\tilde{\xi}_2 = \tilde{\xi}_3 = 0$ one obtains that $V_{\mathrm{FI},SO(8)} \propto V_{\mathrm{FI},SO(2,2)}$, and everything, up to some redefinitions, goes as in the $SO(2, 2)$ symplectic basis.

In the following treatment we will assume the simplifying assumption

$$\tilde{\xi}_0 = \tilde{\xi}_1 = \tilde{\xi}_2 = \tilde{\xi}_3, \tag{4.19}$$

corresponding to the most symmetric implementation of the *FI gauging* in the considered symplectic frame. Under such an assumption, one gets[5]

$$\tilde{V}_{\mathrm{eff,red},SO(8)} = \frac{\sqrt{1 + \frac{\tilde{\alpha}_{SO(8)}}{x^3}(1 + 3x^4)(\frac{1}{x} + x)} - 1}{\tilde{\alpha}_{SO(8)}(1/x + x)}, \tag{4.20}$$

where [without loss of generality, under the assumption (4.19)] we put $x = y = z$, and $\tilde{\alpha}_{SO(8)} \equiv 3\tilde{\xi}_0^2$.

The attractor equation for such a potential yields the following $\tilde{\alpha}_{SO(8)}$-independent solution:

$$\frac{\partial \tilde{V}_{\mathrm{eff,red},SO(8)}}{\partial x} = 0 \Leftrightarrow x = 1, \tag{4.21}$$

and the corresponding BH entropy reads

$$S_{\mathrm{BH},SO(8)} = \pi\sqrt{|q_0 p^1 p^2 p^3|}\,\frac{\sqrt{1 + 8\tilde{\alpha}_{SO(8)}} - 1}{2\tilde{\alpha}_{SO(8)}}. \tag{4.22}$$

Such an expression coincides with the result obtained in [26], up to a redefinition of the constant $\tilde{\alpha}_{SO(8)}$ by considering $q_0 = p^1 = p^2 = p^3$. Consistently, in the ungauged limit $\tilde{\alpha}_{SO(8)} \to 0$ the well-known expression of the BH entropy is recovered [10,35,80]:

$$\lim_{g\to 0} S_{\mathrm{BH,SO(8)}} = 2\pi\sqrt{|q_0 p^1 p^2 p^3|}, \tag{4.23}$$

whereas $S_{\mathrm{BH},SO(8)}$ vanishes in the limit $\tilde{\alpha}_{SO(8)} \to \infty$.

As seen from Eqs. (4.6), (4.7), (4.8), and (4.9), depending on the sign of the charges of the magnetic configuration, either the central charge $Z$ or its covariant derivatives vanish, or we can also obtain the nonvanishing of all $Z$, $D_xZ$, $D_yZ$, and $D_zZ$.

[5]Here and below $\tilde{\alpha}_{SO(8)} \geq 0$, implying asymptotically AdS (flat in the ungauged limit $\tilde{\alpha}_{SO(8)} \to 0$) BHs.

### B. *D*0-*D*6 configuration

Let us now consider the so-called $D0$-$D6$ BH charge configuration, in which [in the $(SO(1, 1))^2$ covariant symplectic basis] all charges vanish but $p^0$ and $q_0$. For such a configuration the symplectic-invariant $V_{\mathrm{BH}}$, defined by Eq. (2.12), reads

$$V_{\mathrm{BH},D0\text{-}D6} = \frac{1}{2x_2y_2z_2}[(q_0)^2 + 2q_0p^0x_1y_1z_1 + p_0^2(x_1^2 + x_2^2)(y_1^2 + y_2^2)(z_1^2 + z_2^2)]. \tag{4.24}$$

By recalling the expressions of the FI potentials given by Eqs. (3.9), (3.18), and (3.34), it is easy to realize that the criticality conditions of $V_{\mathrm{eff},D0\text{-}D6}$ with respect to the real parts of moduli coincide with the analogous ones for $V_{\mathrm{BH},D0\text{-}D6}$:

$$\begin{aligned}\frac{\partial V_{\mathrm{eff},D0\text{-}D6}}{\partial x^1} = 0 \Leftrightarrow \frac{\partial V_{\mathrm{BH},D0\text{-}D6}}{\partial x^1} = 0;\\ \frac{\partial V_{\mathrm{eff},D0\text{-}D6}}{\partial y^1} = 0 \Leftrightarrow \frac{\partial V_{\mathrm{BH},D0\text{-}D6}}{\partial y^1} = 0;\\ \frac{\partial V_{\mathrm{eff},D0\text{-}D6}}{\partial z^1} = 0 \Leftrightarrow \frac{\partial V_{\mathrm{BH},D0\text{-}D6}}{\partial z^1} = 0.\end{aligned} \tag{4.25}$$

It is well known that criticality conditions on the right-hand side of Eq. (4.25) imply in the $D0$-$D6$ BH charge configuration the conditions $x_1 = y_1 = z_1 = 0$ (see e.g. [41]); i.e. they yield the purely imaginary nature of the critical moduli. Thus, by using such *on-shell* conditions for the real part of the moduli, $V_{\mathrm{BH},D0\text{-}D6}$ can be rewritten in the following partially *on-shell*, *reduced* form:

$$V_{\mathrm{BH},D0\text{-}D6,red} = \frac{1}{2x_2y_2z_2}[(q_0)^2 + (p^0)^2x_2^2y_2^2z_2^2]. \tag{4.26}$$

As noticed above, the symplectic invariance of the reduced forms of $V_{\mathrm{BH}}$ depends on the symplectic invariance of the conditions implemented in order to go *on shell* for the variables dropped out. Because of Eqs. (4.25), $V_{\mathrm{BH},D0\text{-}D6,\mathrm{red}}$ given by Eq. (4.26) holds for the *stu* model in the $D0$-$D6$ BH charge configuration and in any symplectic frame.

It is now convenient to introduce the following new coordinates:

$$x_2 \equiv \sqrt[3]{\left|\frac{q_0}{p^0}\right|}\,x, \qquad y_2 \equiv \sqrt[3]{\left|\frac{q_0}{p^0}\right|}\,y, \qquad z_2 \equiv \sqrt[3]{\left|\frac{q_0}{p^0}\right|}\,z. \tag{4.27}$$

By doing so, Eqs. (3.9), (3.18), (3.34), and (4.26), can, respectively, be rewritten as follows (we drop the subscript $D0$-$D6$):

$$\begin{cases} V_{\mathrm{BH,red}} = \frac{|q_0 p^0|}{2} \tilde{V}_{\mathrm{BH,red}}, \\ \tilde{V}_{\mathrm{BH,red}} \equiv \frac{1+x^2y^2z^2}{xyz}; \end{cases} \tag{4.28}$$

and

$$\begin{cases} V_{\mathrm{FI},(SO(1,1))^2} = -\frac{1}{2|q_0 p^0|} \tilde{V}_{\mathrm{FI},(SO(1,1))^2}, \\ \tilde{V}_{\mathrm{FI},(SO(1,1))^2} \equiv \frac{\tilde{\xi}_2\tilde{\xi}_3}{x} + \frac{\tilde{\xi}_1\tilde{\xi}_3}{y} + \frac{\tilde{\xi}_1\tilde{\xi}_2}{z}, \\ \tilde{\xi}_i \equiv \sqrt{2}\sqrt{[3]}(p^0)^2|q_0|\xi_i^x, \qquad i = 1,2,3; \end{cases} \tag{4.29}$$

$$\begin{cases} V_{\mathrm{FI},SO(2,2)} = -\frac{1}{2|q_0 p^0|} \tilde{V}_{\mathrm{FI},SO(2,2)}, \\ \tilde{V}_{\mathrm{FI},SO(2,2)} \equiv \frac{\tilde{\alpha}}{x}, \\ \tilde{\alpha} \equiv \sqrt[3]{(p^0)^4 q_0^2 \xi_\Lambda^x \eta^{\Lambda\Sigma} \xi_\Sigma^x}; \end{cases} \tag{4.30}$$

$$\begin{cases} V_{\mathrm{FI},SO(8)} = -\frac{1}{2|q_0 p^0|} \tilde{V}_{\mathrm{FI},SO(8)}, \\ \tilde{V}_{\mathrm{FI},SO(8)} \equiv \tilde{\xi}_0(\frac{\tilde{\xi}_1}{x} + \frac{\tilde{\xi}_2}{y} + \frac{\tilde{\xi}_3}{z}) + \tilde{\xi}_2\tilde{\xi}_3 x + \tilde{\xi}_1\tilde{\xi}_3 y + \tilde{\xi}_1\tilde{\xi}_2 z, \\ \tilde{\xi}_0 \equiv \sqrt{2}|p^0|\xi_0^x, \tilde{\xi}_i \equiv \sqrt{2}\sqrt{[3]}(q_0)^2|p^0|\xi_i^x, \qquad i = 1,2,3. \end{cases} \tag{4.31}$$

Let us also write here the expressions of the (symplectic-invariant) superpotential and of its covariant derivatives (*matter charges*):

$$W = q_0[1 + \mathrm{sgn}(q_0 p^0) xyz]; \tag{4.32}$$

$$D_x W = -\frac{q_0}{x}; \tag{4.33}$$

$$D_y W = -\frac{q_0}{y}; \tag{4.34}$$

$$D_z W = -\frac{q_0}{z}. \tag{4.35}$$

Now, by recalling the definition (2.22), the effective potential can be written as follows:

$$\begin{cases} V_{\mathrm{eff,red}} = |q_0 p^0| \tilde{V}_{\mathrm{eff,red}}, \\ \tilde{V}_{\mathrm{eff,red}} \equiv \frac{\sqrt{1+\tilde{V}_{\mathrm{BH,red}}\tilde{V}_{\mathrm{FI}}}-1}{\tilde{V}_{\mathrm{FI}}}, \end{cases} \tag{4.36}$$

where $\tilde{V}_{\mathrm{BH,red}}$ is given by Eq. (4.28), and $\tilde{V}_{\mathrm{FI}}$, depending on the symplectic basis, is given by Eqs. (4.29), (4.30), and (4.31).

### 1. $SO(2,2)$

$\tilde{V}_{\mathrm{eff,red},SO(2,2)}$ reads

$$\tilde{V}_{\mathrm{eff,red},SO(2,2)} = x\frac{\sqrt{1 + \frac{\tilde{\alpha}}{x^2yz}(1 + x^2y^2z^2)} - 1}{\tilde{\alpha}}. \tag{4.37}$$

The corresponding attractor equations for the variables $y$ and $z$ yield

$$\frac{\partial \tilde{V}_{\mathrm{eff,red},SO(2,2)}}{\partial y} = \frac{\partial \tilde{V}_{\mathrm{eff,red},SO(2,2)}}{\partial z} = 0 \Leftrightarrow xyz = 1. \tag{4.38}$$

On the other hand, when inserting the attractor equation for the variable $x$ into the condition (4.38), one obtains

$$\tilde{\alpha} yz = 0. \tag{4.39}$$

It is then easy to realize that the attractor equations have solutions only in the ungauged case $\tilde{\alpha} = 0$. In such a case, the corresponding solution is non-BPS $Z \neq 0$, it is stable (up to a non-BPS $Z \neq 0$ moduli space with $\dim_{\mathbb{R}} = 2$ [21,35,37]), and the corresponding BH entropy is given by the well-known result [10,35,80]

$$S_{\mathrm{BH}} = \pi|p^0 q_0|. \tag{4.40}$$

Thus, one can conclude that, in the presence of a FI potential in the $SO(2,2)$ covariant symplectic basis (of the *stu* model), the $D0$-$D6$ BH charge configuration does not support any critical point of the effective potential $\tilde{V}_{\mathrm{eff,red},SO(2,2)}$, and thus no attractor solutions are admitted when only $p^0$ and $q_0$ are nonvanishing [in the $(SO(1,1))^2$ covariant symplectic basis].

### 2. $(SO(1,1))^2$

As it can be seen from Eqs. (4.29) and (4.30), in the case $\xi_1 = 0$ one obtains that $V_{\mathrm{FI},(SO(1,1))^2} \propto V_{\mathrm{FI},SO(2,2)}$, and everything, up to some redefinitions, goes as in the $SO(2,2)$ symplectic frame.

As done for the treatment of the magnetic BH charge configuration, we will make the simplifying assumption (4.14), which we recall to correspond to the most symmetric implementation of the FI potential in the considered symplectic frame. Under such an assumption, one gets

$$\tilde{V}_{\mathrm{eff,red},(SO(1,1))^2} = \frac{\sqrt{x^2y^2z^2 + \frac{g^2}{3}(xy + xz + xy)(1 + x^2y^2z^2)} - xyz}{\frac{g^2}{3}(xy + xz + xy)}. \tag{4.41}$$

Attractor equations for such a potential yield the following solutions:

$$\begin{gathered} \frac{\partial \tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial x} = \frac{\partial \tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial y} = \frac{\partial \tilde{V}_{\mathrm{eff,red},(SO(1,1))^2}}{\partial z} = 0; \\ \Updownarrow \\ \begin{cases} x = y = z; \\ 3x^4(-1 + x^6) + g^2(-1 + 2x^6)^2 = 0. \end{cases} \end{gathered} \tag{4.42}$$

Differently from the case of the magnetic BH charge configuration [given by Eq. (4.16)], the algebraic equation of degree 12 on the right-hand side of Eq. (4.42) does not have an analytical solution. The corresponding BH entropy can be computed in a parametric way as follows:

$$S_{\mathrm{BH},(SO(1,1))^2} = \pi|p^0 q_0| \frac{(2x^6 - 1)}{x^3}, \tag{4.43}$$

yielding the restriction $\frac{1}{2} < x^6 \le 1$ on the solutions of the algebraic equation of degree 12 on the right-hand side of Eq. (4.42).

#### 3. $SO(8)$-truncated

As it can be seen from Eqs. (4.30) and (4.31), in the case $\tilde{\xi}_2 = \tilde{\xi}_3 = 0$ one obtains that $V_{\mathrm{FI},SO(8)} \propto V_{\mathrm{FI},SO(2,2)}$, and everything, up to some redefinitions, goes as in the $SO(2,2)$ symplectic frame.

As done for the treatment of the magnetic BH charge configuration, we will make the simplifying assumption (4.19), which we recall to correspond to the most symmetric implementation of the *FI gauging* in the considered symplectic frame. Under such an assumption, one gets

$$\tilde{V}_{\mathrm{eff,red},SO(8)} = \frac{\sqrt{1 + \frac{\tilde{\alpha}_{SO(8)}}{x^3}(1 + x^6)(\frac{1}{x} + x)} - 1}{\tilde{\alpha}_{SO(8)}(1/x + x)}, \tag{4.44}$$

where [without loss of generality, under the assumption (4.19)] we put $x = y = z$.

The attractor equation for such a potential yields the following $\tilde{\alpha}_{SO(8)}$-independent solution:

$$\frac{\partial \tilde{V}_{\mathrm{eff,red},SO(8)}}{\partial x} = 0 \Leftrightarrow x = 1, \tag{4.45}$$

which is formally the same solution obtained for the magnetic BH charge configuration. The corresponding BH entropy reads

$$S_{\mathrm{BH},SO(8)} = \pi|q_0 p^0| \frac{\sqrt{1 + 4\tilde{\alpha}_{SO(8)}} - 1}{2\tilde{\alpha}_{SO(8)}}. \tag{4.46}$$

Consistently, in the ungauged limit $\tilde{\alpha}_{SO(8)} \to 0$ the well-known expression of the BH entropy is recovered [10,35,80]:

$$\lim_{g\to 0} S_{\mathrm{BH},SO(8)} = \pi|q_0 p^0|, \tag{4.47}$$

whereas $S_{\mathrm{BH},SO(8)}$ vanishes in the limit $\tilde{\alpha}_{SO(8)} \to \infty$.

As it can be seen by recalling Eqs. (4.32), (4.33), (4.34), and (4.35), the critical points of $\tilde{V}_{\mathrm{eff,red},SO(8)}$ necessarily have at least one matter charge vanishing, depending on the sign of charges $p^0$ and $q_0$.

## V. ANALYSIS OF STABILITY

In order to determine the actual stability of the critical points of $V_{\mathrm{eff}}$ found in the previous section for the magnetic and $D0$-$D6$ BH charge configurations in various symplectic bases, one has to study the sign of the eigenvalues of the (real form of the) Hessian of $V_{\mathrm{eff}}$ case by case. It is worth pointing out that this is possible only in the *effective potential formalism* studied here, which is intrinsically *off shell* (indeed, the treatment of Secs. II and III holds in the whole scalar manifold; only when computing the critical points of $V_{\mathrm{eff}}$ in the various symplectic frames, as done in Sec. IV, one goes *on shell*). The so-called *entropy function formalism* (see e.g. [6,8,13,20,26,46,48,54]) is intrinsically *on shell*, and it does not allow one to study analytically the stability of the attractor scalar configurations.

The key feature is the fact that in the (three different symplectic frames of the) *stu* model $V_{\mathrm{FI}}$ is independent of the axions $(x_1, y_1, z_1)$.

By denoting $f \equiv \{x, y, z\}$, the first derivatives of $V_{\mathrm{eff}}$ with respect to the axions $\partial_{f_1} V_{\mathrm{eff}}$ read

$$\frac{\partial V_{\mathrm{eff}}}{\partial f_1} = \frac{\partial V_{\mathrm{eff}}}{\partial V_{\mathrm{BH}}} \frac{\partial V_{\mathrm{BH}}}{\partial f_1}. \tag{5.1}$$

By recalling Eq. (2.22) one gets that $\frac{\partial V_{\mathrm{eff}}}{\partial V_{\mathrm{BH}}} = \frac{1}{\sqrt{1-4VV_{\mathrm{BH}}}}$, which is strictly positive in the region of definition $V_{\mathrm{BH}}V < \frac{1}{4}$ of $V_{\mathrm{eff}}$; consequently, as already pointed out above,

$$\frac{\partial V_{\mathrm{eff}}}{\partial f_1} = 0 \Leftrightarrow \frac{\partial V_{\mathrm{BH}}}{\partial f_1} = 0. \tag{5.2}$$

In the *stu* model such criticality conditions along the axionic directions imply, both in the magnetic and $D0$-$D6$ BH charge configurations, $x_1 = y_1 = z_1 = 0$ (see e.g. [41]), i.e. the vanishing of all axions.

Let us now investigate the second order derivatives of $V_{\mathrm{eff}}$ at its critical points. Using the axionic criticality conditions (5.2) one can easily show that ($g \equiv \{x, y, z\}$)

$$\frac{\partial^2 V_{\mathrm{eff}}}{\partial f_1 \partial g_1}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0} = \frac{\partial V_{\mathrm{eff}}}{\partial V_{\mathrm{BH}}}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0} \frac{\partial^2 V_{\mathrm{BH}}}{\partial f_1 \partial g_1}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0}, \tag{5.3}$$

$$\frac{\partial^2 V_{\mathrm{eff}}}{\partial f_1 \partial g_2}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0} = \frac{\partial V_{\mathrm{eff}}}{\partial V_{\mathrm{BH}}}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0} \frac{\partial^2 V_{\mathrm{BH}}}{\partial f_1 \partial g_2}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0}. \tag{5.4}$$

By straightforward calculations, one can show that all axionic-dilatonic mixed derivatives vanish at $\frac{\partial V_{\mathrm{eff}}}{\partial f_1} = 0$:

$$\frac{\partial^2 V_{\mathrm{BH}}}{\partial f_1 \partial g_2}\bigg|_{\frac{\partial V_{\mathrm{eff}}}{\partial f_1}=0} = 0. \tag{5.5}$$

This implies that the $6 \times 6$ (real form of the) critical Hessian $(\frac{\partial^2 V_{\mathrm{eff}}}{\partial f_i \partial g_j})_{\frac{\partial V_{\mathrm{eff}}}{\partial f_k}=0}$ ($k = 1, 2$) is block diagonal. The first

$3 \times 3$ block is the axionic one $\frac{\partial^2 V_{\rm eff}}{\partial f_1 \partial g_1}|_{\frac{\partial V_{\rm eff}}{\partial f_k}=0}$, whereas the second is the $3 \times 3$ dilatonic one $\frac{\partial^2 V_{\rm eff}}{\partial f_2 \partial g_2}|_{\frac{\partial V_{\rm eff}}{\partial f_k}=0}$. Clearly, the real eigenvalues of such two blocks can be investigated separately.

It is worth pointing out that in the dilatonic block $\frac{\partial^2 V_{\rm eff}}{\partial f_2 \partial g_2}|_{\frac{\partial V_{\rm eff}}{\partial f_k}=0}$ the axions $x_1$, $y_1$, $z_1$ are like parameters, and one can implement the criticality condition $x_1 = y_1 = z_1 = 0$ *before* differentiating with respect to dilatons without losing any generality. This means that one can use the reduced form $V_{\rm eff,red}$ of the effective potential in order to study the sign of the eigenvalues of the dilatonic block of the critical Hessian of $V_{\rm eff}$, which we will denote by $\zeta_1$, $\zeta_2$, and $\zeta_3$.

Concerning the axionic block $\frac{\partial^2 V_{\rm eff}}{\partial f_1 \partial g_1}|_{\frac{\partial V_{\rm eff}}{\partial f_k}=0}$, in the $D0$-$D6$ BH charge configuration it has strictly positive eigenvalues [as it can be shown by straightforward calculations by recalling that $x_2 y_2 z_2 > 0$—see Eq. (3.5)]. On the other hand, in the magnetic BH charge configuration the axionic block is proportional (through a strictly positive coefficient) to the following $3 \times 3$ matrix:

$$\begin{pmatrix} |\frac{q_0 p^2 p^3}{p^1}|(y^2 + z^2) & p^3 q_0[\aleph z^2 - 1] & p^2 q_0[\aleph y^2 - 1] \\ p^3 q_0[\aleph z^2 - 1] & |\frac{q_0 p^1 p^3}{p^2}|(x^2 + z^2) & p^1 q_0[\aleph x^2 - 1] \\ p^2 q_0[\aleph y^2 - 1] & p^1 q_0[\aleph x^2 - 1] & |\frac{q_0 p^1 p^2}{p^3}|(x^2 + y^2). \end{pmatrix}_{\frac{\partial V_{\rm eff}}{\partial f_2}=0}, \tag{5.6}$$

where $\aleph \equiv \mathrm{sgn}(q_0 p^1 p^2 p^3)$, and the rescaled dilatons are defined by Eq. (4.1). In the following we will denote the eigenvalues of the axionic block of the critical Hessian of $V_{\rm eff}$ by $\chi_1$, $\chi_2$, and $\chi_3$.

Let us briefly recall the stability of critical points of $V_{\rm BH}$ in the *ungauged stu* model. The $\frac{1}{2}$-BPS and non-BPS $Z = 0$ critical points are actually attractors in the strict sense, with no vanishing eigenvalues of the Hessian at all: $\chi_1$, $\chi_2$, $\chi_3$, $\zeta_1$, $\zeta_2$, $\zeta_3 > 0$. On the other hand, the non-BPS $Z \neq 0$ critical points are stable, with two flat directions persisting at all orders, and spanning the moduli space $(SO(1,1))^2$ (i.e. the scalar manifold of the *ungauged stu* model in $d = 5$; see [10,21,35,37,41]): one gets $\chi_1 > 0$, $\chi_2 = \chi_3 = 0$, $\zeta_1$, $\zeta_2$, $\zeta_3 > 0$ and $\chi_1$, $\chi_2$, $\chi_3 > 0$, $\zeta_1 > 0$, $\zeta_2 = \zeta_3 = 0$, respectively, for the magnetic and $D0$-$D6$ BH charge configurations. It is also worth recalling that the magnetic BH charge configuration supports, in general, all three classes of critical points of $V_{\rm BH}$, whereas the $D0$-$D6$ BH charge configuration supports only non-BPS $Z \neq 0$ critical points of $V_{\rm BH}$.

In the following we will report the results concerning the stability of the critical points of $V_{\rm eff}$ in the various considered symplectic frames of the *stu* model with FI terms, both in the magnetic and in the $D0$-$D6$ BH charge configurations.

### 1. Magnetic configuration

As given by Eq. (4.12), in the $SO(2,2)$ covariant symplectic frame two sets of critical points of $\tilde{V}_{{\rm eff,red},SO(2,2)}$ exist. For both solutions one gets the following result:

$$SO(2,2)\ \text{sympl.frame:} \begin{cases} q_0 p^1 p^2 p^3 > 0\text{:} \begin{cases} \chi_1, \chi_2, \chi_3 > 0; \\ \zeta_1, \zeta_2, \zeta_3 > 0; \end{cases} \\ q_0 p^1 p^2 p^3 < 0\text{:} \begin{cases} \chi_1, \chi_2 > 0, \chi_3 = 0 (\tilde{\alpha} = 0\text{: } \chi_1 > 0, \chi_2 = \chi_3 = 0); \\ \zeta_1, \zeta_2, \zeta_3 > 0. \end{cases} \end{cases} \tag{5.7}$$

On the other hand, as given by Eq. (4.16), in the $(SO(1,1))^2$ covariant symplectic basis in the simplifying assumption (4.14), only one set of critical points of $\tilde{V}_{{\rm eff,red},(SO(1,1))^2}$ exist, for which it holds that

$$(SO(1,1))^2\ \text{sympl.frame:} \begin{cases} q_0 p^1 p^2 p^3 > 0\text{:} \begin{cases} \chi_1, \chi_2, \chi_3 > 0; \\ \zeta_1, \zeta_2, \zeta_3 > 0; \end{cases} \\ q_0 p^1 p^2 p^3 < 0\text{:} \begin{cases} \chi_1 > 0, \chi_2 < 0, \chi_3 < 0, (g = 0\text{: } \chi_1 > 0, \chi_2 = \chi_3 = 0); \\ \zeta_1, \zeta_2, \zeta_3 > 0. \end{cases} \end{cases} \tag{5.8}$$

Finally, as given by Eq. (4.21), in the $SO(8)$-truncated symplectic basis in the simplifying assumption (4.19), only one set of critical points of $\tilde{V}_{{\rm eff,red},SO(8)}$ exist, and it can be computed that

$$SO(8)\text{-trunc.sympl.frame:}\ \begin{cases} q_0p^1p^2p^3>0\colon \begin{cases}\chi_1,\chi_2,\chi_3>0;\\ \zeta_1,\zeta_2,\zeta_3>0;\end{cases}\\ q_0p^1p^2p^3<0\colon \begin{cases}\chi_1>0,\chi_2=\chi_3=0;\\ \zeta_1,\zeta_2,\zeta_3>0.\end{cases}\end{cases} \tag{5.9}$$

Let us briefly analyze the obtained results (5.7) and (5.8). As intuitively expected, the introduction of the (FI) potential never affects the sign of those eigenvalues strictly positive in the ungauged limit, and thus it does not modify the stability of those gauged solutions (supported by $q_0p^1p^2p^3>0$) which in the ungauged limit are $\frac{1}{2}$-BPS or non-BPS $Z=0$. On the other hand, by looking at those gauged solutions (supported by $q_0p^1p^2p^3<0$) which in the ungauged limit are non-BPS $Z\neq 0$, one realizes that in the presence of the (FI) potential the possible lift of the two ungauged non-BPS $Z\neq 0$ flat directions strictly depends on the symplectic basis being considered. Indeed, such flat directions both persist in the $SO(8)$-truncated frame, only one is lifted up to a positive direction in the $SO(2,2)$ covariant basis, and both are lifted down to tachyonic (i.e. negative) directions in the $(SO(1,1))^2$ covariant basis.

### B. $D0$-$D6$ configuration

As given by Eqs. (4.38) and (4.39), the $SO(2,2)$ covariant symplectic basis does not admit any gauged ($\tilde{\alpha}\neq 0$) critical point of $\tilde{V}_{\text{eff,red},SO(2,2)}$ in the $D0$-$D6$ BH charge configuration.

On the other hand, as given by Eqs. (4.42) and (4.43), despite the fact that analytical critical points of $\tilde{V}_{\text{eff,red},(SO(1,1))^2}$ cannot be computed, in the $(SO(1,1))^2$ symplectic frame one can still study the sign of the eigenvalues of the critical Hessian of $\tilde{V}_{\text{eff,red},(SO(1,1))^2}$, thus obtaining

$$(SO(1,1))^2\ \text{cov. sympl. frame:}\ \begin{cases}\chi_1,\chi_2,\chi_3>0;\\ \zeta_1>0,\zeta_2>0,\zeta_3>0\quad (g=0\colon \zeta_1>0,\zeta_2=\zeta_3=0).\end{cases} \tag{5.10}$$

Finally, as given by Eq. (4.45), in the $SO(8)$-truncated symplectic basis in the simplifying assumption (4.19), only one set of critical points of $\tilde{V}_{\text{eff,red},SO(8)}$ exist, and it can be computed that

$$SO(8)\text{-trunc. sympl. frame:}\ \begin{cases}\chi_1,\chi_2,\chi_3>0;\\ \zeta_1>0,\zeta_2<0,\zeta_3<0\quad (\tilde{\alpha}_{SO(8)}=0\colon \zeta_1>0,\zeta_2=\zeta_3=0).\end{cases} \tag{5.11}$$

Once again, let us briefly analyze the obtained results (5.10) and (5.11). As intuitively expected, also in the $D0$-$D6$ BH charge configuration the introduction of the (FI) potential never affects the sign of those eigenvalues strictly positive in the ungauged limit. Furthermore, by looking at Eqs. (5.10) and (5.11) one can realize once again that in the presence of the (FI) potential the possible lift of the two ungauged non-BPS $Z\neq 0$ flat directions strictly depends on the symplectic frame being considered. Indeed, such flat directions are both lifted up to positive directions in the $(SO(1,1))^2$ covariant basis, whereas they are both lifted down to tachyonic directions in the $SO(8)$-truncated frame [as it was the case for $\chi_2$ and $\chi_3$ in the $(SO(1,1))^2$ basis for the magnetic BH charge configuration].

## VI. FURTHER DEVELOPMENTS

It is clear that the present study paves the way to a number of possible further developments. One of particular relevance is the following.

By considering the issue of stability in the magnetic BH charge configuration, one might wonder whether $\chi_3=0$ in the $SO(2,2)$ frame [see Eq. (5.7)] and $\chi_2=\chi_3=0$ in the $SO(8)$-truncated frame [see Eq. (5.9)] actually are *flat directions* of the corresponding effective potentials, i.e. whether they persist also at higher order in the covariant differentiation of such (not symplectic-invariant) potentials. Since, in general, the geometry of the scalar manifold is not affected by the introduction of the gauging, in the case of a positive answer it would be interesting to determine the moduli space spanned by such *flat directions*. It is clear that the reasoning about *flat directions* performed in [37] should not apply in the considered cases, because in the presence of (FI, but likely for a completely general) gauging $V_{\text{eff}}$ is not symplectic invariant, as instead $V_{\text{BH}}$ is by definition. We leave this issue for future work.

## ACKNOWLEDGMENTS

We would like to warmly acknowledge L. Andrianopoli, R. D'Auria, P. Fré, and M. Trigiante for enlightening discussions and interest in this work. A. M. would also like to thank the Department of Physics, Theory Unit Group at CERN, where part of this work was done, for kind hospitality and a stimulating environment. The work of S. B. has been supported in part by the European

Community Human Potential Program under Contract No. MRTN-CT-2004-005104 "*Constituents, fundamental forces and symmetries of the Universe.*" The work of S. F. has been supported in part by the European Community Human Potential Program under Contract No. MRTN-CT-2004-005104 "*Constituents, fundamental forces and symmetries of the Universe*," in association with INFN Frascati National Laboratories, and by DOE Grant No. DE-FG03-91ER40662, Task C. The work of A. M. has been supported by a Junior Grant of the "*Enrico Fermi*" Centre, Rome, in association with INFN Frascati National Laboratories. The work of A. Y. was supported in part by Grant No. INTAS-05-7928, in association with INFN Frascati National Laboratories.

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[1] S. Ferrara, R. Kallosh, and A. Strominger, Phys. Rev. D **52**, R5412 (1995).
[2] A. Strominger, Phys. Lett. B **383**, 39 (1996).
[3] S. Ferrara and R. Kallosh, Phys. Rev. D **54**, 1514 (1996).
[4] S. Ferrara and R. Kallosh, Phys. Rev. D **54**, 1525 (1996).
[5] S. Ferrara, G. W. Gibbons, and R. Kallosh, Nucl. Phys. **B500**, 75 (1997).
[6] A. Sen, J. High Energy Phys. 09 (2005) 038.
[7] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, Phys. Rev. D **72**, 124021 (2005).
[8] A. Sen, J. High Energy Phys. 03 (2006) 008.
[9] R. Kallosh, J. High Energy Phys. 12 (2005) 022.
[10] P. K. Tripathy and S. P. Trivedi, J. High Energy Phys. 03 (2006) 022.
[11] A. Giryavets, J. High Energy Phys. 03 (2006) 020.
[12] K. Goldstein, R. P. Jena, G. Mandal, and S. P. Trivedi, J. High Energy Phys. 02 (2006) 053.
[13] M. Alishahiha and H. Ebrahim, J. High Energy Phys. 03 (2006) 003.
[14] R. Kallosh, N. Sivanandam, and M. Soroush, J. High Energy Phys. 03 (2006) 060.
[15] B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, J. High Energy Phys. 08 (2006) 004.
[16] J. P. Hsu, A. Maloney, and A. Tomasiello, J. High Energy Phys. 09 (2006) 048.
[17] S. Bellucci, S. Ferrara, and A. Marrani, Phys. Lett. B **635**, 172 (2006).
[18] S. Bellucci, S. Ferrara, and A. Marrani, *Supersymmetric Mechanics–Vol. 2*, Lect. Notes Phys. 701 (Springer, Berlin, Heidelberg, 2006).
[19] S. Ferrara and R. Kallosh, Phys. Rev. D **73**, 125005 (2006).
[20] M. Alishahiha and H. Ebrahim, J. High Energy Phys. 11 (2006) 017.
[21] S. Bellucci, S. Ferrara, M. Günaydin, and A. Marrani, Int. J. Mod. Phys. A **21**, 5043 (2006).
[22] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen, and S. P. Trivedi, J. High Energy Phys. 10 (2006) 058.
[23] R. Kallosh, N. Sivanandam, and M. Soroush, Phys. Rev. D **74**, 065008 (2006).
[24] P. Kaura and A. Misra, Fortschr. Phys. **54**, 1109 (2006).
[25] G. L. Cardoso, V. Grass, D. Lüst, and J. Perz, J. High Energy Phys. 09 (2006) 078.
[26] J. F. Morales and H. Samtleben, J. High Energy Phys. 10 (2006) 074.
[27] S. Bellucci, S. Ferrara, A. Marrani, and A. Yeranyan, Riv. Nuovo Cimento Soc. Ital. Fis. **29N5**, 1 (2006).
[28] G. L. Cardoso, B. de Wit, and S. Mahapatra, J. High Energy Phys. 03 (2007) 085.
[29] R. D'Auria, S. Ferrara, and M. Trigiante, J. High Energy Phys. 03 (2007) 097.
[30] S. Bellucci, S. Ferrara, and A. Marrani, arXiv:hep-th/0702019, to appear in the Proceedings of the XVII SIGRAV Conference, Turin, Italy, 2006.
[31] A. Ceresole and G. Dall'Agata, J. High Energy Phys. 03 (2007) 110.
[32] L. Andrianopoli, R. D'Auria, S. Ferrara, and M. Trigiante, J. High Energy Phys. 07 (2007) 019.
[33] K. Saraikin and C. Vafa, arXiv:hep-th/0703214.
[34] S. Ferrara and A. Marrani, arXiv:0705.3866 [Nucl. Phys. B (to be published)].
[35] S. Nampuri, P. K. Tripathy, and S. P. Trivedi, arXiv:0705.4554.
[36] L. Andrianopoli, R. D'Auria, E. Orazi, and M. Trigiante, arXiv:0706.0712.
[37] S. Ferrara and A. Marrani, Phys. Lett. B **652**, 111 (2007).
[38] D. Astefanesei and H. Yavartanoo, Nucl. Phys. **B794**, 13 (2008).
[39] G. L. Cardoso, A. Ceresole, G. Dall'Agata, J. M. Oberreuter, and J. Perz, arXiv:0706.3373.
[40] A. Misra and P. Shukla, arXiv:0707.0105.
[41] A. Ceresole, S. Ferrara, and A. Marrani, arXiv:0707.0964 [Class. Quant. Grav. (to be published)].
[42] M. M. Anber and D. Kastor, J. High Energy Phys. 10 (2007) 084.
[43] Y. S. Myung, Y.-W. Kim, and Y.-J. Park, Phys. Rev. D **76**, 104045 (2007).
[44] S. Bellucci, A. Marrani, E. Orazi, and A. Shcherbakov, arXiv:0707.2730 [Phys. Lett. B (to be published)].
[45] K. Hotta and T. Kubota, arXiv:0707.4554.
[46] X. Gao, J. High Energy Phys. 11 (2007) 006.
[47] S. Ferrara and A. Marrani, AIP Conf. Proc. **957**, 58 (2007).
[48] A. Sen, arXiv:0708.1270.
[49] A. Belhaj, L. B. Drissi, E. H. Saidi, and A. Segui, arXiv:0709.0398.
[50] L. Andrianopoli, S. Ferrara, A. Marrani, and M. Trigiante, Nucl. Phys. **B795**, 428 (2008).
[51] D. Gaiotto, W. Li, and M. Padi, J. High Energy Phys. 12 (2007) 093.
[52] S. Bellucci, Sergio Ferrara, A. Marrani, and A. Shcherbakov, arXiv:0710.3559.
[53] E. G. Gimon, F. Larsen, and J. Simon, J. High Energy Phys. 01 (2008) 040.
[54] D. Astefanesei, H. Nastase, H. Yavartanoo, and S. Yun,

arXiv:0711.0036.

[55] S. Bellucci, S. Ferrara, R. Kallosh, and A. Marrani, in Proceedings of the Winter School on the Attractor Mechanism 2006 (SAM2006), INFN-LNF, Frascati, Italy.

[56] R.-G. Cai and D.-W. Pang, J. High Energy Phys. 01 (2008) 046.

[57] M. Huebscher, P. Meessen, T. Ortín, and S. Vaulà, arXiv:0712.1530.

[58] G. W. Gibbons and C. M. Hull, Phys. Lett. **109B**, 190 (1982).

[59] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); S. W. Hawking, Phys. Rev. Lett. **26**, 1344 (1971); C. DeWitt and B. S. DeWitt, *Black Holes (Les Houches 1972)* (Gordon and Breach, New York, 1973); S. W. Hawking, Nature (London) **248**, 30 (1974); Commun. Math. Phys. **43**, 199 (1975).

[60] D. Z. Freedman and J. H. Schwarz, Nucl. Phys. **B137**, 333 (1978).

[61] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, and T. Magri, J. Geom. Phys. **23**, 111 (1997); L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, and P. Fré, Nucl. Phys. **B476**, 397 (1996).

[62] T. Levi-Civita, Rend. Acc. Naz. Lincei **26**, 519 (1917); B. Bertotti, Phys. Rev. **116**, 1331 (1959); I. Robinson, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **7**, 351 (1959).

[63] R. Kallosh, Phys. Lett. B **282**, 80 (1992).

[64] R. Kallosh and A. W. Peet, Phys. Rev. D **46**, R5223 (1992).

[65] S. Cacciatori, D. Klemm, and D. Zanon, Classical Quantum Gravity **17**, 1731 (2000).

[66] W. A. Sabra, Phys. Lett. B **458**, 36 (1999).

[67] A. H. Chamseddine and W. A. Sabra, Phys. Lett. B **485**, 301 (2000).

[68] H. Lu, C. N. Pope, and J. F. Vazquez-Poritz, Nucl. Phys. **B709**, 47 (2005).

[69] A. Ceresole, R. D'Auria, and S. Ferrara, Nucl. Phys. B, Proc. Suppl. **46**, 67 (1996).

[70] M. J. Duff, J. T. Liu, and J. Rahmfeld, Nucl. Phys. **B459**, 125 (1996).

[71] K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova, and W. K. Wong, Phys. Rev. D **54**, 6293 (1996).

[72] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit, and L. Girardello, Nucl. Phys. **B250**, 385 (1985).

[73] L. Andrianopoli, R. D'Auria, S. Ferrara, and M. A. Lledó, J. High Energy Phys. 07 (2002) 010.

[74] E. Calabi and E. Visentini, Ann. Math. **71**, 472 (1960).

[75] A. Ceresole, R. D'Auria, S. Ferrara, and A. Van Proeyen, Nucl. Phys. **B444**, 92 (1995).

[76] M. J. Duff and J. T. Liu, Nucl. Phys. **B554**, 237 (1999).

[77] E. Cremmer and B. Julia, Nucl. Phys. **B159**, 141 (1979).

[78] B. de Wit and H. Nicolai, Nucl. Phys. **B208**, 323 (1982).

[79] A. Ceresole, R. D'Auria, S. Ferrara, and A. Van Proeyen, arXiv:hep-th/9412200.

[80] M. Shmakova, Phys. Rev. D **56**, R540 (1997).