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# Parton showers with quantum interference: leading color, spin averaged

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Abstract: We have previously described a mathematical formulation for a parton shower based on the approximation of strongly ordered virtualities of successive parton splittings. Quantum interference, including interference among different color and spin states, is included. In this paper, we add the further approximations of taking only the leading color limit and averaging over spins, as is common in parton shower Monte Carlo event generators. Soft gluon interference effects remain with this approximation. We find that the leading color, spin averaged shower in our formalism is similar to that in other shower formulations. We discuss some of the differences.

Keywords: Phenomenological Models, QCD, Parton Model, Hadronic Colliders.

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## Contents



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## 1. Introduction

In ref. [1], we presented a formalism for a mathematical representation of a parton shower that incorporates interference in both spin and color. In this paper, we analyze this formalism in the approximation that we average over parton spins at each step and keep only the leading contributions in an expansion in powers of $1/N_c^2$, where $N_c = 3$ is the number of colors.[1] Our interest is to elucidate the structure of the full shower formulation of ref. [1] by examining what happens when the spin-averaged and leading color approximations are imposed. We also anticipate that the approximate shower may be of use in implementing successively better approximations to the full shower including spin and color.

[1] More precisely, we average over the spins of incoming partons at each step and sum over the spins of the outgoing partons.

Our main focus is on the splitting functions that would be used to generate the shower in the spin averaged approximation (which is a customary approximation in current parton shower event generators). In our formalism, there are two sorts of splitting functions. The direct splitting functions correspond to the squared amplitude for a parton $l$ to split into daughter partons that, in our notation, carry labels $l$ and $m+1$, where $m+1$ is the total number of final state partons after the splitting. In this paper, we use the spin dependent splitting functions from ref. [1] and simply average over the spins of the mother parton and sum over the spins of the daughter partons. We analyze some of the important properties of these functions. We also need interference splitting functions. These correspond to the interference between the amplitude for a parton $l$ to split into partons with labels $l$ and $m+1$ and the amplitude for another parton $k$ to split into partons with labels $k$ and $m+1$. These functions generate leading singularities when parton $m+1$ is a soft gluon. We improve the specifications of ref. [1] for this by defining a useful form for certain weight functions $A_{lk}$ and $A_{kl}$ that were assigned the default values 1/2 in ref. [1]. We will see that with the improved form for $A_{ij}$, the total splitting probabilities acquire useful properties in the soft gluon limit.

We will see that when we make the spin-averaged and leading color approximations, the parton shower formalism of ref. [1] amounts to something quite similar to standard parton shower event generators. One significant point in common is that the splitting functions are positive. One difference with some standard event generators is that an angular ordering approximation is not needed because the coherence effects that lead to angular ordering are built into the formalism from the beginning, both for initial state and final state splittings. This coherence feature is a natural consequence of a dipole based shower, as in the final state showers of ARIADNE [2] and the $k_T$ option of PYTHIA [3] or the showers [4, 5] based on the Catani-Seymour dipole splitting formalism [6]. Additionally, our formalism differs from others in its splitting functions and its momentum mappings.

## 2. Direct spin-averaged splitting functions

We begin with the splitting functions that correspond to the amplitude for a parton to split times the complex conjugate amplitude for that same parton to split. We follow the notation of ref. [1]. Before the splitting, there are partons that carry the labels $\{\mathrm{a}, \mathrm{b}, 1, \ldots, m\}$, where a and b are the labels of the initial state partons. The momenta and flavors of these partons are denoted by $\{p, f\}_m = \{p_\mathrm{a}, f_\mathrm{a}; \ldots; p_m, f_m\}$. The flavors are $\{\mathrm{g}, \mathrm{u}, \bar{\mathrm{u}}, \mathrm{d}, \ldots\}$, with the initial state flavors $f_\mathrm{a}$ and $f_\mathrm{b}$ denoting the flavors coming out of the hard interaction and thus the opposite of the flavors entering the hard interaction. We let $l$ be the label of the parton that splits. After the splitting, there are $m+1$ final state partons. The momenta and flavors of the partons are $\{\hat{p}, \hat{f}\}_{m+1}$. We use the label $l$ for one of the daughter partons and the label $m+1$ for the other daughter parton.[2] The partons that do not split keep their labels. However, they donate some of their momenta to the daughter partons so that the daughter partons can be on shell. Thus $\hat{p}_i \neq p_i$ in general for a spectator parton.

[2]For a final state $q \to q\mathrm{g}$ splitting, we use $m+1$ for the label of the gluon. For a final state $\mathrm{g} \to q\,\bar{q}$ splitting, we use $m+1$ for the label of the $\bar{q}$.

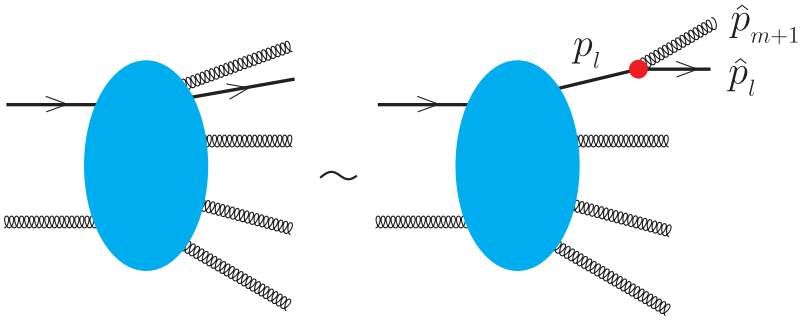


**Figure 1:** Illustration eq. (2.2) for a $qqg$ final state splitting. The small filled circle represents the splitting amplitude $v_l$. The mother parton has momentum $\hat{p}_l + \hat{p}_{m+1}$, but in the amplitude $\big|M(\{p,f\}_m)\big\rangle$, this off-shell momentum is approximated as an on-shell momentum $p_l$.

The momenta and flavors after the splitting, $\{\hat{p},\hat{f}\}_{m+1}$, are determined by the momenta and flavors before the splitting, $\{p,f\}_m$, and variables $\{\zeta_{\rm p},\zeta_{\rm f}\}$ that describe the splitting.[3] A certain mapping

$$\{\hat{p},\hat{f}\}_{m+1} = R_l(\{p,f\}_m,\{\zeta_{\rm p},\zeta_{\rm f}\}) \tag{2.1}$$

defined in ref. [1] gives the relation.

The splitting functions in ref. [1] are based on spin dependent splitting amplitudes $v_l$. One starts with the amplitude $\big|M(\{p,f\}_m)\big\rangle$ to have $m$ partons. The amplitude is a vector in spin⊗color space. After splitting parton $l$, we have a new amplitude $\big|M_l(\{\hat{p},\hat{f}\}_{m+1})\big\rangle$ of the form illustrated in figure 1

$$\big|M_l(\{\hat{p},\hat{f}\}_{m+1})\big\rangle = t_l^\dagger(f_l \to \hat{f}_l + \hat{f}_{m+1})\, V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})\,\big|M(\{p,f\}_m)\big\rangle \ . \tag{2.2}$$

Here $t_l^\dagger$ is an operator on the color space that simply inserts the daughter partons with the correct color structure. The factor $V_l^\dagger$ is a function the momenta and flavors and is an operator on the spin space. It leaves the spins of the partons other than parton $l$ undisturbed and multiplies by a function $v_l$ that depends on the mother spin and the daughter spins:

$$\big\langle\{\hat{s}\}_{m+1}\big|V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})\big|\{s\}_m\big\rangle = \left(\prod_{j\notin\{l,m+1\}}\delta_{\hat{s}_j,s_j}\right) v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l) \ . \tag{2.3}$$

Thus the splitting is defined by the splitting amplitudes $v_l$, which are derived from the QCD vertices.

[3] When a gluon splits, $\zeta_{\rm f}$ determines whether the daughters are a $(g,g)$ pair, a $(u,\bar{u})$ pair, *etc.* In ref. [1], we defined the splitting variables $\zeta_{\rm p}$ in a rather abstract way, but one could imagine using for $\zeta_{\rm p}$ the virtuality of the daughter parton pair, a momentum fraction, and an azimuthal angle.



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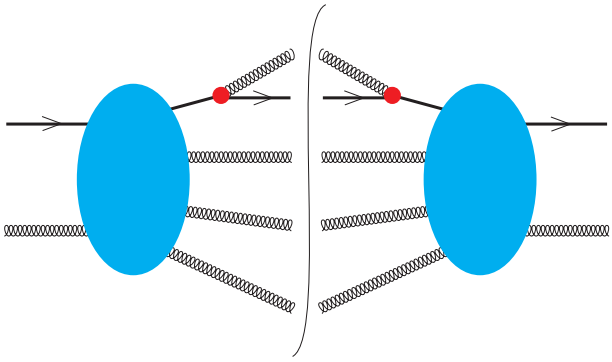

**Figure 2:** Illustration of how $v_l$ times $v_l^*$ appears in the calculation of the approximate matrix element $\left|M_l(\{\hat{p},\hat{f}\}_{m+1})\right\rangle$ times its complex conjugate, $\left\langle M_l(\{\hat{p},\hat{f}\}_{m+1})\right|$. If we average over spins, we need to multiply $\left|M(\{p,f\}_m)\right\rangle$ times $\left\langle M(\{p,f\}_m)\right|$ by $\overline{W}_{ll}$, eq. (2.8).

We can illustrate this for the case of a final state $q \to q + \mathrm{g}$ splitting, for which we define

$$v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l) = \sqrt{4\pi\alpha_s}\,\varepsilon_\mu(\hat{p}_{m+1},\hat{s}_{m+1};\hat{Q})^* \frac{\overline{U}(\hat{p}_l,\hat{s}_l)\gamma^\mu[\hat{\not{p}}_l + \hat{\not{p}}_{m+1} + m(f_l)]\eta_l U(p_l,s_l)}{[(\hat{p}_l+\hat{p}_{m+1})^2 - m^2(f_l)]\,2p_l\cdot n_l} \quad . \tag{2.4}$$

There are spinors for the initial and final quarks. There is a polarization vector for the daughter gluon, defined in timelike axial gauge so that $\hat{p}_{m+1}\cdot\varepsilon = \hat{Q}\cdot\varepsilon = 0$. Here $\hat{Q}$ is the total momentum of the final state partons, which is the same before and after the splitting. There is a vertex $\gamma^\mu$ for the $qqg$ interaction. There is a propagator for the off shell quark with momentum $\hat{p}_l + \hat{p}_{m+1}$. So far, this is exact. Finally, there is an approximation that applies when the splitting is nearly collinear or soft. We approximate $\hat{p}_l + \hat{p}_{m+1}$ by $p_l$ in the hard interaction and insert a projection $\not{n}_l/2p_l\cdot n_l$ onto the "good" components of the Dirac field. This projection uses a lightlike vector $n_l$ that lies in the plane of $\hat{Q}$ and $p_l$,

$$n_l = Q - \frac{Q^2}{Q\cdot p_l + \sqrt{(Q\cdot p_l)^2 - Q^2\,m^2(f_l)}}\,p_l \quad . \tag{2.5}$$

With one exception, the direct splitting functions in ref. [1] are products of a splitting amplitude, $v_l$, times a complex conjugate splitting amplitude, $v_l^*$,

$$v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)\;v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}'_{m+1},\hat{s}'_l,s'_l)^* \quad . \tag{2.6}$$

The calculation of $\left|M_l(\{\hat{p},\hat{f}\}_{m+1})\right\rangle$ times $\left\langle M_l(\{\hat{p},\hat{f}\}_{m+1})\right|$ using $v_l \times v_l^*$ is illustrated in figure 2. In this calculation, in general, we have to keep track of two spin indices, $s$ and $s'$ for each parton in order to describe quantum interference in the spin space. However, in this paper we make an approximation. We set $s' = s$ for each parton, sum over the daughter parton spins and average over the mother parton spins. Thus we use a splitting



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function[4]

$$\overline{W}_{ll} = \frac{1}{2} \sum_{\hat{s}_l, \hat{s}_{m+1}, s_l} |v_l(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l)|^2$$

for any flavor combination allowed with our conventions for assigning the labels $l$ and $m+1$ except for a final state $\mathrm{g} \to \mathrm{g} + \mathrm{g}$ splitting, for which we do something slightly different because the two gluons are identical. We make manifest the definition of which flavor combinations are allowed by defining

$$S_l(\{\hat{f}\}_{m+1}) = \begin{cases} 1/2\ , & l \in \{1, \ldots, m\},\ \hat{f}_l = \hat{f}_{m+1} = \mathrm{g} \\ 1\ , & l \in \{1, \ldots, m\},\ \hat{f}_l \neq \mathrm{g}, \hat{f}_{m+1} = \mathrm{g} \\ 0\ , & l \in \{1, \ldots, m\},\ \hat{f}_l = \mathrm{g}, \hat{f}_{m+1} \neq \mathrm{g} \\ 1\ , & l \in \{1, \ldots, m\},\ \hat{f}_l = q, \hat{f}_{m+1} = \bar{q} \\ 0\ , & l \in \{1, \ldots, m\},\ \hat{f}_l = \bar{q}, \hat{f}_{m+1} = q \\ 1\ , & l \in \{\mathrm{a}, \mathrm{b}\} \end{cases} . \tag{2.7}$$

This is 1 for the allowed combinations, 0 otherwise, with a statistical factor 1/2 for a final state $\mathrm{g} \to \mathrm{g} + \mathrm{g}$ splitting. The complete definition of $\overline{W}_{ll}$ is then

$$\begin{aligned} \overline{W}_{ll} = S_l(\{\hat{f}\}_{m+1})\ \frac{1}{2} \sum_{\hat{s}_l, \hat{s}_{m+1}, s_l} \Big\{ & |v_l(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l)|^2 \\ & + \theta(l \in \{1, \ldots, m\}, \hat{f}_l = \hat{f}_m = \mathrm{g}) \\ & \times \Big[ |v_{2,l}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l)|^2 - |v_{3,l}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l)|^2 \Big] \Big\} \ . \end{aligned} \tag{2.8}$$

The second term applies for a final state $\mathrm{g} \to \mathrm{g} + \mathrm{g}$ splitting and is arranged to keep the total splitting probability the same but associate the leading soft gluon singularity with gluon $m+1$ rather than gluon $l$. The functions $v_{2,l}$ and $v_{3,l}$ are defined in section 2.3.

The form of the splitting amplitude $v_l$ depends on the type of partons that are involved. However, there is a common result in the limit $\hat{p}_{m+1} \to 0$ whenever parton $m+1$ is a gluon. In this limit, $v_l$ is given by the eikonal approximation,

$$v_l^{\mathrm{eikonal}}(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l) = \sqrt{4\pi\alpha_{\mathrm{s}}}\, \delta_{\hat{s}_l, s_l}\, \frac{\varepsilon(\hat{p}_{m+1}, \hat{s}_{m+1}; Q)^* \cdot \hat{p}_l}{\hat{p}_{m+1} \cdot \hat{p}_l} \ . \tag{2.9}$$

The soft gluon limit of $\overline{W}_{ll}$ is then

$$\overline{W}_{ll}^{\mathrm{eikonal}} = 4\pi\alpha_{\mathrm{s}}\, \frac{\hat{p}_l \cdot D(\hat{p}_{m+1}; \hat{Q}) \cdot \hat{p}_l}{(\hat{p}_{m+1} \cdot \hat{p}_l)^2} \ . \tag{2.10}$$

Here $D_{\mu\nu}$ is the sum over $\hat{s}_{m+1}$ of $\varepsilon_\mu \varepsilon_\nu^*$,

$$D_{\mu\nu}(\hat{p}_{m+1}, \hat{Q}) = -g_{\mu\nu} + \frac{\hat{p}_{m+1}^\mu \hat{Q}^\nu + \hat{Q}^\mu \hat{p}_{m+1}^\nu}{\hat{p}_{m+1} \cdot \hat{Q}} - \frac{\hat{Q}^2 \hat{p}_{m+1}^\mu \hat{p}_{m+1}^\nu}{(\hat{p}_{m+1} \cdot \hat{Q})^2} \ . \tag{2.11}$$

The function $\overline{W}_{ll}$ and its approximate form $\overline{W}_{ll}^{\mathrm{eikonal}}$ give the dependence of the splitting operator on momentum and spin for a given set of parton flavors. The partons also carry

[4] The function $\overline{W}_{ll}$ here is the same as $\overline{w}_{ll}$ in ref. [1].

color. In ref. [1] there is a separate factor that gives the color dependence. This factor is an operator on the color space that we can call $t_l^\dagger \otimes t_l$, where $t_l^\dagger$ is the operator in eq. (2.2), which inserts the proper color matrix into the amplitude, and $t_l$ inserts the proper color matrix into the complex conjugate amplitude.[5] So far, we do not make any approximations with respect to color. In section 5, we will make the approximation of keeping only the leading color conributions.

We now turn to a more detailed discussion of $\overline{W}_{ll}$ for particular cases.

### 2.1 Final state $q \to q + \mathrm{g}$ splitting

Let us look at $\overline{W}_{ll}$ for a final state $q \to q + \mathrm{g}$ splitting,

$$\begin{aligned}\overline{W}_{ll} = {} & \frac{4\pi\alpha_{\mathrm{s}}}{2(p_l \cdot n_l)^2} \, \frac{1}{(2\,\hat{p}_l \cdot \hat{p}_{m+1})^2} \, D_{\mu\nu}(\hat{p}_{m+1}, \hat{Q}) \\ & \times \frac{1}{4} \mathrm{Tr}\left[ (\hat{\not{p}}_l + m)\gamma^\mu (\hat{\not{p}}_l + \hat{\not{p}}_{m+1} + m) \not{n}_l (\not{p}_l + m) \not{n}_l (\hat{\not{p}}_l + \hat{\not{p}}_{m+1} + m)\gamma^\nu \right] \quad .\end{aligned} \tag{2.12}$$

Here $m = m(f_l)$ is the quark mass, the lightlike vector $n_l$ is given by eq. (2.5), and $D_{\mu\nu}$ is given by eq. (2.11). It will be convenient to examine the dimensionless function

$$F \equiv \frac{\hat{p}_l \cdot \hat{p}_{m+1}}{4\pi\alpha_{\mathrm{s}}} \, \overline{W}_{ll} \quad . \tag{2.13}$$

The limiting behavior of $F$ as the gluon $m+1$ becomes soft, $\hat{p}_{m+1} \to 0$, is simple. Then the eikonal approximation applies and we obtain from eq. (2.10)

$$F_{\mathrm{eikonal}} = \frac{\hat{p}_l \cdot D(\hat{p}_{m+1}, \hat{Q}) \cdot \hat{p}_l}{\hat{p}_l \cdot \hat{p}_{m+1}} \quad . \tag{2.14}$$

The full behavior of $F$ is more complicated,

$$F = [1 + h(y, a_l, b_l)] \, F_{\mathrm{eikonal}} + \frac{\hat{p}_{m+1} \cdot n_l}{p_l \cdot n_l} \quad . \tag{2.15}$$

Here

$$h(y, a_l, b_l) = \frac{1 + y + \lambda r_l}{1 + r_l} + \frac{2 a_l y}{\lambda r_l (1 + r_l)} - 1 \quad , \tag{2.16}$$

where

$$\begin{aligned} y &= \frac{2\hat{p}_l \cdot \hat{p}_{m+1}}{2 p_l \cdot \hat{Q}} \quad , \\ a_l &= \frac{\hat{Q}^2}{2 p_l \cdot \hat{Q}} \quad , \\ b_l &= \frac{m^2}{2 p_l \cdot \hat{Q}} \quad , \\ r_l &= \sqrt{1 - 4 a_l b_l} \quad , \\ \lambda &= \frac{\sqrt{(1+y)^2 - 4 a_l (y + b_l)}}{r_l} \quad . \end{aligned} \tag{2.17}$$

[5] In ref. [1], we write $t_l^\dagger(f_l \to f_l + \mathrm{g})$ for the operator that we here call just $t_l^\dagger$ and we denote the operator $t_l^\dagger \otimes t_l$ by $\mathcal{G}(l, l)$.

The eikonal approximation to $F$ will turn out to be significant in our analysis when we incorporate the effect of soft-gluon interference graphs. We will find that it is of some importance for the numerical good behavior of the splitting functions including interference that

$$F - F_{\rm eikonal} \geq 0 \ . \tag{2.18}$$

To see that this property holds we note first that $\hat{p}_{m+1} \cdot n_l / p_l \cdot n_l$ is non-negative. Remarkably, $h(y, a_l, b_l) \geq 0$ also. To prove this, we first note that

$$h(y, a_l, 0) = \frac{(\lambda - 1 + y)^2 + 4y}{4\lambda} \quad , \tag{2.19}$$

so that $h(y, a_l, 0) \geq 0$. Then we show that $h(y, a_l, b_l) - h(y, a_l, 0) \geq 0$ by simply making plots of this function. This establishes the positivity property eq. (2.18).

We now examine $F$ further under the assumption that $m = 0$. We write $F$ as a function of the dimensionless virtuality variable $y$, and a momentum fraction[6]

$$z = \frac{\hat{p}_{m+1} \cdot n_l}{(\hat{p}_{m+1} + \hat{p}_l) \cdot n_l} \ . \tag{2.20}$$

It is also convenient to use an auxiliary momentum fraction variable

$$x = \frac{\hat{p}_{m+1} \cdot \hat{Q}}{(\hat{p}_{m+1} + \hat{p}_l) \cdot \hat{Q}} = \frac{\lambda}{1+y} \ z + \frac{2a_l y}{(1+y)(1+y+\lambda)} \quad , \tag{2.21}$$

where, for $m = 0$, $\lambda = \sqrt{(1+y)^2 - 4a_l y}$. Using these variables,

$$F_{\rm eikonal} = 2 \ \frac{1-x}{x} - \frac{2a_l y}{x^2 (1+y)^2} \quad , \tag{2.22}$$

and

$$F = \left[ 1 + \frac{(\lambda - 1 + y)^2 + 4y}{4\lambda} \right] F_{\rm eikonal} + \frac{1}{2} \ z[1 + y + \lambda] \ . \tag{2.23}$$

As $y \to 0$, $F$ must turn into the Altarelli-Parisi function for this splitting,

$$F_{\rm AP}(z) = \frac{1 + (1-z)^2}{z} \ . \tag{2.24}$$

Indeed, the derivation given above is one way to derive the Altarelli-Parisi function. We illustrate how $F(z, y, a_l, b_l)$ at $b_l = 0$ approaches $F_{\rm AP}(z)$ in figure 3.

[6]Note that there are many different ways to define a momentum fraction variable. The value of the splitting function for a given choice of daughter parton momenta does not depend on the momentum fraction variable that one uses to label these momenta. We have taken a simple definition of $z$ in order to display results in a graph.

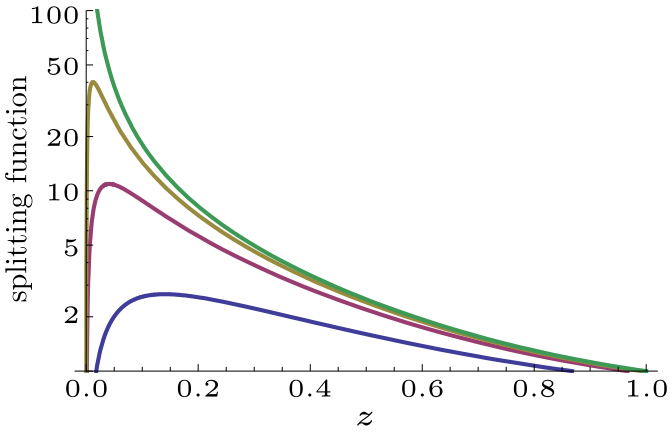


**Figure 3:** The spin averaged splitting function $F$ defined in eq. (2.23) for a final state $q \to q + \mathrm{g}$ splitting, plotted versus the momentum fraction $z$ of the gluon, as defined in eq. (2.20). The quark is taken to be massless and we set $a_l = 4$. The four curves are for, from bottom to top, $y = 0.03, 0.01, 0.001, 0$. For $y = 0$, the result is the Altarelli-Parisi function, $[1 + (1 - z)^2]/z$.

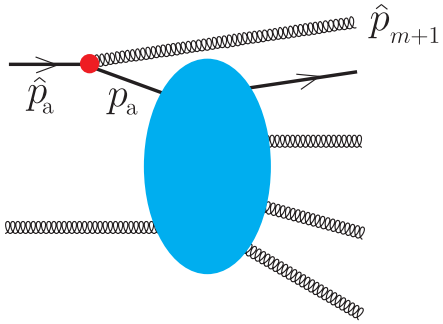


**Figure 4:** An initial state $q \to q + \mathrm{g}$ splitting.

### 2.2 Initial state $q \to q + \mathrm{g}$ splitting

Here we consider an initial state $q \to q+\mathrm{g}$ splitting, as illustrated in figure 4. For notational convenience, we let it be parton "a" that splits, so $l = \mathrm{a}$. We allow both parton "a" and parton "b" to have masses, $m_{\mathrm{a}} \equiv m(f_{\mathrm{a}})$ and $m_{\mathrm{b}} \equiv m(f_{\mathrm{b}})$. One could, of course, choose these masses to be zero. Parton $m+1$ is a (massless) gluon. The shower evolution for initial state particles runs backwards in physical time. Parton "a", which carries momentum $p_{\mathrm{a}}$ into the hard interaction, splits into the final state gluon with momentum $p_{m+1}$ and an initial state parton that carries momentum $\hat{p}_{\mathrm{a}}$ into the splitting. For a nearly collinear splitting, $p_{\mathrm{a}} \approx \hat{p}_{\mathrm{a}} - \hat{p}_{m+1}$. In physical time, it is the initial state parton with momentum $\hat{p}_{\mathrm{a}}$ that splits.

Following ref. [1], we define the kinematics using lightlike vectors $p_{\mathrm{A}}$ and $p_{\mathrm{B}}$ that are

lightlike approximations to the momenta of hadrons A and B, respectively, with $2p_{\rm A}\cdot p_{\rm B} = s$. The momenta of the partons that enter the hard scattering, $p_{\rm a}$ and $p_{\rm b}$, are defined using momentum fractions $\eta_{\rm a}$ and $\eta_{\rm b}$. After the splitting, the momentum fractions are $\hat{\eta}_{\rm a}$ and $\hat{\eta}_{\rm b}$. Because parton "a" splits, $\hat{\eta}_{\rm a} \neq \eta_{\rm a}$. However, with our kinematics, the momentum fraction of parton "b" remains unchanged: $\hat{\eta}_{\rm b} = \eta_{\rm b}$. The initial state parton momenta are defined to be

$$\begin{aligned} p_{\rm a} &= \eta_{\rm a} p_{\rm A} + \frac{m_{\rm a}^2}{\eta_{\rm a} s}\, p_{\rm B} \quad , \\ p_{\rm b} &= \eta_{\rm b} p_{\rm B} + \frac{m_{\rm b}^2}{\eta_{\rm b} s}\, p_{\rm A} \quad , \\ \hat{p}_{\rm a} &= \hat{\eta}_{\rm a} p_{\rm A} + \frac{m_{\rm a}^2}{\hat{\eta}_{\rm a} s}\, p_{\rm B} \quad . \end{aligned} \tag{2.25}$$

The momentum of the final state spectator partons changes in order to make some momentum available to allow both $p_{\rm a}$ and $\hat{p}_{\rm a}$ to be on shell with zero transverse momenta. We denote the total momentum of the final state partons before the splitting by $Q = p_{\rm a} + p_{\rm b}$ and after the splitting by $\hat{Q} = \hat{p}_{\rm a} + p_{\rm b}$. In the splitting function, we make use of a lightlike vector $n_{\rm a}$ in the plane of $p_{\rm a}$ and $Q$. With a convenient choice of normalization, $n_{\rm a} = p_{\rm B}$.

In the following formulas, it will be convenient to define $P_{\rm a} = \hat{p}_{\rm a} - \hat{p}_{m+1}$.

Using the definition eq. (2.8) with the splitting amplitudes $v_{\rm a}$ from table 1 of ref. [1], we write the spin averaged splitting function as

$$\begin{aligned} \overline{W}_{\rm aa} = {} & \frac{4\pi\alpha_{\rm s}}{2(p_{\rm a}\cdot p_{\rm B})^2}\, \frac{1}{(2\,\hat{p}_{\rm a}\cdot \hat{p}_{m+1})^2}\; D_{\mu\nu}(\hat{p}_{m+1}, \hat{Q}) \\ & \times \frac{1}{4}{\rm Tr}\left[(\hat{\not{p}}_{\rm a} + m_{\rm a})\gamma^\mu (\not{P}_{\rm a} + m_{\rm a})\not{n}_{\rm a}(\not{p}_{\rm a} + m_{\rm a})\not{n}_{\rm a}(\not{P}_{\rm a} + m_{\rm a})\gamma^\nu\right] \quad . \end{aligned} \tag{2.26}$$

Here $D_{\mu\nu}$ is given in eq. (2.11).

The spin averaged splitting function can be simplified. Let us adopt the notation

$$\Phi_a = \frac{m_{\rm a}^2}{\hat{\eta}_{\rm a}\eta_{\rm b} s} \quad , \qquad\qquad \Phi_b = \frac{m_{\rm b}^2}{\hat{\eta}_{\rm a}\eta_{\rm b} s} \quad . \tag{2.27}$$

Then the result can conveniently be displayed in terms of the dimensionless function

$$F \equiv \frac{\hat{p}_{\rm a}\cdot\hat{p}_{m+1}}{4\pi\alpha_{\rm s}}\; \overline{W}_{\rm aa} \quad . \tag{2.28}$$

The result is

$$F = F_{\rm eikonal} + \frac{\hat{p}_{m+1}\cdot p_{\rm B}}{p_{\rm a}\cdot p_{\rm B}} + \frac{\Phi_{\rm a}\Phi_{\rm b}\,(\hat{\eta}_{\rm a} - \eta_{\rm a})^2}{(1 - \Phi_{\rm a}\Phi_{\rm b})\,\eta_{\rm a}^2}\, F_{\rm eikonal} \quad . \tag{2.29}$$

Here the first term is the simple eikonal approximation for soft gluon emission,

$$F_{\rm eikonal} = \frac{\hat{p}_{\rm a}\cdot D(\hat{p}_{m+1}, \hat{Q})\cdot \hat{p}_{\rm a}}{\hat{p}_{\rm a}\cdot\hat{p}_{m+1}} \quad . \tag{2.30}$$

The second term is present in the case of massless or massive quarks and is manifestly positive. The third term is present only if $m_{\rm a}$ and $m_{\rm b}$ are both non-zero. It is likewise



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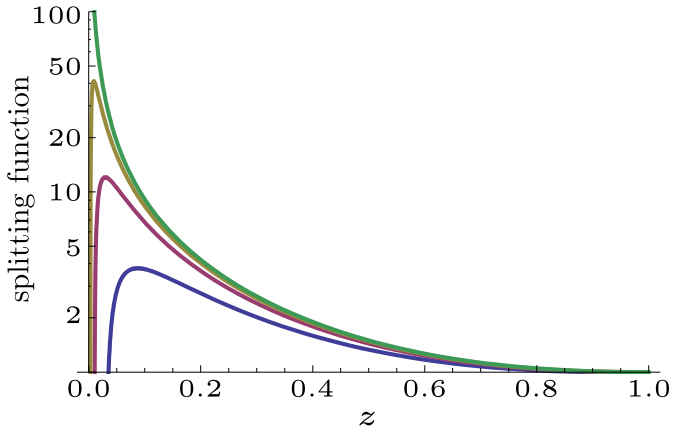


**Figure 5:** The spin averaged splitting function $(\eta_a/\hat{\eta}_a)F$, with $F$ defined in eq. (2.28), for an initial state $q \to q + g$ splitting, plotted versus the momentum fraction $z$ of the gluon, as defined in eq. (2.35). All partons are taken to be massless. The four curves are for, from bottom to top, $y = 0.03, 0.01, 0.001, 0$. For $y = 0$, the result is the Altarelli-Parisi function, $[1 + (1 - z)^2]/z$.

manifestly positive as long as $\Phi_a \Phi_b < 1$. Thus, as for the final state splitting analyzed in the previous section,

$$F - F_{\text{eikonal}} \geq 0 \ . \tag{2.31}$$

Let us now specialize to $m_a = m_b = 0$ and examine the behavior of $F$ in more detail. We define a virtuality variable

$$y = \frac{\hat{p}_{m+1} \cdot \hat{p}_a}{\hat{p}_a \cdot p_b} \tag{2.32}$$

and a variable representing the momentum fraction of the daughter gluon

$$z = \frac{\hat{p}_{m+1} \cdot (p_b - \hat{p}_{m+1})}{\hat{p}_a \cdot (p_b - \hat{p}_{m+1})} \ . \tag{2.33}$$

We can write $z$ in a different form by using the kinematic relation that is used to define the momentum mapping $\mathcal{R}_a$, $(p_a + p_b)^2 = (\hat{p}_a + p_b - \hat{p}_{m+1})^2$, which is equivalent to

$$\hat{p}_a \cdot \hat{p}_{m+1} = (P_a - p_a) \cdot p_b \ . \tag{2.34}$$

This relation gives

$$z = \frac{x - y}{1 - y} \ , \tag{2.35}$$

where

$$x = 1 - \frac{\eta_a}{\hat{\eta}_a} \ . \tag{2.36}$$

Note that $z$ and $x$ are equivalent when $y = 0$ but $z$ varies in the range $0 < z < 1$. The inverse relation is

$$x = z + y(1 - z) \ . \tag{2.37}$$

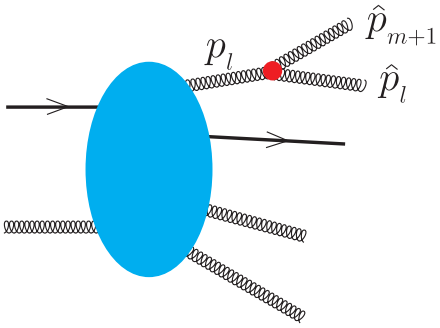


**Figure 6:** A final state $g \to g + g$ splitting.

A simple calculation gives

$$F = F_{\text{eikonal}} + \frac{x}{1-x} - y \ , \tag{2.38}$$

where

$$F_{\text{eikonal}} = \frac{2}{x} - \frac{2y}{x^2} \ . \tag{2.39}$$

As expected, $(1-x)F = (\eta_a/\hat{\eta}_a)F$ approaches the Altarelli-Parisi splitting function, $[1 + (1-z)^2]/z$ as $y \to 0$. The approach to the limit is depicted in figure 5.

### 2.3 Final state $g \to g + g$ splitting

Next we consider a final state $g \to gg$ splitting, as illustrated in figure 6. According ref. [1], the splitting amplitude is built from the ggg vertex,

$$v^{\alpha\beta\gamma}(p_a, p_b, p_c) = v_1^{\alpha\beta\gamma}(p_a, p_b, p_c) + v_2^{\alpha\beta\gamma}(p_a, p_b, p_c) + v_3^{\alpha\beta\gamma}(p_a, p_b, p_c) \ , \tag{2.40}$$

where

$$\begin{aligned} v_1^{\alpha\beta\gamma}(p_a, p_b, p_c) &= g^{\alpha\beta}(p_a - p_b)^\gamma \ , \\ v_2^{\alpha\beta\gamma}(p_a, p_b, p_c) &= g^{\beta\gamma}(p_b - p_c)^\alpha \ , \\ v_3^{\alpha\beta\gamma}(p_a, p_b, p_c) &= g^{\gamma\alpha}(p_c - p_a)^\beta \ . \end{aligned} \tag{2.41}$$

We use $v^{\alpha\beta\gamma}$ to define the splitting amplitude

$$\begin{aligned} &v_l(\{\hat{p}, \hat{f}\}_{m+1}, \hat{s}_{m+1}, \hat{s}_l, s_l) \\ &\qquad = \frac{\sqrt{4\pi\alpha_s}}{2\hat{p}_{m+1}\cdot\hat{p}_l}\, \varepsilon_\alpha(\hat{p}_{m+1}, \hat{s}_{m+1}; \hat{Q})^* \varepsilon_\beta(\hat{p}_l, \hat{s}_l; \hat{Q})^* \varepsilon^\nu(p_l, s_l; \hat{Q}) \\ &\qquad\quad \times v^{\alpha\beta\gamma}(\hat{p}_{m+1}, \hat{p}_l, -\hat{p}_{m+1} - \hat{p}_l)\, D_{\gamma\nu}(\hat{p}_l + \hat{p}_{m+1}; n_l) \ . \end{aligned} \tag{2.42}$$

We have the ggg vertex, polarization vectors for the external particles, and a propagator $D/(2\hat{p}_{m+1}\cdot\hat{p}_l)$ for the off-shell gluon. The numerator $D_{\gamma\nu}(\hat{p}_l + \hat{p}_{m+1}; n_l)$ projects on to the physical polarization states for the off-shell gluon,

$$D_{\mu\nu}(P, n_l) = -g_{\mu\nu} + \frac{P^\mu n_l^\nu + n_l^\mu P^\nu}{\hat{P}\cdot n_l} \ . \tag{2.43}$$



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Here $n_l$ is a lightlike vector in the plane of $p_l$ and $\hat{Q}$, defined as in eq. (2.5). Then $n_l^\gamma D_{\gamma\nu} = 0$.

Following ref. [1], we define the spin averaged splitting function using eq. (2.8),

$$\begin{aligned}\overline{W}_{ll} = \frac{1}{2}\left(\frac{1}{2}\sum_{\hat{s}_l,\hat{s}_{m+1},s_l}\right)\Big\{&\left|v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)\right|^2 \\ &+ |v_{2,l}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)|^2 - |v_{3,l}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)|^2\Big\} \ .\end{aligned} \tag{2.44}$$

Here $v_{2,l}$ and $v_{3,l}$ are defined as in eq. (2.42), but with $v_2^{\alpha\beta\gamma}$ or $v_3^{\alpha\beta\gamma}$, respectively, replacing the full ggg vertex $v^{\alpha\beta\gamma}$. Note first of all the prefactor 1/2, which is a statistical factor for having two identical final state particles in a $g \to g + g$ splitting. This is the factor $S_l$ in eq. (2.8). Then we add $|v_{2,l}|^2 - |v_{3,l}|^2$. This does not change the result when we add this function to the same function with the roles of the two daughter gluons interchanged. With this modification, there is a singularity when daughter gluon $m+1$ becomes soft but not when daughter gluon $l$ becomes soft.

One can evaluate $\overline{W}_{ll}$ as given in eq. (2.44) by using

$$\sum_s \varepsilon_\mu(k,s;\hat{Q})^* \varepsilon_\nu(k,s;\hat{Q}) = D_{\mu\nu}(k,\hat{Q}) \ , \tag{2.45}$$

where $D_{\mu\nu}(k,\hat{Q})$ is defined in eq. (2.11). One might expect a complicated result, but $\overline{W}_{ll}$ is actually quite simple. As in previous subsections, we display the result in terms of the dimensionless function

$$F \equiv \frac{\hat{p}_l \cdot \hat{p}_{m+1}}{4\pi\alpha_s}\ \overline{W}_{ll} \ . \tag{2.46}$$

The result is

$$F = F_{\rm eikonal} + \frac{(-k_\perp^2)[1+(1-\Delta)^2]}{4\,\hat{p}_l\cdot\hat{p}_{m+1}} \ . \tag{2.47}$$

Here $F_{\rm eikonal}$ is the standard eikonal function given in eq. (2.14) and

$$\begin{aligned} k_\perp^\mu &= D(p_l,\hat{Q})^\mu{}_\nu\ \hat{p}_{m+1}^\nu \ , \\ \Delta &= \frac{\hat{Q}^2\,\hat{p}_l\cdot\hat{p}_{m+1}}{\hat{p}_l\cdot\hat{Q}\,\hat{p}_{m+1}\cdot\hat{Q}} \ . \end{aligned} \tag{2.48}$$

Since $k_\perp$, the part of $\hat{p}_{m+1}$ orthogonal to $p_l$ and $\hat{Q}$, is spacelike, we again find

$$F - F_{\rm eikonal} \geq 0 \ . \tag{2.49}$$

We can evaluate $F$ as a function of the variables $y$ and $z$ and the parameter $a_l$, defined as for a final state quark splitting in eqs. (2.17) and (2.20). We find

$$F = F_{\rm eikonal} + \frac{1+(1-\Delta)^2}{2}\ z(1-z) \ . \tag{2.50}$$

Here $F_{\rm eikonal}$ was given in terms of $z$ and $y$ in eq. (2.22) and

$$\Delta = \frac{2a_l y}{x(1-x)(1+y)^2} \ , \tag{2.51}$$

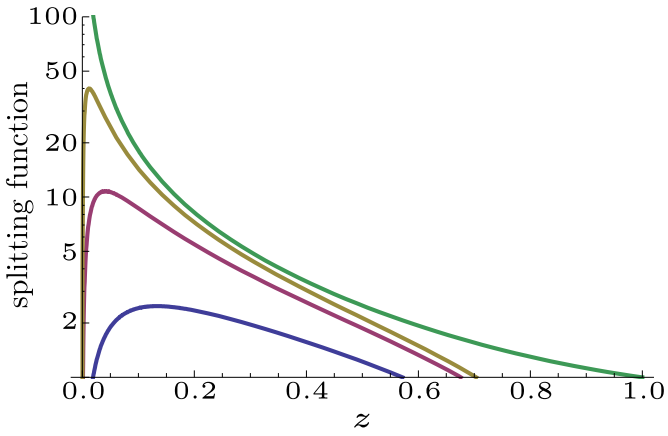


**Figure 7:** The spin averaged splitting function $F$ defined in eq. (2.46) for a final state $g \to g + g$ splitting, plotted versus the momentum fraction $z$ of the gluon, as defined in eq. (2.20). We set $a_l = 4$. The four curves are for, from bottom to top, $y = 0.03, 0.01, 0.001, 0$. For $y = 0$, the result is $2(1-z)/z + z(1-z)$. The sum of this and the same function with $z \to 1-z$ is the standard Altarelli-Parisi function, $2(1-z)/z + 2z/(1-z) + 2z(1-z)$.

where the auxiliary momentum fraction $x$ was given in terms of $z$ and $y$ in eq. (2.21).

For $y \to 0$, $F$ becomes

$$F \to f(z) \equiv \frac{2(1-z)}{z} + z(1-z) \quad . \tag{2.52}$$

The standard Altarelli-Parisi function,

$$f_{\rm AP}(z) = 2\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] \quad , \tag{2.53}$$

is $f(z) + f(1-z)$. Recall from eq. (2.44) that we broke the symmetry in a $g \to g + g$ splitting in such a way that there is a leading singularity for gluon $m+1$ becomming soft but not for gluon $l$ becoming soft. We could have accomplished the same end by using the full ggg vertex but multiplying the splitting function by $\theta(z < 1/2)$. Had we done that, the small $y$ limit of $F$ would have been $f(z) = f_{\rm AP}(z)\,\theta(z < 1/2)$. This would also give $f(z) + f(1-z) = f_{\rm AP}(z)$.

The full function $F(z, y, a_l)$ approaches $f(z)$ as $y \to 0$, as illustrated in figure 7.

### 2.4 Initial state $g \to g + g$ splitting

We now consider an initial state $g \to gg$ splitting, as illustrated in figure 8. According ref. [1], the splitting amplitude is again built from the ggg vertex, $v^{\alpha\beta\gamma}$, eq. (2.40). We use $v^{\alpha\beta\gamma}$ to define the splitting amplitude for the splitting of one of the initial state partons,

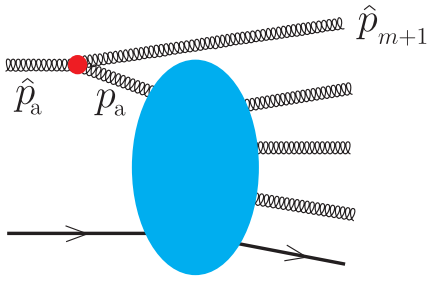


**Figure 8:** An initial state g → g + g splitting.

say parton "a,"

$$\begin{aligned} v_{\rm a}(\{\hat{p},\hat{f}\}_{m+1},&\hat{s}_{m+1},\hat{s}_l,s_l) \\ &= -\frac{\sqrt{4\pi\alpha_{\rm s}}}{2\hat{p}_{m+1}\cdot\hat{p}_{\rm a}}\,\varepsilon_\alpha(\hat{p}_{m+1},\hat{s}_{m+1};\hat{Q})^*\varepsilon_\beta(\hat{p}_{\rm a},\hat{s}_{\rm a};\hat{Q})\varepsilon^\nu(p_{\rm a},s_{\rm a};\hat{Q})^* \\ &\quad\times v^{\alpha\beta\gamma}(\hat{p}_{m+1},-\hat{p}_{\rm a},\hat{p}_{\rm a}-\hat{p}_{m+1})\,D_{\gamma\nu}(\hat{p}_{\rm a}-\hat{p}_{m+1};n_{\rm a}) \ . \end{aligned} \tag{2.54}$$

We have the ggg vertex, polarization vectors for the external particles, and a propagator $D/(2\hat{p}_{m+1}\cdot\hat{p}_{\rm a})$ for the off-shell gluon. The numerator $D_{\gamma\nu}(\hat{p}_{\rm a}-\hat{p}_{m+1};n_{\rm a})$ projects on to the physical polarization states for the off-shell gluon. It is defined using eq. (2.43), with the lightlike vector $n_{\rm a}=p_{\rm B}$. Following ref. [1], we use eq. (2.8) to define the spin averaged splitting function from the square of $v_{\rm a}$,

$$\overline{W}_{\rm aa}(\{\hat{f},\hat{p}\}_{m+1}) = \frac{1}{2}\sum_{\hat{s}_{\rm a},\hat{s}_{m+1},s_{\rm a}} \left|v_{\rm a}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_{\rm a},s_{\rm a})\right|^2 \ . \tag{2.55}$$

Remarkably, $\overline{W}_{\rm aa}$ is rather simple. As in previous subsections, we display the result in terms of the dimensionless function

$$F \equiv \frac{\hat{p}_{\rm a}\cdot\hat{p}_{m+1}}{4\pi\alpha_{\rm s}}\ \overline{W}_{\rm aa} \ . \tag{2.56}$$

The result is

$$F = F_{\rm eikonal} + \frac{-k_\perp^2}{\hat{p}_{\rm a}\cdot\hat{p}_{m+1}}\left[\left(\frac{\hat{p}_{m+1}\cdot n_{\rm a}}{(\hat{p}_{\rm a}-\hat{p}_{m+1})\cdot n_{\rm a}}\right)^2 + \frac{\hat{p}_{\rm a}\cdot n_{\rm a}\ \hat{p}_{m+1}\cdot\hat{Q}+\hat{p}_{m+1}\cdot n_{\rm a}\ \hat{p}_{\rm a}\cdot\hat{Q}}{(\hat{p}_{\rm a}-\hat{p}_{m+1})\cdot n_{\rm a}\ \hat{p}_{m+1}\cdot\hat{Q}}\right] \ . \tag{2.57}$$

Here $F_{\rm eikonal}$ is the eikonal function, eq. (2.30), and $k_\perp^\mu = D(p_{\rm a},\hat{Q})^\mu{}_\nu\ \hat{p}^\nu_{m+1}$ as in eq. (2.48). Examination of eq. (2.57) shows that, as in the previous cases,

$$F - F_{\rm eikonal} \geq 0 \ . \tag{2.58}$$

To see this, one needs to know that splitting kinematics ensures that $(\hat{p}_{\rm a}-\hat{p}_{m+1})\cdot n_{\rm a} > 0$. We note that the splitting kinematics allows non-zero parton masses, although the gluon that splits here is massless.

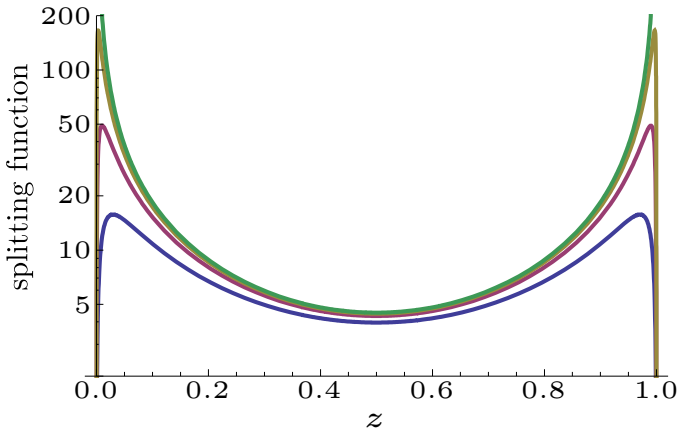


**Figure 9:** The spin averaged splitting function $(\eta_a/\hat{\eta}_a)F$, with $F$ defined in eq. (2.56), for an initial state $g \to g + g$ splitting, plotted versus the momentum fraction $z$ of the gluon, as defined in eq. (2.20). The four curves are for, from bottom to top, $y = 0.03, 0.01, 0.001, 0$. For $y = 0$, the result is the standard Altarelli-Parisi function, $2(1-z)/z + 2z/(1-z) + 2z(1-z)$.

Let us look at this assuming massless partons and using the splitting variables $y$, $z$ and $x = z + y(1-z)$ defined in section (2.2). A straightforward calculation gives

$$F = \frac{2}{x} - \frac{2y}{x^2} + 2(1-y)z \left[ \left( \frac{(1-y)z}{1-(1-y)z} \right)^2 + \frac{2z(1-y)+y}{z(1-z)(1-y)^2+y} \right] . \tag{2.59}$$

As expected, $(1-x)F = (\eta_a/\hat{\eta}_a)F$ approaches the Altarelli-Parisi splitting function, $2z/(1-z) + 2(1-z)/z + 2z(1-z)$ as $y \to 0$. The approach to the limit is depicted in figure 9.

### 2.5 Other cases

We have covered the cases of quark or gluon splittings in which a daughter gluon enters the final state. There is also the possibility of an antiquark splitting replacing a quark spitting, but, because of charge conjugation invariance, these are essentially the same as the quark splitting cases. There are also cases in which no daughter gluon enters the final state: final state and initial state $g \to q + \bar{q}$ and initial state $q \to q + g$ and $\bar{q} \to \bar{q} + g$ in which the gluon enters the hard scattering and the quark or antiquark enters the final state. The spin averaged splitting functions for these cases are manifestly positive. In these cases, there is no leading singularity when a final state daughter parton becomes soft, so we do not need to consider soft gluon singularities. We list the results for these cases in appendix A.

## 3. Interference diagrams

We have analyzed the spin averaged splitting functions $\overline{W}_{ll}$, which correspond to the squared amplitude for a parton $l$ to split into daughter partons with labels $l$ and $m+1$. Now

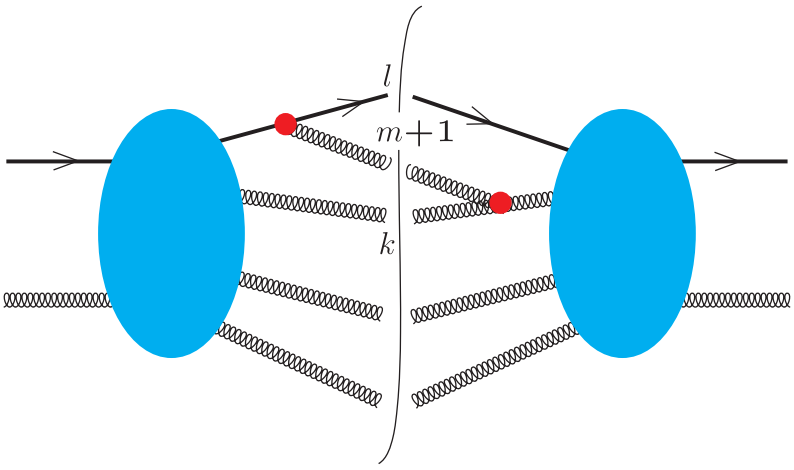


**Figure 10:** An interference diagram. A gluon, labelled $m+1$, is emitted from parton $l$ in the amplitude and from parton $k$ in the complex conjugate amplitude. This diagram has a leading singularity when the gluon is soft.

we need to consider interference diagrams, such as the diagram illustrated in figure 10. In the amplitude, parton $l$ can change into a daughter parton with label $l$ by emitting a gluon with label $m+1$. In the complex conjugate amplitude, parton $k$ can change into a daughter parton with label $k$ by emitting a gluon with label $m+1$. If we were to temporarily ignore questions about how to define the kinematics and were to use the splitting amplitudes $v_l$ and $v_k$ for this, the corresponding contribution to the splitting function would be

$$v_l(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)\,\delta_{\hat{s}_k,s_k}\;v_k(\{\hat{p},\hat{f}\}_{m+1},\hat{s}'_{m+1},\hat{s}'_k,s'_k)^*\,\delta_{\hat{s}'_l,s'_l}\quad. \tag{3.1}$$

This function is singular when gluon $m+1$ is soft, $\hat{p}_{m+1}\to 0$. However it does not have a leading singularity when gluon $m+1$ is collinear with parton $l$ or parton $k$. For this reason, we can use a simple eikonal approximation to the splitting amplitude,

$$v_l^{\rm soft}(\{\hat{p},\hat{f}\}_{m+1},\hat{s}_{m+1},\hat{s}_l,s_l)=\sqrt{4\pi\alpha_{\rm s}}\,\delta_{\hat{s}_l,s_l}\,\frac{\varepsilon(\hat{p}_{m+1},\hat{s}_{m+1};Q)^*\cdot\hat{p}_l}{\hat{p}_{m+1}\cdot\hat{p}_l}\quad, \tag{3.2}$$

if parton $m+1$ is a gluon, with $v_l^{\rm soft}=0$ otherwise.

Making the eikonal approximation, the splitting function is

$$W_{lk}=v_l^{\rm soft}(v_k^{\rm soft})^*\;\delta_{\hat{s}'_l,s'_l}\;\delta_{\hat{s}_k,s_k}\quad. \tag{3.3}$$

This function gives the dependence of the splitting operator on momentum and spin. In ref. [1] there is a separate factor that gives the color dependence. This factor is an operator on the color space that we can call $t_l^\dagger\otimes t_k$, where $t_l^\dagger$ is the operator in eq. (2.2) that inserts the proper color matrix into line $l$ in the amplitude and $t_k$ inserts the proper color matrix

into line $k$ in the complex conjugate amplitude.[7] We do not yet make any approximations with respect to color. In section 5, we will make the approximation of keeping only the leading color conributions.

There is an ambiguity with the prescription (3.3). The functions in $W_{lk}$ are defined in terms of daughter parton momenta and flavors, $\{\hat{p},\hat{f}\}_{m+1}$. However, we want to define $\{\hat{p},\hat{f}\}_{m+1}$ from the momenta and flavors $\{p,f\}_m$ before the splitting together with the splitting variables $\{\zeta_{\rm p},\zeta_{\rm f}\}$. We need to specify what relation to use. One way is to use the kinematic functions that we use for the splitting of parton $l$ into a daughter parton $l$ and the gluon $m+1$, $\{\hat{p},\hat{f}\}_{m+1} = R_l(\{p,f\}_m,\{\zeta_{\rm p},\zeta_{\rm f}\})$. With this mapping, we define a function $W_{lk}^{(l)}$ of the $\{p,f\}_m$ and the splitting variables. Alternatively, we could use the kinematic functions, $R_k$, that we use for the splitting of parton $k$ into a daughter parton $k$ and the gluon $m+1$. With this momentum mapping, we define a function $W_{lk}^{(k)}$ of the $\{p,f\}_m$ and the splitting variables. Instead of using one or the other of these possibilities, we average over them. We use $W_{lk}^{(l)}$ with weight $A_{lk}$ and $W_{lk}^{(k)}$ with weight $A_{kl}$. In ref. [1], we let the weight functions take the default value $A_{lk} = A_{kl} = 1/2$. This choice is certainly conceptually simple. However, we can obtain spin-summed splitting functions that have nicer properties if we define the weights as certain functions $A_{lk}(\{\hat{p}\}_{m+1})$ and $A_{kl}(\{\hat{p}\}_{m+1})$ of the momenta. It is simplest to specify the functional forms of the weight functions using the momenta $\{\hat{p}\}_{m+1}$ after splitting. The momenta after splitting are to be determined by the mapping $\mathcal{R}_l$ for $A_{lk}$ and by the mapping $\mathcal{R}_k$ for $A_{kl}$.[8] The weight functions are non-negative and obey $A_{lk}(\{\hat{p}\}_{m+1}) + A_{kl}(\{\hat{p}\}_{m+1}) = 1$ at fixed momenta $\{\hat{p}\}_{m+1}$. The relation $A_{lk} + A_{kl} \approx 1$ then holds at fixed $\{p,f\}_m$ and splitting variables. This approximate relation becomes exact in the limit of an infinitely soft splitting, for which the mappings $\mathcal{R}_l$ and $\mathcal{R}_k$ become identical.

With the choice of momentum mappings determined by $A_{lk}$ and $A_{kl}$, the net splitting function, including the color factor, summed over the two graphs arising from interference of soft gluons emitted from partons $l$ and $k$, is

$$\begin{aligned}\Big[A_{lk}W_{lk}^{(l)} + A_{kl}W_{lk}^{(k)}\Big]\; t_l^\dagger\otimes t_k + \Big[A_{kl}W_{kl}^{(k)} + A_{lk}W_{kl}^{(l)}\Big]\; t_k^\dagger\otimes t_l = \\ A_{lk}\left[W_{lk}^{(l)}\, t_l^\dagger\otimes t_k + W_{kl}^{(l)}\, t_k^\dagger\otimes t_l\right] + A_{kl}\left[W_{lk}^{(k)}\, t_l^\dagger\otimes t_k + W_{kl}^{(k)}\, t_k^\dagger\otimes t_l\right]\;.\end{aligned} \tag{3.4}$$

We will see in the following section that we obtain spin-summed splitting functions that have nice properties if we define $A_{lk}$ as a ratio,

$$A_{lk}(\{\hat{p}\}_{m+1}) = \frac{B_{lk}(\{\hat{p}\}_{m+1})}{B_{lk}(\{\hat{p}\}_{m+1}) + B_{kl}(\{\hat{p}\}_{m+1})}\;, \tag{3.5}$$

where

$$B_{lk}(\{\hat{p}\}_{m+1}) = \frac{\hat{p}_{m+1}\cdot\hat{p}_k}{\hat{p}_{m+1}\cdot\hat{p}_l}\;\hat{p}_l\cdot D(\hat{p}_{m+1},\hat{Q})\cdot\hat{p}_l\;. \tag{3.6}$$

Here $D(\hat{p}_{m+1},\hat{Q})$ is the transverse projection tensor defined in eq. (2.11).

[7] In ref. [1], we write $t_l^\dagger(f_l \to f_l + {\rm g})$ for the operator that we here call just $t_l^\dagger$ and we denote the operator $t_l^\dagger\otimes t_k$ by $\mathcal{G}(l,k)$.

[8] This is expressed most precisely using the operator language of eq. (8.26) of ref. [1].

## 4. Spin-averaged interference graph splitting functions

The part of the soft splitting function representing $l$-$k$ interference that is associated with the kinematic mapping $\mathcal{R}_l$ is

$$A_{lk}\left[W_{lk}^{(l)}\, t_l^\dagger \otimes t_k + W_{kl}^{(l)}\, t_k^\dagger \otimes t_l\right] \quad . \tag{4.1}$$

We now make the approximation of setting $s' = s$ for each parton, summing over the daughter parton spins, and averaging over the mother parton spins. The sum over spins of $W_{lk}^{(l)}$ is the same as the sum over spins of $W_{kl}^{(l)}$. Thus the spin averaged splitting function, including the color factor, becomes[9]

$$\frac{1}{2}\left[t_l^\dagger \otimes t_k + t_k^\dagger \otimes t_l\right]\, \overline{W}_{lk} \quad , \tag{4.2}$$

where

$$\overline{W}_{lk} = \frac{1}{4} \sum_{s_l,\hat{s}_l,s_k,\hat{s}_k,\hat{s}_{m+1}} A_{lk}\left[W_{lk}^{(l)} + W_{kl}^{(l)}\right]_{\{s'\}=\{s\}} \quad . \tag{4.3}$$

Here we have used the notation $\{s'\} = \{s\}$ to indicate the instruction to set $s'_l = s_l$, $\hat{s}'_l = \hat{s}_l$, $s'_k = s_k$, $\hat{s}'_k = \hat{s}_k$, and $\hat{s}'_{m+1} = \hat{s}_{m+1}$. The structure of $\overline{W}_{lk}$ is quite simple,

$$\overline{W}_{lk} = 4\pi\alpha_{\rm s}\; 2A_{lk}\; \frac{\hat{p}_l \cdot D(\hat{p}_{m+1};\hat{Q}) \cdot \hat{p}_k}{\hat{p}_{m+1}\cdot\hat{p}_l\; \hat{p}_{m+1}\cdot\hat{p}_k} \quad . \tag{4.4}$$

We can associate $\overline{W}_{lk}$ with the splitting of parton $l$, since it uses the kinematic mapping $\mathcal{R}_l$. Then we are led to consider the relation of $\overline{W}_{lk}$ to the direct splitting function $\overline{W}_{ll}$. Now, the color factor that multiplies $\overline{W}_{ll}$ is $t_l^\dagger \otimes t_l$. However, as discussed in ref. [1], the invariance of the matrix element under color rotations implies that

$$t_l^\dagger \otimes t_l = -\sum_{k\neq l} \frac{1}{2}\left[t_l^\dagger \otimes t_k + t_k^\dagger \otimes t_l\right] \quad . \tag{4.5}$$

Thus we can combine the direct and interference graphs to give

$$\left(-\frac{1}{2}\left[t_l^\dagger \otimes t_k + t_k^\dagger \otimes t_l\right]\right)\; \left[\overline{W}_{ll} - \overline{W}_{lk}\right] \quad . \tag{4.6}$$

We will see in section 5 that the color factor here is very simple in the leading color limit, essentially amounting to multiplying by $C_{\rm F}$ or zero. We are thus motivated to investigate the coefficient of this color operator, $\overline{W}_{ll} - \overline{W}_{lk}$.

It is useful to break $\overline{W}_{ll} - \overline{W}_{lk}$ into two pieces,

$$\overline{W}_{ll} - \overline{W}_{lk} = (\overline{W}_{ll} - \overline{W}_{ll}^{\rm eikonal}) + (\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk}) \quad . \tag{4.7}$$

Here, we recall from eq. (2.10),

$$\overline{W}_{ll}^{\rm eikonal} = 4\pi\alpha_{\rm s}\; \frac{\hat{p}_l \cdot D(\hat{p}_{m+1};\hat{Q}) \cdot \hat{p}_l}{(\hat{p}_{m+1}\cdot\hat{p}_l)^2} \quad . \tag{4.8}$$

[9] The function $\overline{W}_{lk}$ here equals the product $2A_{lk}\overline{w}_{lk}$ of functions in ref. [1].

We have investigated $(\overline{W}_{ll} - \overline{W}_{ll}^{\rm eikonal})$ in section 2 and found that

$$\overline{W}_{ll} - \overline{W}_{ll}^{\rm eikonal} \geq 0 \ . \tag{4.9}$$

Thus we should consider $\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk}$. We have

$$\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk} = \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l} \left\{ \frac{\hat{p}_l \cdot D(\hat{p}_{m+1};\hat{Q})\cdot\hat{p}_l}{\hat{p}_{m+1}\cdot\hat{p}_l} - 2A_{lk}\, \frac{\hat{p}_l \cdot D(\hat{p}_{m+1};\hat{Q})\cdot\hat{p}_k}{\hat{p}_{m+1}\cdot\hat{p}_k} \right\} \ . \tag{4.10}$$

We can simplify this if we use the definitions (3.5) of $A_{lk}$ and (3.6) of $B_{lk}$,

$$\begin{aligned}
&\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk} = \\
&\qquad \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{1}{\hat{p}_{m+1}\cdot\hat{p}_k} \left\{ \frac{\hat{p}_{m+1}\cdot\hat{p}_k}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \hat{p}_l\cdot D\cdot\hat{p}_l - 2A_{lk}\ \hat{p}_l\cdot D\cdot\hat{p}_k \right\} \\
&= \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot p_l}\ \frac{1}{\hat{p}_{m+1}\cdot\hat{p}_k} \left\{ B_{lk} - 2A_{lk}\ \hat{p}_l\cdot D\cdot\hat{p}_k \right\} \\
&= \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{A_{lk}}{\hat{p}_{m+1}\cdot\hat{p}_k} \left\{ B_{lk} + B_{kl} - 2\ \hat{p}_l\cdot D\cdot\hat{p}_k \right\} \\
&= \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{A_{lk}}{\hat{p}_{m+1}\cdot\hat{p}_k} \left\{ \frac{\hat{p}_{m+1}\cdot p_k}{\hat{p}_{m+1}\cdot p_l}\ \hat{p}_l\cdot D\cdot\hat{p}_l + \frac{\hat{p}_{m+1}\cdot p_l}{\hat{p}_{m+1}\cdot p_k}\ \hat{p}_k\cdot D\cdot\hat{p}_k \right. \\
&\qquad\qquad \left. - 2\ \hat{p}_l\cdot D\cdot\hat{p}_k \right\} \\
&= \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{A_{lk}}{\hat{p}_{m+1}\cdot\hat{p}_k}\ \frac{1}{\hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_{m+1}\cdot\hat{p}_k} \left\{ (\hat{p}_{m+1}\cdot\hat{p}_k)^2\ \hat{p}_l\cdot D\cdot\hat{p}_l \right. \\
&\qquad + (\hat{p}_{m+1}\cdot\hat{p}_l)^2\ \hat{p}_k\cdot D\cdot\hat{p}_k - 2\ (\hat{p}_{m+1}\cdot\hat{p}_k)(\hat{p}_{m+1}\cdot\hat{p}_l)\ \hat{p}_l\cdot D\cdot\hat{p}_k \Big\} \\
&= \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{A_{lk}}{\hat{p}_{m+1}\cdot\hat{p}_k}\ \frac{1}{\hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_{m+1}\cdot\hat{p}_k} \\
&\qquad \times (\hat{p}_{m+1}\cdot\hat{p}_k\ \hat{p}_l - \hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_k)\cdot D\cdot(\hat{p}_{m+1}\cdot\hat{p}_k\ \hat{p}_l - \hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_k) \ .
\end{aligned} \tag{4.11}$$

We can simplify this further by noting that the vector $\hat{p}_{m+1}\cdot\hat{p}_k\ \hat{p}_l - \hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_k$ is orthogonal to $\hat{p}_{m+1}$, so that only the term $-g^{\mu\nu}$ in $D^{\mu\nu}$ contributes. Thus

$$\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk} = \frac{4\pi\alpha_{\rm s}}{\hat{p}_{m+1}\cdot\hat{p}_l}\ \frac{A_{lk}}{\hat{p}_{m+1}\cdot\hat{p}_k}\ \frac{-(\hat{p}_{m+1}\cdot\hat{p}_k\ \hat{p}_l - \hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_k)^2}{\hat{p}_{m+1}\cdot\hat{p}_l\ \hat{p}_{m+1}\cdot\hat{p}_k} \ . \tag{4.12}$$

Since the vector $\hat{p}_{m+1}\cdot p_k\ p_l - \hat{p}_{m+1}\cdot p_l\ p_k$ is orthogonal to the lightlike vector $\hat{p}_{m+1}$, it is either lightlike or spacelike. Furthermore, $A_{lk} \geq 0$. Thus

$$\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk} \geq 0 \ . \tag{4.13}$$

Thus both parts of our splitting function, $\overline{W}_{ll} - \overline{W}_{ll}^{\rm eikonal}$ and $\overline{W}_{ll}^{\rm eikonal} - \overline{W}_{lk}$, are non-negative. This means that we can use these functions as probabilities in constructing a parton shower Monte Carlo program without needing separate weight functions. We discuss this further in section 6.

The analysis so far has allowed partons $l$ and $k$ to have non-zero masses. Let us now consider the case of massless partons, $\hat{p}_l^2 = \hat{p}_k^2 = 0$. The massless result can be understood

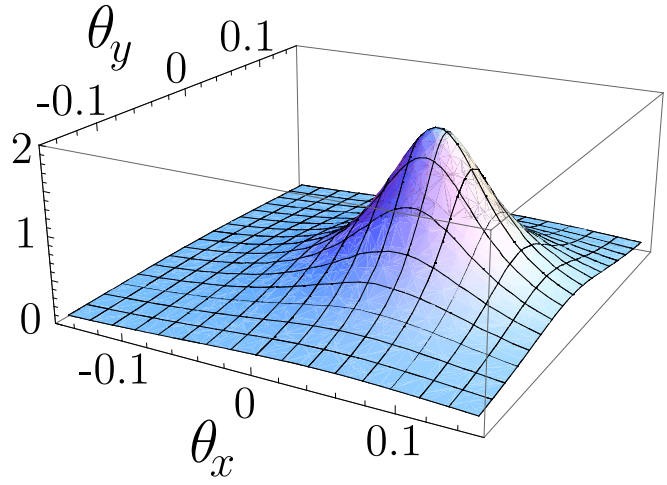


**Figure 11:** The function $g$ defined in eq. (4.15) that serves to suppress soft gluon radiation outside of the "angle ordered" region. The plot coordinates are $\theta_x = \theta\cos\phi$ and $\theta_y = \theta\sin\phi$, where $\theta, \phi$ are the polar angles of $\vec{u}_{m+1}$. The vector $\vec{u}_l$ is at $\theta = 0$ and the vector $\vec{u}_k$ is at $\theta = 0.1$, $\phi = 0$.

in more detail if we write it in terms of three-vectors in the frame in which $\vec{Q} = 0$. We define $\vec{u}_k$, $\vec{u}_l$, and $\vec{u}_{m+1}$ to be unit three-vectors in the directions of the space parts of $\hat{p}_l$, $\hat{p}_k$, and $\hat{p}_{m+1}$ respectively. Then

$$\overline{W}_{ll}^{\text{eikonal}} - \overline{W}_{lk} = \frac{4\pi\alpha_s \; 2\hat{Q}^2}{(\hat{Q}\cdot\hat{p}_{m+1})^2 \; (1 - \vec{u}_{m+1}\cdot\vec{u}_l)} \; g(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k) \;\; , \tag{4.14}$$

where

$$g(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k) = \frac{(1 + \vec{u}_{m+1}\cdot\vec{u}_l)(1 - \vec{u}_l\cdot\vec{u}_k)}{(1 - \vec{u}_{m+1}\cdot\vec{u}_l)(1 + \vec{u}_{m+1}\cdot\vec{u}_k) + (1 - \vec{u}_{m+1}\cdot\vec{u}_k)(1 + \vec{u}_{m+1}\cdot\vec{u}_l)} \; . \tag{4.15}$$

We can make some comments about this. First, the splitting probability is singular when the angle between $\vec{u}_{m+1}$ and $\vec{u}_l$ approaches zero, $(1 - \vec{u}_{m+1}\cdot\vec{u}_l) \to 0$. This is the standard collinear singularity, seen in the soft limit. Second, when $(1 - \vec{u}_{m+1}\cdot\vec{u}_l) \ll (1 - \vec{u}_k\cdot\vec{u}_l) \ll 1$, $\overline{W}_{ll}^{\text{eikonal}} - \overline{W}_{lk}$ behaves like $1/(1 - \vec{u}_{m+1}\cdot\vec{u}_l)$. If we integrate over the angle of $\vec{u}_{m+1}$ with a lower cutoff on the angle between $\vec{u}_{m+1}$ and $\vec{u}_l$, the integral is logarithmically sensitive to the cutoff. Third, when $(1 - \vec{u}_k\cdot\vec{u}_l) \ll (1 - \vec{u}_{m+1}\cdot\vec{u}_l) \ll 1$, $\overline{W}_{ll}^{\text{eikonal}} - \overline{W}_{lk}$ behaves like $1/(1 - \vec{u}_{m+1}\cdot\vec{u}_l)^2$. If we were to put an upper cutoff on the angular integration, there would be no logarithmic sensitivity to this cutoff. Thus, only the angle ordered region $(1 - \vec{u}_{m+1}\cdot\vec{u}_l) \lesssim (1 - \vec{u}_k\cdot\vec{u}_l)$ is important in the integral over angles. There is a smooth decrease in the splitting probability when the angle between $\vec{u}_{m+1}$ and $\vec{u}_l$ becomes greater than the angle between $\vec{u}_k$ and $\vec{u}_l$. There is no sharp cutoff.

We illustrate this in figure 11. We take the polar angles of $\vec{u}_{m+1}$ to be $\theta, \phi$ where $\vec{u}_l$ is along the $\theta = 0$ axis. We choose $\vec{u}_k$ to have polar angles $\theta_k = 0.1$ and $\phi_k = 0$. Then we plot $g(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k)$ versus $\theta_x = \theta\cos\phi$ and $\theta_y = \theta\sin\phi$. Since $\overline{W}_{ll}^{\text{eikonal}} - \overline{W}_{lk} \propto g(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k)/(1 - \vec{u}_{m+1}\cdot\vec{u}_l)$, the main feature of $\overline{W}_{ll}^{\text{eikonal}} - \overline{W}_{lk}$ is a singularity at



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$\theta_x = \theta_y = 0$. We see that the factor $g$ that multiplies the singular factor is a smooth function with a gentle peak between $\vec{u}_l$ and $\vec{u}_k$. This peak above $g = 1$ represents constructive interference. When $\vec{u}_{m+1}$ moves outside the "angle ordered" region $(1 - \vec{u}_{m+1} \cdot \vec{u}_l) < (1 - \vec{u}_{m+1} \cdot \vec{u}_k)$, the factor $g$ drops below 1 and decreases to zero, representing destructive interference. We notice in figure 11 that there is an enhancement of soft gluon radiation in the region between the directions of parton $l$ and parton $k$. This enhancement is known as the string effect and has been observed experimentally [7].

## 5. The leading color limit

We have studied the spin-averaged splitting function $[\overline{W}_{ll} - \overline{W}_{lk}]$. Here $\overline{W}_{ll}$ describes the square of the graph for emission of a gluon from parton $l$. There are also interference graphs between emitting the gluon from parton $l$ and emitting the same gluon from parton $k$. The function $\overline{W}_{lk}$ describes the part of the interference graphs that we group with parton $l$. These functions give the momentum dependence. They multiply a color operator as given in eq. (4.6),

$$-\frac{1}{2}\left[t_l^\dagger \otimes t_k + t_k^\dagger \otimes t_l\right] \quad . \tag{5.1}$$

We have so far not made any approximations with respect to color. Let us now take the leading color approximation. To do that, recall from ref. [1] that we use color states based on color string configurations. For instance, we could have a state $[4, 5, 2, 3, 1]$ in which 4 labels a quark, 1 labels an antiquark, and 5, 2, and 3 label gluons that connect, in that order, to a color string between the quark and antiquark. One can also have a closed string such as $(4, 5, 2, 3, 1)$ in which all of the partons are gluons. A color basis state can also consist of more than one color string connecting the partons. In general, the amplitude can have one color state $\big|c\big\rangle$ and the complex conjugate amplitude can have a color state $\big\langle c'\big|$ with $c' \neq c$. However, in the leading color approximation we can only have $c' = c$. Additionally, in the leading color approximation we have

$$-t_l^\dagger \otimes t_k = -t_k^\dagger \otimes t_l = -\frac{1}{2}\left[t_l^\dagger \otimes t_k + t_k^\dagger \otimes t_l\right] \sim C_{\rm F}\, a_{lk}^\dagger \otimes a_{lk} \quad . \tag{5.2}$$

Here $a_{lk}^\dagger$ represents the operator that inserts gluon $m+1$ between partons $l$ and $k$ on the color string if these partons are adjacent to each other on the same color string, that is, if partons $l$ and $k$ are color connected. When $a_{lk}^\dagger$ is applied to a state $\big|c\big\rangle$ in which $l$ and $k$ are not color connected, we define $a_{lk}^\dagger\big|c\big\rangle = 0$.[10] For the complex conjugate amplitude, $\big\langle c\big|\, a_{lk}$ again gives a state with the soft gluon inserted between partons $l$ and $k$. Thus, starting with a color state $\big|c\big\rangle$ in the amplitude and $\big\langle c\big|$ in the complex conjugate amplitude, we get zero if partons $l$ and $k$ are not color connected and we get a new color state with the soft gluon inserted between $l$ and $k$ if $l$ and $k$ are color connected. The bookkeeping on

[10] We here adapt the notation of ref. [1], where we had gluon insertion operators $a_+^\dagger(l)$ and $a_-^\dagger(l)$ that insert the gluon to the right or the left of parton $l$, respectively. If partons $l$ and $k$ are color connected, we have $a_{lk}^\dagger|c\rangle = a_+^\dagger(l)|c\rangle$ or $a_{lk}^\dagger|c\rangle = a_-^\dagger(l)|c\rangle$, depending on whether parton $k$ was to the right or left of parton $l$ along the string.

color connections is a standard part of parton shower event generators. The momentum dependent numerical factor $[\overline{W}_{ll} - \overline{W}_{lk}]$ is multiplied by a color factor $C_{\rm F}$.

This analysis has covered the case in which parton $m+1$ is a gluon, so that there are interference graphs arising from this gluon being emitted from parton $l$ in the amplitude and from parton $k$ in the complex conjugate amplitude (or the other way around). There are also graphs for which parton $m+1$ is a quark or antiquark, as described in section 2.5. In these cases, we have just the splitting function $\overline{W}_{ll}$, which multiplies the color operator $t_l^\dagger \otimes t_l$. This operator is very simple in the leading color limit.

Consider first the case of an initial state splitting in which $f_l = q$ and $\{\hat{f}_l, \hat{f}_{m+1}\} = \{\mathrm{g}, q\}$, where $q$ is a quark flavor (u, ū, d, ...). In physical time, this is a splitting $\mathrm{g} \to q + \bar{q}$, while in shower time it is a splitting $q \to \mathrm{g} + q$. As discussed in section 7.3 of ref. [1],

$$t_l^\dagger \otimes t_l = C_{\rm F}\, a_{\rm g}^\dagger(l) \otimes a_{\rm g}(l) \ . \tag{5.3}$$

Here $a_{\rm g}^\dagger(l)$ represents the operator that inserts the gluon at the end of the string terminated by quark $l$ before the splitting and terminated by quark $m+1$ after the splitting.[11] Similarly, in the case of an initial state splitting in which $f_l = \bar{q}$ and $\{\hat{f}_l, \hat{f}_{m+1}\} = \{\mathrm{g}, \bar{q}\}$, we have the same result, where now $a_{\rm g}^\dagger(l)$ represents the operator that inserts the gluon at the end of the string terminated by antiquark $l$ before the splitting and terminated by antiquark $m+1$ after the splitting.

Consider next the case of an initial state splitting in which $f_l = \mathrm{g}$ and $\{\hat{f}_l, \hat{f}_{m+1}\} = \{\bar{q}, q\}$. In physical time, this is a splitting $q \to q + \mathrm{g}$, while in shower time it is a splitting $\mathrm{g} \to q + \bar{q}$. As discussed in section 7.3 of ref. [1], in the leading color limit,

$$t_l^\dagger \otimes t_l \sim T_{\rm R}\ a_q^\dagger(l) \otimes a_q(l) \ , \tag{5.4}$$

where $T_{\rm R} = 1/2$ and $a_q^\dagger(l)$ splits the color string at the point at which gluon $l$ attaches, creating new string ends corresponding to the quark and the antiquark. The same analysis applies for an initial state splitting with $f_l = \mathrm{g}$ and $\{\hat{f}_l, \hat{f}_{m+1}\} = \{q, \bar{q}\}$ and for a final state $\mathrm{g} \to q + \bar{q}$ splitting, for which $\{\hat{f}_l, \hat{f}_{m+1}\} = \{q, \bar{q}\}$.

## 6. Evolution equation

We now have the information that we need to present the formulas from ref. [1] for parton shower evolution specialized to the spin averaged, leading color approximation. In the general case, we had basis states $\big|\{p, f, s', c', s, c\}_m\big)$ with two color configurations $\{c\}_m$ and $\{c'\}_m$, representing the color state in the amplitude and the color state in the complex conjugate amplitude, respectively, and two spin color configurations $\{s\}_m$ and $\{s'\}_m$. In this paper, we have averaged over spins, so that we can describe the evolution of the states without referring to spin at all. We also use the leading color approximation, so that we always work with states with $\{c\}_m = \{c'\}_m$. Thus our description is vastly simplified and we can work with basis states $\big|\{p, f, c\}_m\big)$.

[11] This operator is denoted $a_+^\dagger(l)$ in ref. [1].

As in ref. [1], we use the logarithm of the virtuality of a splitting as the evolution variable, so that a splitting of parton $l$ is assigned to a shower time $t = T_l(\{\hat{p}, \hat{f}\}_{m+1})$,

$$T_l(\{\hat{p}, \hat{f}\}_{m+1}) = \log\left(\frac{Q_0^2}{|(\hat{p}_l + (-1)^{\delta_{l,\mathrm{a}}+\delta_{l,\mathrm{b}}}\hat{p}_{m+1})^2 - m^2(f_l)|}\right) \quad , \tag{6.1}$$

where $f_l = \hat{f}_l + \hat{f}_{m+1}$ and $Q_0^2$ is the starting virtuality scale. Shower evolution is based on the probability that, at shower time $t$, a state $\big|\{p, f, c\}_m\big)$ that had not already split now splits to make a new state $\big|\{\hat{p}, \hat{f}, \hat{c}\}_{m+1}\big)$ with one more parton. This probability is represented as a matrix element of a splitting operator $\mathcal{H}_{\mathrm{I}}^{(0)}(t)$, which is similar to the splitting operator $\mathcal{H}_{\mathrm{I}}(t)$ of ref. [1] except that the spin averaged, leading color approximations ("(0)") have been applied. Then $\mathcal{H}_{\mathrm{I}}^{(0)}(t)$ operates on states $\big|\{p, f, c\}_m\big)$ instead of the states of the full theory. We write

$$\begin{aligned}
&\big(\{\hat{p}, \hat{f}, \hat{c}\}_{m+1}\big|\mathcal{H}_{\mathrm{I}}^{(0)}(t)\big|\{p, f, c\}_m\big) \\
&\quad = \sum_l (m+1)\,\frac{n_{\mathrm{c}}(a)n_{\mathrm{c}}(b)\,\eta_{\mathrm{a}}\eta_{\mathrm{b}}}{n_{\mathrm{c}}(\hat{a})n_{\mathrm{c}}(\hat{b})\,\hat{\eta}_{\mathrm{a}}\hat{\eta}_{\mathrm{b}}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}}, \mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}}, \mu_F^2)}{f_{a/A}(\eta_{\mathrm{a}}, \mu_F^2)f_{b/B}(\eta_{\mathrm{b}}, \mu_F^2)} \\
&\qquad \times \big(\{\hat{p}, \hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p, f\}_m\big)\,\delta\Big(t - T_l(\{\hat{p}, \hat{f}\}_{m+1})\Big) \\
&\qquad \times \Big\{\theta(\hat{f}_{m+1} = \mathrm{g})\sum_{\substack{k \\ k\neq l}}\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^\dagger\big|\{c\}_m\big\rangle\,\Phi_{lk}(\{\hat{p}, \hat{f}\}_{m+1}) \\
&\qquad\quad + \theta(\hat{f}_{m+1} \neq \mathrm{g})\,\theta(\hat{f}_l = \mathrm{g})\,\big\langle\{\hat{c}\}_{m+1}\big|a_{\mathrm{g}}^\dagger(l)\big|\{c\}_m\big\rangle\,\Phi_{ll}(\{\hat{p}, \hat{f}\}_{m+1}) \\
&\qquad\quad + \theta(\hat{f}_{m+1} \neq \mathrm{g})\,\theta(f_l = \mathrm{g})\,\big\langle\{\hat{c}\}_{m+1}\big|a_q^\dagger(l)\big|\{c\}_m\big\rangle\,\Phi_{ll}(\{\hat{p}, \hat{f}\}_{m+1})\Big\} \quad .
\end{aligned} \tag{6.2}$$

The first line on the right hand side of this formula contains factors copied directly from ref. [1]. There is a sum over the index $l$ of the parton that splits. Then there is a ratio of parton distribution functions. This ratio is 1 for a final state splitting but different from 1 for an initial state splitting. The next line concerns the relation of the variables $\{\hat{p}, \hat{f}\}_{m+1}$ and $t$ to the variables $\{p, f\}_m$. For the flavors, this factor vanishes unless there is a QCD vertex for $f_l \to \hat{f}_l + \hat{f}_{m+1}$ and it vanishes unless $\hat{f}_j = f_j$ for the other partons. For an allowed relationship between $\{\hat{f}\}_{m+1}$ and $\{f\}_m$, the flavor factor is 1. There is a similar factor for the momenta. Given the momenta $\{p\}_m$, the momenta $\{\hat{p}\}_{m+1}$ must lie on a certain three dimensional surface specified by the momentum mapping $\mathcal{R}_l$ defined in ref. [1]. The function $\big(\{\hat{p}, \hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p, f\}_m\big)$ contains a delta function on this surface. There is also a delta function that defines the shower time $t$. Thus if we integrate $\big(\{\hat{p}, \hat{f}, \hat{c}\}_{m+1}\big|\mathcal{H}_{\mathrm{I}}^{(0)}(t)\big|\{p, f, c\}_m\big)$ over $t$ and the momenta $\{\hat{p}\}_{m+1}$, we are really integrating over three variables that describe the splitting of parton $l$.

The final factor in eq. (6.2) contains three terms. Our main interest is in the first term, for $\hat{f}_{m+1} = \mathrm{g}$. There is a sum over the index $k$ of other partons in the process. These are the partons that might be connected with parton $l$ in an interference diagram. The remaining factors are rather complicated in the general case described in ref. [1], but are quite simple in the spin averaged, leading color approximation. The factor $\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^\dagger\big|\{c\}_m\big\rangle$ embodies

the color considerations described in section 5. It equals 1 provided two conditions hold. First, partons $l$ and $k$ must be color connected in the initial color state $\{c\}_m$. Second, the new color state $\{\hat{c}\}_{m+1}$ must be the same as $\{c\}_m$ with the gluon with label $m+1$ inserted between partons $l$ and $k$. If either of these conditions fails, this factor vanishes. The remaining factor is the splitting function

$$\Phi_{lk} \equiv C_{\rm F}\,[\overline{W}_{ll} - \overline{W}_{lk}] \quad . \tag{6.3}$$

We have seen explicitly what this factor is, and have noted that $\Phi_{lk}$ is positive.

The next term in the braces in eq. (6.2) applies to an initial state splitting in which $\hat{f}_l = {\rm g}$ and $\{f_l, \hat{f}_{m+1}\}$ is either $\{q,q\}$ or $\{\bar{q},\bar{q}\}$. The color factor $\big\langle\{\hat{c}\}_{m+1}\big|a^\dagger_{\rm g}(l)\big|\{c\}_m\big\rangle$ is 1 if the new color state $\{\hat{c}\}_{m+1}$ is the same as $\{c\}_m$ with the end of the string at quark or antiquark $l$ now terminated at quark or antiquark $m+1$ and the new the gluon with label $l$ inserted just next to the end of the string. Otherwise, this factor vanishes. The corresponding splitting function is

$$\Phi_{ll} \equiv C_{\rm F}\,\overline{W}_{ll} \quad . \tag{6.4}$$

The final term in the braces in eq. (6.2) applies to a splitting in which $f_l = {\rm g}$ and $\{\hat{f}_l, \hat{f}_{m+1}\}$ is either $\{q,\bar{q}\}$ or $\{\bar{q},q\}$. The color factor $\big\langle\{\hat{c}\}_{m+1}\big|a^\dagger_{\rm q}(l)\big|\{c\}_m\big\rangle$ is 1 if the color state $\{\hat{c}\}_{m+1}$ is related to $\{c\}_m$ by cutting the color string on which parton $l$ (a gluon) lies into two strings, terminating at the new quark and antiquark. Otherwise, this factor vanishes. The corresponding splitting function is

$$\Phi_{ll} \equiv T_{\rm R}\,\overline{W}_{ll} \quad . \tag{6.5}$$

We have now specified the probability that a state $\big|\{p,f,c\}_m\big)$ splits. The probability that this state does *not* split between shower times $t$ and $t'$ is

$$\Delta^{(0)}(t,t';\{p,f,c\}_m) = \exp\left(-\int_{t'}^{t} d\tau\ \big(1\big|\mathcal{H}^{(0)}_{\rm I}(\tau)\big|\{p,f,c\}_m\big)\right) \quad . \tag{6.6}$$

Here $\big(1\big|\mathcal{H}^{(0)}_{\rm I}(\tau)\big|\{p,f,c\}_m\big)$ is the inclusive probability for the state $\big|\{p,f,c\}_m\big)$ to split at time $\tau$,

$$\big(1\big|\mathcal{H}^{(0)}_{\rm I}(\tau)\big|\{p,f,c\}_m\big) = \frac{1}{(m+1)!}\int \big[d\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big]\ \big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big|\mathcal{H}^{(0)}_{\rm I}(t)\big|\{p,f,c\}_m\big) \quad . \tag{6.7}$$

To get the inclusive splitting probability, we have integrated over the momenta $\{\hat{p}\}_{m+1}$ after the splitting and summed over the flavors and colors, using the integration measure in Eq, (3.15) of ref. [1], supplemented by a sum over color states.[12]

With these ingredients, we can describe shower evolution using the evolution equation (14.1) from ref. [1]. The evolution from a shower time $t'$ to a final time $t_{\rm f}$ at which showering

[12] According to eq. (3.15) of ref. [1], there is an extra normalization factor $\big\langle\{\hat{c}\}_{m+1}\big|\{\hat{c}\}_{m+1}\big\rangle$ in eq. (6.7). With our choice of the normalization of color states, this factor is not exactly 1, but it is 1 in the leading color limit.

is terminated is given by an operator $\mathcal{U}^{(0)}(t_{\rm f}, t')$ that obeys[13]

$$\mathcal{U}^{(0)}(t_{\rm f}, t') = \mathcal{N}^{(0)}(t_{\rm f}, t') + \int_{t'}^{t_{\rm f}} d\tau\, \mathcal{U}^{(0)}(t_{\rm f}, \tau)\, \mathcal{H}_{\rm I}^{(0)}(\tau)\, \mathcal{N}^{(0)}(\tau, t') \ . \tag{6.8}$$

Here $\mathcal{N}^{(0)}(t, t')$ is a no-splitting operator defined by

$$\mathcal{N}^{(0)}(t, t') \big| \{p, f, c\}_m \big) = \Delta^{(0)}(t, t'; \{p, f, c\}_m) \big| \{p, f, c\}_m \big) \ . \tag{6.9}$$

If we apply this to a state $\big|\{p, f, c\}_m\big)$ that exists at shower time $t'$, we have

$$\begin{aligned} \mathcal{U}^{(0)}(t_{\rm f}, t') \big| \{p, f, c\}_m \big) &= \Delta^{(0)}(t_{\rm f}, t'; \{p, f, c\}_m) \big| \{p, f, c\}_m \big) \\ &\quad + \int_{t'}^{t_{\rm f}} d\tau\, \frac{1}{(m+1)!} \int \big[ d\{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big]\, \mathcal{U}^{(0)}(t_{\rm f}, \tau) \, \big| \{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big) \\ &\qquad \times \big( \{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big| \mathcal{H}_{\rm I}^{(0)}(\tau) \big| \{p, f, c\}_m \big)\, \Delta^{(0)}(\tau, t'; \{p, f, c\}_m) \ . \end{aligned} \tag{6.10}$$

The first term gives the probability that the state does not split before shower time $t_{\rm f}$. The main evolution is represented by the second term. There is an integration over the shower time $\tau$ of the next splitting and over the splitting parameters. In an implementation of this equation, the integration would be performed by Monte Carlo integration. That is, we would choose $\tau$ and $\{\hat{p}, \hat{f}, \hat{c}\}_{m+1}$ with some probability density $\rho$ that contains delta functions that restrict $\tau$ and $\{\hat{p}, \hat{f}, \hat{c}\}_{m+1}$ to the allowed surface defined by the eq. (6.1) for $\tau$ and the momentum mapping $\mathcal{R}_l$. Then we multiply by a weight $w$ defined by

$$\frac{1}{(m+1)!} \, \big( \{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big| \mathcal{H}_{\rm I}^{(0)}(\tau) \big| \{p, f, c\}_m \big)\, \Delta^{(0)}(\tau, t'; \{p, f, c\}_m) = w \times \rho \ . \tag{6.11}$$

In the present case, the integrand has two welcome features. First, it is positive. Second, using the definition of $\Delta^{(0)}$,

$$\begin{aligned} &\frac{1}{(m+1)!} \int_{t'}^{\infty} d\tau \int \big[ d\{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big] \big( \{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big| \mathcal{H}_{\rm I}^{(0)}(\tau) \big| \{p, f, c\}_m \big)\, \Delta^{(0)}(\tau, t'; \{p, f, c\}_m) \\ &\qquad = 1 \ . \end{aligned} \tag{6.12}$$

Thus the function

$$\rho = \frac{\big( \{\hat{p}, \hat{f}, \hat{c}\}_{m+1} \big| \mathcal{H}_{\rm I}^{(0)}(\tau) \big| \{p, f, c\}_m \big)\, \Delta^{(0)}(\tau, t'; \{p, f, c\}_m)}{(m+1)!} \tag{6.13}$$

is positive and properly normalized to be a probability density. Using standard methods from shower Monte Carlo algorithms [8–11], we can choose points with this probability density. Then $w = 1$. With a probability $\Delta^{(0)}(t_{\rm f}, t'; \{p, f, c\}_m)$, the point selected will be in the range $t_{\rm f} < \tau < \infty$. In this case, there is no splitting and we simply keep the state

[13] In ref. [1], $[\mathcal{H}_{\rm I}(\tau) - \mathcal{V}_{\rm S}(\tau)]$ appears in place of $\mathcal{H}_{\rm I}^{(0)}(\tau)$ here. With the leading color approximation, $\mathcal{V}_{\rm S}(\tau) = 0$.

$\big|\{p,f,c\}_m\big)$. This corresponds to the no splitting term in eq. (6.10). If $\tau < t_{\rm f}$, the state splits to $\{\hat{p},\hat{f},\hat{c}\}_{m+1}$. Then, according to eq. (6.10), we should apply $\mathcal{U}^{(0)}(t_{\rm f},\tau)$ to this state, repeating the process. Thus the evolution proceeds by what is known as a Markov chain.

The starting point for evolution is a state that is a mixture of the basis states $\big|\{p,f,c\}_m\big)$ for $m=2$, assuming that we start with a $2\to 2$ hard process,

$$\big|\rho^{(0)}(0)\big) = \frac{1}{2!}\int \big[d\{p,f,c\}_2\big]\ \big|\{p,f,c\}_2\big)\big(\{p,f,c\}_2\big|\rho^{(0)}(0)\big)\ . \tag{6.14}$$

Here $\big(\{p,f,c\}_2\big|\rho^{(0)}(0)\big)$ is obtained from the $2\to 2$ matrix element summed over spins,[14]

$$\big(\{p,f,c\}_2\big|\rho^{(0)}(0)\big) = \frac{f_{a/A}(\eta_{\rm a},\mu_F^2)f_{b/B}(\eta_{\rm b},\mu_F^2)}{4n_{\rm c}(a)n_{\rm c}(b)\,2\eta_{\rm a}\eta_{\rm b}p_{\rm A}\cdot p_{\rm B}}\sum_{\{s\}_2}\big|\big\langle\{s,c\}_2\big|M(\{p,f\}_2)\big\rangle\big|^2\ . \tag{6.15}$$

To implement eq. (6.14), one would choose points $\{p,f,c\}_2$ by Monte Carlo methods. This gives the starting point for the shower evolution. The state $\big|\rho^{(0)}(0)\big)$ then evolves into a state

$$\big|\rho^{(0)}(t_{\rm f})\big) = \mathcal{U}^{(0)}(t_{\rm f},0)\big|\rho^{(0)}(0)\big) \tag{6.16}$$

at the shower time $t_{\rm f}$ at which we choose to terminate shower evolution. At this point, as described in ref. [1], the desired cross section is obtained by applying a hadronization model to the component states $\big|\{p,f,c\}_N\big)$ in $\big|\rho^{(0)}(t_{\rm f})\big)$, producing a hadronic state $\mathcal{U}^{\rm had}(\infty,t_{\rm f})\big|\rho^{(0)}(t_{\rm f})\big)$. Then the desired cross section $\sigma[F_{\rm h}]$ results from applying the measurement function $F_{\rm h}$ to the hadronic states produced. Thus

$$\begin{aligned}\sigma^{(0)}[F_{\rm h}] &= \big(F_{\rm h}\big|\mathcal{U}^{\rm had}(\infty,t_{\rm f})\big|\rho^{(0)}(t_{\rm f})\big)\\ &= \sum_N \frac{1}{N!}\int\big[d\{p,f,c\}_N\big]\ \big(F_{\rm h}\big|\mathcal{U}^{\rm had}(\infty,t_{\rm f})\big|\{p,f,c\}_N\big)\big(\{p,f,c\}_N\big|\rho^{(0)}(t_{\rm f})\big)\ .\end{aligned} \tag{6.17}$$

Just as in the parton shower evolution, the integration in eq. (6.17) can be implemented by simply taking the states $\big|\{p,f,c\}_N\big)$ generated by the shower evolution and passing them to a Monte Carlo implementation of a hadronization model. Then application of the measurement function is acheived by, for instance, putting the events into desired bins according to the momenta of the resulting hadrons.

## 7. Other approaches

In this section, we sketch the relation of the shower evolution of this paper to some other approaches to the description of parton showers. For the shake of the simplicity we work only with massless partons in this section but it is still allowed for the non-QCD particles to have non-zero masses.

[14] As explained in ref. [1], we should most properly project out the component of $\big|M(\{p,f\}_2)\big\rangle$ that is proportional to a color basis state $\big|\{c\}_2\big\rangle$ by using a dual basis state ${}_D\big\langle\{c\}_2\big|$, but in the leading color limit there is no distinction between the dual basis states and the ordinary basis states.



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### 7.1 Dipole shower

One possibility for organizing the gluon radiation in a (spin averaged, leading color) parton shower is to use the same functions that are used for organizing the subtractions in a next-to-leading order perturbative calculation. In particular, the dipole subtraction scheme of Catani and Seymour [6] is an attractive possibility [12] that has been developed as the basis for parton shower programs by Schumann and Krauss [4] and by Dinsdale, Ternick and Weinzierl [5].

To see how this can work, consider the case that the emitted parton $m+1$ is a gluon, so that the splitting operator is given by the main term in eq. (6.2),

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big|\mathcal{H}_{\rm I}^{(0)}(t)\big|\{p,f,c\}_m\big) \\
&\quad= \sum_l \sum_{\substack{k\\k\neq l}} (m+1)\,\frac{n_{\rm c}(a)n_{\rm c}(b)\,\eta_{\rm a}\eta_{\rm b}}{n_{\rm c}(\hat{a})n_{\rm c}(\hat{b})\,\hat{\eta}_{\rm a}\hat{\eta}_{\rm b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\rm a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\rm b},\mu_F^2)}{f_{a/A}(\eta_{\rm a},\mu_F^2)f_{b/B}(\eta_{\rm b},\mu_F^2)} \\
&\qquad\times \big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big)\,\delta\Big(t-T_l(\{\hat{p},\hat{f}\}_{m+1})\Big)\,\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^\dagger\big|\{c\}_m\big\rangle \\
&\qquad\times \Phi_{lk}(\{\hat{p},\hat{f}\}_{m+1})\ .
\end{aligned} \tag{7.1}$$

The term $l,k$ generates gluons predominately soft or collinear with parton $l$. That is because $\Phi_{lk}$ is singular when $\hat{p}_{m+1}$ is soft or collinear with $\hat{p}_l$ but finite when $\hat{p}_{m+1}$ is collinear with $\hat{p}_k$. Each term is defined with its own phase space mapping $\mathcal{P}_l$ and evolution parameter $t$. Now we can use the momentum mappings $\mathcal{P}_{lk}^{\rm cs}$ of Catani and Seymour. These obey

$$\begin{aligned}
\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_{lk}^{\rm cs}\big|\{p,f\}_m\big) &\sim \big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big) \quad \text{when } \hat{p}_{m+1}\to\lambda\hat{p}_l \\
\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_{lk}^{\rm cs}\big|\{p,f\}_m\big) &\sim \big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_l\big|\{p,f\}_m\big) \\
&\sim \big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_k\big|\{p,f\}_m\big) \quad \text{when } \hat{p}_{m+1}\to 0\ .
\end{aligned} \tag{7.2}$$

We can also use the splitting functions $\Phi_{lk}^{\rm cs}$ of Catani and Seymour. These substitutions give

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big|\mathcal{H}_{\rm I}^{\rm cs}(t)\big|\{p,f,c\}_m\big) \\
&\quad= \sum_l \sum_{\substack{k\\k\neq l}} (m+1)\,\frac{n_{\rm c}(a)n_{\rm c}(b)\,\eta_{\rm a}\eta_{\rm b}}{n_{\rm c}(\hat{a})n_{\rm c}(\hat{b})\,\hat{\eta}_{\rm a}\hat{\eta}_{\rm b}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\rm a},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\rm b},\mu_F^2)}{f_{a/A}(\eta_{\rm a},\mu_F^2)f_{b/B}(\eta_{\rm b},\mu_F^2)} \\
&\qquad\times \big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_{lk}^{\rm cs}\big|\{p,f\}_m\big)\,\delta\Big(t-T_l(\{\hat{p},\hat{f}\}_{m+1})\Big)\,\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^\dagger\big|\{c\}_m\big\rangle \\
&\qquad\times \Phi_{lk}^{\rm cs}(\{\hat{p},\hat{f}\}_{m+1})\ .
\end{aligned} \tag{7.3}$$

The splitting operator $\mathcal{H}_{\rm I}^{\rm cs}(t)$ matches $\mathcal{H}_{\rm I}^{(0)}(t)$ in the collinear and soft limits.

We see that the structure of shower generation using the Catani-Seymour functions is quite similar to that of this paper. It is of interest to compare the splitting functions in the soft limit, $\hat{p}_{m+1}\to 0$. Using the definitions in ref. [6], we have

$$\Phi_{lk}^{\rm cs}(\{\hat{p},\hat{f}\}_{m+1}) \sim \frac{4\pi\alpha_{\rm s}\,C_{\rm F}\,2\hat{Q}^2}{(\hat{Q}\cdot\hat{p}_{m+1})^2\,(1-\vec{u}_{m+1}\cdot\vec{u}_l)}\;g^{\rm cs}(\vec{u}_{m+1},\vec{u}_l,\vec{u}_k;E_l/E_k)\ , \tag{7.4}$$

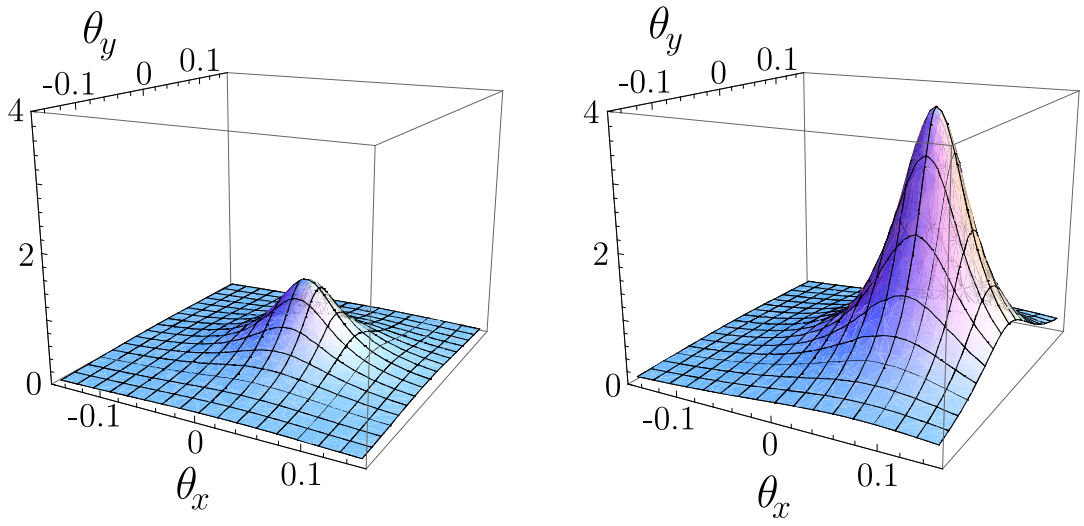


**Figure 12:** The soft radiation function for parton $l$ corresponding to Catani-Seymour dipole splitting. The function $g$ defined in eq. (4.15) and plotted in figure 11 is replaced by the function $g_{\text{c.s.}}$, eq. (7.5), plotted here. In the plot on the left, $E_l/E_k = 3$. In the plots on the right, $E_l/E_k = 1/3$. The plot coordinates and value of $\theta_{lk}$ are as in figure 11.

for $\hat{p}_{m+1} \to 0$, where

$$g^{\text{cs}}(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k; E_l/E_k) = \frac{(1 - \vec{u}_l \cdot \vec{u}_k)}{(E_l/E_k)(1 - \vec{u}_{m+1} \cdot \vec{u}_l) + (1 - \vec{u}_{m+1} \cdot \vec{u}_k)} \,. \tag{7.5}$$

Here $E_l$ and $E_k$ are the energies of partons $l$ and $k$, respectively, in the rest frame of $\hat{Q}$, the total momentum of the final state partons. Thus $E_l/E_k = \hat{p}_l \cdot \hat{Q}/\hat{p}_k \cdot \hat{Q}$. This function is similar in form to the function $g$ of this paper, plotted in figure 11, but it depends on the ratio $E_l/E_k$. We plot it in figure 12 for $E_l/E_k = 3$ and $E_l/E_k = 1/3$. We see that the Catani-Seymour functions assign little soft radiation to the more energetic of partons $l$ and $k$. More soft radiation is assigned to the less energetic parton of $l$ and $k$, with quite a lot of the radiation going in approximately the direction of the more energetic parton.

The the final state shower in the latest version (version 8.1) of Pythia [3, 10] is essentially a dipole shower as described above. In particular, the splitting function describing gluon emission in the soft limit $\hat{p}_{m+1} \to 0$ is that in eq. (7.4) with the same function $g$ as given in eq. (7.5).

### 7.2 Antenna shower

In the method of this paper and in a dipole shower following the Catani-Seymour scheme, the creation of a new gluon is attributed to the splitting of one of the previously existing partons. This requires that for the interference graph between the amplitude for emitting the gluon from parton $l$ and the amplitude for emitting the gluon from parton $k$, one assigns a certain fraction $A_{lk}$ of the graph to the splitting of parton $l$ and a fraction $1 - A_{lk}$ to the

splitting of parton $k$. In an antenna shower, one treats the pair of color connected partons, $l, k$ as a unit. The $l, k$ dipole constitutes an antenna that radiates the daughter gluon.[15] The pioneering development along these lines is the final state shower of ARIADNE[2]. More recent examples include those in refs. [13, 14]. There is a corresponding subtraction scheme for next-to-leading order calculations, antenna subtraction [15].

To define an antenna shower, we choose a momentum mapping $\mathcal{P}_{lk}^{\text{ant}}$ with the properties previously defined and with the symmetry property

$$\mathcal{P}_{lk}^{\text{ant}} = \mathcal{P}_{kl}^{\text{ant}} \quad . \tag{7.6}$$

We also redefine the shower evolution variable to be symmetric under $l \leftrightarrow k$ interchange. For instance, we could take

$$t = \log\left(\frac{Q_0^2}{2\min[\hat{p}_l\cdot\hat{p}_{m+1},\, \hat{p}_k\cdot\hat{p}_{m+1}]}\right) \quad . \tag{7.7}$$

Then we can rewrite the sum over $l$ and $k$ as a sum over pairs $l, k$, with each pair counted once, giving

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{c}\}_{m+1}\big|\mathcal{H}_{\text{I}}^{\text{ant}}(t)\big|\{p,f,c\}_m\big) \\
&\quad= \sum_{\substack{l,k\\ \text{pairs}}} (m+1)\,\frac{n_{\text{c}}(a)n_{\text{c}}(b)\,\eta_{\text{a}}\eta_{\text{b}}}{n_{\text{c}}(\hat{a})n_{\text{c}}(\hat{b})\,\hat{\eta}_{\text{a}}\hat{\eta}_{\text{b}}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\text{a}},\mu_F^2)f_{\hat{b}/B}(\hat{\eta}_{\text{b}},\mu_F^2)}{f_{a/A}(\eta_{\text{a}},\mu_F^2)f_{b/B}(\eta_{\text{b}},\mu_F^2)} \\
&\qquad\times\big(\{\hat{p},\hat{f}\}_{m+1}\big|\mathcal{P}_{lk}^{\text{ant}}\big|\{p,f\}_m\big)\,\delta\!\left(t-\log\left(\frac{Q_0^2}{2\min[\hat{p}_l\cdot\hat{p}_{m+1},\, \hat{p}_k\cdot\hat{p}_{m+1}]}\right)\right) \\
&\qquad\times\big\langle\{\hat{c}\}_{m+1}\big|a_{lk}^{\dagger}\big|\{c\}_m\big\rangle\;\Phi_{lk}^{\text{ant}}(\{\hat{p},\hat{f}\}_{m+1}) \quad .
\end{aligned} \tag{7.8}$$

Here $\Phi_{lk}^{\text{ant}}$ can be

$$\Phi_{lk}^{\text{ant}} = \Phi_{lk} + \Phi_{kl} \tag{7.9}$$

or any function that matches it in the soft and collinear limits.

In the soft limit, $\hat{p}_{m+1} \to 0$, $\Phi_{lk}^{\text{ant}}$ approaches the soft limit of the sum $\Phi_{lk} + \Phi_{kl}$, which is

$$\Phi_{lk}^{\text{ant}}(\{\hat{p},\hat{f}\}_{m+1}) \sim \frac{4\pi\alpha_{\text{s}}\,C_{\text{F}}\,2\hat{Q}^2}{(\hat{Q}\cdot\hat{p}_{m+1})^2}\;\frac{(1-\vec{u}_l\cdot\vec{u}_k)}{(1-\vec{u}_{m+1}\cdot\vec{u}_l)(1-\vec{u}_{m+1}\cdot\vec{u}_k)} \quad . \tag{7.10}$$

There is no function $g$ here. The function $g$ in the previous subsections arises from separating this into two terms, one that remains finite when $(1-\vec{u}_{m+1}\cdot\vec{u}_k) \to 0$ and the other that remains finite when $(1-\vec{u}_{m+1}\cdot\vec{u}_l) \to 0$.

### 7.3 Angular ordering approximation

With massless kinematics, the distribution of soft radiation that is kinematically of the form for a splitting of parton $l$ is proportional to $g(\vec{u}_{m+1},\vec{u}_l,\vec{u}_k)/(1-\vec{u}_{m+1}\cdot\vec{u}_l)$, as given in eq. (4.14). From the plot of $g$ in figure 11, we see that the soft gluon radiation from

[15] One ought to call this a dipole shower, but then one would need a new name for the kind of shower described in the previous subsection.

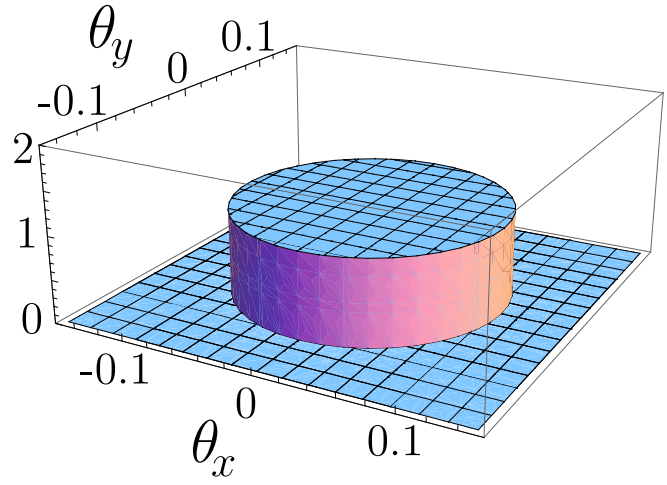


**Figure 13:** The angular ordering approximation. The function $g$ defined in eq. (4.15) and plotted in figure 11 is replaced by the function $g_{\text{a.o.}}$, eq. (7.11), plotted here. The plot coordinates and value of $\theta_{lk}$ are as in figure 11.

partons $l$ and $k$ is approximately confined to a cone between $\vec{p}_l$ and $\vec{p}_k$. This is called "angular ordering." There is also an angular ordering approximation [16] that is sometimes used for parton showers and, in particular, lies at the heart of HERWIG [11]. With this approximation, the function $g$ in figure 11 is approximated by the function plotted in figure 13,

$$g_{\text{a.o.}}(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k) = \theta(\vec{u}_{m+1} \cdot \vec{u}_l > \vec{u}_k \cdot \vec{u}_l) \ . \tag{7.11}$$

We see that in the angular region between the two hard parton directions ($\theta_x \approx 0.5, \theta_y \approx 0$ in the figures), the angular distribution of the soft radiation determined by the exact function $g$ is about twice as large as that determined by $g_{\text{a.o.}}$. In other angular regions $g$ gives less soft radiation than $g_{\text{a.o.}}$. The angular ordering approximation has the good feature that it gets the total amount of soft radiation right,

$$\int d\Omega_{m+1} \ \frac{g(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k) - g_{\text{a.o.}}(\vec{u}_{m+1}, \vec{u}_l, \vec{u}_k)}{1 - \vec{u}_{m+1} \cdot \vec{u}_l} = 0 \ . \tag{7.12}$$

This result follows from the original construction of refs. [16]. We note, however, that the original construction involved only an integration over the azimuthal angle $\phi$, while eq. (7.12) requires an integral over both $\theta$ and $\phi$. We have also checked eq. (7.12) by numerical integration.

One should note that the theta function in $g_{\text{a.o.}}$ restricts the emission angle of a soft gluon to be smaller than the angle between $\vec{u}_k$ and $\vec{u}_l$, where $k$ is a parton that is color connected to parton $l$. If parton $l$ is a quark, then there is only one choice for $k$. However, if parton $l$ is a gluon, then there are two color connected partons. Then there are two contributions with separate angle restrictions.



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## 8. Conclusions

In ref. [1], we presented evolution equations that represent a leading order parton shower including quantum interference, spin, and color. We did not, however, present a way to implement the integrations implied by these equations in a fashion that would be practical for more than a few partons. The idea behind the evolution equations was to make just one approximation: that the virtualities in successive splittings are strongly ordered.

Typical Monte Carlo event generators, such as PYTHIA [10], ARIADNE [2], HERWIG [11], and SHERPA [17], make additional approximations. In particular, they typically average over parton spins and take the leading term in an expansion in $1/N_c^2$, where $N_c = 3$ is the number of colors.[16] Our aim in this paper has been to work out how the general formalism could work as a practical calculation if we make the further approximations of averaging over parton spins and of keeping only the leading order in $1/N_c^2$. We do, however, keep some aspects of quantum interference in that the interference graphs between the emission of a soft gluon from parton $l$ and the emission of the soft gluon from another parton $k$ are accounted for.

The result is an algorithm that is similar to what is done in widely used parton shower event generators in that the calculation can be implemented as a Markov chain, as described in section 6. The form of the evolution is perhaps most similar to that in the dipole showers of refs. [4] and [5] and is also similar to the $k_\perp$ version of PYTHIA [3]. One can think of the basic object that splits as not one parton, but two partons, $l$ and $k$, that are next to each other along a color string. This basic object is often referred to as a color dipole. When we incorporate the joint splitting of partons $l$ and $k$, there is a contribution to the splitting probability that corresponds to the square of the amplitude for parton $l$ to split. There is another contribution to the splitting probability that corresponds to the square of the amplitude for parton $k$ to split. Then there are two contributions that correspond to the interference of these amplitudes. We reorganize the four terms into two terms. One is kinematically of the form for a splitting of parton $l$, while the other is kinematically of the form for a splitting of parton $k$. This is rather similar to the structure of the dipole subtraction scheme for next-to-leading order calculations proposed by Catani and Seymour [6], which has been implemented for parton showers in two recent papers [4, 5].

There are differences between the shower formulation used here and that in, say, the dipole showers of refs. [4] and [5]. The splitting functions are different. In particular, we have separate formulations for the interference graphs (based on the simple eikonal approximation) and for the direct graphs, for which our splitting functions are quite directly read off from the Feynman graphs with a minimal approximation applied where an off-shell mother parton attaches to a hard scattering amplitude. The momentum mapping functions, which were presented in ref. [1], are also different. They are similar to the Catani-Seymour momentum mappings in that they are systematically defined, invertible mappings, but

[16] More precisely, one averages over the spins of a parton before it splits and sums over the spins of daughter partons. For colors, the color accounting by which one assigns color factors $C_F$ or $C_A$ to splittings is based on treating each gluon as carrying color $\mathbf{3} \otimes \overline{\mathbf{3}}$ instead of $\mathbf{8}$.



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they have the advantage that the form of the mapping depends on the parton index $l$ but not on the index $k$ of the partner parton.

We have seen that the leading color, spin averaged shower of this paper has a structure similar to that implemented in standard parton shower event generators. In particular, this simple shower can be implemented using a Markov chain. The full shower formalism of ref. [1] is more general than the simple shower in that parton spin and color correlations are included. We anticipate that the full formalism will be more difficult than the simple version to implement in a practical fashion. However, we anticipate that one can use the simple shower as a basis for a systematically improvable approximation to the full shower. The idea would be to start with the simple shower and provide parameters that remove the approximations gradually, so that the result is still approximate but the approximation is systematically improvable as computer resources allow. We expect to return to this subject in future papers.


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## A. The remaining splitting functions

In this section we record the spin averaged splitting functions $\overline{W}_{ll}$ for the cases in which $\hat{f}_{m+1} \neq \mathrm{g}$, which were not covered in the main body of the paper. We use the general definition (2.8) of $\overline{W}_{ll}$ together with the formulas from ref. [1] for the splitting amplitudes $v_l$.

We first consider a final state splitting with $\{f_l, \hat{f}_l, \hat{f}_{m+1}\} = \{\mathrm{g}, q, \bar{q}\}$ where $q$ is a quark flavor and $\bar{q}$ is the corresponding antiflavor. A straightforward calculation gives

$$\overline{W}_{ll}(\{\hat{f},\hat{p}\}_{m+1}) = \frac{8\pi\alpha_{\mathrm{s}}}{(\hat{p}_l+\hat{p}_{m+1})^2}\left(1+\frac{2\;\hat{p}_l\cdot D(p_l,\hat{Q})\cdot\hat{p}_{m+1}}{(\hat{p}_l+\hat{p}_{m+1})^2}\right) \quad . \tag{A.1}$$

For an initial state splitting with $\{f_l, \hat{f}_l, \hat{f}_{m+1}\} = \{\mathrm{g}, q, \bar{q}\}$, we find

$$\overline{W}_{ll}(\{\hat{f},\hat{p}\}_{m+1}) = \frac{8\pi\alpha_{\mathrm{s}}}{(\hat{p}_l-\hat{p}_{m+1})^2}\left(-1+\left(\frac{\hat{p}_l\cdot n_l}{(\hat{p}_l-\hat{p}_{m+1})\cdot n_l}\right)^2\frac{2\;\hat{p}_{m+1}\cdot D(p_l,\hat{Q})\cdot\hat{p}_{m+1}}{(\hat{p}_l-\hat{p}_{m+1})^2}\right) \quad . \tag{A.2}$$

Here $n_l = p_{\mathrm{B}}$ for $l = \mathrm{a}$ and $n_l = p_{\mathrm{A}}$ for $l = \mathrm{b}$. The same result holds for an initial state splitting with $\{f_l, \hat{f}_l, \hat{f}_{m+1}\} = \{\mathrm{g}, \bar{q}, q\}$.

We consider next an initial state splitting with $\{f_l, \hat{f}_l, \hat{f}_{m+1}\} = \{q, \mathrm{g}, q\}$. A straightforward calculation gives

$$\overline{w}_{ll}(\{\hat{f},\hat{p}\}_{m+1}) = \frac{4\pi\alpha_{\mathrm{s}}}{\hat{p}_l\cdot\hat{p}_{m+1}}\left(\frac{\hat{p}_l\cdot n_l}{p_l\cdot n_l}-\frac{(\hat{p}_l-\hat{p}_{m+1})\cdot n_l}{p_l\cdot n_l}\,\frac{\hat{p}_{m+1}\cdot D(\hat{p}_l,\hat{Q})\cdot\hat{p}_{m+1}}{\hat{p}_l\cdot\hat{p}_{m+1}}\right) \quad . \tag{A.3}$$

Again, $n_{\mathrm{a}} = p_{\mathrm{B}}$ and $n_{\mathrm{b}} = p_{\mathrm{A}}$. The same result holds for an initial state splitting with $\{f_l, \hat{f}_l, \hat{f}_{m+1}\} = \{\bar{q}, \mathrm{g}, \bar{q}\}$.

This completes the analysis of $\overline{W}_{ll}$ for cases in which $\hat{f}_{m+1} \neq \mathrm{g}$.

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