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ON THE STUDY OF INVARIANTS OF EQUATIONS OF MOTION IN CALCULATIONS OF ABERRATION COEFFICIENTS

YU. P. SIVKOV†

D. V. Efremov Research Institute of Electrophysical Instrumentation, Leningrad, U.S.S.R.

In this paper we study aberration coefficients, i.e., the coefficients in the expansion of the equations for the trajectories of charged particles

$$q_i = q_i(q_{10}, p_{10}, \ldots q_{k0}, p_{k0} \ldots q_{n0}, p_{n0}, s)$$
$$p_i = p_i(q_{10}, p_{10}, \ldots q_{k0}, p_{k0} \ldots q_{n0}, p_{n0}, s)$$

in powers of the initial conditions q_{k0}, p_{k0}. It is shown that, owing to the invariants of the equations of dynamics, these coefficients are related to each other by certain identities, for whose construction a method is given. In particular, a system of identities relating the second and third order aberration coefficients is constructed for a stationary field possessing one or two planes of symmetry.

In certain problems of ion optics, specifically, problems of ion-beam transport, equations for the trajectories of particles are frequently used in the form

$$\begin{aligned} q_i &= q_i(q_{10}, p_{10}, \cdots q_{k0}, p_{k0} \cdots q_{n0}, p_{n0}, s) \\ p_i &= p_i(q_{10}, p_{10}, \cdots q_{k0}, p_{k0} \cdots q_{n0}, p_{n0}, s) \end{aligned} \tag{1}$$

considered as a transformation of phase space. Here q_i, p_i are canonically conjugate coordinates of the particle, q_{k0}, p_{k0} are the initial values of the corresponding coordinates, and s is an independent variable.

If the interactions of the particles, due to the discrete character of the beam, may be neglected, then $n = 3$ in the general case. In special cases, the number of variables may be reduced.

A well-studied case is that in which the equations of motion are linear and q_i, p_i are linear in the initial values. This corresponds to the study of first-order terms in the expansion of q_i, p_i in powers of q_{k0}, p_{k0}:

$$\begin{aligned} q_i = {} & \sum_{k=1}^{n} {}_i a_k\, q_{k0} + \sum_{k=1}^{n} {}_i a^k p_{k0} + \sum_{k,l=1}^{n} {}_i b_{kl}\, q_{k0}\, q_{l0} + \sum_{k,l=1}^{n} {}_i b_k^l\, q_{k0}\, p_{l0} + \sum_{k,l=1}^{n} {}_i b^{kl} p_{k0}\, p_{l0} + \sum_{k,l,m=1}^{n} {}_i c_{klm}\, q_{k0}\, q_{l0}\, q_{m0} \\ & + \sum_{k,l,m=1}^{n} {}_i c_{kl}^m\, q_{k0}\, q_{l0}\, p_{m0} + \sum_{k,l,m=1}^{n} {}_i c_k^{lm} q_{k0}\, p_{l0}\, p_{m0} + \sum_{k,l,m=1}^{n} {}_i c^{klm} p_{k0}\, p_{l0}\, p_{m0} + \cdots \\ p_i = {} & \sum_{k=1}^{n} {}^i a_k\, q_{k0} + \sum_{k=1}^{n} {}^i a^k p_{k0} + \sum_{k,l=1}^{n} {}^i b_{kl}\, q_{k0}\, q_{l0} + \sum_{k,l=1}^{n} {}^i b_k^l\, q_{k0}\, p_{l0} + \sum_{k=1}^{n} {}^i b^{kl} p_{k0}\, p_{l0} + \sum_{k,l,m=1}^{n} {}^i_i c_{klm}\, q_{k0}\, q_{l0}\, q_{m0} \\ & + \sum_{k,l,m=1}^{n} {}^i c_{kl}^m\, q_{k0}\, q_{l0}\, p_{m0} + \sum_{k,l,m=1}^{n} {}^i c_k^{lm} q_{k0}\, p_{l0}\, p_{m0} + \sum_{k,l,m=1}^{n} {}^i c^{klm} p_{k0}\, p_{l0}\, p_{m0} + \cdots \end{aligned} \tag{2}$$

In Eqs. (2), and in the sequel, left indices refer to dynamical variables, right indices to the corresponding initial values. The subscripts refer to the coordinates, the superscripts to momenta.

In a series of papers,[1-5] higher-order terms were determined by successive iterations of simple nonlinear systems.

Among the coefficients of the expansion (2) aberration coefficients hold certain relations, which result from integral invariants of the equations of dynamics, irrespective of the specific form of the focusing fields. The present paper is devoted to a systematic study of the constraints which formulas

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(1) satisfy as a result of the integral invariants of the equations of dynamics.

When the aberration coefficients are found by successive iterations, these constraints may serve as a side check on the resulting solutions.

The aberration coefficients may also be found by numerical calculation or experimental determination of separate trajectories. From some power on, the terms of the expansion (2) are neglected in these calculations. The relations given below may be used to check the validity of the approximation, to any order, for describing the motion of particles whose phase-space coordinates lie in some region at the initial time.

Finally, some of the aberration coefficients can be expressed directly in terms of the others.

It is well known[6] that the matrix constructed from the partial derivatives

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix} = \begin{bmatrix} \frac{\partial q_1}{\partial q_{10}} & \frac{\partial q_1}{\partial p_{10}} & \frac{\partial q_1}{\partial q_{20}} & \frac{\partial q_1}{\partial p_{20}} & \frac{\partial q_1}{\partial q_{30}} & \frac{\partial q_1}{\partial p_{30}} \\ \frac{\partial p_1}{\partial q_{10}} & \frac{\partial p_1}{\partial p_{10}} & \frac{\partial p_1}{\partial q_{20}} & \frac{\partial p_1}{\partial p_{20}} & \frac{\partial p_1}{\partial q_{30}} & \frac{\partial p_1}{\partial p_{30}} \\ \frac{\partial q_2}{\partial q_{10}} & \frac{\partial q_2}{\partial p_{10}} & \frac{\partial q_2}{\partial q_{20}} & \frac{\partial q_2}{\partial p_{20}} & \frac{\partial q_2}{\partial q_{30}} & \frac{\partial q_2}{\partial p_{30}} \\ \frac{\partial p_2}{\partial q_{10}} & \frac{\partial p_2}{\partial p_{10}} & \frac{\partial p_2}{\partial q_{20}} & \frac{\partial p_2}{\partial p_{20}} & \frac{\partial p_2}{\partial q_{30}} & \frac{\partial p_2}{\partial p_{30}} \\ \frac{\partial q_3}{\partial q_{10}} & \frac{\partial q_3}{\partial p_{10}} & \frac{\partial q_3}{\partial q_{20}} & \frac{\partial q_3}{\partial p_{20}} & \frac{\partial q_3}{\partial q_{30}} & \frac{\partial q_3}{\partial p_{30}} \\ \frac{\partial p_3}{\partial q_{10}} & \frac{\partial p_3}{\partial p_{10}} & \frac{\partial p_3}{\partial q_{20}} & \frac{\partial p_3}{\partial p_{20}} & \frac{\partial p_3}{\partial q_{30}} & \frac{\partial p_3}{\partial p_{30}} \end{bmatrix} \quad (3)$$

is symplectic, i.e., it satisfies the identity

$$\tilde{A}SA = S, \quad (4)$$

where $\tilde{A}$ is the transpose of A and

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (5)$$

In particular, it follows from Eq. (4) that the determinant of A is equal to 1, which means that for any A the inverse A^{-1} is easily calculated. This, however, also follows from physical considerations. The matrix A^{-1} may be obtained by the usual inversion procedure and, in view of condition (4), assumes the form

$$A^{-1} = \begin{bmatrix} A_{22} & -A_{12} & A_{24} & -A_{14} & A_{26} & -A_{16} \\ -A_{21} & A_{11} & -A_{23} & A_{13} & -A_{25} & A_{15} \\ A_{42} & -A_{32} & A_{44} & -A_{34} & A_{46} & -A_{36} \\ -A_{41} & A_{31} & -A_{43} & A_{33} & -A_{45} & A_{35} \\ A_{62} & -A_{52} & A_{64} & -A_{54} & A_{66} & -A_{56} \\ -A_{61} & A_{51} & -A_{63} & A_{53} & -A_{65} & A_{55} \end{bmatrix} \quad (6)$$

This expression is well known for one-dimensional motion.

Condition (4), which involves antisymmetric matrices of rank $2n$, corresponds to $n(2n-1)$ scalar equations. Calculation of the partial derivatives in the Jacobian (3) and their substitution into Eq. (4) leads to $n(2n-1)$ expressions, which are identities in q_{i0}, p_{i0} and bilinear in the aberration coefficients. In these identities all coefficients in the expansion in q_{i0}, p_{i0} are equal to zero.

The relations thus obtained enable one to calculate certain aberration coefficients in terms of others. Not all the equations, however, are independent. One reason for this is that for high-order aberration the number of equations exceeds the number of aberration coefficients.

Let N of the total M aberration coefficients of order at most k be mutually independent. Clearly, the others can be obtained from these with the aid of M–N linearly independent equations (M–$N \leqq Q$, where Q is the total number of equations). This procedure is possible if the Jacobian of the partial derivatives, with respect to the dependent coefficients, of the left-hand sides of the $M-N$ independent equations is different from zero. Since these left-hand sides are bilinear functions of the aberration coefficients, all the elements of the Jacobian are also aberration coefficients (possibly multiplied by integers).

The problem of the minimal set of independent aberration coefficients may be solved in the following way. Construct a matrix of M columns and Q rows from all the aberration coefficients. Each row (corresponding to one of the equations) contains any aberration coefficient multiplying any other in the same equation, in the column corresponding to the latter coefficient. All other row elements are zero. We must then find the rank of the matrix, that is, the maximum order of a non-vanishing minor of this matrix. Obviously, the rank is $M-N$. The aberration coefficients corresponding to the columns of such a determinant are linearly dependent, and can be expressed in terms of the others. Rows not occurring in the above determinant correspond to linearly dependent equations.

Generally speaking, there may well be different sets of linearly dependent aberration coefficients, for which the Jacobian of order $M-N$ does not vanish. Hence the choice of a specific set of independent aberration coefficients may be guided by practical considerations of convenience or by their experimental determination.

Below we consider a set of identities which relate the aberration coefficients of the second and third order, for a stationary field, to one or two planes of symmetry. One of the most practical possibilities for a set of independent aberration coefficients for this case will also be discussed.

If the electric and magnetic fields acting on the particle are time-independent, a useful choice for the independent variable is the length of arc of a coordinate-line coinciding with one of the possible trajectories.[7] The transverse coordinates q_1 and q_2 are then taken along the binormal and normal, respectively, to the coordinate line, the time t is a cyclic coordinate, and the total energy H of the particle becomes an integral of the motion. In this case all elements of the matrix A which belong to the row corresponding to $p_3 = -H$ vanish, with the exception of $\partial p_3/\partial p_{30} = 1$.

The system of equalities (4) for the matrix elements (3) becomes

$$\left.\begin{aligned}
A_{11}A_{22}-A_{12}A_{21}+A_{31}A_{42}-A_{32}A_{41} &= 1\\
A_{13}A_{24}-A_{14}A_{23}+A_{33}A_{44}-A_{34}A_{43} &= 1\\
A_{11}A_{42}-A_{12}A_{41}+A_{13}A_{44}-A_{14}A_{43} &= 0\\
A_{11}A_{32}-A_{12}A_{31}+A_{13}A_{34}-A_{14}A_{33} &= 0\\
A_{12}A_{42}-A_{22}A_{41}+A_{23}A_{44}-A_{24}A_{43} &= 0\\
A_{12}A_{32}-A_{22}A_{31}+A_{23}A_{34}-A_{24}A_{33} &= 0
\end{aligned}\right\} \quad (7a)$$

$$\left.\begin{aligned}
A_{21}A_{16}-A_{11}A_{26}+A_{41}A_{36}-A_{31}A_{46} &= A_{51}\\
A_{22}A_{16}-A_{12}A_{26}+A_{42}A_{36}-A_{32}A_{46} &= A_{52}\\
A_{23}A_{16}-A_{13}A_{26}+A_{43}A_{36}-A_{33}A_{46} &= A_{53}\\
A_{24}A_{16}-A_{14}A_{26}+A_{44}A_{36}-A_{34}A_{46} &= A_{54}
\end{aligned}\right\} \quad (7b)$$

$$\left.\begin{aligned}
A_{21}A_{15}-A_{11}A_{25}+A_{41}A_{35}-A_{31}A_{45} &= 0\\
A_{22}A_{15}-A_{12}A_{25}+A_{42}A_{35}-A_{32}A_{45} &= 0\\
A_{23}A_{15}-A_{13}A_{25}+A_{43}A_{35}-A_{33}A_{45} &= 0\\
A_{24}A_{15}-A_{14}A_{25}+A_{44}A_{35}-A_{34}A_{45} &= 0
\end{aligned}\right\} \quad (7c)$$

Equations (7a) are conditions for the 4×4 minor in the upper left corner of the matrix (3), corresponding to a two-dimensional motion, to be a symplectic matrix. Conditions (7b) uniquely determine the 'time-like' elements A_{5i} in terms of A_{k6} (and the other elements) and vice versa. Because of conditions (7a), the determinants of the systems (7b) and (7c) with respect to the unknowns A_{k6} and A_{k5}, respectively, are equal to 1. Finally, the last four equations mean that all elements of the column corresponding to q_{30} (except $A_{55} = \partial q_3/\partial q_{30} = 1$) vanish; the matrix A becomes

$$A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & 0 & A_{16}\\
A_{21} & A_{22} & A_{23} & A_{24} & 0 & A_{26}\\
A_{31} & A_{32} & A_{33} & A_{34} & 0 & A_{36}\\
A_{41} & A_{42} & A_{43} & A_{44} & 0 & A_{46}\\
A_{51} & A_{52} & A_{53} & A_{54} & 1 & A_{56}\\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (8)$$

The majority of practically applicable focusing systems possess a plane of symmetry $q_2 s$ (defined by the electric field) or a plane of antisymmetry (defined by the magnetic field).

In that case, upon simultaneous change of sign in the initial values q_{10} and p_{10}, the quantities q_1 and p_1 also change sign, while the other variables remain unchanged. It then turns out that all coefficients in the expansion of q_1 and p_1, where the index 1 to the right is repeated an even number of times, vanish, and the same holds for the coefficients in the expansion of the other variables, where the index 1 is repeated an odd number of times. The matrix of the linear approximation then assumes the form

$$\begin{bmatrix} {}_1a_1 & {}_1a^1 & 0 & 0 & 0 & 0 \\ {}^1a_1 & {}^1a^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & {}_2a_2 & {}_2a^2 & 0 & {}_2a^3 \\ 0 & 0 & {}^2a_2 & {}^2a^2 & 0 & {}^2a^3 \\ 0 & 0 & {}_3a_2 & {}_3a^2 & 1 & {}_3a^3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (9)$$

For the linear coefficients, there are two known relations for the elements involved in transverse motion,

$$\begin{aligned} {}_1a_1\,{}^1a^1 - {}_1a^1\,{}^1a_1 &= 1 \\ {}_2a_2\,{}^2a^2 - {}_2a^2\,{}^2a_2 &= 1 \end{aligned} \qquad (10)$$

The total number of second-order aberration coefficients is 30, 20 of which describe spatial aberrations and 10 chromatic aberrations.

Conditions (7a) for the second-order aberration coefficients have the form

$$\left.\begin{aligned} {}_1a_1\,{}^1b^1_2 + {}^1a^1\,{}_1b_{12} - {}_1a^1\,{}^1b_{12} - {}^1a_1\,{}_1b^1_2 &= 0 \\ {}_1a_1\,{}^1b^{12} + {}^1a^1\,{}_1b^2_1 - {}_1a^1\,{}^1b^2_1 - {}^1a_1\,{}_1b^{12} &= 0 \\ {}_1a_1\,{}^2b^1_1 - 2\,{}_1a^1\,{}^2b_{11} + {}^2a^2\,{}_1b_{12} - {}^2a_2\,{}_1b^2_1 &= 0 \\ {}_1a_1\,{}_2b^1_1 - 2\,{}_1a^1\,{}_2b_{11} + {}_2a^2\,{}_1b_{12} - {}_2a_2\,{}_1b^2_1 &= 0 \\ 2\,{}_1a_1\,{}^2b^{11} - {}_1a^1\,{}^2b^1_1 + {}^2a^2\,{}_1b^1_2 - {}^2a_2\,{}_1b^{12} &= 0 \\ 2\,{}_1a_1\,{}_2b^{11} - {}_1a^1\,{}_2b^1_1 + {}_2a^2\,{}_1b^1_2 - {}_2a_2\,{}_1b^{12} &= 0 \\ {}^1a_1\,{}^2b^1_1 - 2\,{}^1a^1\,{}^2b_{11} + {}^2a^2\,{}^1b_{12} - {}^2a_2\,{}^1b^2_1 &= 0 \\ {}^1a_1\,{}_2b^1_1 - 2\,{}^1a^1\,{}_2b_{11} + {}_2a^2\,{}^1b_{12} - {}_2a_2\,{}^1b^2_1 &= 0 \\ 2\,{}^1a_1\,{}^2b^{11} - {}^1a^1\,{}^2b^1_1 + {}^2a^2\,{}^1b^1_2 - {}^2a_2\,{}^1b^{12} &= 0 \\ 2\,{}^1a_1\,{}_2b^{11} - {}^1a^1\,{}_2b^1_1 + {}_2a^2\,{}^1b^1_2 - {}_2a_2\,{}^1b^{12} &= 0 \end{aligned}\right\} \quad (11a)$$

$$\left.\begin{aligned} {}_2a_2\,{}^2b^2_2 + 2\,{}^2a^2\,{}_2b_{22} - 2\,{}_2a^2\,{}^2b_{22} - {}^2a_2\,{}_2b^2_2 &= 0 \\ 2\,{}_2a_2\,{}^2b^{22} + {}^2a^2\,{}_2b^2_2 - {}_2a^2\,{}^2b^2_2 - 2\,{}^2a_2\,{}_2b^{22} &= 0 \end{aligned}\right\} \quad (11b)$$

$$\left.\begin{aligned} {}_1a_1\,{}^1b^{13} + {}^1a^1\,{}_1b^3_1 - {}_1a^1\,{}^1b^3_1 - {}^1a_1\,{}_1b^{13} &= 0 \\ {}_2a_2\,{}^2b^{23} + {}^2a^2\,{}_2b^3_2 - {}_2a^2\,{}^2b^3_2 - {}^2a_2\,{}_2b^{23} &= 0 \end{aligned}\right\} \quad (11c)$$

The chromatic coefficients ${}_2b^{33}$ and ${}^2b^{33}$ are independent, since neither appear in any of the equations. The other eight chromatic coefficients are related to each other by the two Eqs. (11c), so that any six of them may be considered independent. Of the six coefficients ${}_2b_{22}$, ${}^2b_{22}$, ${}^2b^2_2$, ${}_2b^2_2$, ${}_2b^{22}$, ${}^2b^{22}$

TABLE I

		Stationary field with one symmetry plane: Spatial	Stationary field with one symmetry plane: Chromatic	Stationary field with two symmetry planes: Spatial	Stationary field with two symmetry planes: Chromatic
Linear approximation	Number of coefficients	8	2	8	0
	Number of equations	2	0	2	0
	Number of independent coefficients	6	2	6	0
Second order approximation	Number of coefficients	20	10	0	8
	Number of independent equations	10	2	0	2
	Number of independent coefficients	10	8	0	6
Third order approximation	Number of coefficients	40	30	40	8
	Number of independent equations	21	12	21	2
	Number of independent coefficients	19	18	19	6

which characterize aberrations in the plane of symmetry, any two can be determined from Eqs. (11b). It is particularly convenient to determine the coefficients ${}_2b_2^2$ and ${}^2b_2^2$, whose determinant is identically unity. The 14 coefficients characterizing the motion outside the plane of symmetry are related to each other by Eqs. (11a). It is easy to show that the first two of these follow from the others. The latter eight are easily solved for ${}_1b_{12}$, ${}^1b_{12}$, ${}^1b_1^2$, ${}_1b_1^2$, ${}_1b^{12}$, ${}^1b^{12}$, ${}_1b_2^1$, ${}^1b_2^1$. The independent coefficients are then ${}_2b_{11}$, ${}^2b_{11}$, ${}_2b_1^1$, ${}^2b_1^1$, ${}_2b^{11}$, ${}^2b^{11}$, which characterize the projection of trajectory onto the plane of symmetry for nonzero q_{20}, p_{20}.

If the system possesses two planes of symmetry in the above sense, all spatial aberration coefficients vanish, and Eqs. (11a) and (11b) hold identically.

Of the chromatic coefficients, ${}_2b^{33}$ and ${}^2b^{33}$ vanish, and thus there are six independent second-order aberration coefficients.

The total number of spatial and chromatic third-order aberration coefficients for a stationary field with one plane of symmetry is 70. The equations which contain spatial aberrations only are of the form

$$\left.\begin{aligned}
2\,{}_2b_{11}\,{}^2b_1^1 - 2\,{}^2b_{11}\,{}_2b_{11}^1 - {}_1a_1\,{}^1c_{11}^1 + 3\,{}^1a^1\,{}_1c_{111} - 3\,{}_1a^1\,{}^1c_{111} - {}^1a_1\,{}_1c_{11}^1 &= 0\\
2\,{}_2b_{11}\,{}^2b^{11} - 2\,{}^2b_{11}\,{}_2b^{11} + {}_1a_1\,{}^1c_1^{11} + {}^1a^1\,{}_1c_{11}^1 - {}_1a^1\,{}^1c_{11}^1 - {}^1a_1\,{}_1c_1^{11} &= 0\\
2\,{}_2b_1^1\,{}^2b^{11} - 2\,{}_2b^{11}\,{}^2b_1^1 + 3\,{}_1a_1\,{}^1c^{111} + {}^1a^1\,{}_1c_1^{11} - {}_1a^1\,{}^1c_1^{11} - 3\,{}^1a_1\,{}_1c^{111} &= 0
\end{aligned}\right\} \quad (12a)$$

$$\left.\begin{aligned}
2\,{}_2b_{22}\,{}^2b_2^2 - 2\,{}^2b_{22}\,{}_2b_2^2 + {}_2a_2\,{}^2c_{22}^2 + 3\,{}^2a^2\,{}_2c_{222} - 3\,{}_2a^2\,{}^2c_{222} - {}^2a_2\,{}_2c_{22}^2 &= 0\\
2\,{}_2b_{22}\,{}^2b^{22} - 2\,{}^2b_{22}\,{}_2b^{22} + {}_2a_2\,{}^2c_2^{22} + {}^2a^2\,{}_2c_{22}^2 - {}_2a^2\,{}^2c_{22}^2 - {}^2a_2\,{}_2c_2^{22} &= 0\\
2\,{}_2b_2^2\,{}^2b_{22} - 2\,{}^2b_2^2\,{}_2b^{22} + 3\,{}_2a_2\,{}^2c^{222} + {}^2a^2\,{}_2c_2^{22} - {}_2a^2\,{}^2c_2^{22} - 3\,{}^2a_2\,{}_2c^{222} &= 0
\end{aligned}\right\} \quad (12b)$$

$$\left.\begin{aligned}
{}_1b_{12}({}^2b_1^1 + {}^2b_2^2) - 2\,{}_1b_2^1\,{}^2b_{11} - 2\,{}_1b_1^2\,{}^2b_{22} + {}_1a_1\,{}^2c_{12}^1 - 2\,{}_1a^1\,{}^2c_{112} + 2\,{}^2a^2\,{}_1c_{122} - {}^2a_2\,{}_1c_{12}^2 &= 0\\
2\,{}_1b_{12}\,{}^2b^{22} + {}_1b_1^2({}^2b_1^1 - {}^2b_2^2) - 2\,{}_1b^{12}\,{}^2b_{11} + {}_1a_1\,{}^2c_1^{12} - 2\,{}_1a^1\,{}^2c_{11}^2 + {}^2a^2\,{}_1c_{12}^2 - 2\,{}^2a_2\,{}_1c_1^{22} &= 0\\
2\,{}_1b_{12}\,{}^2b^{11} + {}_1b_2^1({}^2b_2^2 - {}^2b_1^1) - 2\,{}_1b^{12}\,{}^2b_{22} + 2\,{}_1a_1\,{}^2c_2^{11} - {}_1a^1\,{}^2c_{12}^1 + 2\,{}^2a^2\,{}_1c_{22}^1 - {}^2a_2\,{}_1c_2^{12} &= 0\\
2\,{}_1b_1^2\,{}^2b^{11} + 2\,{}_1b_2^1\,{}^2b^{22} - {}_1b^{12}({}^2b_1^1 + {}^2b_2^2) + 2\,{}_1a_1\,{}^2c^{112} - {}_1a^1\,{}^2c_1^{12} + {}^2a^2\,{}_1c_2^{12} - 2\,{}^2a_2\,{}_1c^{122} &= 0\\
{}_1b_{12}({}_2b_1^1 + {}_2b_2^2) - 2\,{}_1b_2^1\,{}_2b_{11} - 2\,{}_1b_1^2\,{}_2b_{22} + {}_1a_1\,{}_2c_{12}^1 - 2\,{}_1a^1\,{}_2c_{112} + 2\,{}_2a^2\,{}_1c_{122} - {}_2a_2\,{}_1c_{12}^2 &= 0\\
2\,{}_1b_{12}\,{}_2b^{22} + {}_1b_1^2({}_2b_1^1 - {}_2b_2^2) - 2\,{}_1b^{12}\,{}_2b_{11} + {}_1a_1\,{}_2c_1^{12} - 2\,{}_1a^1\,{}_2c_{11}^2 + {}_2a^2\,{}_1c_{12}^2 - 2\,{}_2a_2\,{}_1c_1^{22} &= 0\\
2\,{}_1b_{12}\,{}_2b^{11} - {}_1b_2^1({}_2b_1^1 - {}_2b_2^2) - 2\,{}_1b^{12}\,{}_2b_{22} + 2\,{}_1a_1\,{}_2c_2^{11} - {}_1a^1\,{}_2c_{12}^1 + 2\,{}_2a^2\,{}_1c_{22}^1 - {}_2a_2\,{}_1c_2^{12} &= 0\\
2\,{}_1b_1^2\,{}_2b^{11} + 2\,{}_1b_2^1\,{}_2b^{22} - {}_1b^{12}({}_2b_1^1 + {}_2b_2^2) + 2\,{}_1a_1\,{}_2c^{112} - {}_1a^1\,{}_2c_1^{12} + {}_2a^2\,{}_1c_2^{12} - 2\,{}_2a_2\,{}_1c^{122} &= 0\\
{}^1b_{12}({}^2b_1^1 + {}^2b_2^2) - 2\,{}^1b_1^2\,{}^2b_{22} - 2\,{}^1b_2^1\,{}^2b_{11} + {}^1a_1\,{}^2c_{12}^1 - 2\,{}^1a^1\,{}^2c_{112} + 2\,{}^2a^2\,{}^1c_{122} - {}^2a_2\,{}^1c_{12}^2 &= 0\\
2\,{}^1b_{12}\,{}^2b^{22} + {}^1b_1^2({}^2b_1^1 - {}^2b_2^2) - 2\,{}^1b^{12}\,{}^2b_{11} + {}^1a_1\,{}^2c_1^{12} - 2\,{}^1a^1\,{}^2c_{11}^2 + {}^2a^2\,{}^1c_{12}^2 - 2\,{}^2a_2\,{}^1c_1^{22} &= 0\\
2\,{}^1b_{12}\,{}^2b^{11} - {}^1b_2^1({}^2b_1^1 - {}^2b_2^2) - 2\,{}^1b^{12}\,{}^2b_{22} + 2\,{}^1a_1\,{}^2c_2^{11} - {}^1a^1\,{}^2c_{12}^1 + 2\,{}^2a^2\,{}^1c_{22}^1 - {}^2a_2\,{}^1c_2^{12} &= 0\\
2\,{}^1b_1^2\,{}^2b^{11} + 2\,{}^1b_2^1\,{}^2b^{22} - {}^1b^{12}({}^2b_1^1 + {}^2b_2^2) + 2\,{}^1a_1\,{}^2c^{112} - {}^1a^1\,{}^2c_1^{12} + {}^2a^2\,{}^1c_2^{12} - 2\,{}^2a_2\,{}^1c^{122} &= 0\\
{}^1b_{12}({}_2b_1^1 + {}_2b_2^2) - 2\,{}^1b_1^2\,{}_2b_{22} - 2\,{}^1b_2^1\,{}_2b_{11} + {}^1a_1\,{}_2c_{12}^1 - 2\,{}^1a^1\,{}_2c_{112} + 2\,{}_2a^2\,{}^1c_{122} - {}_2a_2\,{}^1c_{12}^2 &= 0\\
2\,{}^1b_{12}\,{}_2b^{22} + {}^1b_1^2({}_2b_1^1 - {}_2b_2^2) - 2\,{}^1b^{12}\,{}_2b_{11} + {}^1a_1\,{}_2c_1^{12} - 2\,{}^1a^1\,{}_2c_{11}^2 + {}_2a^2\,{}^1c_{12}^2 - 2\,{}_2a_2\,{}^1c_1^{22} &= 0\\
2\,{}^1b_{12}\,{}_2b^{11} - {}^1b_2^1({}_2b_1^1 - {}_2b_2^2) - 2\,{}^1b^{12}\,{}_2b_{22} + 2\,{}^1a_1\,{}_2c_2^{11} - {}^1a^1\,{}_2c_{12}^1 + 2\,{}_2a^2\,{}^1c_{22}^1 - {}_2a_2\,{}^1c_2^{12} &= 0\\
2\,{}^1b_1^2\,{}_2b^{11} + 2\,{}^1b_2^1\,{}_2b^{22} - {}^1b^{12}({}_2b_1^1 + {}_2b_2^2) + 2\,{}^1a_1\,{}_2c^{112} - {}^1a^1\,{}_2c_1^{12} + {}_2a^2\,{}^1c_2^{12} - 2\,{}_2a_2\,{}^1c^{122} &= 0
\end{aligned}\right\} \quad (12c)$$

$$\left.\begin{aligned}{}_1b_{12}{}^1b^1_2-{}_1b^1_2{}^1b_{12}+{}_1a_1{}^1c^1_{22}+{}^1a^1{}_1c_{122}-{}_1a^1{}^1c_{122}-{}^1a_1{}_1c^1_{22}&=0\\ {}_1b_{12}{}^1b^{12}+{}_1b^2_1{}^1b^1_2-{}_1b^1_2{}^1b^2_1-{}_1b^{12}{}^1b_{12}+{}_1a_1{}^1c^{12}_2+{}^1a^1{}_1c^2_{12}-{}_1a^1{}^1c^2_{12}-{}^1a_1{}_1c^{12}_2&=0\\ {}_1b^2_1{}^1b^{12}-{}_1b^{12}{}^1b^2_1+{}_1a_1{}^1c^{122}+{}^1a^1{}_1c^{22}_1-{}_1a^1{}^1c^{22}_1-{}^1a_1{}_1c^{122}&=0\end{aligned}\right\}\tag{12d}$$

$$\left.\begin{aligned}{}_1b_{12}{}^1b^2_1-{}_1b^2_1{}^1b_{12}+{}_2a_2{}^2c^2_{11}+{}^2a^2{}_2c_{112}-{}_2a^2{}^2c_{112}-{}^2a_2{}_2c^2_{11}&=0\\ {}_1b_{12}{}^1b^{12}+{}_1b^1_2{}^1b^2_1-{}_1b^2_1{}^1b^1_2-{}_1b^{12}{}^1b_{12}+{}_2a_2{}^2c^{12}_1+{}^2a^2{}_2c^1_{12}-{}_2a^2{}^2c^1_{12}-{}^2a_2{}_2c^{12}_1&=0\\ {}_1b^1_{12}{}^1b^{12}-{}_1b^{12}{}^1b^1_2+{}_2a_2{}^2c^{112}+{}^2a^2{}_2c^{11}_2-{}_2a^2{}^2c^{11}_2-{}^2a_2{}_2c^{112}&=0\end{aligned}\right\}\tag{12e}$$

Equations (12c) (twelve of which are independent) give rise to twelve aberration coefficients, characterizing the deviation (normal to the plane of symmetry) ${}_1c_{122}$, ${}^1c_{122}$, ${}_1c^2_{12}$, ${}^1c^2_{12}$, ${}_1c^1_{22}$, ${}^1c^1_{22}$, ${}_1c^{22}_1$, ${}^1c^{22}_1$, ${}_1c^{12}_2$, ${}^1c^{12}_2$, ${}_1c^{122}$, ${}^1c^{122}$ in terms of the twelve coefficients ${}_2c_{112}$, ${}^2c_{112}$, ${}_2c^1_{12}$, ${}^2c^1_{12}$, ${}_2c^{11}_2$, ${}^2c^{11}_2$, ${}_2c^2_{11}$, ${}^2c^2_{11}$, ${}_2c^{12}_1$, ${}^2c^{12}_1$, ${}_2c^{112}$, ${}^2c^{112}$. The latter are related to each other by Eqs. (12e). Equations (12d) become identities upon substitution of the evaluated coefficients. The coefficients ${}_1c_{111}$, ${}^1c_{111}$, ${}_1c^1_{11}$, ${}^1c^1_{11}$, ${}_1c^{11}_1$, ${}^1c^{11}_1$, ${}_1c^{111}$, ${}^1c^{111}$ and ${}_2c_{222}$, ${}^2c_{222}$, ${}_2c^2_{22}$, ${}^2c^2_{22}$, ${}_2c^{22}_2$, ${}^2c^{22}_2$, ${}_2c^{222}$, ${}^2c^{222}$ figure in Eqs. (12a) and (12b), respectively.

The following relations hold for the 30 chromatic aberration coefficients:

$$\left.\begin{aligned}{}_1b_{12}{}^1b^{13}+{}^1b^1_2{}_1b^3_1-{}^1b_{12}{}_1b^{13}-{}_1b^1_2{}^1b^3_1+{}_1a_1{}^1c^{13}_2+{}^1a^1{}_1c^3_{12}-{}^1a_1{}_1c^{13}_2-{}_1a^1{}^1c^3_{12}&=0\\ {}_1b^2_1{}^1b^{13}+{}^1b^{12}{}_1b^3_1-{}^1b^2_1{}_1b^{13}-{}_1b^{12}{}^1b^3_1+{}_1a_1{}^1c^{123}+{}^1a^1{}_1c^{23}_1-{}^1a_1{}_1c^{123}-{}_1a^1{}^1c^{23}_1&=0\\ {}_1b_{12}{}^1b^3_1+2{}^2b_{11}{}_2b^3_2-{}^1b_{12}{}_1b^3_1-2{}_2b_{11}{}^2b^3_2+{}_1a_1{}^1c^3_{12}+2{}_2a_2{}^2c^3_{11}-{}^1a_1{}_1c^3_{12}-2{}^2a_2{}_2c^3_{11}&=0\\ {}_1b^1_2{}^1b^3_1+{}^2b^1_1{}_2b^3_2-{}^1b^1_2{}_1b^3_1-{}_2b^1_1{}^2b^3_2+{}_1a_1{}^1c^{13}_2+{}_2a_2{}^2c^{13}_1-{}^1a_1{}_1c^{13}_2-{}^2a_2{}_2c^{13}_1&=0\\ {}_1b^2_1{}^1b^3_1+2{}^2b_{11}{}_2b^{23}-{}^1b^2_1{}_1b^3_1-2{}_2b_{11}{}^2b^{23}+{}_1a_1{}^1c^{23}_1+2{}_2a^2{}^2c^3_{11}-{}^1a_1{}_1c^{23}_1-{}^2a^2{}_2c^3_{11}&=0\\ {}_1b^{12}{}^1b^3_1+{}^2b^1_1{}_2b^{23}-{}^1b^{12}{}_1b^3_1-{}_2b^1_1{}^2b^{23}+{}_1a_1{}^1c^{123}+{}_2a^2{}^2c^{13}_1-{}^1a_1{}_1c^{123}-{}^2a^2{}_2c^{13}_1&=0\\ {}_1b_{12}{}^1b^{13}+{}^2b^1_1{}_2b^3_2-{}^1b_{12}{}_1b^{13}-{}_2b^1_1{}^2b^3_2+{}^1a^1{}_1c^3_{12}+{}_2a_2{}^2c^{13}_1-{}_1a^1{}^1c^3_{12}-{}^2a_2{}_2c^{13}_1&=0\\ {}_1b^1_2{}^1b^{13}+2{}^2b^{11}{}_2b^3_2-{}^1b^1_2{}_1b^{13}-2{}_2b^{11}{}^2b^3_2+{}^1a^1{}_1c^{13}_2+2{}_2a_2{}^2c^{13}_1-{}_1a^1{}^1c^{13}_2-2{}^2a_2{}^2c^{113}&=0\\ {}_1b^2_1{}^1b^{13}+{}^2b^1_1{}_2b^{23}-{}^1b^2_1{}_1b^{13}-{}_2b^1_1{}^2b^{23}+{}^1a^1{}_1c^{23}_1+{}_2a^2{}^2c^{13}_1-{}_1a^1{}^1c^{23}_1-{}^2a^2{}_2c^{13}_1&=0\\ {}_1b^{12}{}^1b^{13}+2{}^2b^{11}{}_2b^{23}-{}^1b^{12}{}_1b^{13}-2{}_2b^{11}{}^2b^{23}+{}^1a^1{}_1c^{123}+2{}_2a^2{}^2c^{113}-{}_1a^1{}^1c^{123}-2{}^2a^2{}_2c^{113}&=0\end{aligned}\right\}\tag{13a}$$

$$\left.\begin{aligned}{}_2b^2_2{}^2b^3_2+2{}^2b_{22}{}_2b^{23}-2{}_2b_{22}{}^2b^{23}-{}^2b^2_2{}_2b^3_2+{}^2a_2{}_2c^{23}_2+2{}_2a^2{}^2c^3_{22}-{}_2a_2{}^2c^{23}_2-2{}^2a^2{}_2c^3_{22}&=0\\ 2{}_2b^{22}{}^2b^3_2+{}^2b^2_2{}_2b^{23}-{}_2b^2_2{}^2b^{23}-2{}^2b^{22}{}_2b^3_2+{}_2a^2{}^2c^{23}_2+2{}^2a_2{}_2c^{223}-2{}_2a_2{}^2c^{223}-{}^2a^2{}_2c^{23}_2&=0\end{aligned}\right\}\tag{13b}$$

$${}_1b^3_1{}^1b^{13}-{}^1b^3_1{}_1b^{13}+2{}_1a_1{}^1c^{133}+2{}^1a^1{}_1c^{33}_1-2{}^1a_1{}_1c^{133}-2{}_1a^1{}^1c^{33}_1=0\tag{13c}$$

$${}^2b^3_2{}_2b^{23}-{}_2b^3_2{}^2b^{23}+{}^2a_2{}_2c^{233}+{}_2a^2{}^2c^{33}_2-{}_2a_2{}^2c^{233}-{}^2a^2{}_2c^{33}_2=0\tag{13d}$$

The first two equations of (13a) are dependent. With the aid of the next eight equations of (13a) one can determine the coefficients ${}_1c^3_{12}$, ${}^1c^3_{12}$, ${}_1c^{23}_1$, ${}^1c^{23}_1$, ${}_1c^{13}_2$, ${}^1c^{13}_2$, ${}_1c^{123}$, ${}^1c^{123}$ in terms of the other six ${}_2c^3_{11}$, ${}^2c^3_{11}$, ${}_2c^{13}_1$, ${}^2c^{13}_1$, ${}_2c^{113}$, ${}^2c^{113}$. The six coefficients ${}_2c^3_{22}$, ${}^2c^3_{22}$, ${}_2c^{23}_2$, ${}^2c^{23}_2$, ${}_2c^{223}$, ${}^2c^{223}$ are related to each other by Eqs. (13b) and the coefficients ${}_1c_1^{33}$, ${}^1c_1^{33}$, ${}_1c^{133}$, ${}^1c^{133}$ and ${}_2c_2^{33}$, ${}^2c_2^{33}$, ${}_2c^{233}$, ${}^2c^{233}$ by Eqs. (13c) and (13d), respectively. The coefficients ${}_2c^{333}$, ${}^2c^{333}$ do not appear in any of the equations and are hence independent.

For the case of two planes of symmetry, all transverse third-order aberration coefficients and all Eqs. (12) are the same as before. For the non-vanishing chromatic coefficients ${}_1c_1^{33}$, ${}^1c_1^{33}$, ${}_1c^{133}$, ${}^1c^{133}$, ${}_2c_2^{33}$, ${}^2c_2^{33}$, ${}_2c^{233}$, ${}^2c^{233}$, the two equations (13c) and (13d) still hold.

In conclusion, we present a table summarizing the number of independent first to third order aberration coefficients for systems with one or two planes of symmetry.

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