Particle Accelerators, 1971, Vol. 2, pp. 121–131

SINGLE PARTICLE THEORY OF PLASMA BETATRONS

L. A. FERRARI
Queens College of the City University of New York, New York, N.Y., U.S.A.
AND
M. S. ZUCKER
Brookhaven National Laboratory, Upton, New York, U.S.A.

Particle orbits in combined betatron and azimuthal magnetic fields are computed using the guiding center approximation as well as the single particle equations directly. Particular attention is given to the start of the acceleration cycle. The problem of capture into betatron orbits in a plasma betatron without azimuthal field is considered. It is found that a finite electron temperature places severe restrictions on the beam current. It is shown that a plasma must be formed with a static azimuthal magnetic field already established and that beam loss via scattering is negligible. Finally some new types of plasma betatron experiments are suggested.

1. INTRODUCTION

In the past several years there has been renewed interest in the concept of the collective effect accelerator. These devices utilize dense charge clusters to produce confining fields that are considerably larger than those available from external sources. Protons are trapped and subsequently accelerated along with the cluster, their final velocity being determined by the electron mass. The 'Electron Ring Accelerator' utilizes this concept and if successful will be a new type of high energy proton accelerator.[1]

A type of collective effect accelerator, the plasma betatron, was proposed by Budker[2] in 1956. By using a plasma as the electron source space-charge forces could be drastically reduced thereby increasing the beam current. A circulating current of 17,000 A was envisioned. Damping of the transverse motion of the electrons would result in a beam with a small diameter and hence large self fields which then could be used to guide protons.

Following Budker's suggestion several plasma betatron experiments were started.[3–9] Since the electrons were to be accelerated from a plasma the magnet was to have several features not found in conventional betatrons. Air-core designs were chosen in order that both high frequency operation (60–100 kHz) and high accelerating field strengths (≈ 50 V/cm or larger) could be obtained. Two types of air core magnets have evolved: in the first, termed low inductance, the betatron guide field is generated by eddy current field shaping[10] (flux concentrators), the second type, termed high inductance, uses nested single turns of wire.[11]

Experiments quickly revealed that no electrons were accelerated unless the betatron field was supplemented with an additional magnetic field component directed along the betatron equilibrium orbit. It was also observed that the charges were not accelerated to the full energy capability of the device. After approximately one-fourth of the acceleration cycle had elapsed most of the beam current struck the vacuum chamber walls. The magnitude of the beam current was disappointingly small, about 10–20 A, far below the Budker figure and far below the figure predicted by the Schmidt theory[12] of beam current limitation (≈ 400 A). Interest in these devices subsequently waned.

The results of these plasma betatron experiments have since been studied in considerable detail[13–16] and while the analysis is somewhat lengthy the results can be summarized quite simply: in all of the experiments conditions at the start of the acceleration cycle were not proper for establishment of a high beam current hence none could have succeeded. These experiments failed, according to our analysis,[15] either due to the ignoring of some principle of plasma physics, or not having conditions proper for the establishment of stable orbits at least on the basis of a single particle model. None of these two basic faults involve any necessary compromise in design, construction, or operation. These difficulties can be obviated by a plasma betatron design based on a detailed analysis of the single particle motion as well as careful attention to the preparation of the plasma prior to the start of the acceleration cycle.

It is our supposition that the onset of collective effects causing instabilities will occur at currents perhaps an order of magnitude higher than instabilities that can be accounted for with a single particle model.

In an equilibrium mode of operation a small plasma betatron (orbit radius $\approx$ 18 cm) should be capable of producing a 400 A, 8 MeV electron beam. A nonequilibrium mode of operation may produce a 10,000 to 30,000 A beam with energy between fifty and several hundred keV. While these parameters fall far short of those proposed by Budker this device, while no longer a collective effect proton accelerator, may prove to be a relatively small, inexpensive radiation or electron source. Accordingly the single particle orbits in combined betatron and azimuthal magnetic fields are considered in the subsequent sections of this paper. Collective effects are ignored since at present they are not well understood, at least as they apply to this accelerator concept. The treatment is nonrelativistic since most of the difficulties encountered so far occur at the start of the acceleration cycle where the electronic motion is nonrelativistic.

2. PARTICLE MOTION IN COMBINED BETATRON AND AZIMUTHAL MAGNETIC FIELDS

In this section, single particle motion in combined betatron and time-independent azimuthal magnetic fields is studied via the guiding center approximation as well as the single particle equations directly. Since the B_θ field is time-independent while the betatron field starts from zero, it is obvious that the nonlinear terms due to the gradient in B_θ will be larger than the betatron focusing forces for a certain fraction of the acceleration cycle. This is considered and an orbit for a typical set of parameters is computed. The displacement of an electron located at the equilibrium orbit initially is shown to be much smaller than the minor diameter of acceleration chamber. Its displacement relative to the acceptance diameter of the field will be larger but just how much larger is not known. The B_θ turn on problem for the low inductance plasma betatron is also considered.

A. *Single Particle Equations of Motion*

The equations of motion for a particle of mass m and charge q moving in a quasistatic magnetic field with components B_r, B_θ, and B_z (cylindrical coordinates, r, θ, z) are:

$$m\ddot{r} = mr\dot{\theta}^2 - q\dot{z}B_\theta + qr\dot{\theta}B_z \tag{1}$$

$$m\ddot{z} = q\dot{r}B_\theta - qr\dot{\theta}B_r \tag{2}$$

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -q\dot{r}B_z + q\dot{z}B_r. \tag{3}$$

By quasistatic we mean that the fields do not change appreciably during one revolution. These equations are also relativistically correct if the energy gained by the particle in one revolution is small, and if the relativistic mass is used appropriately in place of m. In combined betatron and azimuthal magnetic fields the field components are given by (approximately):

$$B_z = B_0\left(\frac{r_0}{r}\right)^n, \qquad 0 < n < 1$$

$$B_r = -\frac{n}{r}B_0\left(\frac{r_0}{r}\right)^n z$$

$$B_\theta = B_{\theta 0}\frac{r_0}{r}.$$

Here r_0 is the position of the equilibrium orbit and

$$n \equiv \frac{\partial \ln B_z}{\partial \ln r}$$

is the field index. To investigate the motion about the position of the equilibrium orbit the substitution $r = r_0 + x$, with $x \ll r_0$, $z \ll r_0$ is made. Then, neglecting nonlinear terms, the equations of motion reduce to:

$$\ddot{x} + \omega_0^2(1-n)x = \omega_c\dot{z} \tag{4}$$

$$\ddot{z} + \omega_0^2 nz = -\omega_c\dot{x}, \tag{5}$$

where

$$\omega_0 = \left|\frac{qB_0}{m}\right|$$

and

$$\omega_c = \left|\frac{qB_{\theta 0}}{m}\right|.$$

If the usual assumption that x, $z \propto e^{i\omega t}$ is made, it is found that the frequencies of the motion are given by

$$\left(\frac{\omega}{\omega_0}\right)^2 = \frac{1}{2}\left[1 + \left(\frac{\omega_c^2}{\omega_0^2}\right)\right] \pm \frac{1}{2}\left\{\left[1 + \left(\frac{\omega_c^2}{\omega_0^2}\right) - 4n(1-n)\right]\right\}^{1/2}. \tag{6}$$

The ω's determined from this equation are always positive so that the motion about r_0 is stable. The usual experimental situation has $\omega_c \gg \omega_0$ at least during the start-up phase. In this limit, it can be shown from (6) that the motion consists of a rapid gyration with $\omega_1 \approx \omega_c$ superimposed on a slow elliptic motion of the so-called guiding center, with $\omega_2 \approx (\omega_0^2/\omega_c)[n(1-n)]^{1/2}$.

Equations (4) and (5) have been derived assuming that the guiding fields are essentially unchanged during a period of revolution, which is to say that they are comparatively slowly varying functions of time. To the extent that this is the case, the equations could be solved using the WKB approximation for the weak focusing situation ($\omega_c = 0$) except in the neighborhood of $t = 0$, where the WKB method is no longer applicable to the present plasma betatron calculation. The latter is in contrast to the usual accelerator calculation in that in the usual circular accelerator particles are injected at $t = 0$ with a finite value of B_z already established; in the present case the plasma electrons are already present at $t = 0$ with B_z being zero at that instant. Thus, in the neighborhood of $t = 0$, the motion is not adiabatic.

This in itself does not cause difficulty in solving the equations of motion in this region, for putting $B_z = 0$ in Eqs. (4) and (5) would just reduce them to the simpler forms

$$\ddot{x} = \omega_c \dot{z} \tag{7a}$$

$$\ddot{z} = -\omega_c \dot{x} \tag{7b}$$

which are the equations of motion for a single particle in a B_θ field whose magnitude is independent of position. Such difficulty as there is arises from the fact that higher order nonlinear terms that were dropped in writing Eqs. (4) and (5) are no longer negligible when $B_z \approx 0$. These nonlinear terms arise from the fact that the toroidal B_θ field cannot by its nature be constant, but must have a gradient in the radial direction. This radial gradient of the B_θ field causes a charged particle to have a net average or drift velocity in the z-direction rather than the zero $\dot{z}$ predicted by Eqs. (7a) and (7b). Corrected Eqs. (4) and (5) now become

$$\ddot{x} + \omega_0^2(1-n)x = \omega_c \dot{z}\left[1 - \frac{x}{r_0} + \ldots\right] \tag{8}$$

$$\ddot{z} + \omega_0^2 n z = -\omega_c \dot{x}\left[1 - \frac{x}{r_0} + \ldots\right]. \tag{9}$$

In these equations the term in brackets on the right-hand side represents the r^{-1} dependence of the B_θ field. Since ω_0 starts from zero while ω_c is time independent, it is obvious that the nonlinear terms on the right-hand side of Eqs. (8) and (9) will be larger than the betatron focusing terms for some initial fraction of the acceleration cycle. As time progresses, the ratio $(\omega_0 n z/\omega_c \dot{x} x r_0^{-1}) \equiv \rho$, for example, becomes larger and larger so that the linearized equations eventually apply. The particle displacements during this interim period are of especial importance because it may be possible for them to drift out of the acceptance region of the field and thus be lost.

It is not apparent that Eqs. (8) and (9) with $\omega_0 = \omega_0(t)$ can be solved by any but numerical means, and to do so would not be particularly enlightening. Instead of that, therefore, the range of integration is broken into three regions and appropriate approximations made such that simple analytic results valid for the respective regions can be found. These regions can be defined in terms of the ratio ρ.

$\rho \ll 1$: This is the region including and shortly after the start of the acceleration cycle, when the betatron field is zero or small, so that the betatron focusing terms are comparable with the nonlinear B_θ terms. We define a dimensionless parameter ϵ to be of first order and solve to second order

$$\ddot{x} + \epsilon\omega_0^2(1-n)x = \omega_c \dot{z}\left[1 - \frac{x}{r_0}\right]$$

$$\ddot{z} + \epsilon\omega_0^2 n z = -\omega_c \dot{x}\left[1 - \frac{x}{r_0}\right]. \tag{10}$$

Since the products ϵx and ϵz are second order, the first order equations become (7a, b) with solutions

$$x_1 = a_1 \sin \omega_1 t + a_2 \cos \omega_1 t$$

$$z_1 = a_3 \sin \omega_1 t + a_4 \cos \omega_1 t, \tag{11}$$

where $\omega_1 = \omega_c$ and a_1, a_2, etc., are constants.

Using the expansion procedure $x = x_1 + x_2$, $z = z_1 + z_2$ we obtain the second order equations:

$$\ddot{x}_2 - \omega_c \dot{z}_2 = -\omega_c \dot{z}_1 \frac{x_1}{r_0} - \epsilon\omega_0^2(1-n)x_1$$

$$\ddot{z}_2 + \omega_c \dot{x}_2 = \frac{\omega_c}{r_0} \dot{x}_1 x_1 - \epsilon\omega_0^2 n z_1. \tag{12}$$

The particular solutions of these equations can be shown to be

$$x_2 = b_1 \sin 2\omega_1 t + b_2 \cos 2\omega_1 t + b_3 \sin \omega_1 t + b_4 \cos \omega_1 t$$

$$z_2 = d_1 \sin 2\omega_1 t + d_2 \cos 2\omega_1 t + d_3 \sin \omega_1 t + d_4 \cos \omega_1 t + d_5 t,$$

where b_1, b_2, d_1, d_2, etc. are constants. The constant d_5 is most important since the $d_5 t$ term gives a net displacement in the z-direction. The value of d_5 can be shown to be

$$d_5 = \frac{1}{2r_0}(a_2 a_3 - a_4 a_1)\omega_1. \tag{13}$$

It is found that d_5 is the velocity of the so-called guiding center of the particle, which will be discussed later.

$\rho \gtrsim 1$: In this region the betatron field is large enough to be appreciable, but not so large that the first nonlinear terms in the B_θ field can be neglected. A power series (in x and z) representation of the vector potential which is accurate to an order appropriate to this region has been derived by Landau[17] by requiring that $\nabla \cdot \mathbf{B} = 0$ and $\nabla x \mathbf{B} = 0$ to that order:

$$A_\theta = B_0 r_0 \left[1 + (1-n)\frac{x^2}{2r_0^2} + n\frac{z^2}{2r_0^2} + \frac{n'+n-3}{6}\frac{x^3}{r_0^3} + \frac{n'}{2}\frac{xz^2}{r_0^3}\right]. \tag{14}$$

The equations of motion, correct to second order, then become

$$\ddot{z} + \omega_0^2 n\left(1 - \frac{n'}{n}\frac{x}{r_0}\right)z = \omega_c \dot{x}\left(1 - \frac{x}{r_0}\right), \tag{15}$$

$$\ddot{x} + \omega_0^2(1-n)\left(1 + \frac{\alpha x}{r_0}\right)x = -\omega_c \dot{z}\left(1 - \frac{x}{r_0}\right), \tag{16}$$

where $\alpha \equiv (n'+n-3)/2(1-n)$ and $n' \equiv \partial n/\partial r$.

The solution to these equations was obtained by Landau by means of the expansion procedure described in the previous section. The solutions are written as:

$$x = x_1 + x_2$$
$$z = z_1 + z_2, \tag{17}$$

where x_1, z_1 are the solutions of (4) and (5). The subscripts 1 and 2 denote first and second order, respectively, as before. Thus substituting (17) into (15) and (16) we obtain to second order

$$\ddot{z}_2 + \omega_0^2 n z_2 - \omega_c \dot{x}_2 = \omega_0^2 n \frac{n'}{n}\frac{x_1}{r_0} z_1 - \omega_c \frac{\dot{x}_1}{r_0} x_1 \tag{18}$$

$$\ddot{x}_2 + \omega_0^2(1-n)x_2 + \omega_c \dot{z}_2 = -\omega_0^2(1-n)\alpha\frac{x_1^2}{r_0} + \omega_c \dot{z}_1 \frac{x_1}{r_0}. \tag{19}$$

The solutions of the first order equations can be written as

$$x_1 = c_1 \sin \omega_1(t - c_3) + c_2 \sin \omega_2(t - c_4)$$
$$z_1 = K_1 C_1 \cos \omega_1(t - C_3) + K_2 C_2 \cos \omega_2(t - C_4), \tag{20}$$

where C_1, C_2, C_3, C_4, K_1, and K_2 are constants and ω_1 and ω_2 are given by

$$2\omega_1^2 = \omega_0^2(1-n) + \omega_0^2 n + \omega_c^2 + [(\omega_0^2 + \omega_c^2)^2 - 4\omega_0^4 n(1-n)]^{1/2} \tag{21a}$$

$$2\omega_2^2 = \omega_0^2 + \omega_c^2 - [(\omega_0^2 + \omega_c^2)^2 - 4\omega_0^4 n(1-n)]^{1/2}. \tag{21b}$$

The particular solutions of (18) and (19) are

$$z_2 = A_z \sin 2\omega_1\tau_1 + B_z \sin 2\omega_2\tau_2 + C_z \sin(\omega_1\tau_1 + \omega_2\tau_2) + D_z \sin(\omega_1\tau_1 - \omega_2\tau_2)$$

$$x_2 = E_x + A_x \cos 2\omega_1\tau_1 + B_x \cos 2\omega_2\tau_2 + \cos(\omega_1\tau_1 + \omega_2\tau_2) + D_x \cos(\omega_1\tau_1 - \omega_1\tau_2), \tag{22}$$

where A, B, C, D, and E are constants independent of time and

$$\tau_1 \equiv t - C_3, \qquad \tau_2 = t - C_4.$$

The constant E_x is given by

$$\omega_0^2(1-n)E_x = -\omega_0^2(1-n)\frac{\alpha}{r_0}\left[\frac{C_1^2}{2} + \frac{C_2^2}{2}\right] + \frac{\omega_c}{r_0}\left[\frac{-K_1\omega_1 C^2}{2} - \frac{K_2\omega_2 C_2^2}{2}\right].$$

To compute a particle orbit the constants C_1, C_2, etc., must be evaluated. However, since this solution (22) is not valid near $t = 0$, there is no unambiguous choice for the initial conditions needed to evaluate the constants.

$\rho \gg 1$: Here the focusing term is large enough for the higher order terms in B_θ to be negligible. The equations are then given by (4) and (5), with solutions as discussed earlier.

B. *Average Particle Motion or Guiding Center Approximation*

When $\omega_0 \ll \omega_c$, the electronic motion consists of a rapid circular gyration with $\omega \approx \omega_c$ about the B_θ lines of force superimposed on a much slower motion that we wish to investigate. If the motion were to be averaged over a period $\tau \sim 2\pi/\omega_c$ the position of the center of a circle of radius $r_c = v_\perp/\omega_c$ would be obtained, where $v_\perp$ is the component of particle velocity perpendicular to the B_θ lines of force. This center of curvature is called the guiding center and its path as a function of time will be the time average of the individual particle motion of the time interval over which the average is taken is long compared with $1/\omega_c$ but short compared to $1/\omega_0$. The velocity of the guiding center associated with a particle of mass m and charge q is to a second order of approximation given by[18]

$$\mathbf{v}_0 = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} + \frac{(mv_\parallel^2 + \frac{1}{2}mv_\perp^2)}{qB^4}\mathbf{B}\times\frac{\nabla B^2}{2}, \quad (23)$$

where $\mathbf{F}$ denotes some external force field of non-magnetic origin and $v_\parallel$ is the component of particle velocity along the line of force $\mathbf{B}$. The guiding center approximation requires that the value of the field of the particle be only slightly different from its value at the guiding center and that its magnitude measured in a coordinate system moving with the particle remain essentially unchanged during a complete revolution, i.e., the field components must satisfy[19]

$$\frac{v}{\omega_c}\left|\frac{\partial B_j}{\partial x_k}\right| |B_j| \ll 1, \qquad \frac{1}{\omega_c}\left|\frac{\mathrm{d}B_j}{\mathrm{d}t}\right| |B_j| \ll 1. \quad (24)$$

The origin of each term in (23) requires more explanation before this approximation may be applied to the problem of interest. If an external force field $\mathbf{F}$ is superimposed on a homogeneous magnetic field $\mathbf{B}$ then in a coordinate system moving with velocity $(\mathbf{F}\times\mathbf{B})/qB^2$ the particle motion will be circular. Thus for this problem the proper force to insert in this expression is the betatron focusing force with components

$$F_r = -m\omega_0^2(1-n)x$$
$$F_z = -m\omega_0^2 nz \quad (25)$$

provided $(v_\perp/\omega_c)1/x$ or $(v_\perp/\omega_c)1/z \ll 1$. Thus neglecting curvature effects the components of the guiding center velocity are:

$$\dot{x} = \frac{\omega_0^2}{\omega_c}nz$$
$$\dot{z} = -\frac{\omega_0^2}{\omega_c}(1-n)x. \quad (26)$$

Differentiating with respect to time one obtains

$$\ddot{x} + \frac{\omega_0^4}{\omega_c^2}n(1-n)x = 0$$
$$\ddot{z} + \frac{\omega_0^4}{\omega_c^2}n(1-n)z = 0. \quad (27)$$

The solution of these equations is harmonic with frequency $\omega = (\omega_0^2/\omega_c)[n(1-n)]^{1/2}$ and is identical with one of the roots of (6) in the $\omega_0/\omega_c \ll 1$ limit.

If the external magnetic field has a small gradient, there will be an interaction between the rapid gyrating motion and the magnetic field gradient (a dipole interacting with a gradient in the field) resulting in a net force acting on the particle. If this force is inserted into $(\mathbf{F}\times\mathbf{B})/qB^2$ the $v_\perp^2$ term in (23) is obtained. Similarly a particle moving along a curved magnetic line with $v_\parallel$ experiences a centrifugal force $mv_\parallel^2/R$ which when substituted into $(\mathbf{F}\times\mathbf{B})/qB^2$ results in the $v_\parallel^2$ term in Eq. (23). In this problem a particle moving along the equilibrium orbit r_0 has no net force acting on it since the centrifugal $mv_\parallel^2/R$ force is exactly counter-balanced by the centripetal magnetic force $qv_\parallel B_0$. Hence the $v_\parallel^2$ term will not appear. Proceeding in this fashion, the components of the guiding center velocity are

$$\dot{x} = \frac{\omega_0^2 n}{\omega_c}z$$
$$\dot{z} = -\frac{\omega_0^2(1-n)x}{\omega_c} + \frac{1}{2}\frac{v_\perp^2}{r_0\omega_c} \quad (28)$$

to first order. Differentiating with respect to time and substituting one obtains:

$$\ddot{x} + \lambda t^4 x = r_0\eta t^2 \quad (29a)$$
$$\ddot{z} + \lambda t^4 z = 0, \quad (29b)$$

where it is assumed that $\omega_0 \sim \omega_{0m}\Omega t$ and λ and η are given by

$$\lambda \equiv \frac{\omega_{0m}^4\Omega^4 n(1-n)}{\omega_c^2}$$
$$\eta = \frac{1}{2}\frac{\omega_{0m}^2 nv_\perp^2\Omega^2}{r_0^2\omega_c^2}.$$

In the derivation of (29) the terms involving $\mathrm{d}\omega_0/\mathrm{d}t$ have been neglected. Inspection shows that this approximation is valid provided

$$2\frac{\dot{\omega}_0}{\omega_0} \ll \frac{\dot{z}}{z}. \quad (30)$$

Using $(\dot{z})_{t=0}$ and z values obtained by integrating (28) numerically, it is found that (30) is reasonably well satisfied for the times of interest. The solution to these equations can be described qualitatively as follows: at very early instances of time $r_0\eta t^2$ will be larger than λt^4, thus the motion is unstable and the particles drift in the x and z direction; their displacement increasing without bound. As time progresses the λt^4 dependence term will increase rapidly because of the t^4 dependence. Hence, for large values of t, we will have $\lambda t^4(x/r_0) \gg \eta t^2$. For this situation the solutions to (29) will be ordinary Bessel functions (fractional order) and so x and z motions will be damped oscillations. Thus, assuming the particles start at $z = x = 0$, the displacement will increase to some maximum value and then decrease, oscillating about the starting point.

Since Eq. (29a) has a driving term that increases with time, the displacement in the radial direction will be larger than that in the z. Thus, our analytical efforts will be concentrated here. We assume that particles are located at $x = 0$ with zero velocity at $t = 0$. Then a particular solution of (29a) can be found in the form of a power series, i.e.,

$$x = \sum_{s=0}^{\infty} a_s t^s. \tag{31}$$

The coefficients a_s can be found by substituting (31) into (29a) and then equating coefficients of the same power of t, i.e.,

$$\frac{x}{r_0} = \frac{1}{12}\eta t^4 \left[1 - \frac{(\lambda^{1/2}t^3)^2}{10\cdot 9} + \frac{(\lambda^{1/2}t^3)^4}{16\cdot 15\cdot 10\cdot 9} - \frac{(\lambda^{1/2}t^3)^6}{22\cdot 21\cdot 16\cdot 15\cdot 10\cdot 9} + \dots\right].$$

This series converges very slowly; however, it was noticed that it can be approximated closely by the tabulated function $J_{21}(\lambda^{1/2}t^3)$. Therefore, as an approximate solution to (29a) we take

$$\frac{x}{r_0} = \frac{1}{12}\eta \frac{(21)!\,2^{21}}{t^{58}\lambda^{21/2}} J_{21}(\lambda^{1/2}t^3). \tag{34}$$

This solution exhibits the properties that the exact solution to (32) will have, namely, a steady increase from $x = 0$ to a maximum displacement followed by a damped oscillation about $x = 0$. The validity of this approximation solution is verified by numerical integration of (28) and (29a).

An orbit for a typical set of low inductance plasma betatron parameters [$B_{0\,\max} = 800$ G, $B_\theta = 2900$ G, $\Omega = 0.39 \times 10^6$ sec^{-1}, $r_0 = 4.85$ cm, $v_\perp = 1.87 \times 10^8$ cm/sec] has been obtained by solving Eq. (28) with time dependent ω_0. It is found that the maximum displacement relative to the equilibrium orbit is $(x/r_0)_{\max} \gg 3 \times 10^{-3}$. At the vacuum chamber wall, (x/r_0) is approximately 80 times larger and this value would appear to be negligible, however, the acceptance diameter of the field is not known. If the displacement relative to this parameter is large, some charged particles will be lost, limiting the beam current, and may result in disastrous electrostatic results. An orbit for a high inductance device has also been computed ($B_{\text{om}} = 1360$ G, $B_\theta = 1500$ G, $\Omega = 0.68 \times 10^6$ sec^{-1}, $r_0 = 20$ cm, $v_\perp = 1.87 \times 10^8$ cm/sec) and it is found that the maximum displacement is about an order of magnitude less, or $x/r_0 \approx 3 \times 10^{-4}$.

C. *The B_θ Turn-on Problem*

It has previously been noted that in the low inductance plasma betatron experiments the plasma was formed prior to application of the B_θ field. In this section we consider the B_θ turn-on and its effect on the initial plasma.

We assume that the B_θ field is uniform and directed along the z-axis of a Cartesian coordinate system. Thus we neglect curvature effects while B_θ is increasing from zero to some steady value. At the instant B_θ becomes steady, we 'turn on' the curvature effects. Following a delay of typically 10 μsec, the betatron field is applied. Thus, during the interim period it will be possible for charges to drift to the vacuum chamber walls and be lost.

It is easily shown that after B_θ has reached its steady value the paths of the charged particles will be circular, all passing through the origin, with radius of curvature R given by (assuming all particles start from rest)

$$R = \frac{mv_\theta}{qB_\theta} = \frac{P}{2},$$

where v_θ is the velocity imparted by the induction electric field and $P = (x_0^2 + y_0^2)^{1/2}$ is the initial displacement from the origin. Turning on the curvature effects it is found that the electrons drift across the B_θ field with velocity given by

$$v_0 = \frac{1}{8}\omega_c \frac{P^2}{r_0}.$$

Denoting by t the interim period before the betatron field is applied, and by $b/2$ the position of the vacuum chamber wall one obtains

$$P_c^{\,2} = \frac{4nr_0}{\omega_c t}.$$

Thus, all particles with initial $P < P_c$ may be captured into betatron orbits while those with initial $P > P_c$ cannot. Inserting appropriate parameters ($B_\theta = 2900$ G, $r_0 = 5$ cm, $t = 10^{-5}$ sec, $b = 1.6$ cm) it is found that

$$P_c \approx 8 \times 10^{-3}\ \text{cm}.$$

If all the particles with initial $P < P_c$ are captured into stable orbits the beam current will be

$$i = 2\pi^2 ne P_c^2 r_0 C,$$

where C is the velocity of light. Assuming $n \approx 10^{12}$ cm^{-3} the maximum current is

$$i \approx 0.3\ \text{A}.$$

This is an optimistic estimate since the drift during the rise time of the B_θ field ($\approx 10\ \mu$sec) was neglected. Thus the B_θ field must be constant during the initial ionization.

3. CAPTURE INTO BETATRON ORBITS IN A PLASMA BETATRON WITHOUT B_θ

In this section we discuss the effect that a non-zero electron temperature has on the amount of runaway current that can be generated in a plasma betatron with no B_θ field. This current will be due to the acceleration of those electrons present in the plasma at time $t = 0$ which have velocities and positions that are within the range of those possible initial conditions leading to solutions of the equations of motion which are stable with time.

This important problem has not been solved in all generality and various approximations have had to be made in each of the two existing studies, the present one, and that of Lukasik *et al.*,[20] results of which will also be quoted.

In the present analysis the particle is assumed to start out at time $t = 0$ from a position on the orbit with only a radial component of thermal velocity, i.e., $v_\theta = v_z = 0$, and with $B_\theta = 0$; however, the solution obtained is analytical. The treatment by Lukasik *et al.* is a purely numerical (computer) solution in which there is no restriction as to initial position, all components of thermal velocity are assumed present, and in some calculations a $B_\theta \neq 0$ is assumed; these last calculations are unfortunately believed to be incomplete.

The nonrelativistic Hamiltonian for a charged particle moving in a cylindrically symmetric magnetic field is given by

$$H = \frac{p_r^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2m}\left[\frac{p_\theta}{r} - (qA_\theta)\right]^2, \qquad (37)$$

where p_θ is a constant of motion. The terms $[(p_r^2/2m) + (p_z^2/2m)] \equiv K_T$ represent the transverse kinetic energy of the particle, while the term $1/2m[(p_\theta/r) - (qA_\theta)]^2 \equiv \psi$ which is a function of r alone can be shown to play the part of an effective potential. In order for an orbit to be stable there must be a minimum in ψ in the neighbourhood of that orbit and the total energy H must be less than the smallest neighboring maxima. During a typical acceleration cycle in the time of which H increases so slowly as to be essentially constant, a plot of ψ and H versus radial position r is expected to qualitatively have the appearance sketched in Fig. 1.

The difference between H and ψ represents K_T, which approaches zero at the classical turning points a and b, corresponding to the minimum and maximum radial excursions. For capture into a stable orbit about r_0, we must therefore have

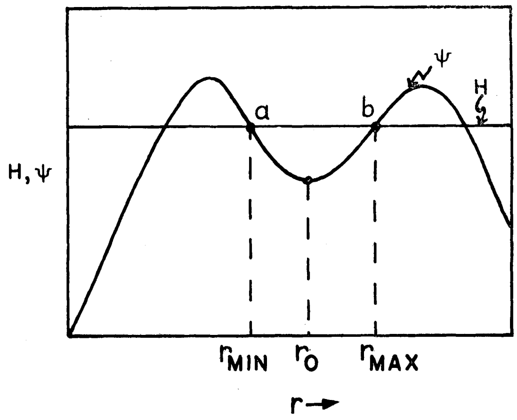

FIG. 1. Radial variation of the effective potential ψ in the neighborhood of the equilibrium orbit.

$K_T < (\psi_{\max} - \psi_{\min})$, with ψ being evaluated at $t = 0$. The maximum transverse speed that the particle can have at $t = 0$, will be that allowing the above inequality to be satisfied.

Assuming that at $t = 0$, $v_\theta = 0$, then $p_\theta = 0$, and the Hamiltonian reduces to

$$H = \frac{p_z^2}{2m} + \frac{p_r^2}{2m} + \frac{q^2 A_\theta^2}{2m}$$

so that the radial and axial equations of motion become

$$\ddot{r} = -\left(\frac{q^2}{2m^2}\right)\left(\frac{\partial A_\theta^2}{\partial r}\right),$$

$$\ddot{z} = -\left(\frac{q^2}{2m^2}\right)\left(\frac{\partial A_\theta^2}{\partial z}\right). \qquad (39)$$

For simplicity, A_θ will be considered a function of r alone. Then the z equation of motion can be ignored; in the neighborhood of the equilibrium orbit A_θ^2 can be approximated by a parabola, i.e.,

$$A_\theta = \alpha^{1/2}\left[1 + \frac{\beta}{\alpha}(r - r_0)^2\right]^{1/2} \sin\omega t, \qquad (40)$$

where α and β are constants. Thus the radial equation motion becomes

$$\ddot{r} + \frac{q^2}{m^2}\beta(r - r_0)\sin^2\omega t = 0 \qquad (41)$$

or

$$\ddot{r} + \omega_{\text{om}}^2 \frac{\beta}{B_{\text{om}}^2}(r - r_0)\sin^2\omega t = 0, \qquad (42)$$

where $\omega_{\text{om}} = q(B_{\text{om}}/m)$ and B_{om} is the peak value of the magnetic field at the equilibrium orbit.

We will be interested in the solutions of this equation when $\Omega t \ll 1$. Since $\sin \Omega t \sim \Omega t$ for

$\Omega t \ll 1$, Eq. (42) now becomes

$$\ddot{r} + (\omega_{om}^2 \Omega^2) \frac{\beta}{B_{om}^2} (r - r_0) t^2 = 0 .$$

Again introducing the coordinate $x = (r - r_0)$, which measures the displacement from the equilibrium position, and letting $\eta = (\omega_{om}^2 \Omega^2)(\beta / B_{om}^2)$ the equation of motion becomes

$$\ddot{x} + \eta^2 t^2 x = 0 . \tag{43}$$

The general solution of this equation is given by

$$x = t^{1/2}[A_1 J_{1/4}(\tfrac{1}{2}\eta t^2) + B_1 J_{-1/4}(\tfrac{1}{2}\eta t^2)] , \tag{44}$$

where A_1 and B_1 are constants, and $J_{\pm 1/4}$ is the ordinary Bessel function. If we assume that at $t = 0$, $x = x_0$, and $dx/dt = v_0$, Eq. (9) becomes

$$x = -\frac{v_0}{\rho} t^{1/2} J_{1/4}(\tfrac{1}{2}\eta t^2) + \frac{x_0}{\gamma} t^{1/2} K_{-1/4}(\tfrac{1}{2}\eta t^2) , \tag{45}$$

where

$$\gamma = \frac{1}{(\frac{1}{4}\eta)^{1/4}\Gamma(\frac{3}{4})} , \qquad \rho = \frac{(\frac{1}{4}\eta)^{1/4}}{\Gamma(\frac{5}{4})}$$

and Γ is the gamma function. Now with $\beta/\alpha = \lambda$ and $\alpha^{1/2} = A_0 = r_0 B_0$, we have

$$\eta = \omega_{om} \Omega r_0 \lambda^{1/2} .$$

If the particles start out at $x_0 = 0$, Eq. (9) reduces to

$$x = -\frac{v_0}{\rho} t^{1/2} J_{1/4}(\tfrac{1}{2}\eta t^2) .$$

The particle velocity is

$$v = \frac{dx}{dt} = \frac{x}{t} + \frac{v_0}{\rho} \eta t^{3/2} J_{5/4}(\tfrac{1}{2}\eta t^2) . \tag{47}$$

There will be turning points at the zeros of Eq. (47), i.e., whenever

$$t_c^{-1/2} J_{1/4}(\tfrac{1}{2}\eta t_c^2) = \eta t_c^{3/2} J_{5/4}(\tfrac{1}{2}\eta t_c^2) . \tag{48}$$

We consider the solution of Eq. (48) when the argument is small compared with unity. If only the first term in the series expansion of the Bessel function is retained the time t_c to reach the first turning point is

$$t_c = (\tfrac{5}{4})^{1/4} 2^{1/2} \eta^{-1/2} . \tag{49}$$

At this time the argument of the Bessel function is larger than one since

$$\tfrac{1}{2}\eta t_c^2 = (\tfrac{5}{4})^{1/2} = 1.12 .$$

Thus the first term does not represent the series very well. A somewhat better approximation is obtained if the first two terms of the series expansion are used. For this case t_c is given by

$$t_c = 2^{1/4} \eta^{-1/4} \tag{50}$$

and

$$\tfrac{1}{2}\eta t_c^2 = \frac{\sqrt{2}}{2} = 0.707 .$$

The particle displacement (x_m) at the first turning point is given by

$$x_m = -\frac{v_0}{\rho} t_c^{1/2} J_{1/4}(\tfrac{1}{2}\eta t_c^2) . \tag{51}$$

With t_c as given by Eq. (50), x_m becomes

$$x_m = -\frac{v_0}{\rho} 2^{1/8} \eta^{-1/4} J_{1/4}(\sqrt{2}/2) = -0.389 \frac{v_0}{\rho} \eta^{-1/4} . \tag{52}$$

The negative sign indicates that x_m lies radially inward from the equilibrium orbit. We must now examine the solutions of Eq. (46) for the conditions $x = 0$, $dx/dt = +v_0$ at $t = 0$. The turning point occurs at a positive value of x given by

$$x_m = +\frac{v_0}{\rho} 2^{1/8} \eta^{-1/4} J_{1/4}(\sqrt{2}/2) \tag{53}$$

which is the same as (x_m) for the case when $(dx/dt)_{t=0} = v_0$. This is as it should be since the effective potential is parabolic. The fractional energy well depth is given by

$$\delta = \frac{A_\theta^2(x_m) - A_0^2}{A_0^2} = \frac{A_0^2[1 + \lambda x_m^2] - A_0^2}{A_0^2}$$

or

$$\delta = \lambda x_m^2 . \tag{54}$$

We can now solve for v_0. Recalling that $\rho^2 = (\frac{1}{4}\eta)^{1/2}/[\Gamma(\frac{5}{4})]^2$ and $\eta = \omega_0 \Omega r_0 \lambda^{1/2}$, the limiting transverse velocity is

$$v_m = \frac{\delta^{1/4}(\omega_{om} \Omega r_0 x_m)^{1/2}}{\xi^{1/2}} , \tag{55}$$

where

$$\xi = 2^{5/4} [\Gamma(\tfrac{5}{4})]^2 [J_{1/4}(\sqrt{2}/2)]^2 \approx 0.5 .$$

The velocity v_m is the maximum transverse velocity a particle located on the equilibrium orbit can have at injection time ($t = 0$) and still be captured by the rising betatron field. For this calculation we have neglected the finite azimuthal velocity at $t = 0$, i.e., we have placed $p_\theta = 0$.

The transverse temperature T_m that corresponds to v_m, obtained via $\frac{1}{2}mv_m^2 = \frac{1}{2}kT_m$, has been calculated for the case $\delta = 8 \times 10^{-3}$, $x_m = 0.5$ cm, $r_0 = 4.8$ cm, and $\omega = 0.39 \times 10^6$, and is listed in Table I.

TABLE I

B_{om} (G)	T_m (eV)
500	0.88
1000	1.77
1500	2.64
2000	3.72

Amplitude of magnetic field at the orbit versus maximum transverse temperature of electrons that can be captured into stable orbits.

An extensive series of numerical calculations of the single particle trajectories of low energy electrons during the start-up phase of a betatron have been performed by Lukasik *et al.*[20] They have examined the condition necessary for capture into a stable betatron orbit; these have included the effect of the initial position of the electron, the depth of the vector potential well, the rate of rise and the strength of the betatron field, and the allowable initial velocity of the particle. The analytical model for the magnetic field configuration chosen for the computation is shown in Fig. 2.

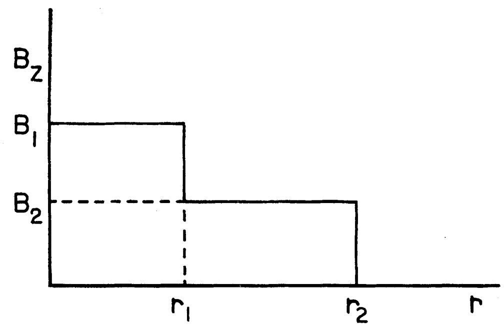

FIG. 2. Analytical model of magnetic field configuration used in numerical study of betatron capture calculations.

This field distribution leads to a vector potential that possesses the most important property for this study, i.e., a radial potential well. The B field is uniform ($n = 0$) so that there is no vertical stability of single particle orbits.

The results of the calculation show that the maximum transverse and azimuthal velocities that can be captured are nearly equal for particles starting in the center of the well. In a magnetic field shaped to provide vertical as well as radial focusing the maximum vertical velocity that can be captured will be approximately the same as the limiting radial and azimuthal velocities for capture. Setting these three velocities equal then permits an estimation of the fraction F of particles that can be captured from an isotropic Maxwellian velocity distribution. This fraction is just the fraction of particles in the distribution with speeds less than v_m. It is readily shown[20] for this model that

$$F = \operatorname{erf}\left(\frac{T_m}{2T}\right)^{1/2} - \frac{2}{\pi^{1/2}}\left(\frac{T_m}{2T}\right)^{1/2} \exp\left(\frac{-T_m}{2T}\right),$$

where $T_m = mv_m^2/k$ and T is the temperature of the electron gas. The fraction F has been calculated as a function of the particle temperature for a typical set of experimental parameters ($B_2 = 1000$ G, $\delta = 8 \times 10^{-3}$, $x_m = 0.5$ cm, $r_0 = 4.8$ cm, $\omega = 0.39 \times 10^6$) and is listed in Table II.

TABLE II

T (eV)	F (capture fraction)
0	1
1	0.38
3	0.11
5	0.06
7	0.04

Fraction of electrons that can be captured into stable orbits versus initial electron temperature.

Thus with a beam diameter of 1.86 cm, $Te \approx 5$ eV, $n_e \approx 10^{10}/\text{cm}^3$, and electron energy of 100 keV, the maximum runaway current is only ≈ 0.50 A.

The results of this section indicate that the electron temperature prior to the application of the betatron field is an important factor in determining the runaway current and can easily be responsible for the failure of $B_\theta = 0$ plasma betatrons in this regard.

4. BEAM LOSS BY GAS SCATTERING

The standard theoretical treatment of the scattering of charged particles in a circular accelerator is that of Blackman and Courant.[21] They arrive at an expression which gives the maximum allowable pressure in the acceleration chamber, p, in terms of a quantity η which relates to the fraction of particles lost and various parameters of the system, namely, the magnetic field index n, the vertical semi-aperture A, the orbit radius R, the kinetic energy at injection T_i, and the energy gain per revolution near injection (eV): the constant K refers to the nature of the residual gas and its ambient temperature:

$$p = Kn\left(\frac{A^2}{R^3}\right) T_i(\text{eV})\eta .$$

With p expressed in Torr, A and R in cm, T_i and (eV) in keV, then the constant $K = 303/\sum_i n_i Z_i^2 \ln(183 Z_i^{-1/3})$ where n_i is the number of atoms of atomic number Z_i present in one molecule of the gas, and it is assumed that the ambient gas temperature is of the order of 300 °K.

The Blackman–Courant derivation is based on the assumption that Rutherford scattering dominates by many orders of magnitude any other scattering or loss mechanism. This assumption is valid at the injection energies involved in the usual accelerators, but is clearly not valid for the plasma betatron, in which particles start out at essentially thermal energies.

A conservative upper bound to the loss in particles for the plasma betatron can however be established in the following manner: The energy interval between thermal and the maximum accelerated energy is divided into two regions, from thermal to an intermediate region such that the *B–C* assumption becomes valid, and from this intermediate energy on up to the maximum accelerated energy. In the latter region the *B–C* formula can be applied without difficulty, while in the low energy region the following approximation will be used. It will be assumed for the lower energy region that any particle which suffers even one collision will be lost, even though it is possible (and even most probable) that it merely suffers a shift in phase and amplitude of the betatron oscillation, which still allows the particle to remain well within the acceleration chamber walls. With this assumption, the attenuation of the beam current, I, with time will be exponential:

$$\frac{I(t)}{I(0)} = \exp\left(-\int_0^t N_0 \sigma v \, \mathrm{d}t\right),$$

$$= \exp\left(-\frac{mN_0}{e}\int_0^v \frac{\sigma(v)v}{E}\,\mathrm{d}v\right),$$

where σ is the total cross section, N_0 is the density of neutral gas molecules or atoms, and E is the electric field, which certainly may be considered constant during the early part of the acceleration cycle which is of concern here. At the highest energy reached in this early part of the cycle the motion is only slightly relativistic, making it easy to transform to the kinetic energy W as the variable of integration:

$$\frac{I(t)}{I(0)} \approx \left(-\frac{N_0}{eE}\int_0^W \sigma(W)\,\mathrm{d}W\right).$$

The average total cross section for electrons scattering from molecular hydrogen is less than 10^{-15} cm^2 for $0 \leqslant W < 100$ eV. For 100 eV $< W < 1{,}000$ eV, the dominant process is ionization which behaves roughly as $\sigma = -7.8\times10^{-20}\ W + 1.08\times10^{-16}$ for molecular hydrogen. For $W >$ 1,000 eV, this formula overestimates σ, which actually falls more rapidly, until in the neighborhood of $W \sim 10{,}000$ eV, σ is essentially given by the Rayleigh formula. Thus the scattering from 0 to 10,000 eV will be overestimated by taking $\sigma = 1\times10^{-15}$ cm^2 up to $W = 100$ eV, and the above linear form for 100 eV $\leqslant W \leqslant$ 10,000 eV. With a pressure of 10^{-4} Torr of hydrogen, it is then found that this overestimated loss is less than 30 per cent.

The region above 10 keV, where the *B–C* formula does apply, is found to contribute negligibly to the loss already calculated.

Even at the high beam currents forecast for the plasma betatron, scattering by ions will be negligible, so that Lawson's(22) calculation of beam scattering by a neutral plasma does not enter.

Thus it appears that scattering losses will not prevent the plasma betatron from operating as intended.

5. SUGGESTED FUTURE INVESTIGATIONS

The calculations of the preceding sections indicate that a beam should be formed in a plasma betatron providing (a) an equilibrium orbit exists, and (b) that there are sufficient charges present. The self magnetic field of an intense beam alters the shape of the betatron guide field causing the equilibrium orbit to shift to a smaller radius. This effect limits the maximum beam current in practical air core coils to roughly 400 A.

We note that this figure is obtained assuming the vacuum field satisfies the betatron 2:1 condition. Stable betatron orbits require that the *net* field satisfy the usual conditions. This suggests that beam currents in excess of 400 A may be obtained if the self field of the beam is used in conjunction with an external field to achieve stability. Obviously since the electrons become relativistic rather quickly this process cannot exist for the full duration of the acceleration cycle. As the electrons reach relativistic velocities the beam current ceases to change significantly and the relative effect of the self field diminishes. Since the vacuum field does not admit stable orbits the beam will be driven to the wall. This would be the nonequilibrium mode of operation mentioned earlier.

There is some experimental evidence that this mode of operation is feasible. Drees and Trinks[23] have noted that by balancing the conduction current against the induction magnetic field it is possible to obtain intense bursts of low energy electrons (2000 A, 50 keV). These results were obtained in a plasma betatron that uses a high frequency quadrupole electric field for plasma generation and stabilization rather than the B_θ field. However as the plasma density builds up shielding occurs reducing the effectiveness of the rf field. In the B_θ plasma betatron the charge motion to the walls is showed by the B_θ field so that even in the absence of a stable orbit some acceleration is still possible. Calculations of the energy and currents expected this way are in progress.

The experimental evidence gathered so far also does not show that the acceleration must be 'rapid' to overcome slowly growing beam plasma instabilities. In fact the negative-mass-instability, the only collective effect for which there appears to be some evidence, predicts instability at certain values of electron momentum. The energy corresponding to these values is in the several hundred keV range for modest plasma betatron parameters. This suggests that a conventional betatron magnet be used instead of air core providing of course that the electric field strength is large enough to yield copious runaway electrons. Operation at several hundred cycles per second would then substantially increase the X-ray or electron flux over that available from conventionally dulsed air core coils.

Finally we note that the present knowledge regarding the development of runaway electron beams in a thermal plasma and the interactions of runaway electrons with plasma is rather primitive; clearly much further study is needed.

REFERENCES

1. D. Keefe, *Particle Accelerators* **1,** 1 (1970), An extensive list of references on the Electron Ring Accelerator will also be found here.
2. G. I. Budker, in *Proc. CERN Symp. on High-Energy Accelerators and Pion Physics, Geneva*, 1956 (CERN Scientific Information Service, Geneva, Switzerland, 1956), p. 68.
3. J. G. Linhart, *Proc. 4th Intern. Conf. on Ionization Phenomena in Gases, Uppsala* (North-Holland Publishing Co., Amsterdam, 1960) p. 981.
4. K. C. Rogers, D. Finkelstein, L. Ferrari, D. Caulfield, I. Mansfield and G. Brucker, *Proc. Intern. Conf. on High Energy Accelerators and Instrumentation, CERN, Geneva*, 1959 (CERN Scientific Information Service, Geneva, Switzerland, 1959), p. 145.
5. P. Reynolds and H. M. Skarsgard, *J. Nuc. Energy* Part **C1,** 36 (1959).
6. J. Drees, Diplomarbeit, Bonn., July 1960.
7. L. T. Shepherd and H. M. Skarsgard, *Phys. Rev. Letters*, 1, **10,** 121 (1963).
8. J. Drees and W. Paul, *Z. Physik*, **180,** 340 (1964).
9. L. A. Ferrari, Ph.D. Thesis, Stevens Inst. of Tech. (1965). L. A. Ferrari and K. C. Rogers, *Physics of Fluids*, **10,** 1319 (1967).
10. L. A. Ferrari, K. C. Rogers and A. Jermakian, *Rev. Sci. Inst.*, **38,** 1697 (1967).
11. J. V. Gore and H. M. Skarsgard, *Rev. Sci. Inst.*, **36,** 1807 (1965).
12. G. Schmidt, *J. Nuc. Energy*, Part **C2,** 156 (1961).
13. L. A. Ferrari and M. S. Zucker, *Physics of Fluids*, **12,** 1312 (1969).
14. L. A. Ferrari, *Physics of Fluids*, **12,** 2203 (1969).
15. L. A. Ferrari and M. S. Zucker, Brookhaven National Laboratory Report BNL 13274 (Dec. 1968).
16. L. A. Ferrari, K. C. Rogers and R. W. Landau, *Physics of Fluids*, **11,** 691 (1968).
17. R. W. Landau, *Physics of Fluids*, **11,** 205 (1968).
18. G. Schmidt, *Physics of High Temperature Plasmas* (Academic Press, Inc., New York, N.Y. 1966) p. 50.
19. B. Lehnert, *Dynamics of Charged Particles* (North Holland Publishing Co., Amsterdam, 1964) p. 43.
20. S. J. Lukasik, K. C. Rogers, G. W. Zepko and G. J. Brucker, *Nucl. Instr. and Methods*, **24,** 365 (1963). S. J. Lukasik and K. C. Rogers, Stevens Institute of Technology Report SIT-P147, June 1965.
21. M. M. Blackman and E. D. Courant, *Phys. Rev.*, **74,** 140 (1948).
22. J. D. Lawson, *Jour. of Electronics and Control*, **3,** 587 (1957).
23. J. Drees and H. W. Trinks, *Zeitschrift fur Physik*, **200,** 410 (1967).

Received 16 October 1970