Date	J. D.	$-10^8\,\bar{x}_0$	$-10^8\,\bar{y}_0$	$-10^8\,\bar{z}_0$
1959 July 21	2436770.5	+ 92.42	—405.38	—177.13
31	780.5	+ 183.74	—375.81	—167.32
Aug. 10	790.5	+ 266.22	—328.12	—149.20
20	800.5	+ 335.39	—264.67	—123.63
30	810.5	+ 387.48	—188.77	— 91.89
Sept. 9	820.5	+ 419.65	—104.43	— 55.64
19	830.5	+ 430.23	— 16.19	— 16.82
29	840.5	+ 418.72	+ 71.19	+ 22.44
Oct. 9	850.5	+ 385.98	+ 153.00	+ 60.00
19	860.5	+ 334.05	+ 224.87	+ 93.79
29	870.5	+ 266.14	+ 283.07	+ 121.99
Nov. 8	880.5	+ 186.38	+ 324.72	+ 143.10
18	890.5	+ 99.52	+ 348.05	+ 156.10
28	900.5	+ 10.66	+ 352.42	+ 160.49
Dec. 8	910.5	— 75.17	+ 338.43	+ 156.30
18	920.5	—153.31	+ 307.78	+ 144.11
28	930.5	—219.84	+ 263.16	+ 125.01
1960 Jan. 7	940.5	—271.75	+ 207.96	+ 100.43
17	950.5	—307.21	+ 146.06	+ 72.08
27	960.5	—325.54	+ 81.45	+ 41.82
Feb. 6	970.5	—327.22	+ 17.98	+ 11.49
16	980.5	—313.74	— 40.92	— 17.21
26	2436990.5	—287.48	— 92.41	— 42.82
Mar. 7	2437000.5	—251.41	—134.47	— 64.21
17	010.5	—208.86	—165.89	— 80.63
27	020.5	—163.26	—186.39	— 91.74
Apr. 6	030.5	—117.86	—196.48	— 97.61
16	040.5	— 75.50	—197.44	— 98.68
26	050.5	— 38.44	—191.13	— 95.68
May 6	060.5	— 8.16	—179.80	— 89.64
16	070.5	+ 14.68	—165.90	— 81.69
26	080.5	+ 30.30	—151.81	— 73.01
June 5	090.5	+ 39.74	—139.64	— 64.70
15	100.5	+ 44.71	—130.99	— 57.69
25	110.5	+ 47.42	—126.80	— 52.64
July 5	120.5	+ 50.31	—127.25	— 49.87
15	130.5	+ 55.79	—131.69	— 49.34
25	140.5	+ 65.93	—138.74	— 50.63
Aug. 4	150.5	+ 82.20	—146.43	— 53.02
14	160.5	+ 105.31	—152.36	— 55.53
24	170.5	+ 135.04	—154.00	— 57.05
Sept. 3	180.5	+ 170.26	—148.95	— 56.48
13	190.5	+ 208.96	—135.20	— 52.78
23	200.5	+ 248.45	—111.34	— 45.18
Oct. 3	210.5	+ 285.54	— 76.73	— 33.15
13	220.5	+ 316.82	— 31.65	— 16.59
23	230.5	+ 338.90	+ 22.74	+ 4.25
Nov. 2	240.5	+ 348.75	+ 84.41	+ 28.71

Det Kongelige Danske Videnskabernes Selskab

Matematisk-fysiske Meddelelser, bind **28**, nr. 11

Dan. Mat. Fys. Medd. **28**, no. 11 (1954)

MESON PRODUCTION IN MESON-NUCLEON COLLISIONS

BY

B. d'ESPAGNAT

København

i kommission hos Ejnar Munksgaard

1954

CONTENTS

Printed in Denmark
Bianco Lunos Bogtrykkeri A-S

The production of one meson in meson-nucleon collisions is investigated, with the purpose of expressing the differential and total cross section for the process in terms of the known parameters of elastic meson-nucleon scattering. As little use as possible is made of the questionable approximations of field theory. Particular attention is paid to the reaction in which two positive mesons emerge.

A comparison is made with nuclear reactions. Although the two processes are very different, such a comparison still makes it likely that the l, J contribution to meson production (l, relative orbital, J, total angular momentum) is in some rough way proportional to the l, J contribution to elastic scattering. The same conclusion is reached independently through a field theoretical approach; it is therefore believed to be true, although neither of the two methods is entirely cogent by itself.

Formulae are derived for the total cross section for production and the angular distribution of the emergent mesons. The first is considered to give only an order of magnitude, and agrees roughly with what a purely statistical approach to the problem would give. The angular distribution result is further supported by an argument of invariance under rotation of the coordinate system and is therefore considered to have good reliability as long as the l, J expansion remains workable.

I. Introduction.

In the last few years, the elastic scattering of π mesons by nucleons has been considered as providing one of the most direct experimental approaches to the problem of π meson-nucleon coupling and as such has been, and is, extensively studied from the experimental side (ANDERSON, FERMI, LONG, MARTIN, and NAGLE, 1952; ANDERSON, FERMI, LONG, and NAGLE, 1952; ANDERSON, FERMI, NAGLE, and YODH, 1952; ANDERSON, FERMI, MARTIN, and NAGLE, 1953; CHEDESTER, ISAACS, SACHS, and STEINBERGER, 1951; FOWLER, FOWLER, SHUTT, THORNDYKE, and WHITTEMORE, 1952; ISAACS, SACHS, and STEINBERGER, 1952).

The first conclusion manifest from these experiments was that a perturbation treatment of the meson-nucleon scattering fails entirely to give agreement with the observations. Several kinds of approximations were then proposed, some based on strong coupling methods (BRUECKNER, 1952; BRUECKNER and

WATSON, 1952 WENTZEL, 1953), some on a Tamm-Dancoff approach (CHEW, 1953; BETHE and DYSON, 1953; FUBINI, 1953; LÉVY and MARSHAK, 1954), and others (DRELL and HENLEY, 1952). These theories showed on the whole a great improvement over the Born approximation (weak coupling), though the agreement with empirical data still remains partly qualitative. Also, from a theoretical point of view, none of these theories seems to be entirely justified, except perhaps when rather unnatural assumptions (e. g., cut-off momentum $\ll M$) are made at the start.

In the present paper, we are dealing with another effect which could provide useful information, viz. reactions of the type

$$\pi + \mathfrak{N} \to \mathfrak{N} + \pi + \pi, \tag{1}$$

where $\mathfrak{N}$ means nucleon. Such reactions can now be studied experimentally with the help of the big accelerating machines, and it might be useful to have some rough theoretical estimate of their relative frequencies and principal aspects*.

One could, in principle, make an attempt at developing an independent theory of such processes, following, for example, a Tamm-Dancoff approach. In view of the theoretical difficulties that make the Tamm-Dancoff approach not really consistent even in the elastic scattering problem, this would, however, be a rather ambitious programme. It might be more rewarding to look, first, for some formula connecting the cross sections for these effects to the phase shifts of elastic π meson scattering and, subsequently, to make use of the experimental values of these phase shifts.

In Chapter II, a general theory of processes such as (1) is given and a comparison is made with ordinary nuclear reactions. In Chapters III and IV, the special case

$$\pi^+ + P \to N + \pi^+ + \pi^+ \tag{2}$$

is examined, this time from the point of view of conventional meson field theory. In Chapter V, results concerning the total cross section and the angular distribution of the emitted mesons are given. It is shown, moreover, that the characteristic features of the latter are independent of any field theoretical approximation.

* Note added in proof: See also BETHE and NELKIN, Bull. Am. Phys. Soc. **29**, 30 (1954).

II. General theory.

1) Mathematical formalism.

To present a convenient formalism for both elastic meson scattering and mesoproduction of mesons, let us define $\psi^{(+)}_{(a)}$ and $\psi^{(1)}_{(a)}$ by means of the integral equations (LIPPMAN and SCHWINGER, 1950)

$$\psi^{(+)}_{(a)} = \varphi_{(a)} + \frac{1}{E_{(a)} + i\eta - H_0} H' \psi^{(+)}_{(a)}, \tag{3}$$

$$\psi^{(1)}_{(a)} = \varphi_{(a)} + P \frac{1}{E_{(a)} - H_0} H' \psi^{(1)}_{(a)}. \tag{4}$$

Here, H_0 is the free Hamiltonian, H' the interaction between meson and nucleon fields, $\varphi_{(a)}$ is an eigenvector of H_0 pertaining to state (a), P means Cauchy principal value, $\eta \to 0^+$.

Then, one can show (GOLDBERGER, 1951; BELINFANTE and MØLLER, 1954)

$$\psi^{(+)}_{(a)} = \psi^{(1)}_{(a)} - i\pi \sum_{(c)} \psi^{(1)}_{(c)} \delta(E_{(a)} - E_{(c)}) \boldsymbol{T}_{ca} \tag{5}$$

with the definition

$$\boldsymbol{T}_{ca} = (\varphi_{(c)}, H' \psi^{(+)}_{(a)}), \tag{6}$$

when $E_{(c)} = E_{(a)}$. $\boldsymbol{T}_{ca}$ is thus defined only on the energy shell. The matrix element $\boldsymbol{T}_{ca}$ is connected to the S or $R = S - 1$ matrix (MØLLER, 1945) by means of the relation

$$R_{ba} = -2\pi i \delta(E_{(a)} - E_{(b)}) \boldsymbol{T}_{ba}, \tag{7}$$

its most important property being that the rate of transition from state (a) to state (b) is given by (LIPPMAN and SCHWINGER, 1950)

$$w_{ba} = 2\pi \hbar^{-1} |\boldsymbol{T}_{ba}|^2 \varrho(E_{(a)}), \tag{8}$$

$\varrho(E_{(a)})$ being the density of states. It thus plays the same role in an exact theory as the lowest possible order perturbation matrix element in a perturbation theory.

As shown by GOLDBERGER (1951), instead of trying to compute $\boldsymbol{T}_{ba}$ directly, it is convenient to introduce the so-called reaction matrix

$$K_{ba} = (\varphi_{(b)},\ H'\ \psi^{(1)}_{(a)}) \tag{9}$$

which is thus defined both on and out of the energy shell. Of course, states (a) and (b) can differ not only in the momenta and spin of the particles involved, but also in the *number* of these particles. (4) gives

$$K_{ba} = H'_{ba} + P \sum_{(c)} \frac{H'_{bc} K_{ca}}{E_{(a)} - E_{(c)}}, \tag{10}$$

where the sum is to be taken over all possible states and (5) gives

$$\boldsymbol{T}_{ba} = \boldsymbol{K}_{ba} - i\pi \sum_{(c)} \boldsymbol{K}_{bc} \delta (E_{(a)} - E_{(c)})\, \boldsymbol{T}_{ca}, \tag{11}$$

where $\boldsymbol{K}_{ba}$ means just K_{ba} when $E_{(a)} = E_{(b)}$; the sum $\sum_{(c)}$ is here to be taken only over states (c) which have the same energy as the initial and final ones. (11) is usually known as the Heitler integral equation. It can be shown (GOLDBERGER, 1951) that $\boldsymbol{K}_{ba}$ is a Hermitian matrix (on the energy shell only).

The problem of solving eq. (10) is very complicated, involving self-energies, elimination of divergences, etc. We shall thus focus our attention, in this section, on equation (11), which is much simpler because of the energy shell condition, and we shall attempt to derive as many results as possible from this equation only.

2) Solution of Heitler equation for energies lying below the threshold for meson production.

Equation (11) is especially simple in this case because the intermediate states (c) can contain only one meson.

We work in the system where the total momentum of the colliding particles is zero. States (a) (or (b) or (c)) are then specified by the momentum $\boldsymbol{k}_a = k_a \boldsymbol{n}_a$ (or $\boldsymbol{k}_b$ or $\boldsymbol{k}_c$) of, say, the meson and the value m_a (or m_b or m_c) of the z-component of the nucleon spin.

We first look for the most general form for $\boldsymbol{K}_{ba}$, and to that effect we consider the eigenvalue problem

$$\sum_a \boldsymbol{K}_{ba} f^A_a = K'^A f^A_b. \tag{12}$$

$\boldsymbol{K}_{ba}$ is, of course, an invariant with respect to rotations of the coordinate system. Applying a rotation $\Re$ to both sides, we thus get

$$\sum_a \boldsymbol{K}_{ba} (\Re f^A_a) = K'^A (\Re f^A_b); \tag{13}$$

therefore, if the eigenvalue K'^A is p–fold degenerate, with the eigenfunctions $f^{A,1}_a \cdots f^{A,q}_a \cdots f^{A,p}_a$, we have

$$\Re f^{A,q}_a = \sum_{q'=1}^{p} c^A_{qq'} f^{A,q'}_a. \tag{14}$$

Applying successively two rotations, we get, by a well-known argument, that the matrix $c^A_{qq'}$ is just the irreducible representation of dimension p of the rotation group. Setting $p = 2J + 1$, $q = M + J + 1$, and using J instead of A as a label for the corresponding eigenvalue, we thus have

$$\Re f^{JM}_a = \sum_{M'} \mathfrak{D}^J_{MM'} f^{JM'}_a, \tag{15}$$

which means that f^{JM}_a can be written as

$$f^{JM}_a = \sum_{l=J-1/2}^{J+1/2} d_{Jl}(k_a)\, \mathcal{Y}^M_{Jl\,1/2}(\boldsymbol{n}_a, m_a) \tag{16}$$

with (notations of [BLATT and WEISSKOPF, 1952] for the Clebsch-Gordan coefficients)

$$\mathcal{Y}^M_{Jl\,1/2}(\boldsymbol{n}_a\, m_a) = \sum_{\nu\nu'} C_{l\,1/2}(J, M, \nu, \nu')\, Y^\nu_l(\boldsymbol{n}_a)\, Y^{\nu'}_{1/2}(m_a); \tag{17}$$

here, $Y^{\nu'}_{1/2}(m_a)$ is the amplitude for finding the nucleon of state (a) with a spin whose projection on the z-axis is ν'. m_a having been chosen as the projection of spin of the now fixed z-axis, one has simply

$$Y^{\nu'}_{l\,1/2}(m_a) = \delta_{\nu', m_a}. \tag{18}$$

Returning now to eq. (12), multiplying both sides by f_a^{A*} and summing over $A \equiv J, M$, one has, owing to the closure relation ($\boldsymbol{K}_{ba}$ Hermitian):

$$\boldsymbol{K}_{ba} = \sum_{lJ} K_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a, m_a). \tag{19}$$

This is thus the most general form for $\boldsymbol{K}_{ba}$. We write $K_{lJ}(k_a, k_a)$ since $k_b = k_a$ (energy shell). J can take both values $l + 1/2$ and $l - 1/2$. The reason that l is a good quantum number is the conservation of parity which forbids $\Delta l = \pm 1$.

So far, we have not introduced the charge coordinates. In fact, $\boldsymbol{K}_{ba}$ depends also on the values t_3, T_3 of the third component of the isotopic spin of nucleon and meson. If isotopic spin I is a good quantum number it is easy to see that $\boldsymbol{T}_{ba}$ and $\boldsymbol{K}_{ba}$ both can be expanded by

$$\left.\begin{aligned} &(\boldsymbol{K} \text{ or } \boldsymbol{T})_{t_{3b}, T_{3b}, t_{3a} T_{3a}} \\ &= \sum_{I=1/2}^{3/2} C_{l\,1/2}(I, N, t_{3b}, T_{3b})\, (\boldsymbol{K}^I_{ba} \text{ or } \boldsymbol{T}^I_{ba})\, C_{l\,1/2}(I, N, t_{3a}\, T_{3a}) \\ &\text{with } N = t_{3b} + T_{3b} = t_{3a} + T_{3a}; \end{aligned}\right\} \tag{20}$$

when carried into (11) these expansions give readily an equation for $\boldsymbol{T}^I_{ba}$ in terms of $\boldsymbol{K}^I_{ba}$, which is just equation (11) with $\boldsymbol{K}_{ba}$ replaced by $\boldsymbol{K}^I_{ba}$, and $\boldsymbol{T}_{ba}$ by $\boldsymbol{T}^I_{ba}$. The separation into total isotopic spin eigenvalues offers thus no difficulty. We can rewrite (19) as:

$$\boldsymbol{K}^I_{ba} = \sum_{lJ} K^I_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a, m_a). \tag{21}$$

Eq. (11) can now easily be solved by assuming for $\boldsymbol{T}^I_{ba}$ an expansion

$$\boldsymbol{T}^I_{ba} = \sum_{lJ} T^I_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a, m_a) \tag{22}$$

of the same form as (21). Carrying these expansions into (11), and making use of the orthonormality relations

$$\sum_{m_c} \int d\omega_c\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_c, m_c)\, \mathcal{Y}^{M'}_{J'l'\,1/2}(\boldsymbol{n}_c, m_c) = \delta_{J,J'}\, \delta_{M,M'}\, \delta_{l,l'}, \tag{23}$$

we get

$$T^I_{lJ}(k_a, k_a) = K^I_{lJ}(k_a, k_a) - i\pi\varrho\, K^I_{lJ}(k_a, k_a)\, T^I_{lJ}(k_a, k_a), \tag{24}$$

where $\varrho\, d\omega$ is the density in phase space. As is well known, relation (24) can be expressed parametrically by

$$\left.\begin{aligned} K^I_{lJ}(k_a, k_a) &= -(\pi\varrho)^{-1}\, tg\, \delta^I_{lJ}(k_a); \\ T^I_{lJ}(k_a, k_a) &= -(\pi\varrho)^{-1}\, e^{i\delta^I_{lJ}(k_a)} \sin \delta^I_{lJ}(k_a). \end{aligned}\right\} \tag{25}$$

Use of (8) and (25) readily shows that the δ are just the usual phase shifts.

Of course, one can express also the S-matrix in terms of the $\mathcal{Y}$-functions. Using (7) together with

$$\delta_{ba} = \sum_{lJM} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a, m_a), \tag{26}$$

we get

$$\left.\begin{aligned} S_{ba} = \delta_{ba} + R_{ba} &= \sum_{lJ} [1 - 2\, i\pi\varrho\, T^I_{lJ}(k_a, k_a)] \\ &\times \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a, m_a) \end{aligned}\right\} \tag{27}$$

$$= \sum_{lJ} S^I_{lJ}(k_a, k_a) \sum_{M} \mathcal{Y}^{M}_{Jl\,1/2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl\,1/2}(\boldsymbol{n}_a\, m_a) \tag{28}$$

with

$$S^I_{lJ} = 1 - 2\, i\pi\varrho\, T^I_{lJ} = \frac{1 - i\pi\varrho\, K^I_{lJ}}{1 + i\pi\varrho\, K^I_{lJ}} = e^{2i\delta^I_{lJ}}. \tag{29}$$

It may happen that $K^I_{lJ}(k_a, k_a)$ goes to infinity for a special value k_s of k_a; then, with the general notation

$$\varepsilon_a = (\mu^2 c^4 + k_a^2)^{1/2}, \tag{30}$$

one can set, in the neighbourhood of ε_s,

$$\frac{1}{K^I_{lJ}(k_a, k_a)} = \lambda(\varepsilon_a - \varepsilon_s) + \cdots; \tag{31}$$

λ being a constant, this gives

$$S^I_{lJ}(k_a, k_a) = 1 - 2i\frac{\pi\varrho\lambda^{-1}}{\varepsilon_a - \varepsilon_s + i\pi\varrho\lambda^{-1}}. \tag{32}$$

(32) corresponds to the Breit-Wigner one-level formula when there is only one channel open (elastic scattering). The resonance width is $\Gamma = 2\pi\varrho\lambda^{-1}$. Factors "$\omega$" accounting in the Breit-Wigner formula for the finite nuclear radii are, of course, not present in (32). Although one could introduce them formally by a slight modification of ε_s and Γ, such a formal modification of (29) and (32) seems to have no physical significance in our case and we therefore keep (32) as it is. Consequently, we have here no "potential" or "hard sphere" scattering. One can also interpret the resonance as being due to the formation of an excited nucleon of binding energy ε_s and mean lifetime $\hbar\,\Gamma^{-1}$ (MØLLER, 1946). It should perhaps be mentioned that one has a resonance in the usual sense only if δ crosses the value $\pi/2$ (K^I_{lJ} infinite).

3) Solution of Heitler equation for energies lying somewhat higher than the threshold for one-meson production.

(11) is now a set of two coupled integral equations. It will be convenient to use Greek indices for states containing two mesons and to reserve Latin indices for states containing one meson; then,

$$\boldsymbol{T}_{ba} = \boldsymbol{K}_{ba} - i\pi\sum_c \boldsymbol{K}_{bc}\,\delta(E_{(a)} - E_{(c)})\,\boldsymbol{T}_{ca} - i\pi\sum_\gamma \boldsymbol{K}_{b\gamma}\,\delta(E_{(a)} - E_{(\gamma)})\,\boldsymbol{T}_{\gamma a} \tag{33}$$

$$\boldsymbol{T}_{\beta a} = \boldsymbol{K}_{\beta a} - i\pi\sum_c \boldsymbol{K}_{\beta c}\,\delta(E_{(a)} - E_{(c)})\,\boldsymbol{T}_{ca} - i\pi\sum_\gamma \boldsymbol{K}_{\beta\gamma}\,\delta(E_{(a)} - E_{(\gamma)})\,\boldsymbol{T}_{\gamma a}. \tag{34}$$

Let us restrict our investigations to the domain of energies where $|\boldsymbol{T}_{\beta a}| \ll |\boldsymbol{T}_{ba}|$, i. e., where the elastic scattering is much more important than the meson production (apart from the difference in phase space densities). It seems probable that this domain

extends to rather high energies. The third terms in (33) and (34) are then comparatively very small. In the system of equations (33), (34), $\sum_c$ represents a summation which, owing to the energy shell condition, can extend only over the angles defining the direction of the intermediate meson; $\sum_\gamma$ is a summation both over the directions of the two mesons contained in state γ and over the magnitude of their momenta, this summation being restricted, however, to values satisfying the energy shell condition. In actual calculations, the δ-functions should of course be replaced by the densities in phase-space.

We first give an approximate solution of (33), (34) by neglecting the third term in the right-hand side of (34) and introducing (34) into (33), thus neglecting a term of order $|\boldsymbol{T}_{\gamma a}|^2$. One easily sees that it is possible to write

$$\boldsymbol{T}_{ba} = \mathfrak{K}_{ba} - i\pi\sum_c \mathfrak{K}_{bc}\,\delta(E_{(a)} - E_{(c)})\,\boldsymbol{T}_{ca} \tag{35}$$

with

$$\mathfrak{K}_{ba} = \boldsymbol{K}_{ba} - i\pi\sum_\gamma \boldsymbol{K}_{b\gamma}\,\delta(E_{(a)} - E_{(\gamma)})\,\boldsymbol{K}_{\gamma a}. \tag{36}$$

(36) is expected to be a good approximation as long as the imaginary part is smaller than the real part.

The case where $\boldsymbol{K}_{ba}$ becomes large ("resonance") deserves special consideration. A-priori we know nothing about the behaviour of $\boldsymbol{K}_{b\gamma}$, $\boldsymbol{K}_{\gamma a}$, so we cannot exclude the possibility that these quantities become large for the same energy as $\boldsymbol{K}_{ba}$. A simplified picture of this case is obtained by writing the $\boldsymbol{K}$'s as products:

$$\boldsymbol{K}_{ba} = f_b \cdot f_a; \qquad \boldsymbol{K}_{b\gamma} = f_b \cdot \varphi_\gamma, \tag{37}$$

where we assume that f_b, f_a, φ_γ all become very large in the neighbourhood of an energy ε_s. Then, (36) does not hold, but we can easily get a better approximation by iterating (34) any number of time before we introduce the result into (33). Thus, one gets (35) again, now with

$$\left.\begin{aligned}\mathfrak{K}_{ba} = \boldsymbol{K}_{ba} - i\pi\sum_\gamma \boldsymbol{K}_{b\gamma}\,\delta(\cdots)\,\boldsymbol{K}_{\gamma a} \\ + (-i\pi)^2\sum_{\gamma\varepsilon} \boldsymbol{K}_{b\gamma}\,\delta(\cdots)\,\boldsymbol{K}_{\gamma\varepsilon}\,\delta(\cdots)\,\boldsymbol{K}_{\varepsilon a} + \cdots.\end{aligned}\right\} \tag{38}$$

With (37), this gives

$$\mathfrak{K}_{ba} = \frac{\boldsymbol{K}_{ba}}{1 + i\pi \sum_\gamma \varphi_\gamma \delta(\cdots)\varphi_\gamma}. \tag{39}$$

It seems therefore a reasonable approximation to write, in a general way,

$$\mathfrak{K}_{ba} = \frac{\boldsymbol{K}_{ba}}{1 + i a} \tag{40}$$

instead of (36), a being a real function of the incident energy which we do not specify any further. (36) is a particular case of (40) when a is small.

(35) can now be treated in the same way as (11) (§§ 1 and 3) and we get:

$$T^I_{lJ}(k_a, k_a) = \frac{\mathfrak{K}^I_{lJ}(k_a, k_a)}{1 + i\pi\varrho\, \mathfrak{K}^I_{lJ}(k_a, k_a)} \tag{41}$$

with

$$\mathfrak{K}^I_{lJ}(k_a, k_a) = \frac{K^I_{lJ}(k_a, k_a)}{1 + i a}. \tag{42}$$

In fact, the assumption (37) is somewhat oversimplified for our purpose. A more accurate treatment can be made if we anticipate over a general result proved in § 4 below and which states that expansions of the form

$$\boldsymbol{K}_{ba} = \sum_{lJ} K_{lJ} \sum_M \mathcal{Y}^M_{Jl^{\prime}_2}(\boldsymbol{n}_b, m_b)\, \mathcal{Y}^{M*}_{Jl^{\prime}_2}(\boldsymbol{n}_a, m_a)$$

$$\boldsymbol{K}_{\gamma a} = \sum_{lJl'l''} K_{lJl'l''} \sum_M \Phi^M_{l'l''(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)\, \mathcal{Y}^{M*}_{Jl^{\prime}_2}(\boldsymbol{n}_a, m_a)$$

$$\boldsymbol{K}_{\beta\gamma} = \sum_{LJl_1l_2l'l''} K_{l_1l_2LJl'l''} \sum_M \Phi^M_{l_1l_2(L)J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_\beta)\, \Phi^{M*}_{l'l''(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)$$

are always possible, with the orthonormality property (use eq. (60) below)

$$\sum_{m\gamma} \int d\omega' \int d\omega'\, \Phi^M_{l_1l_2(L)J}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma)\, \Phi^{M'*}_{l'l''(L')J'}(\boldsymbol{n}', \boldsymbol{n}'', m_\gamma) = \delta_{l_1l'}\delta_{l_2l''}\delta_{L,L'}\delta_{J,J'}\delta_{M,M'}.$$

K_{lJ} is a function of k_a and it is convenient to write it in the form $K_{lJ} = [f_{lJ}(k_a)]^2$, $K_{lJl'l''}$ is a function of k_a, k', k''. We replace assumption (37) by

$$K_{lJl'l''} = f_{lJ}(k_a) \cdot \varphi_{lJl'l''}(k', k'') \tag{37a}$$

and, similarly,

$$K_{l_1l_2LJl'l'''} = \varphi_{LJl_1l_2}(k_1, k_2) \cdot \varphi_{LJl'l'''}(k', k''). \tag{37b}$$

Equations (33) and (34) then give, by substitution,

$$\left.\begin{aligned} T_{lJ}(k_a, k_a) = f_{lJ}(k_a) \Big[& f_{lJ}(k_a) - i\pi\varrho\, f_{lJ}(k_a)\, T_{lJ}(k_a, k_a) \\ & - i\pi \sum_{k'k''l'l''} \varphi_{lJl'l''}(k', k'')\, T_{lJl'l''}(k', k'', k_a)\, \varrho(k', k'') \Big] \end{aligned}\right\} \tag{33a}$$

$$T_{lJl_1l_2}(k_1, k_2, k_a) = \varphi_{lJl_1l_2}(k_1, k_2)\ \text{[same bracket]}; \tag{34a}$$

therefore,

$$T_{lJl_1l_2}(k_1, k_2, k_a) = \frac{\varphi_{lJl_1l_2}(k_1, k_2)}{f_{lJ}(k_a)}\, T_{lJ}(k_a, k_a). \tag{34b}$$

Substitution in the right-hand side of (33a) and solution with respect to $T_{lJ}(k_a, k_a)$ give again eq. (41), now however with

$$\mathfrak{K}^I_{lJ}(k_a, k_a) = \frac{K^I_{lJ}(k_a, k_a)}{1 + i a^I_{lJ}} \tag{42a}$$

$$a^I_{lJ} = \pi \sum_{k'k''l'l''} \varphi_{lJl'l''}(k', k'')\, \varphi_{lJl'l''}(k', k'')\, \varrho(k', k''). \tag{42b}$$

We get rid of the factors involving $\pi\varrho$ by defining new quantities:

$$L = -\pi\varrho K; \quad \mathfrak{L} = -\pi\varrho\mathfrak{K}; \quad R = -2\pi i\varrho T, \tag{43a, b, c}$$

(43c) being just a rewriting of (7). (41) is then

$$R^I_{lJ}(k_a, k_a) = 2i \frac{\mathfrak{L}^I_{lJ}(k_a, k_a)}{1 - i\mathfrak{L}^I_{lJ}(k_a, k_a)}, \tag{44}$$

and one has:

$$S^I_{lJ}(k_a, k_a) = 1 + R^I_{lJ}(k_a, k_a) = \frac{1 + i\,\mathfrak{L}^I_{lJ}(k_a, k_a)}{1 - i\,\mathfrak{L}^I_{lJ}(k_a, k_a)}. \tag{45}$$

Let us define

$$f^I_{lJ}(k_a) = \frac{1}{\mathfrak{L}^I_{lJ}(k_a, k_a)} = \frac{1}{L^I_{lJ}(k_a, k_a)} + i\,\frac{a^I_{lJ}}{L^I_{lJ}(k_a, k_a)}. \tag{46}$$

(45) can be written:

$$S^I_{lJ}(k_a, k_a) = \frac{f^I_{lJ}(k_a) + i}{f^I_{lJ}(k_a) - i}, \tag{47}$$

where the notations are somewhat similar to those used in nuclear reactions*. Here, as in the theory of nuclear reactions, the fact that $f^I_{lJ}(k_a)$ has now a (negative) imaginary part makes $|S^I_{lJ}|^2 < 1$, thus accounting for the possibility of processes ("reactions") different from elastic scattering. Here, these processes are the meson production. The total elastic cross section for l, J, I waves is

$$\sigma^I_{lJ\ \mathrm{el.}} \sim |S^I_{lJ} - 1|^2 = \frac{4}{[\Re f^I_{lJ}]^2 + [1 - \Im f^I_{lJ}]^2}, \tag{48}$$

the total reaction cross section being (unitarity of S-matrix):

$$\sigma^I_{lJ\ \mathrm{prod.}} \sim 1 - |S^I_{lJ}|^2 = \frac{-4\,\Im f^I_{lJ}}{[\Re f^I_{lJ}]^2 + [1 - \Im f^I_{lJ}]^2}. \tag{49}$$

(48), (49) are still rather formal because the ratio of the two cross sections depends (eq. (46)) on the quantity a^I_{lJ} which we have so far no means of evaluating.

In order to test the analogy with nuclear reactions it will be useful to study the case where a resonance should occur. In the theory of nuclear reactions, a resonance is said to take place for a certain energy $\varepsilon_s = (\mu^2 c^4 + k_s^2)^{1/2}$ if $\Re f^I_{lJ}(k_a) \to o$ when $k_a \to k_s$. Another important condition, however, should be fulfilled if

* Our f_{lJ} are, in fact, equivalent to $\frac{f_l - \Delta_l}{s_l}$ in the notations of BLATT and WEISSKOPF. Similarly, our ε_s are their ε'_s ("actual resonance energies"), see below.

we want the scattering and reaction cross section to obey the usual Breit-Wigner formulae for resonance processes. This condition is that $\Im f^I_{lJ}(k_a)$ should be a finite, smoothly varying function in the neighbourhood of k_s (see formulae below). Therefore, as an inspection of (46) shows, we shall have a real analogy with nuclear reaction processes if, and only if, a^I_{lJ} tends to infinity for the same energy as L^I_{lJ}. This in turn implies that the reaction matrix elements, such as $\boldsymbol{K}_{\gamma a}$, for production processes increase proportionally to elastic ones, $\boldsymbol{K}_{ba}$, when the latter approach a resonance. Only if this condition is fulfilled will the analogy with nuclear reactions be somewhat more than a purely superficial one.

Then, defining $\Gamma^I_{lJ\ \mathrm{el.}}(k_a)$, $\Gamma^I_{lJ\ \mathrm{prod.}}(k_a)$ by

$$\frac{1}{L^I_{lJ}(k_a)} = \Re f^I_{lJ}(k_a) = \frac{-2}{\Gamma^I_{lJ\ \mathrm{el.}}(k_a)}(\varepsilon_a - \varepsilon_s) + \cdots \tag{50}$$

$$\Im f^I_{lJ}(k_a) = -\frac{\Gamma^I_{lJ\ \mathrm{prod.}}(k_a)}{\Gamma^I_{lJ\ \mathrm{el.}}(k_a)}, \tag{51}$$

and replacing in (48), (49), one gets the usual Breit-Wigner formulae

$$\sigma^I_{lJ\ \mathrm{el.}} \sim \frac{[\Gamma^I_{lJ\ \mathrm{el.}}(k_a)]^2}{(\varepsilon_a - \varepsilon_s)^2 + \frac{1}{4}[\Gamma^I_{lJ\ \mathrm{el.}}(k_a) + \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)]^2}, \tag{52}$$

$$\sigma^I_{lJ\ \mathrm{prod.}} \sim \frac{\Gamma^I_{lJ\ \mathrm{el.}}(k_a) \cdot \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)}{(\varepsilon_a - \varepsilon_s)^2 + \frac{1}{4}[\Gamma^I_{lJ\ \mathrm{el.}}(k_a) + \Gamma^I_{lJ\ \mathrm{prod.}}(k_a)]^2}, \tag{53}$$

which, as is well known, can be written (BLATT and WEISSKOPF, chapter X) in the compact from

$$\sigma^I_{lJ\ \mathrm{el.}}(k_a) = \sigma^I_{lJ\,(c)}(k_a) \cdot \frac{\Gamma^I_{lJ\ \mathrm{el.}}}{\Gamma^I_{lJ}}, \tag{54 a}$$

$$\sigma^I_{lJ\ \mathrm{prod.}}(k_a) = \sigma^I_{lJ\,(c)}(k_a) \cdot \frac{\Gamma^I_{lJ\ \mathrm{prod.}}}{\Gamma^I_{lJ}}, \tag{54 b}$$

$$\sigma^I_{lJ\,(c)}(k_a) \propto \frac{\Gamma^I_{lJ}\cdot\Gamma^I_{lJ\,\mathrm{el.}}}{(\varepsilon_a-\varepsilon_s)^2+\frac{1}{4}(\Gamma^I_{lJ})^2}\,;\quad \Gamma^I_{lJ}=\Gamma^I_{lJ\,\mathrm{el.}}+\Gamma^I_{lJ\,\mathrm{prod.}}. \qquad (54\,\mathrm{c\,d})$$

The second factors in (54a), (54b) are the branching ratios for the two possible phenomena, elastic scattering and meson production, (54d) is the total width, (54c) may be interpreted as the cross section for the formation of an excited nucleon.

These analogies with resonance nuclear reactions are not quite general: one can see that they depend on the possibility of making the approximation (42a) for $\mathfrak{K}^I_{lJ}$ and that they would break down if, for example, the correct approximation had been $\mathfrak{K}^I_{lJ} = K^I_{lJ}(1 - i\,a^I_{lJ})$, *whatever energy dependence we should then choose for* a^I_{lJ}. They can only be partial analogies, however, for, quite apart from the different nature of the involved particles, there are important differences between the two phenomena: the main difference is that, in mesoproduction, three particles emerge at the same time as compared to only two in nuclear reactions. This particular feature, which makes very non-physical the consideration of the inverse processes, prevents us, for example, from giving to (54a, b, c, d) the simple interpretation that it proves Bohr's hypothesis at resonances: it is true that (54a, b, c, d) show that formation and decay of the excited nucleon take place as independent processes, but we must keep in mind that here we have only one way of forming the excited nucleon; to say that the excited nucleon does not remember how it was formed has therefore no great meaning.

As mentioned before, any analogy with nuclear reactions would break down if the matrix elements $\boldsymbol{K}_{\gamma a}$ for production of a meson happened not to go to infinity precisely for the same energies ε_s as the elastic matrix elements $\boldsymbol{K}_{ba}$ if, of course, these energies exist. Equations (52), (53) would still be valid in that case, but $\Gamma^I_{lJ\,\mathrm{prod.}}$ would go to zero at ε_s, thus making $\sigma^I_{lJ\,\mathrm{prod.}}(\varepsilon_s) \to 0$ instead of going through a maximum.

In this chapter, we have not made any attempt to calculate explicitly the matrix elements of the K-matrix, i. e. to solve eq. (10); solution of eq. (10) always requires some more or less drastic assumptions in order to restrict the number of significant graphs (intermediate states). Although such assumptions are

customary in field theory (Born approximation, Tamm-Dancoff method) one should be very cautious in using them and not neglect any possible check of their results. Here we make also an assumption, though of a quite different kind: we assume that an analogy with nuclear reactions does exist. This leads us to the conclusion that some kind of proportionality between the production and elastic matrix elements of the K-matrix should exist. In Chapter IV, we shall find the same kind of proportionality by means of some plausible, but not altogether cogent, arguments concerning which of the graphs are most significant. In the author's opinion, the plausibility of both assumptions is somewhat reinforced by the fact that, though very different, they lead to the same conclusion in that respect.

As regards the scattering cross section, it is clear that, for energies above the meson production threshold, the conventional analysis in phase shifts cannot be rigorously valid any more. We quote the following relevant formulae:

$$S^I_{lJ}(k_a, k_a) = e^{-2q^I_{lJ}(k_a)}\, e^{2i\delta^I_{lJ}(k_a)} \qquad (55\,\mathrm{a})$$

with

$$e^{-2q^I_{lJ}} = \frac{\cos\delta^{(+)}}{\cos\delta^{(-)}};\quad 2\,\delta^I_{lJ} = \delta^{(+)} + \delta^{(-)} \qquad (55\,\mathrm{b,c})$$

and

$$\mathrm{tg}\,\delta^{(\pm)} = -\pi\varrho\, K^I_{lJ}\left(1 \pm \frac{\sigma^I_{lJ\,\mathrm{prod.}}}{\sigma^I_{lJ\,\mathrm{el.}}}\right). \qquad (55\,\mathrm{d})$$

4. Remarks on the matrix element T for production.

We first neglect the third term in the right hand side not only of (34), but also of (33): the phaseshift analysis of the scattering process is then possible as in § 2 (this is also apparent from formulae (55) with σ^I_{lJ} prod. $\ll \sigma^I_{lJ}$ el.).

In order to perform the integration (34) we look, as in § 2, for the most general form for the production matrix element $\boldsymbol{K}_{\beta a}$. This can be done easily by a slight extension of the argument in § 2. It will be convenient here to change somewhat our notations: states $a, b, c \ldots$ with one meson only will be referred

to by suffixes N_a, a with $N_a = 1$, states $\alpha, \beta, \gamma \ldots$ which contain two mesons by suffixes N_a, a with $N_a = 2, \ldots$ We then ask for the solution of the eigenvalue problem:

$$\sum_{N_a} \sum_{a} \boldsymbol{K}_{N_b,b;N_a,a} f^{A}_{N_a,a} = K'^{A} f_{N_b,b}. \tag{56}$$

The same argument as in § 2 shows that $f^{A}_{N_a,a} = f^{J,M}_{N_a,a}$ must satisfy

$$\Re f^{J,M}_{N_a,a} = \sum_{M'} \mathfrak{D}^{J}_{MM'} f^{J,M'}_{N_a,a}. \tag{57}$$

For the particular case $N_a = 1$, (57) reduces to (15) so that $f^{J,M}_{1,a}$ has the form (16) with (17). For the case $N_b = 2$, one has

$$\left.\begin{aligned} f^{JM}_{2,b} = \sum_{l_1 l_2 L} g_{l_1 l_2}(k_1, k_2 : k_a) \sum_{m_1 m_2 \mu m_s} C_{L^{1}/_{2}}(J, M, \mu, m_s) \\ \times C_{l_1 l_2}(L, \mu, m_1, m_2)\, Y^{m_1}_{l_1}(\boldsymbol{n}_1)\, Y^{m_2}_{l_2}(\boldsymbol{n}_2)\, Y^{m_s}_{1/2}(m_b), \end{aligned}\right\} \tag{58}$$

where $\boldsymbol{k}_1 = k_1 \boldsymbol{n}_1$, $\boldsymbol{k}_2 = k_2 \boldsymbol{n}_2$ are the momenta of the emerging mesons in state b. Therefore,

$$\boldsymbol{K}_{2b;1a} = \sum_{l J l_1 l_2} \boldsymbol{K}_{l J l_1 l_2}(k_1, k_2; k_a) \sum_{M} \Phi^{M}_{l_1 l_2 (L) J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_b\, \mathfrak{Y}^{M*}_{J l^{1}/_{2}}(\boldsymbol{n}_a, m_a) \tag{59}$$

with

$$\Phi^{M}_{l_1 l_2 (L) J} = \sum_{m_1 m_2 \mu m_s} C_{L^{1}/_{2}}(J, M, \mu, m_s) C_{l_1 l_2}(L, \mu, m_1 m_2)\, Y^{m_1}_{l_1}(\boldsymbol{n}_1)\, Y^{m_2}_{l_2}(\boldsymbol{n}_2)\, Y^{m_s}_{1/2}(m_b) \tag{60}$$

The summation in (59) should, from general symmetry requirements, be restricted to $l_1 + l_2 - l$ odd. For two emergent π^+ this gives:

$$L = l + 1 \text{ if } J = l + \frac{1}{2};\ L = l - 1 \text{ if } J = l - \frac{1}{2}. \tag{61}$$

when one keeps only the terms of (59) whose radial and angular dependences are both symmetrical. The remaining terms vanish for $k_1 = k_2$, and will be neglected.

No problem arises here with regard to isotopic spin indices since we are primarily interested in reaction (2): $P + \pi^+$ is a

pure $I = {}^{3}/_{2}$ state so that in (33) and (34) only $\boldsymbol{T}^{3/2}_{ba}$, $\boldsymbol{K}^{3/2}_{ba}$ can enter. We introduce (59) and (22) into (34) and make use of the orthonormality relation of the $\mathfrak{Y}$'s. This gives (we note $\boldsymbol{T}^{++}$, the reaction where two π^+ come out)

$$\boldsymbol{T}^{++}_{2b1a} = \sum_{l J l_1 l_2} T^{++}_{l J l_1 l_2}(k_1, k_2; k_a) \sum_{M} \Phi^{M}_{l_1 l_2 (L) J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_b)\, \mathfrak{Y}^{M*}_{J l^{1}/_{2}}(\boldsymbol{n}_a, \boldsymbol{n}_a) \tag{62}$$

with

$$T^{++}_{l J l_1 l_2} = K^{++}_{l J l_1 l_2}(k_1, k_2; k_a) - i\pi\varrho\, K^{++}_{l J l_1 l_2}(k_1, k_2; k_a)\, T^{3/2}_{lJ}(k_a, k_a) \tag{64}$$

$$= K^{++}_{l J l_1 l_2}(k_1, k_2; k_a)\,[1 - i\pi\varrho\, T^{3/2}_{lJ}(k_a, k_a)], \tag{64a}$$

$$T^{++}_{l J l_1 l_2}(k_1, k_2; k_a) = \frac{K^{++}_{l J l_1 l_2}(k_1, k_2; k_a)}{K^{3/2}_{lJ}(k_a, k_a)}\, T^{3/2}_{lJ}(k_a, k_a) \tag{65}$$

(65) is here derived for $\boldsymbol{T}_{\gamma a}$ small; under the assumptions (37a, b) it is, however, valid for arbitrary $\boldsymbol{T}_{\gamma a}$ (cf. (34b)).

Relation (65) is a first step toward a solution of the problem stated in Chapter I, which is to express the matrix element of the T-matrix for production, and therefore the production cross section, in terms of the matrix elements of the T-matrix for scattering, i. e. in terms of the scattering phase shifts. Relation (65) is especially significant if we cling to the conclusion arrived at in § 3 from a comparison with nuclear reactions, that the K-matrix elements for production should show some proportionality with the K-matrix elements for scattering. We then expect that, if a particular $K^{3/2}_{lJ}$ becomes much larger than the others in some domain of energies, the corresponding K-matrix elements for production will also become large for these energies, thus making the ratio of the two K's in (65) roughly a constant. $T^{3/2}_{lJ}$ being large, we then expect $T^{++}_{l J l_1 l_2}$ to be large. Therefore it is probable that the main contribution to the production cross section will be given by the l, J waves in this domain of energies.

In that case, one can use (62) together with (60) to make some predictions on the angular distribution of the emitted mesons. Of course, in order to get definite results, it will be necessary to limit oneself to low values of l_1 and l_2: it is, how-

ever, interesting to note that such predictions are possible without any approximation of a field theoretical nature, e. g., limiting the number of graphs. We defer the detailed study of angular distribution to Chapter V, where some possible experimental tests of the theory will be given.

III. The reaction $\pi^+ + P \to N + \pi^+ + \pi^+$ (2). Field theoretical approach.

While the general considerations of Chapter II have the great advantage of not being dependent on any simplification of the kind used in field theory, i. e. on any reduction of the number of graphs, it is clear that they are too formal to give any answer to many important questions; for example, they can lead to no conclusion concerning the relative magnitude of the production cross section. We now attack the problem from a different angle and look for a way of approach more connected to the ordinary methods of elementary quantum field theory.

The method that first suggests itself is, of course, the Feynman-Dyson covariant approach. Let us therefore write the two Feynman graphs (A) and ($\tilde{A}$) (Fig. 1) that represent the reaction to lowest order in the coupling constant G.

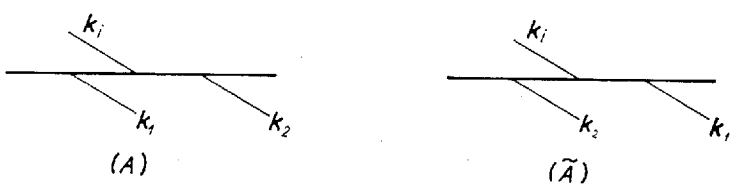

Fig. 1. The two covariant Feynman graphs of lowest order for $\pi^+ + P \to N + \pi^+ + \pi^+$.

These graphs are symmetric in $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, the momenta of the emerging mesons, so that it is sufficient to calculate only one of them. The calculation of the matrix elements pertaining to these graphs follows the well-known pattern and does not give rise to any difficulty but, of course, in view of the failure of lowest order calculations in the elastic scattering problem, one cannot have much confidence in the result.

Instead of using the covariant method one can just as well use the old non-covariant lowest order approach. This method gives, of course, the same end-result as the covariant one, but it is instructive to see how the matrix elements pertaining to each of the non-covariant graphs combine with each other.

The six non-covariant graphs corresponding to the covariant graph (A) are (α), (β), (γ), (δ), (ε), (ζ) (Fig. 2).

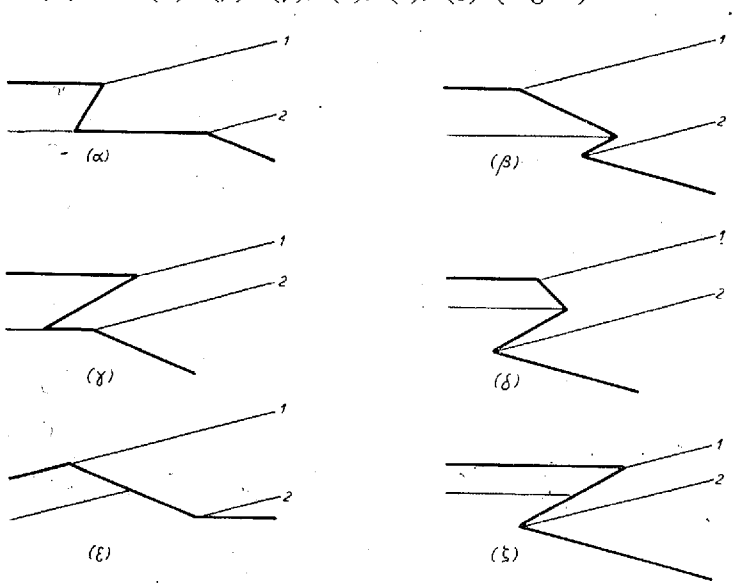

Fig. 2. The six non-covariant graphs corresponding to the covariant graph (A).

The six graphs corresponding to ($\tilde{A}$) are just the symmetrics of those with respect to interchange of k_1 and k_2 (1 and 2 on Fig. 2) and will be denoted by ($\tilde{\alpha}$), ($\tilde{\beta}$), ($\tilde{\gamma}$), ($\tilde{\delta}$), ($\tilde{\varepsilon}$), ($\tilde{\zeta}$).

Let us now choose a not too large incident energy, so that k_i, k_1, k_2, $\varepsilon_i = (\mu^2 c^4 + k_i^2)^{1/2}$, $\varepsilon_1 = (\mu^2 c^4 + k_1^2)^{1/2}$, $\varepsilon_2 = (\mu^2 c^4 + k_2^2)^{1/2}$ are all of order μ compared to M (μ = meson mass, M = nucleon mass); roughly speaking, this is satisfied when the incident energy of the π meson in laboratory system is significantly smaller than 1 Bev. Then a quite straightforward calculation of the matrix elements (we assume γ_5 coupling throughout) gives the following results:

(i) both graphs (α) and (β) (and of course ($\tilde{\alpha}$) and ($\tilde{\beta}$)) give contributions larger than the contributions of the various other

graphs by a factor of order M/μ; (ii) however, the contributions of (α) and $(\tilde{\beta})$ (and of $(\tilde{\alpha})$ and (β)) cancel to leading order in M/μ.

As a consequence of this latter fact, all 12 graphs contribute significantly (in spite of (i)) to the total matrix element of lowest order. Of course, this cancellation occurs so-to-speak automatically in the covariant method.

As mentioned before, there are reasons for not believing in the lowest third order calculations; therefore the question arises: how can we hope to improve them?

To get a plausible answer let us focus our attention on graphs (α) and $(\tilde{\beta})$. Graph (α) can be described by saying that it represents the main second order contribution of a scattering of the incident meson by a proton *followed* by the emission of meson "2" (the intermediary state being, of course, virtual). In the same way, graph $(\tilde{\beta})$ can be said to represent a scattering of the incident meson by the final neutron which was *preceded* by the emission of meson "2". Now, in graphs (α) and $(\tilde{\beta})$ the scattering part is treated to lowest order in the coupling constant G, but we can more generally consider the subsets of graphs, which we shall denote by (α_p) and $(\tilde{\beta}_p)$, where the scattering parts are treated exactly to, say, 2 p^{th} order*; the sum over p of all (α_p) and $(\tilde{\beta}_p)$ we call $(\underline{\alpha})$ and $(\underline{\tilde{\beta}})$, respectively (Fig. 3).

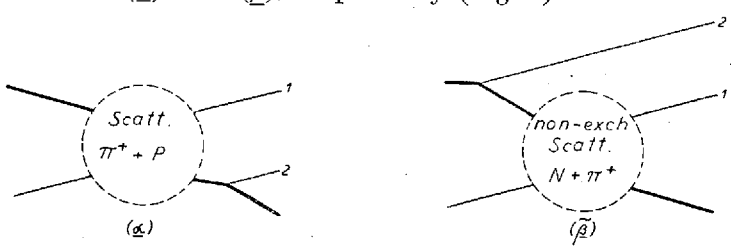

Fig. 3. The "improved" graphs. (α) and (β) are sums over all the graphs one would get by drawing the "scattering parts" (hidden by circles on the figure) to any order.

For $p = 1$, what has been said before shows that the matrix elements of (α_p), $(\tilde{\alpha}_p)$, (β_p), $(\tilde{\beta}_p)$, taken separately, are larger than all other matrix elements of lowest order in G. Our hypothesis will be that this is still true more generally for $(\underline{\alpha})$, $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$, $(\underline{\tilde{\beta}})$

* Note that some of the graphs entering in (α_p) may be identical with some entering in (β_p), as would be the case for (ε); then they should be taken only once. This, however, will not modify the general argument.

compared separately to the sum of all other possible graphs. This hypothesis, which constitutes the whole of our approximation, is of course largely arbitrary, as is the case in all approximations where one selects particular sets of a finite or infinite number of graphs and neglects others. It seems, however, a less drastic simplification than that introduced, for example, by the Tamm-Dancoff method, because the number of simultaneous meson lines in the scattering parts is here in no way limited. At the same time the condition is preserved that the set of graphs which is kept includes the largest graphs of lowest order in G, a condition which it is, somehow, desirable to keep.

This hypothesis is not sufficient in itself to allow for neglecting other graphs than $(\underline{\alpha})$, $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$ and $(\underline{\tilde{\beta}})$. The question which now arises is: does a cancellation of the leading terms in set $(\underline{\alpha})$ with the leading terms in set $(\underline{\tilde{\beta}})$ still occur here as it occurred in lowest order?

To answer this question let us call $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_1)$ the K-matrix element for the scattering part of graph $(\underline{\alpha})$ (spin suffixes being omitted). The suffixes $++$ are intended to recall the fact that it is necessarily a $\pi^+ \to \pi^+$ scattering. By analogy with (19) (Chapter II) one may write

$$K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i) = K^{3/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \sum_{lJ} K^{3/2}_{lJ}(k_1, k_i) \sum_{M} \mathcal{Y}^{M}_{Jl^{1/2}}(\boldsymbol{n}_1, m') \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_i, m_i). \quad (66)$$

The matrix element of the scattering part of graph $(\underline{\tilde{\beta}})$ is not taken exactly in the c. m. system; however, its main properties will be the same as if it were and, to alleviate notations, we call it $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$. The suffixes $--$ recall the fact that this matrix element pertains also to the experimentally studied $P + \pi^-$ non-exchange scattering. Expansion (20) in total isotopic spin eigenvalues, with use of the appropriate Clebsch-Gordan coefficients, gives

$$K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \frac{1}{3}\left[K^{3/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) + 2\, K^{1/2}(\boldsymbol{k}_1, \boldsymbol{k}_i)\right] \quad (67)$$

with

$$K^{1/2}(\boldsymbol{k}_1, \boldsymbol{k}_i) = \sum_{lJ} K^{1/2}_{lJ}(k_1, k_i) \sum_{M} \mathcal{Y}^{M}_{Jl^{1/2}}(\boldsymbol{n}_1, m_l) \mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_i, m'). \quad (68)$$

Now, the quantities $K^I_{lJ}(k_1, k_i)$ appearing in (66) are not taken on the energy shell ($k_1 \neq k_i$). In order to compare them with their values $K^I_{lJ}(k_i, k_i)$ on the energy shell, we must make use of a very rough and qualitative result of the Tamm-Dancoff method applied to the scattering K-matrix (see Appendix I for a formulation of this method). This will be the only place where we make use of a result derived by the Tamm-Dancoff method and, even if the Tamm-Dancoff method is not quantitatively reliable for scattering, it is felt that the qualitative and rough result of it used here has a high degree of probability. Let us call $B^I_{lJ}(k_1, k_i)$ the lowest order approximation of $K^I_{lJ}(k_1, k_i)$ in an expansion in the coupling constant: i. e. the l, J, I coefficient in an expansion, similar to (66), of $B(\boldsymbol{k}_1, \boldsymbol{k}_i)$, $B(\boldsymbol{k}_1, \boldsymbol{k}_i)$ being the Born approximation matrix element. The result just referred to is that the ratios

$$y^I_{lJ} = \frac{K^I_{lJ}(k_1, k_i)}{B^I_{lJ}(k_1, k_i)} \tag{69}$$

should not be extremely different off the energy shell from what they are on the energy shell ($k_1 = k_i$). This means that

$$K^I_{lJ}(k_1, k_i) \approx K^I_{lJ}(k_i, k_i) \cdot F_{lJ}(k_1, k_i) \text{ with } F_{lJ}(k_1, k_i) = \frac{B^I_{lJ}(k_1, k_i)}{B^I_{lJ}(k_i, k_i)} \tag{70}$$

as the dependence of $B^I_{lJ}(k_1, k_i)$ on k_1 is independent of I. $F_{lJ}(k_1, k_i)$ is the same for $I = {}^3/_2$ and $I = {}^1/_2$. Therefore,

$$\frac{K^{3/2}_{lJ}(k_1, k_i)}{K^{1/2}_{lJ}(k_1, k_i)} \approx \frac{K^{3/2}_{lJ}(k_i, k_i)}{K^{1/2}_{lJ}(k_i, k_i)}. \tag{71}$$

(71) means that in order to compare $K^{3/2}_{lJ}$ and $K^{1/2}_{lJ}$ we can take their values on the energy shell, which are experimentally known through the corresponding phase shifts δ (eq. (25)).

In order to compare $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ and $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ we carry (66) and (68) into (67), thus obtaining an expansion of $K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$ of the same form as (66), $K^{3/2}_{lJ}$ being replaced by $\frac{1}{3}(K^{3/2}_{lJ} + 2K^{1/2}_{lJ})$. Now, for energies studied so far it happens

(Anderson, Fermi, Martin, and Nagle, 1953) that all $\delta^{1/2}_{lJ}$ are of opposite sign to the corresponding $\delta^{3/2}_{lJ}$ and of such relative values as to make

$$\frac{1}{3}(K^{3/2}_{lJ} + 2K^{1/2}_{lJ}) \ll K^{3/2}_{lJ}. \tag{72}$$

Therefore experimental evidence leads us to the conclusion that $K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_i) \gg K^{--}(\boldsymbol{k}_1, \boldsymbol{k}_i)$, at least up to k_i values corresponding to energies of 135 Mev. Although these energies are still somewhat below the threshold for meson production (176 Mev), it seems rather likely that the extrapolation to this energy and beyond is correct. Also, detailed inspection of the phase shift values given by Anderson et. al. shows that $y^I_{lJ}(k_1, k_i)$ given by (69) should be very different indeed from its value on the energy shell in order to invalidate the present conclusion.

We conclude that the cancellation between the $(\underline{\alpha})$ and $(\underline{\tilde{\beta}})$ terms which appeared when we kept only the lowest order in G may very likely be an accidental one, and a particular feature of the lowest order approximation; it corresponds to the fact that the cross sections for $P + \pi^+$ and for $P + \pi^-$ non-exchange scattering are equal in the lowest order approximation, whereas they are quite different in reality. We conclude moreover that, the hypothesis having been made that (α), $(\underline{\tilde{\alpha}})$, $(\underline{\beta})$, $(\underline{\tilde{\beta}})$ are separately more important than all others, we are then justified in finally keeping only the $(\underline{\alpha})$ and $(\underline{\tilde{\alpha}})$ terms.

IV. The reaction $\pi^+ + P \to N + \pi^+ + \pi^+$ (2). Matrix elements and cross sections.

1. Calculation of the K-matrix element.

According to formula (10) of Chapter II the (exact) K-matrix element for the reaction is

$$\left.\begin{aligned} K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) \\ &+ \sum_{m_c} \frac{H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_1, m_c) K^{3/2}(\boldsymbol{k}_1, m_c; \boldsymbol{k}_i, m_i)}{E - E_1 - \varepsilon_1} \\ &+ \text{sym. terms } 1 \rightleftarrows 2 + \text{other terms.} \end{aligned}\right\} \tag{73}$$

Here, $E_1 = (M^2c^4 + k_1^2)^{1/2}$; $\varepsilon_1 = (\mu^2 c^4 + k_1^2)^{1/2}$; $E = E_{(i)} = E_i + \varepsilon_i$
$= (M^2c^4 + k_i^2)^{1/2} + (\mu^2 c^4 + k_i^2)^{1/2}$.

The first term in (73) is zero on the energy shell, the second term is represented by the set of graphs labelled $(\underline{\alpha})$ in the previous chapter, the third term by its symmetric $(\underline{\tilde{\alpha}})$ in $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. The "other terms" correspond to the graphs which we neglect according to Chapter III.

Taking as interaction Hamiltonian

$$H' = iG\,\bar{\psi}\gamma_5\tau_\alpha\psi\varphi_\alpha, \tag{74}$$

and writing $\varepsilon_2 = (\mu^2 c^4 + k_2^2)^{1/2}$; $E_{12} = [M^2c^4 + (\boldsymbol{k}_1 + \boldsymbol{k}_2)^2]^{1/2}$, (75)

we have*

$$\left.\begin{aligned} &H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_1 m_c) = \\ &\frac{iG\sqrt{2}}{V^{1/2}}\,\frac{\hbar}{(2\varepsilon_2)^{1/2}}\left(\frac{M+E_{12}}{2E_{12}}\right)^{1/2}\left(\frac{M+E_1}{2E_1}\right)^{1/2}\left(u_f, \sigma\left[\frac{\boldsymbol{k}_1+\boldsymbol{k}_2}{M+E_{12}} - \frac{\boldsymbol{k}_1}{M+E_1}\right]u_c\right), \end{aligned}\right\} \tag{76}$$

where u_f and u_c are the normalized spin wave functions of the nucleon in final and intermediate states, E_{12} is the final nucleon energy, V is as usual the volume of the normalizing box. For k_1 and $k_2 \ll M$, (76) reduces to

$$H'(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_1, m_c) = \frac{iG\hbar}{V^{1/2}}\,\frac{1}{2M}\,\frac{k_2}{\varepsilon_2^{1/2}}\,(u_f, \sigma\,\boldsymbol{n}_2\,u_c), \tag{77}$$

while, to the same approximation, the denominator in (73) reduces to

$$E - E_1 - \varepsilon_1 = E_i + \varepsilon_i - E_1 - \varepsilon_1 \approx \varepsilon_2. \tag{78}$$

For $K^{3/2}(\boldsymbol{k}_1, m_c;\ \boldsymbol{k}_i\, m_i)$ we have the expansion (66). Therefore

$$\left.\begin{aligned} &K^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f;\ \boldsymbol{k}_i, m_i) = \frac{iG\hbar}{V^{1/2}}\,\frac{1}{2M} \\ &\times\left\{\frac{k_2}{\varepsilon_2^{3/2}}\left[\sum_{m_c}(u_f, \sigma\,\boldsymbol{n}_2\,u_c)\sum_{lJ}K^{3/2}_{lJ}(k_1, k_i)\sum_{M}\mathcal{Y}^{M}_{J l^{1/2}}(\boldsymbol{n}_1, m_c)\,\mathcal{Y}^{M*}_{J l^{1/2}}(\boldsymbol{n}_i, m_i)\right]\right. \\ &\left. + \text{sym. term } 1 \rightleftarrows 2\right\}. \end{aligned}\right\} \tag{79}$$

* We take units such that $c = 1$ in the following formulae.

2. From the K-matrix to the T-matrix.

The transition from the K-matrix to the T-matrix follows the pattern of Chapter II, § 4. With the same assumptions as in this paragraph ($|\boldsymbol{T}_{\beta\alpha}|$ small), (34) gives

$$\left.\begin{aligned} &\boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) = \boldsymbol{K}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) \\ &- i\pi\sum_{m_c}\int d\omega_c\,\boldsymbol{K}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_c, m_c)\,\boldsymbol{T}^{3/2}(\boldsymbol{k}_c, m_c; \boldsymbol{k}_i, m_i)\,\varrho, \end{aligned}\right\} \tag{80}$$

where the density of the one-meson states in phase space is $\varrho\, d\omega$.

Expression (80), where the summation occurs only over the angles and the spin, is easily handled by using expansions (22) and (79), together with the orthonormality relation

$$\sum_{m_c}\int d\omega_c\,\mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_c, m_c)\,\mathcal{Y}^{M'}_{J'l'^{1/2}}(\boldsymbol{n}_c, m_c) = \delta_{J,J'}\,\delta_{l,l'}\,\delta_{M,M'} \tag{81}$$

which follows directly from the definition (17) of $\mathcal{Y}$. The result is

$$\left.\begin{aligned} &\boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) = \frac{iG\hbar}{V^{1/2}}\,\frac{1}{2M} \\ &\times\left\{\frac{k_2}{\varepsilon_2^{3/2}}\left[\sum_{m_c}(u_f, \sigma\,\boldsymbol{n}_2\,u_c)\sum_{lJ}\mathfrak{X}^{3/2}_{lJ}(k_1, k_i)\sum_{M}\mathcal{Y}^{M}_{Jl^{1/2}}(\boldsymbol{n}_1, m_c)\,\mathcal{Y}^{M*}_{Jl^{1/2}}(\boldsymbol{n}_i, m_i)\right]\right. \\ &\left. + \text{Sym. term } 1 \rightleftarrows 2\right\} \end{aligned}\right\} \tag{82}$$

with

$$\mathfrak{X}^{(I)}_{lJ}(k_1, k_i) = [1 - i\pi\varrho\,T^I_{lJ}(k_i, k_i)]\,K^I_{lJ}(k_i, k_i) \tag{83a}$$

$$= \frac{K^I_{lJ}(k_1, k_i)}{K^I_{lJ}(k_i, k_i)}\,T^I_{lJ}(k_i, k_i) \tag{83b}$$

by virtue of (24). (82) and (83b) give the matrix element of the T-matrix (which is what we need) for the reaction in terms of the phase shifts for elastic scattering (via T^I_{lJ} and (25)) which are known from experiment, and also of the ratios $K^I_{lJ}(k_1, k_i)/K^I_{lJ}(k_i, k_i)$ which represent the change in the K-matrix element

for scattering when we proceed out of the energy shell*. These ratios are not experimentally measurable quantities. To get an estimate of their values one can approximate them by replacing the K's by their values in lowest order in G, i. e. by the B's (cf. Chapter III, eq. (70) and also Appendix I). Thus one gets

$$\frac{K^I_{0^1/_2}(k_1,k_i)}{K^I_{0^1/_2}(k_i,k_i)} \approx \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2}; \quad \frac{K^I_{1J}(k_1,k_i)}{K^I_{1J}(k_i,k_i)} \approx \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{3/2} \frac{k_1}{k_i}. \tag{84}$$

3. Expression for the differential cross section in c. m. system.

The differential cross section for emission of one positive meson in the solid angle $d\omega_1$ with energy ε_1 and one positive meson in the solid angle $d\omega_2$ is, from (8),

$$d\sigma = \left(\frac{2\pi V}{\hbar v_i}\right) \cdot \left(\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i)\right|^2\right) \cdot \varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1, \tag{85}$$

where v_i is the relative velocity and $\varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1$ is the density of the two-meson states in phase space. The calculation of the second factor in (85) is most easily done by first writing (82) in a somewhat different form which makes more apparent the invariance of (86) with respect to rotations of the coordinate system. In (82) we note the factors

$$X = \sum_{lJ} \mathfrak{X}^{(3/2)}_{lJ}(k_1,k_i) \sum_M \mathcal{Y}^M_{Jl^1/_2}(\boldsymbol{n}_1, m_c)\, \mathcal{Y}^{M*}_{Jl^1/_2} \boldsymbol{n}_i, m_i) \tag{86}$$

which are expansions of the same kind as (19), (22) or (66). It is shown in detail in Appendix II that such expressions can always be put in the form

$$\left.\begin{aligned} X = \sum_{lm} [U_l(k_1,k_i)\,(u_c, u_i) \\ + V_l(k_1,k_i)\,(u_c, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)\,u_i)]\; Y^m_l(\boldsymbol{n}_1)\; Y^{m*}_l(\boldsymbol{n}_i) \end{aligned}\right\} \tag{87}$$

with

$$U_l + V_{l+1} = \mathfrak{X}^{(3/2)}_{l,l+1/2}; \quad U_l + V_{l-1} = \mathfrak{X}^{(3/2)}_{l,l-1/2}. \tag{88}$$

* Cf. also *B.* d'Espagnat (1953).

It should be noted that (88) does not determine uniquely the U_l and V_l if the X's are known: we must add the requirement that U_l and V_l decrease with increasing l in the same way as do the X's themselves. One then gets the general formulae

$$\left.\begin{aligned} U_l &= \sum_{q=0}^{\infty} (\mathfrak{X}^{(3/2)}_{l+2q,\,l+2q+1/2} - \mathfrak{X}^{(3/2)}_{l+2q+2,\,l+2q+2-1/2}), \\ V_l &= \sum_{q=0}^{\infty} (\mathfrak{X}^{(3/2)}_{l+2q+1,\,l+2q+1-1/2} - \mathfrak{X}^{(3/2)}_{l+2q+1,\,l+2q+1+1/2}). \end{aligned}\right\} \tag{89}$$

Using (87) instead of (86) for X, the calculation of the second factor in (85) becomes quite trivial. Here, however, we shall give the explicit result only for the case where phase shifts of $l \geqslant 2$ are negligible compared to S and P phase shifts. This seems to be experimentally correct up to 135 Mev incident energy and we assume that, for the higher energies considered here, it is still approximately true. If this were contradicted by further experiments it would be easy to calculate (85) keeping $l = 2$.

The summation over m_c in (82) gives, with (87):

$$\left.\begin{aligned} \boldsymbol{T}^{++} = \frac{iG\hbar}{V^{1/2}} \frac{1}{2M} \Bigg\{ \frac{k_2}{\varepsilon_2^{3/2}} \sum_{lm} \Big[U_l(k_1,k_i)\,(u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)\,u_i) \\ + V_l(k_1,k_i)\,(u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)\,u_i)\Big] Y^m_l(\boldsymbol{n}_1)\, Y^{m*}_l(\boldsymbol{n}_i) \\ + \text{Sym. term } 1 \rightleftarrows 2 \Bigg\}, \end{aligned}\right\} \tag{90}$$

while, on our assumption, (89) reduce to

$$U_0 = \mathfrak{X}^{(3/2)}_{0^1/_2}; \quad U_1 = \mathfrak{X}^{(3/2)}_{1^3/_2}; \quad V_0 = \mathfrak{X}^{(3/2)}_{1^1/_2} - \mathfrak{X}^{(3/2)}_{1^3/_2}, \tag{91}$$

other U_l and V_l being zero.

Now, by using (83b), (84) and (91), one sees that the coefficients of $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)$ and of $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)$, which appear in the symmetric term, are equal; therefore, using the identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1) + (\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2) = 2\,(\boldsymbol{n}_1\cdot\boldsymbol{n}_2), \tag{92}$$

(90) reduces to

$$\left.\begin{aligned} \boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i, m_i) &= [\lambda_2 + \mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)](\boldsymbol{\sigma} \cdot \boldsymbol{n}_2) \\ &+ [\lambda_1 + \mu(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)](\boldsymbol{\sigma} \cdot \boldsymbol{n}_1) + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)(\boldsymbol{\sigma} \cdot \boldsymbol{n}_i) \end{aligned}\right\} \quad (93)$$

with

$$\lambda_2 = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{k_2}{\varepsilon_2^{3/2}} \left(\frac{\varepsilon_i}{\varepsilon_1}\right)^{1/2} \frac{1}{2\pi\varrho} e_3, \quad (94\,a)$$

$$\lambda_1 = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{k_1}{\varepsilon_1^{3/2}} \left(\frac{\varepsilon_i}{\varepsilon_2}\right)^{1/2} \frac{1}{2\pi\varrho} e_3, \quad (94\,b)$$

$$\mu = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{3 k_1 k_2}{(\varepsilon_1 \varepsilon_2)^{3/2}} \frac{\varepsilon_i^{3/2}}{k_i} \frac{1}{2\pi\varrho} e_{33}, \quad (94\,c)$$

$$\nu = -\frac{G\hbar}{4\pi V^{1/2}} \frac{1}{2M} \frac{2 k_1 k_2}{(\varepsilon_1 \varepsilon_2)^{3/2}} \frac{\varepsilon_i^{3/2}}{k_i} \frac{1}{2\pi\varrho} (e_{31} - e_{33}). \quad (94\,d)$$

In (94 a, b, c, d) the notations (ANDERSON, FERMI, MARTIN, and NAGLE, 1953):

$$\left.\begin{aligned} e_3 &= e^{2i\delta^{3/2}_{01/2}} - 1 \equiv -2\pi i \varrho\, T^{3/2}_{01/2}; \\ e_{3p} &= e^{2i\delta^{3/2}_{1p/2}} - 1 \equiv -2i\pi\varrho\, T^{3/2}_{1p/2};\ p = 1; 3 \end{aligned}\right\} \quad (95)$$

have been used.

Then

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} |\boldsymbol{T}^{++}(\boldsymbol{k}_1, \boldsymbol{k}_2, m_f; \boldsymbol{k}_i\, m_i)|^2 \\ &= |\lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \mu[(\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)\boldsymbol{n}_1 + (\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)\boldsymbol{n}_2] + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\boldsymbol{n}_i|^2. \end{aligned}\right\} \quad (96)$$

(96) and (85) give the required differential cross section for the process.

V. Results and discussion.

1. Total cross section.

We call, as before, $\varrho\, d\omega$ the density in phase space of the $1\,\pi$, $1\,\mathfrak{N}$ states (elastic scattering); $d\omega$ is the element of solid angle into which the meson is scattered. Similarly, let us call

$$\varrho_{12}\, d\omega_1\, d\omega_2\, d\varepsilon_1$$

the density in phase space of the states in which two mesons are present (meson production). The energy of one of the mesons is between ε_1 and $\varepsilon_1 + d\varepsilon_1$ and its momentum is contained in the solid angle $d\omega_1$; the momentum of the second meson is contained in $d\omega_2$; the energy of the second meson is thereby fixed by the conservation laws.

The total cross section σ^{++} for two positive outgoing mesons is given by (85) integrated over $d\varepsilon_1$, $d\omega_1$, $d\omega_2$. Similarly, the total cross section $\sigma_{\text{el.}}$ for positive meson scattering is given by

$$\sigma_{\text{el.}} = \left(\frac{2\pi V}{\hbar v_i}\right) \cdot \int d\omega \frac{1}{2} \sum_{m_i m_f} |\boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i, m_i)|^2 \varrho. \quad (97)$$

To perform the summation over spins one can, in analogy to the procedure of Chapter IV § 3, proceed as follows. First expand $\boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i, m_i)$ by formula (22) and use the above mentioned equivalence between expansions (86) and (87): this, using

$$\sum_m Y_l^m(\boldsymbol{n})\, Y_l^m(\boldsymbol{n}_i) = \frac{2l+1}{4\pi} P_l(\boldsymbol{n} \cdot \boldsymbol{n}_i), \quad (98)$$

gives

$$\left.\begin{aligned} &\boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i, m_i) \\ &= \left(u_f, \sum_l [U_l' + V_l'(\boldsymbol{\sigma} \cdot \boldsymbol{n})(\boldsymbol{\sigma} \cdot \boldsymbol{n}_i)] \frac{2l+1}{4\pi} P_l(\boldsymbol{n} \cdot \boldsymbol{n}_i) u_i\right), \end{aligned}\right\} \quad (99)$$

U_l' and V_l' being given in terms of $T^{3/2}_{lJ}$ by formula (89) (with $\mathfrak{X}$ replaced by T). In the case where only the S and P phase shifts are important, these formulae reduce to

$$\left.\begin{aligned} U_0' = T^{3/2}_{0\,1/2} = (2\pi\varrho)^{-1} ie_3;\ \ U_1' = T^{3/2}_{1\,3/2} = (2\pi\varrho)^{-1} ie_{33};\\ V_0' = T^{3/2}_{1\,1/2} - T^{3/2}_{1\,3/2} = (2\pi\varrho)^{-1} i(e_{31} - e_{33}), \end{aligned}\right\} \quad (100)$$

others being zero.
(99) now gives

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\cdots)\right|^2 \\ &\frac{1}{2}\,\mathrm{Trace}\left\{\left[\sum_l [\cdots]\frac{2l+1}{4\pi}P_l(\cdots)\right]\cdot\left[\sum_l [\cdots]^*\frac{2l+1}{4\pi}P_l(\cdots)\right]\right\}. \end{aligned}\right\} \quad (101)$$

Multiplying in the braces by $(\sigma\cdot\boldsymbol{n})^2 \equiv 1$, this simplifies to

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f;\ \boldsymbol{k}_i, m_i)\right|^2 \\ &= \left[\sum_l] U_l' \boldsymbol{n} + V_l' \boldsymbol{n}_i]\,\frac{2l+1}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i)\right] \\ &\times\left[\sum_l [U_l'^{*}\boldsymbol{n} + V_l'^{*}\boldsymbol{n}_i]\,\frac{2l+1}{4\pi}P_l(\boldsymbol{n}\cdot\boldsymbol{n}_i)\right]; \end{aligned}\right\} \quad (102)$$

in the case where only S and P phase shifts are important (100) and (102) give*

$$\left.\begin{aligned} &\frac{1}{2}\sum_{m_i m_f} \left| \boldsymbol{T}^{3/2}(\boldsymbol{k}, m_f; \boldsymbol{k}_i m_i)\right|^2 \\ &= (8\pi^2\varrho)^{-2}\left|[e_3 + 3(\boldsymbol{n}\cdot\boldsymbol{n}_i)e_{33}]\boldsymbol{n} + (e_{31} - e_{33})\boldsymbol{n}_i\right|^2. \end{aligned}\right\} \quad (103)$$

From now on we shall assume that, as seems likely in view of present experimental evidence, the S and $P_{1/2}$ phase shifts can also be neglected compared to the $P_{3/2}$ phase shift. (85) and (97) then give, with the help of (96), (94), (103) and (25),

* The reader will have no difficulty in verifying that (103) is a compact expression entirely equivalent to the expressions (12) of Anderson, Fermi, Martin, and Nagle (1953).

$$\left.\begin{aligned} &\frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} = \frac{G^2\hbar^2}{V\cdot 4M^2} \\ &\frac{\int d\varepsilon_1 d\omega_1 d\omega_2 \dfrac{\varepsilon_i^3}{k_i^2}\dfrac{k_1^2 k_2^2}{\varepsilon_1^3\varepsilon_2^3}[3(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)\boldsymbol{n}_1 + 3(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)\boldsymbol{n}_2 - 2(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\boldsymbol{n}_i]^2\varrho_{12}}{\varrho\int d\omega\,[3(\boldsymbol{n}\cdot\boldsymbol{n}_i)\boldsymbol{n} - \boldsymbol{n}_i]^2}. \end{aligned}\right\} \quad (104)$$

In (104) the density of states ϱ_{12} is, because of the conservation of momentum, a complicated function of the directions of the two emitted mesons. If, however, following Fermi (1950), we use the approximation that the meson mass is much smaller than the nucleon mass, we can restrict the conservation of momentum to the nucleon only (cf. also, β-decay theory). Then,

$$\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2 = \frac{V^2}{(2\pi\hbar)^6 c^4} k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1\,d\omega_1\,d\omega_2. \quad (105)$$

Performing now the integrations over the angles in (104) and defining a "mean value" $\langle k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\rangle$ by

$$\langle k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\rangle = \int k_1^2\varepsilon_1^{-3}k_2^2\varepsilon_2^{-3}\cdot k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1 \Big/ \int k_2\varepsilon_2 k_1\varepsilon_1\,d\varepsilon_1 \quad (106)$$

(both integrations performed on the energy shell: ε_2 is a function of ε_1) we have

$$\frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} = \frac{G^2\hbar^2}{V\cdot 4M^2}\,\frac{10}{3}\,\frac{\varepsilon_i^3}{k_i^2}\left\langle\frac{k_1^2k_2^2}{\varepsilon_1^3\varepsilon_2^3}\right\rangle\frac{\int\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2}{\varrho\int d\omega}. \quad (107)$$

(107) has been given a form which makes very easy a comparison with the Fermi approach (Fermi, 1950). In fact Fermi's $S(1, 2)$, $S(1, 1)$ (whose ratio should, in a Fermi approach, be equal to $\sigma_{++}/\sigma_{\mathrm{el.}}$) can be written*

$$S(1,1) = \Omega V^{-1}\int\varrho\,d\omega;\ \ S(1,2) = \Omega^2 V^{-2}\int\varrho_{12}\,d\varepsilon_1\,d\omega_1\,d\omega_2. \quad (108)$$

It may be seen that, also in the present theory, one may write

$$\frac{\sigma_{++}}{\sigma_{\mathrm{el.}}} = \frac{S(1,2)}{S(1,1)} \quad (109)$$

* $S(s, n)$ is the statistical weight for the emergence of s nucleons and n mesons, Ω being the interaction volume.

with the same definition for $S(1,2)$ and $S(1,1)$, provided Ω is defined by

$$\Omega = \frac{G^2\hbar^2}{4M^2}\,\frac{10}{3}\,\frac{\varepsilon_i^3}{k_i^2}\Big/\left\langle\frac{k_1^2 k_2^2}{\varepsilon_1^3\varepsilon_2^3}\right\rangle. \tag{110}$$

It will be readily verified that this Ω is of roughly the same order of magnitude as Fermi's Ω; its dependence on the energy is not, however, exactly that which one would expect on the basis of the Fermi statistical approach: if, namely, we assumed that the statistical method proposed by FERMI for nucleon-nucleon collisions is also valid for meson-nucleon collisions, we should apparently be led to an Ω inversely proportional to the total energy (including rest mass) of the nucleon (Lorentz contraction of the nucleon). Instead, we have for Ω the expression (110) with (106). To evaluate (106) we make, according to Fermi, the extreme relativistic approximation for the mesons, ($\varepsilon_1 \approx c\,k_1$, $\varepsilon_2 \approx c\,k_2$). (106) then gives

$$\left\langle\frac{k_1^2 k_2^2}{\varepsilon_1^3\varepsilon_2^3}\right\rangle \approx c^{-4}\int_0^T (T-\varepsilon_1)\,\varepsilon_1\,d\varepsilon_1 \Big/ \int_0^T (T-\varepsilon_1)^2\,\varepsilon_1^2\,d\varepsilon_1, \tag{111}$$

T being the total kinetic energy of the meson-nucleon system in the centre of mass system before collision:

$$T = \varepsilon_i - \mu c^2 + E_i - M c^2 \approx \varepsilon_i - \mu c^2 \quad \text{if} \quad \varepsilon_i \ll M c^2 \tag{112}$$

$$\left\langle\frac{k_1^2 k_2^2}{\varepsilon_1^3\varepsilon_2^3}\right\rangle \approx 5\,c^{-4}\,T^{-2}. \tag{113}$$

Therefore (in a very crude approximation, however),

$$\Omega \sim \frac{G^2}{4\pi\hbar c}\cdot\frac{50}{4}\left(\frac{\mu}{M}\right)^2\cdot\frac{4\pi}{3}\left(\frac{\hbar}{\mu c}\right)^3\cdot\frac{\mu c^2}{T}. \tag{114}$$

(114) shows that the energy dependence of Ω is roughly T^{-1}, T being the energy of the incident meson. In conclusion, the ratio $\sigma_{++}/\sigma_{\text{el.}}$ predicted by our theory agrees roughly with

that which the Fermi statistical theory would give if it were applied to this problem, as far as orders of magnitude are concerned. However, our theory predicts an increase in the ratio $\sigma_{++}/\sigma_{\text{el.}}$ with increasing energy somewhat slower than that which the pure statistical approach would give, this being due to the more rapid decrease of Ω.

2. Angular distribution of emitted mesons.

The angular distribution is given by (96). Here also we first write down the simplified formulae obtained by neglecting all phase shifts except $P_{3/2}$: the relative probability that one π^+ should be emitted in the solid angle $d\omega_1$, and the other π^+ in the solid angle $d\omega_2$ is then, in the c.m. system,

$$\left[(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)\,\boldsymbol{n}_1 + (\boldsymbol{n}_1\cdot\boldsymbol{n}_i)\,\boldsymbol{n}_2 - \frac{2}{3}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)\,\boldsymbol{n}_i\right]^2 d\omega_1\,d\omega_2, \tag{115}$$

$\boldsymbol{n}_i$, $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ being unit vectors in the directions of the incident and the two emergent π^+. This formula shows that

a) the distribution admits of a centre of symmetry, and therefore is also symmetrical with respect to a plane perpendicular to the direction of the incident meson (change $\boldsymbol{n}_1$ in $-\boldsymbol{n}_1$ *and* $\boldsymbol{n}_2$ in $-\boldsymbol{n}_2$);

b) the distribution of one emergent meson, the direction of the other being given, also possesses property a) (change $\boldsymbol{n}_1$ in $-\boldsymbol{n}_1$ *or* $\boldsymbol{n}_2$ in $-\boldsymbol{n}_2$);

c) the mesons have a tendency to be emitted in a direction collinear with the direction $\boldsymbol{n}_i$ of the incident meson: $\pm\boldsymbol{n}_1 = \pm\boldsymbol{n}_2 = \boldsymbol{n}_i$.

Properties a) b) c) are expected to hold only approximately for they are valid only if the S and D ect. phase shifts are negligible.

Let us choose the z-axis in the direction of $\boldsymbol{n}_i$ and call θ_1 and θ_2 the angles between the direction $\boldsymbol{n}_i$ of the incident π^+ and the directions $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ of each of the emitted π^+. Then, the relative probability that θ_1 lies between θ_1 and $\theta_1 + d\theta_1$ and that simultaneously θ_2 lies between θ_2 and $\theta_2 + d\theta_2$, irrespective of the angle that both emitted mesons make with each other, is

$$P_r\{\theta_1, \theta_2\}\, d\theta_1\, d\theta_2 = \left[1 + \frac{7}{2}(\cos^2\theta_1 + \cos^2\theta_2)\right] \sin\theta_1\, d\theta_1 \cdot \sin\theta_2\, d\theta_2 \qquad (116)$$

and the angular distribution of one meson, irrespective of the direction of the other, is

$$P_r\{\theta\}\, d\theta = (13 + 21\cos^2\theta)\sin\theta\, d\theta. \qquad (117)$$

It should be noted that these angular distributions are independent of the distribution of kinetic energy between the two emergent mesons.

These simple results, however, are only valid if all the other $T^{3/2}_{lJ}$ are negligibly small compared to $T^{3/2}_{1\,3/2}$, i.e. if the sines of all the other phase-shifts are negligibly small compared to $\sin\delta^{3/2}_{1\,3/2}$. At the energies we are considering this may well not be the case: for instance, $\delta^{3/2}_{1\,3/2}$ may go on increasing with increasing energy and cross the value $\pi/2$ (resonance); then $\sin\delta^{3/2}_{1\,3/2}$ starts decreasing while other $\sin\delta^{3/2}_{lJ}$ probably still increase — or, possibly, $\delta^{3/2}_{1\,3/2}$ goes through a maximum and then decreases before having attained the value $\pi/2$. On account of these possibilities we quote here the general formula that one gets if one keeps the S, $P_{1/2}$ and $P_{3/2}$ phase shifts. We use the notation e_3, e_{31}, e_{33} of ANDERSON, FERMI, MARTIN and NAGLE (1953) (see chapter IV, eq. (95)).

Then, the relative probability that θ_1 lies between θ_1 and $\theta_1 + d\theta_1$ and that, simultaneously, θ_2 lies between θ_2 and $\theta_2 + d\theta_2$, irrespective of the angle that both emitted mesons make with each other, is

$$\begin{aligned} P_r\{\theta_1, \theta_2\}\, d\theta_1 d\theta_2 = {} & \{|a_2|^2 + |a_1|^2 + 2|e_{31} - e_{33}|^2 \\ & + [a_2 e^*_{33} + a^*_2 e_{33}]\, 3\cos\theta_1 \\ & + [a_1 e^*_{33} + a^*_1 e_{33}]\, 3\cos\theta_2 \\ & + [9|e_{33}|^2 - 2|e_{31} - e_{33}|^2](\cos^2\theta_1 + \cos^2\theta_2) \\ & + [a_1(e^*_{33} + 2e^*_{31}) + a^*_1(e_{33} + 2e_{31})]\cos^2\theta_1\cos\theta_2 \\ & + [a_2(e^*_{33} + 2e^*_{31}) + a^*_2(e_{33} + 2e_{31})]\cos\theta_1\cos^2\theta_2 \\ & + [|e_{31}|^2 + e^*_{31}e_{33} + e_{31}e^*_{33}]\, 6\cos^2\theta_1\cos^2\theta_2\}\sin\theta_1\, d\theta_1 \cdot \sin\theta_2\, d\theta_2, \end{aligned} \qquad (118)$$

where

$$a_1 = \frac{k_i}{\varepsilon_i}\frac{\varepsilon_2}{k_2}e_3; \quad a_2 = \frac{k_i}{\varepsilon_i}\frac{\varepsilon_1}{k_1}e_3, \qquad (119)$$

ε_i, ε_1, ε_2 being as before the energies of the incident and emergent mesons in the c.m. system. The D phase shifts have not been included in formula (118), which should therefore be corrected if the D phase shifts happen not to be negligible at our energies.

3. Angular distribution when no field approximation is used.

We may first summarize the foregoing investigations as follows. An attempt has been made to study the inelastic meson-nucleon collisions that give rise to two emergent mesons, paying particular attention to the reaction in which a π^+ and a proton give rise to a neutron and two π^+: this process has, compared with the others, some simplifying features which are used in part of the present theory.

In Chapter II a rather general approach is developed that does not involve any field theoretical approximation, with the aim of throwing some light on analogies that may exist between such processes and ordinary nuclear reactions. Qualitatively the main conclusions of this chapter are:

a) The "damping" (i. e. the transition from the reaction matrix to the scattering matrix) does *not* appear to disfavour meson production as compared to elastic scattering; in fact, both processes seem to be damped in almost the same proportion, at least at energies where meson production is relatively small (cf. eq. (65)).

b) An analogy with nuclear reactions seems to require that the reaction matrix elements for meson production should be roughly proportional to the corresponding reaction matrix elements for elastic scattering, i. e. that the latter should become large at precisely the same energies for which the former happen to become large.

Since the approach of Chapter II is still somewhat formal we have supplemented it with a treatment involving some approximation of the field theory: this forms the subject of Chapters III and IV. In Chapter III the approximations are stated and discussed; although they appear rather natural, and less drastic

than the approximations of, e. g., the Tamm-Dancoff method*, they should nevertheless be considered a working hypothesis, as their reliability cannot, of course, be proved in any rigorous way. In particular, the emission process enters in the theory in a way that somewhat recalls weak-coupling approximations: this is certainly a not quite satisfactory feature of the theory and, consequently, the numerical values of the expressions containing explicitly the coupling constant G should not be taken too seriously. This applies especially to the total cross-section: formulae (108), (109), (110) . . . . . should be considered as giving nothing more than orders of magnitude.

Therefore, the question arises: "are there any results of the theory that can be considered to be fairly independent of the field theoretical approximation of Chapter III?" We show that, indeed, the results concerning the differential cross section (angular distributions) are likely to hold, even if the approximations of Chapter III appeared to be too crude.

To this purpose we want to restrict our investigations to the case in which both π^+ mesons are emitted with approximately equal energies in the c.m. system. Then, the most general pseudoscalar that can be formed with the three vectors $\boldsymbol{k}_i$, $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and the pseudovector $\boldsymbol{\sigma}$, and that is symmetric with respect to interchange of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ (with $k_1 = k_2$), is

$$\left.\begin{aligned} &(\boldsymbol{\sigma}\cdot\boldsymbol{k}_2)\, f[(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)] \\ &+(\boldsymbol{\sigma}\cdot\boldsymbol{k}_1)\, f[(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)] \\ &+(\boldsymbol{\sigma}\cdot\boldsymbol{k}_i)\, g[(\boldsymbol{k}_1\cdot\boldsymbol{k}_i),(\boldsymbol{k}_2\cdot\boldsymbol{k}_i),(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)], \end{aligned}\right\} \tag{120}$$

f and g being two functions of their three arguments. Now, suppose that f is expanded in a sum of products of three Legendre polynomials $P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)\cdot P_{l'}(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)\cdot P_{l''}(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)$. We express $P_l(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)$ as a sum of products of $Y_l^{m*}(\boldsymbol{n}_1)$ and $Y_l^m(\boldsymbol{n}_i)$, and similarly with $P_{l'}(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)$, and, finally, the products $Y_l^m(\boldsymbol{n}_i)\, Y_{l'}^{m'}(\boldsymbol{n}_i)$ we express as a sum of terms involving only one $Y_L^M(\boldsymbol{n}_i)$ with $|l-l'| \leqslant L \leqslant l+l'$. Now we make the rather natural assumption that, at not too high energies, only incident

* Some a posteriori justification for these approximations may be found in the fact that they yield the same result (b) as the theory of Chapter II.

waves with small angular momenta, say $L \leqslant 1$, take a significant part in the process: this gives $l + l' \leqslant 1$.

The same assumption, made on the emergent meson waves, gives $l' + l'' \leqslant 1$, $l'' + l \leqslant 1$; therefore, either $l = l' = l'' = 0$ or one only of l, l' and l'' is 1, both others being zero, which gives

$$f = a + b(\boldsymbol{n}_1\cdot\boldsymbol{n}_i) + c(\boldsymbol{n}_2\cdot\boldsymbol{n}_i) + d(\boldsymbol{n}_1\cdot\boldsymbol{n}_2), \tag{121}$$

a, b, c, d being some functions of $k_1 = k_2$ and k_i. Similarly,

$$g = \alpha + \beta(\boldsymbol{n}_1\cdot\boldsymbol{n}_i) + \gamma(\boldsymbol{n}_2\cdot\boldsymbol{n}_i) + \delta(\boldsymbol{n}_1\cdot\boldsymbol{n}_2). \tag{122}$$

Now it is easy to see by a similar argumentation that $(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)$ involves in fact spherical harmonics of $\boldsymbol{n}_2$ of order two, $Y_2^M(\boldsymbol{n}_2)$, which we must also reject: therefore $c = 0$ and, similarly, $d = 0$, $\beta = 0$, $\gamma = 0$. Finally, we are left with

$$\left.\begin{aligned} &(\boldsymbol{\sigma}\cdot\boldsymbol{n}_2)[a + b(\boldsymbol{n}_1\cdot\boldsymbol{n}_i)] \\ &+(\boldsymbol{\sigma}\cdot\boldsymbol{n}_1)[a + b(\boldsymbol{n}_2\cdot\boldsymbol{n}_i)] + (\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)[\alpha + \delta(\boldsymbol{n}_1\cdot\boldsymbol{n}_2)]. \end{aligned}\right\} \tag{123}$$

Except for the term $\alpha(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)$, this is just the expression (93) with

$$\lambda_1 = \lambda_2 = a; \quad \mu = b; \quad \nu = \delta. \tag{124}$$

Working backward from expression (93) one then sees that the matrix element of the T-matrix can quite generally be put into the form exhibited by the square bracket in (82), plus a term proportional to $(\boldsymbol{\sigma}\cdot\boldsymbol{k}_i)$.

This result can also be obtained by starting from expansion (62) and limiting the possible values of l, l_1, l_2 to small numbers, say 0 and 1. The possible combinations of l, J, L, l_1, l_2 are then, on account of (61) and the symmetry between $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$,

$$\begin{aligned} &l = 0 \quad J = 1/2 \quad L = 1 \qquad l_1 = 0, l_2 = 1 \text{ or } l_1 = 1, l_2 = 0 \\ &l = 1 \quad J = 1/2 \quad L = 0 \qquad \begin{cases} l_1 = l_2 = 0 \\ l_1 = l_2 = 1 \end{cases} \\ &l = 1 \quad J = 3/2 \quad L = 2 \qquad l_1 = l_2 = 1. \end{aligned}$$

We now express the function $\Phi^M_{l_1 l_2(L)J}$ appearing in (62) in terms of the functions $\Psi^M_{l_1,l_2 s(j),J}$ when l_2 and the spin s are coupled together to give j, and then l_1 and j coupled together to give J (RACAH, 1943). This gives

$$\left.\begin{aligned} -\left(\frac{4}{3}\right)^{1/2} \Phi^M_{11(0)1/2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) \\ = \Psi^M_{1,1^1/_2(1/_2),1/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) + \Psi^M_{1,1^1/_2(1/_2),1/_2}(\boldsymbol{n}_2, \boldsymbol{n}_1, m_f); \\ \left(\frac{10}{3}\right)^{1/2} \Phi^M_{11(2)3/2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) \\ = \Psi_{1,1^1/_2(1/_2),3/_2}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) + \Psi^M_{1,1^1/_2(1/_2),3/_2}(\boldsymbol{n}_2, \boldsymbol{n}_1, m_f). \end{aligned}\right\} \quad (124)$$

Moreover, it is easily shown that

$$\Psi^M_{1,1^1/_2(1/_2),J}(\boldsymbol{n}_1, \boldsymbol{n}_2, m_f) = -(4\pi)^{-1/2} \sum_{m'} (u_f, (\sigma \cdot \boldsymbol{n}_2)\, u')\, \mathcal{Y}^M_{J1^1/_2}(\boldsymbol{n}_1, m'). \quad (125)$$

Use of the general formula of Appendix II then finally gives a general expression for the cross section which includes no approximation, except the neglect of D etc. waves*, and which is

$$\left.\begin{aligned} d\sigma = \frac{2\pi V_i}{\hbar v} \frac{1}{(4\pi)^3} \big| a_2 \boldsymbol{n}_2 + a_1 \boldsymbol{n}_1 + \alpha \cdot \boldsymbol{n}_i + b\,[(\boldsymbol{n}_1 \cdot \boldsymbol{n}_i)\boldsymbol{n}_2 + (\boldsymbol{n}_2 \cdot \boldsymbol{n}_i)\boldsymbol{n}_1] \\ + \delta\,(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2)\,\boldsymbol{n}_i \big|^2 \varrho_{12}\, d\varepsilon_1\, d\omega_1\, d\omega_2, \end{aligned}\right\} \quad (126)$$

where a_1, a_2, α, b, δ are functions of the $T_{lJ\,l_1 l_2}$.

If one writes

$$\frac{T_{lJl_1l_2}(k_1, k_2; k_i) + T_{lJl_2l_1}(k_2, k_1; k_i)}{T_{lJ}(k_i; k_i)} = c_{lJl_1l_2}(k_1, k_2, k_i) \quad (127)$$

and

$$\frac{\sigma^{++}_{lJ}}{\sigma_{lJ\,\mathrm{el.}}} = \frac{1}{\varrho} \sum_{l_1 l_2} \int (c_{lJl_1l_2})^2\, \varrho_{12}\, d\varepsilon_1, \quad (128)$$

the integration being performed on the energy shell. In (128), σ^{++}_{lJ} is the contribution to σ^{++} arising from the incident l, J wave. If the approach of Chapter II is valid, then the functions $c_{lJl_1l_2}(k_1, k_2, k_i)$ are real and smoothly varying.

* See, however, text following eq. (61), p. 18.

The calculations leading to (126) yield

$$\left.\begin{aligned} 2\pi i \varrho \cdot a_1 &= e_3\, c_{0^1/_2 10}(k_1, k_2, k_i) \\ 2\pi i \varrho \cdot a_1 &= e_3\, c_{0^1/_2 01}(k_1, k_2, k_i) \\ 2\pi i \varrho \cdot \alpha &= e_3\, c_{0^1/_2 00}(k_1, k_2, k_i) \\ 2\pi i \varrho \cdot b &= 3\left(\frac{3}{10}\right)^{1/2} e_{33}\, c_{1^3/_2 11}(k_1, k_2, k_i) \\ 2\pi i \varrho \cdot \delta &= -\frac{2}{3}(2\pi i \varrho \cdot b) - 3^{1/2} e_{31} c_{1^1/_2 11}(k_1, k_2, k_i). \end{aligned}\right\} \quad (129)$$

It is easily seen that formula (96), giving the angular distribution derived from the field theoretical treatment, is a special case of (126). Whether it is a good approximation or not depends largely on whether the $\sigma^{++}_{lJ}/\sigma_{lJ\,\mathrm{el.}}$ are all of the same order of magnitude or not. If only the $P_{3/_2}$ wave is significant, then again (116) and (117) follow; these equations therefore do not depend on the approximation of Chapter III.

4. Concluding remarks.

(126), (128), (129) seem to yield a convenient and mathematically consistent formalism for the analysis of the phenomena where two positive mesons emerge at not too high energies and a possibility of connecting them with elastic scattering data; moreover, it could be extended to all meson-nucleon collisions in which two mesons emerge with only minor modifications.

As regards the total cross section, the situation is less satisfactory, though it seems that a statistical approach of the Fermi type might yield correct orders of magnitude. It is, however, quite possible that the value of Ω should be changed when processes involving more than two emergent mesons are taken into consideration.

The present investigation has been carried out during my stay with the Theoretical Study Group of CERN, and I have greatly benefited from frequent discussions with the other members of the Group as well as with members and guests of the

Institute for Theoretical Physics. Especially I am grateful to Professor NIELS BOHR not only for his kind interest in my work, but also for making my participation in the activities of the Group possible through his kind hospitality at the Institute. I also want to express my appreciation to Professor MØLLER for enlightening comments and discussions. The support of the Centre National de la Recherche Scientifique, Paris, over previous years is gratefully acknowledged.

CERN (European Council for Nuclear Research)
Theoretical Study Group
at the
Institute for Theoretical Physics,
University of Copenhagen.

Appendix I.

Formulation of the Tamm-Dancoff method using the reaction matrix.

Since the basic assumption of the Tamm-Dancoff method is not very reliable, even for elastic scattering problems, it was felt necessary to avoid the use of this method in the main body of this paper. However, for reference, a short summary of the Tamm-Dancoff method applied to the K-matrix is given in the following.

Let us call $(p\,\pi, q\,\mathfrak{N})$ a state containing p mesons and q nucleons. The basic assumption of the (lowest order) Tamm-Dancoff method applied to meson scattering is that only states $(1\,\pi, 1\,\mathfrak{N})$, $(0\,\pi, 1\,\mathfrak{N})$, $(2\,\pi, 1\,\mathfrak{N})$, $(0\,\pi, 3\,\mathfrak{N})$ $(2\,\pi, 3\,\mathfrak{N})$ interact directly with each other (cf. Fig. (I. 1)).

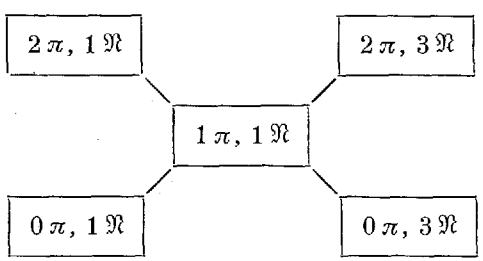

Fig. (I. 1).

Write eq. (10) of the text choosing for (a) a $(1\,\pi, 1\,\mathfrak{N})$ state and for (b), successively, each of the five kinds of states just mentioned; then, between these five equations eliminate all K_{ba} except those where (b) is also a $(1\,\pi, 1\,\mathfrak{N})$ state. The result is, (a), (b), (c) being $(1\,\pi, 1\,\mathfrak{N})$ states,

$$K_{ba} = B_{ba} + P \sum_{(a)} \frac{B_{bc}\, K_{ca}}{E_{(c)} - E_{(a)}} + \lambda\, K_{ba}, \qquad \text{(I. 1)}$$

where λ is an infinite quantity of the nature of a renormalization which (without much justification) we discard. B_{ba} is given by

$$B_{ba} = \sum_{(d)} \frac{H'_{bd}\, H'_{da}}{E_{(a)} - E_{(d)}}. \qquad \text{(I. 2)}$$

Denote by $\boldsymbol{B}_{ba}$ the quantity B_{ba} for $E_{(a)} = E_{(b)}$; then $\boldsymbol{B}_{ba}$ is just the ordinary lowest order matrix element for scattering.

The next step is the separation of (I. 1) according to total isotopic spin I = 1/2 or 3/2. This is easily done with the help of expansions such as (20), with the result that

$$K^I_{ba} = B^I_{ba} + P \sum_{(c)} \frac{B^I_{bc} K^I_{ca}}{E_{(a)} - E_{(c)}} \qquad \text{(I. 3)}$$

$$\left.\begin{aligned} &B(\pi^+ + P \to \pi^+ + P) = B^{3/2} \\ &B(\pi^- + P \to \pi^- + P) = \frac{1}{3} B^{3/2} + \frac{2}{3} B^{1/2} \\ &B(\pi^- + P \to \pi^0 + N) = \frac{\sqrt{2}}{3}(B^{3/2} - B^{1/2}) \\ &\text{id. for the } K'_s \text{ and } T'_s. \end{aligned}\right\} \qquad \text{(I. 4)}$$

We now have to separate each eq. (I. 3) according to eigenvalues l and J of the orbital and total angular momenta. For this purpose, make the explicit calculation of (I. 2) in c.m. system (assuming pseudoscalar coupling) and express the small components of the nucleon wave functions in terms of the normalized big components u_i and u_f (u_i, u_f are 2 row, 1 column-matrices $|u_i|^2 = |u_f|^2 = 1$). The result is

$$B^I_{fi} = f[k_f, k_i, (\boldsymbol{n}_f \cdot \boldsymbol{n}_i)]\,(u_f, u_i) + g[k_f, k_i, (\boldsymbol{n}_f \cdot \boldsymbol{n}_i)]\,(u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_f)(\boldsymbol{\sigma}\cdot\boldsymbol{n}_i)\, u_i),$$

k_i, k_f being the magnitude of the relative momenta in initial and final states (not necessarily of the same total energy), $\boldsymbol{n}_i$, $\boldsymbol{n}_f$ their directions (unit vectors), f and g two known expressions. f and g can both be expanded in Legendre polynomials $P_l(\boldsymbol{n}_f \cdot \boldsymbol{n}_i) = 4\pi(2l+1)^{-1} \sum_m Y^m_l(\boldsymbol{n}_f)\, Y^{m*}_l(\boldsymbol{n}_i)$. Finally, one has

$$B^I_{fi} = \sum_{lm} [U_l \cdot (u_f, u_i) + V_l \cdot (u_f, (\boldsymbol{\sigma}\cdot\boldsymbol{n}_f)(\sigma\cdot\boldsymbol{n}_i)\, u_i)]\, Y^m_l(\boldsymbol{n}_f)\, Y^{m*}_l(\boldsymbol{n}_i) \qquad \text{(I 5)}$$

which, using the result of Appendix II, can be written

$$B^I_{fi} = \sum_{lJ} B^I_{lJ}(k_f, k_i) \sum_M \mathcal{Y}^M_{Jl_{1/2}}(\boldsymbol{n}_f, m_f)\, \mathcal{Y}^{M*}_{Jl_{1/2}}(\boldsymbol{n}_i, m_i) \qquad \text{(I. 6)}$$

with

$$B^I_{l,l\pm 1/2}(k_f, k_i) = U_l(k_f, k_i) + V_{l\pm 1}(k_f, k_i). \qquad \text{(I. 7)}$$

Introducing (I. 6) and (21) in (I. 3) gives a one variable integral equation for each K^I_{lJ} (k_i is a parameter):

$$K^I_{lJ}(k_f, k_i) = B^I_{lJ}(k_f, k_i) + P \int \frac{Vk^2\, dk}{(2\pi\hbar)^3} \frac{B^I_{lJ}(k_f, k)\, K^I_{lJ}(k, k_i)}{E_k + \varepsilon_k - E_i - \varepsilon_i} \qquad \text{(I. 8)}$$

with $\varepsilon_k = (\mu^2 c^4 + k^2)^{1/2}$; $E_k = (M^2 c^4 + k^2)^{1/2}$; $\varepsilon_i \equiv \varepsilon_{k_i}$; $E_i \equiv E_{k_i}$; V being the volume of the normalizing box. Only the equations with I = 3/2 can have a meaning (with I = 1/2 they have no finite solution). For I = 3/2 the actual expressions for U_l and V_l are (we take c = 1):

$$\left.\begin{aligned} &U_l(k_f, k) = \\ &G^2\hbar^2\pi V^{-1}(M+E_f)^{1/2}(M+E_k)^{1/2}(E_f E_k \varepsilon_f \varepsilon_k)^{-1}[\mathfrak{H}_l + (E_f + E_k - M)\mathfrak{k}_l] \\ &V_l(k_f, k) = \\ &G^2\hbar^2\pi V^{-1} k_f k (M+E_f)^{-1/2}(M+E_k)^{-1/2}(E_f E_k \varepsilon_f \varepsilon_k)^{-1}[\mathfrak{H}_l + (E_f + E_k + M)\mathfrak{k}_l], \end{aligned}\right\} \qquad \text{(I 9)}$$

$\mathfrak{H}$ and $\mathfrak{k}$ being quantities defined by Bethe and Dyson (1953) which we rewrite here for convenience:

$$\left.\begin{aligned} \mathfrak{H}_l &= \frac{1}{2\bar{E} r} \int_{1-r}^{1+r} \left(\frac{c}{c+z} - \frac{b}{b+z}\right) P_l\left(\frac{z^2 - 1 - r^2}{2r}\right) dz \\ \mathfrak{k}_l &= \frac{1}{2\bar{E}^2 r} \int_{1-r}^{1+r} \left(\frac{1}{c+z} + \frac{1}{b+z}\right) P_l\left(\frac{z^2 - 1 - r^2}{2r}\right) dz \end{aligned}\right\} \qquad \text{(I 10)}$$

$$\left.\begin{aligned} &E_s = [M^2 + (k_f + k)^2]^{1/2};\quad E_d = [M^2 + (k_f - k)^2]^{1/2};\quad r = \frac{E_s - E_d}{E_s + E_d};\quad \bar{E} = \frac{E_s + E_d}{2} \\ &b = (\varepsilon_f + \varepsilon_k - E_i - \varepsilon_i)\,\bar{E}^{-1};\quad c = (E_f + E_k - E_i - \varepsilon_i)\,\bar{E}^{-1}. \end{aligned}\right\} \qquad \text{(I 11)}$$

To leading order in the small quantity r,

$$\left.\begin{aligned}
\mathfrak{H}_0 &= \bar{E}^{-1}[(1+b)^{-1}-(1+c)^{-1}];\\
\mathfrak{k}_0 &= \bar{E}^{-2}[(1+b)^{-1}+(1+c)^{-1}];\\
\mathfrak{H}_1 &= \frac{1}{3}k_f k\,\bar{E}^{-3}[(1+c)^{-2}-(1+b)^{-2}];\\
\mathfrak{k}_1 &= -\frac{1}{3}\bar{E}^{-4}[(2+c)(1+c)^{-2}+(2+b)(1+b)^{-2}].
\end{aligned}\right\}\text{(I. 12)}$$

Equations (I. 8) can be very much simplified if one introduces a cut-off $\ll M$ on the integration over k; $v(k)$ being a cut-off function with $v(0) = 1$ and y^I_{lJ} being defined by (69) the integral equations for the y^I_{lJ} are ($\mathrm{I} = {}^3/_2$)

$$\left.\begin{aligned}
&y_{0^1/_2}(k_f,k_i) =\\
&1-(4\pi^2\hbar)^{-1}G^2\left(M-\frac{\varepsilon_i}{2}\right)^{-1}P\int_\mu^\infty d\varepsilon_k\frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i}v^2(k)\,y_{0^1/_2}(k,k_i),\\
&y_{1^3/_2}(k_f,k_i) =\\
&1+(12\pi^2\hbar)^{-1}G^2M^{-2}P\int_\mu^\infty d\varepsilon_k\frac{k^3}{\varepsilon_k(\varepsilon_k+\varepsilon_f-\varepsilon_i)}\frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i}v^2(k)\,y_{1^3/_2}(k,k_i),\\
&y_{1^1/_2}(k_f,k_i) =\\
&1-(24\pi^2\hbar)^{-1}G^2M^2P\int_\mu^\infty d\varepsilon_k\frac{k^3}{\varepsilon_k(\varepsilon_k+\varepsilon_f-\varepsilon_i)}\frac{\varepsilon_k}{\varepsilon_k-\varepsilon_i}v^2(k)\,y_{1^1/_2}(k,k_i).
\end{aligned}\right\}\text{(I. 13)}$$

The first one shows that $y_{0^1/_2}(k_f,k_i)$ is in fact independent of k_f and thus equal to $y_{0^1/_2}(k_i,k_i)$. The two others are "true" integral equations. Chew (1953) has solved them approximately by a variation method (cf. also Fubini, 1953).

Appendix II.

Let (m = component of nucleon spin on z-axis):

$$u=\begin{pmatrix}u_+\\u_-\end{pmatrix};\quad u_+=\delta_{m,{}^1/_2};\quad u_-=\delta_{m,-{}^1/_2}$$

and

$$\left.\begin{aligned}
\mathcal{Y}^M_{Jl^1/_2}(\boldsymbol{n},m) &= \sum_{\nu\nu'}C_{l_1}(J,M,\nu,\nu')\,Y^\nu_l(\boldsymbol{n})\,\delta_{\nu,m}\\
&\equiv C_{l_1}(J,M,M-m,m)\,Y^{M-m}_l(\boldsymbol{n}).
\end{aligned}\right\}\text{(II. 1)}$$

Theoretical calculations often give the scattering matrix elements as functions of the angles and the spin operator $\boldsymbol{\sigma}$. The following formula (II. 2) is a convenient and general tool for expanding such results according to eigenvalues of l and J (phase shift analysis).

$$\left.\begin{aligned}
&(u,\sum_l[U_l+V_l(\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n}')]\sum_\nu Y^\nu_l(\boldsymbol{n})\,Y^{\nu*}_l(\boldsymbol{n}')\,u')\\
&\quad=\sum_{l,J,M}\mathfrak{X}_{l,J}\,\mathcal{Y}^M_{Jl^1/_2}(\boldsymbol{n},m)\,\mathcal{Y}^{M*}_{Jl^1/_2}(\boldsymbol{n}',m')\\
&\text{with}\\
&\quad J=l\pm\frac{1}{2};\quad \mathfrak{X}_{l,l\pm{}^1/_2}=U_l+V_{l\pm1}.
\end{aligned}\right\}\text{(II. 2)}$$

Conversely, (II. 2) also provides a convenient means for the evaluation of, e.g., spin averages over quantities given as expansions in l and J (see the compact expression (102) or (103) for the angular distribution in $\pi^+ + P$ scattering).

(II. 2) can be proved by elementary calculation. The first step is to write $(u,(\boldsymbol{\sigma}\cdot\boldsymbol{n})\,u'')$ as a linear combination of $Y^\mu_1(\boldsymbol{n})$. $(u',(\boldsymbol{\sigma}\cdot\boldsymbol{n})\,u'')\,Y^\nu_l(\boldsymbol{n})$ can then be expressed as a linear combination of $Y^M_L(\boldsymbol{n})$ with $L = l\pm1$. The resulting expression for the product $(u,(\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{\sigma}\cdot\boldsymbol{n}')\,u')\sum_\nu Y^\nu_l(\boldsymbol{n})\,Y^{\nu*}_l(\boldsymbol{n}')$ is such that by using simple numerical equalities between products of Clebsch-

Gordan coefficients it can be brought into the form

$$\sum_M [\mathcal{Y}^M_{l+1/2,l+1,1/2}(\boldsymbol{n}, m)\, \mathcal{Y}^{M*}_{l+1/2,l+1,1/2}(\boldsymbol{n}', m') + \mathcal{Y}^M_{l-1/2,l-1,1/2}(\boldsymbol{n}, m)\, \mathcal{Y}^{M*}_{l-1/2,l-1,1/2}(\boldsymbol{n}', m')]$$

from which (II. 2) is easily derived.

References.

H. L. Anderson, E. Fermi, E. A. Long, R. Martin & D. E. Nagle (1952), Phys. Rev. **85**, 934.

H. L. Anderson, E. Fermi, E. A. Long & D. E. Nagle (1952), Phys. Rev. **85**, 936.

H. L. Anderson, E. Fermi, D. E. Nagle & G. B. Yodh (1952), Phys. Rev. **86**, 793.

H. L. Anderson, E. Fermi, R. Martin & D. E. Nagle (1953), Phys. Rev. **91** 155.

F. J. Belinfante & C. Møller (1954), Dan. Mat. Fys. Medd. **28**. no. 6.

H. A. Bethe and F. J. Dyson (1953), Proceedings of the 3rd Rochester Conference (Interscience Publ., New York).

J. M. Blatt & V. F. Weisskopf (1952), Theoretical Nuclear Physics, J. Wiley, New York.

K. A. Brueckner (1952), Phys. Rev. **86**, 106.

K. A. Brueckner & K. M. Watson (1952), Phys. Rev. **86**, 923.

C. Chedester, P. Isaacs, A. Sachs & J. Steinberger (1951), Phys. Rev. **82**, 958.

G. F. Chew (1953), Phys. Rev. **89**, 591.

S. S. Drell & E. M. Henley (1952), Phys. Rev. 88, 1053.

B. d'Espagnat (1953), C. R. **237**, 26, 139.

E. Fermi (1950), Progress of Theoretical Physics **5**, No. 4, 570.

E. C. Fowler, W. B. Fowler, R. P. Shutt, A. M. Thorndike & W. L. Whittemore (1952), Phys. Rev. **86**, 1053.

S. Fubini (1953), Il Nuovo Cimento **10**, 564.

M. L. Goldberger (1951), Phys. Rev. **84**, 929.

P. J. Isaacs, A. M. Sachs, & J. Steinberger (1952), Phys. Rev. **85**, 803.

B. Lippman & J. Schwinger (1950), Phys. Rev. **79**, 469.

C. Møller (1945), Dan. Mat. Fys. Medd. **13**, No. 1.

C. Møller (1946), Dan. Mat. Fys. Medd. **12**, No. 19.

G. Racah (1943), Phys. Rev. **63**, 367.

S. I. Tomonaga (1946), Progress of Theoretical Physics **1**, No. 3, 83.

G. Wentzel (1953), Phys. Rev. **92**, 173.

Indleveret til selskabet den 15. januar 1954.
Færdig fra trykkerlet den 21. juni 1954.

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser, bind **28**, nr. 12

Dan. Mat. Fys. Medd. **28**, no. 12 (1954)

CONFIGURATION SPACE REPRESENTATION FOR NON-LINEAR FIELDS

BY

P. KRISTENSEN

København 1954
i kommission hos Ejnar Munksgaard