On the Role of Feedback in Network Coding

by

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Abstract

Network coding has emerged as a new approach to operating communication networks, with a promise of improved efficiency in the form of higher throughput, especially in lossy conditions. In order to realize this promise in practice, the interfacing of network coding with existing network protocols must be understood well. Most current protocols make use of feedback in the form of acknowledgments (ACKs) for reliability, rate control and/or delay control. In this work, we propose a way to incorporate network coding within such a feedback-based framework, and study the various benefits of using feedback in a network-coded system.

More specifically, we propose a mechanism that provides a clean interface between network coding and TCP with only minor changes to the protocol stack, thereby allowing incremental deployment. In our scheme, the source transmits random linear combinations of packets currently in the TCP congestion window. At the heart of our scheme is a new interpretation of ACKs – the receiver acknowledges every degree of freedom (i.e., a linear combination that reveals one unit of new information) even if it does not reveal an original packet immediately. Such ACKs enable a TCP-compatible sliding-window implementation of network coding. Thus, with feedback, network coding can be performed in a completely online manner, without the need for batches or generations.

Our scheme has the nice feature that packet losses on the link can be essentially masked from the congestion control algorithm by adding enough redundancy in the encoding process. This results in a novel and effective approach for congestion control over networks involving lossy links such as wireless links. Our scheme also allows intermediate nodes to perform re-encoding of the data packets. This in turn leads to a natural way of running TCP flows over networks that use multipath opportunistic routing along with network coding.

We use the new type of ACKs to develop queue management algorithms for coded networks, which allow the queue size at nodes to track the true backlog in information with respect to the destination. We also propose feedback-based adaptive coding techniques that are aimed at reducing the decoding delay at the receivers. Different notions of decoding delay are considered, including an order-sensitive notion which assumes that packets are useful only when delivered in order. We study the asymptotic behavior of the expected queue size and delay, in the limit of heavy traffic.

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To my mother

Ms. Vasantha Sundararajan

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Chapter 1

Introduction

Advances in communication technology impact the lives of human beings in many significant ways. In return, people's lifestyle and needs play a key role in defining the direction in which the technology should evolve, in terms of the types of applications and services it is expected to support. This interplay generates a wide variety of fascinating challenges for the communication engineer.

The main goal of communication system design is to guarantee the fair and efficient use of the available resources. Different kinds of engineering questions arise from this goal, depending on the type of application as well as the environment in which it has to be implemented. For example, while a large file transfer is mostly concerned with long-term average throughput, a video-conferencing application requires strict delay guarantees. Similarly, data networks over the wireless medium have to be robust to channel variability and packet losses, which are usually not an issue in a wired network.

In addressing these challenges, it is useful to keep in mind that the problem of communicating information is quite different from the seemingly related problem of transporting physical commodities because information can be transformed in ways that commodities cannot. For instance, it is easy to imagine a node replicating incoming data onto multiple outgoing links. More generally, a node can code the data in different ways. By coding, we mean that the node can view the incoming data as realizations of some variables, and can transmit the output of a function applied to these variables, evaluated at the incoming data. Finding the "most useful" functions (codes) and establishing the fundamental limits on the

benefit that coding can provide over simple uncoded transmission have been the focus of the field of information theory [2].

The concept of coding data has been put to good use in today's communication systems at the link level. For instance, practical coding schemes are known that achieve data rates very close to the fundamental limit (capacity) of the additive white Gaussian noise channel [3]. However, extending this idea to the network setting in a practical way has been a challenging task. On the theoretical front, although the fundamental limits for many multi-user information theory problems are yet to be established, it is well known that there are significant benefits to coding beyond the link level (for instance from the results in the field of network coding). Yet, today's networks seldom apply coding ideas beyond the link level. Even replication is seldom used – the most common way to implement a multicast connection is to initiate multiple unicast connections, one for each receiver.

A possible explanation is the fact that once we go to the network setting, the control aspect of the communication problem becomes more significant. Several new control problems show up just to ensure that the network is up and running and that all users get fair access to the resources. These are arguably more critical goals than the problem of improving the overall speed of communication. While the main question in the point-to-point setting is one of how best to encode the data to combat channel errors and variability (the coding question), the network setting leads to new questions like who should send when (scheduling), how fast to send (congestion control), and how to find the best path to the destination (routing). To ensure simple and robust coordination and management of the network, the conventional approach has been to group data into packets that are then processed like commodities, without much coding inside the network.

In such a framework, the most popular approach to addressing these control questions has been the use of feedback. Most network protocols today are built around some form of an acknowledgment mechanism. Therefore, in order to realize the theoretically proven benefits of coding in the network setting, we have to find a way to incorporate coding into the existing network protocols, without disrupting the feedback-based control operations. In other words, we have to deal with the deployment problem – a solution that requires significant changes to the existing communication protocols will be very difficult to deploy

in a large scale, for practical reasons. Also, the modification must be *simple*, as otherwise, its interaction with the existing dynamics would be very hard to understand and control.

To summarize, coding is generally associated with improving the efficiency of the communication process, and its robustness to errors. Feedback, on the other hand, is critical for controlling various aspects of the communication system such as delay or congestion. We would definitely like to make use of the flexibility of coding in order to make the communication process more efficient. At the same time, we also have to conform with existing feedback-based protocols.

This thesis explores the option of integrating these two fundamental concepts, namely *feedback* and *coding*, in the context of a packetized data network. We investigate the problems associated with implementing this approach within the framework of existing systems and propose a practical solution. We also demonstrate the potential benefits of combining these two concepts.

1.1 The role of feedback in communication protocols

Feedback plays a crucial role in communication networks [4]. It is well known that feedback can significantly reduce communication delay (error exponents) and the computational complexity of the encoding and decoding process. It can even be used to predict and correct noise if the noise is correlated across time, resulting in an improvement in capacity [5]. In addition, feedback can be used to determine the channel at the transmitter side, and accordingly adapt the coding strategy. In addition to these applications, feedback plays a key role in managing the communication network, by enabling simple protocols for congestion control and coordination among the users. In this work, we do not study the use of feedback for noise prediction or channel learning. Instead, we focus on acknowledgment-based schemes and their use for reliability, congestion control and delay control.

The most common type of feedback signal used in practice is an acknowledgment packet that indicates successful reception of some part of the transmitted data. The simplest protocol that makes use of acknowledgments (ACKs) is the Automatic Repeat reQuest (ARQ) protocol. Reliable point-to-point communication over a lossy packet link or

network with perfect feedback can be achieved using the ARQ scheme. It uses the idea that the sender can interpret the absence of an ACK to indicate the erasure of the corresponding packet within the network, and in this case, the sender simply retransmits the lost packet. Thus, we can ensure the reliability of the protocol.

In the ARQ scheme, if the feedback link is perfect and delay-free, then every successful reception conveys a new packet, implying throughput optimality. Moreover, this new packet is always the next unknown packet, which implies the lowest possible packet delay. Since there is feedback, the sender never stores anything the receiver already knows, implying optimal queue size. Thus, this simple scheme simultaneously achieves the optimal throughput along with minimal delay and queue size. Moreover, the scheme is completely online and not block-based. The ARQ scheme can be generalized to situations that have imperfections in the feedback link, in the form of either losses or delay in the ACKs. Reference [6] contains a summary of various protocols based on ARQ.

In addition to reliability and delay benefits, feedback plays another critical role in the communication system, namely, it enables congestion control. In today's internet, the Transmission Control Protocol (TCP) makes use of an acknowledgment mechanism to throttle the transmission rate of the sender in order to prevent congestion inside the network [7, 8]. The main idea is to use ACKs to infer losses, and to use losses as a congestion indicator, which in turn triggers a reduction in the packet transmission rate.

Thus, acknowledgment mechanisms are very important for at least the following two reasons:

1. Providing reliability guarantees in spite of losses and errors in the network

2. Controlling various aspects of the communication system, such as delay, queue sizes and congestion

1.2 Coding across packets

However, acknowledgment schemes cannot solve all problems. First of all, the feedback may itself be expensive, unreliable or very delayed. This happens, for instance, in satellite

links. In such situations, one has to rely on coding across packets to ensure reliability.

Even if the system has the capability to deliver ACKs reliably and quickly, simple link-by-link ARQ is not sufficient in general, especially if we go beyond a single point-to-point link. The scenario and the requirements of the application may require something more than simple retransmission. A case in point is multicast over a network of broadcast-mode links, for instance, in wireless systems. If a packet it transmitted, it is likely to be received by several nearby nodes. If one of the nodes experienced a bad channel state and thereby lost the packet, then a retransmission strategy is not the best option, since the retransmission is useless from the viewpoint of the other receivers that have already received the packet. Instead, if we allowed coding across packets, then it is possible to convey simultaneously new information to all connected receivers. Reference [9] highlights the need for coding for the case of multicast traffic, even if feedback is present.

Another scenario where coding across packets can make a difference is in certain network topologies where multiple flows have to traverse a bottleneck link. A good example is the butterfly network from [1], which is shown in Figure 1-1. Here, simple ARQ applied to each link can ensure there are no losses on the links. However, even if the links are error-free, the node D has to code (bitwise XOR) across packets in order to achieve the highest possible multicast throughput of 2 packets per time-slot. In the absence of coding, the highest achievable rate is 1.5 packets per time-slot.

Through this example, [1] introduced the field of network coding. The key idea is that nodes inside the network are allowed to perform coding operations on incoming packets, and send out coded versions of packets instead of simply routing the incoming packets onto outgoing links. An algebraic framework for network coding was proposed by Koetter and Médard in [10]. Network coding has been shown to achieve the multicast capacity of any network. In fact, [11] showed that linear coding suffices if all the multicast sessions have the same destination set. Reference [12] presented a random linear network coding approach for this problem. This approach is easy to implement, and yet, does not compromise on throughput. The problem of network coding based multicast with a cost criterion has also been studied, and distributed algorithms have been proposed to solve this problem [13], [14]. Network coding also readily extends to networks with broadcast-mode links or lossy

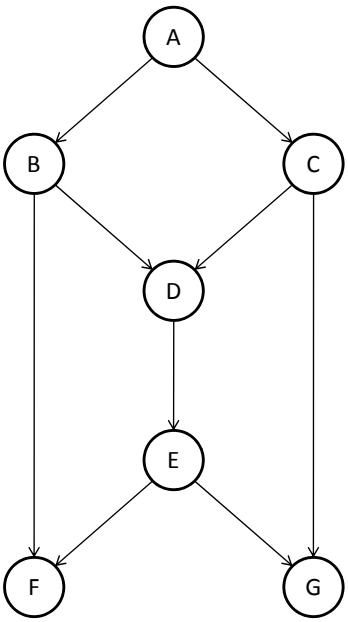


Figure 1-1: The butterfly network of [1]

links [15], [16], [17]. Thus, there are situations where coding is indispensable from a throughput perspective.

Besides improving throughput, network coding can also be used to simplify network management. The work by Bhadra and Shakkottai [18] proposed an interesting scheme for large multi-hop networks, where intermediate nodes in the network have no queues. Only the source and destination nodes maintain buffers to store packets. The packet losses that occur due to the absence of buffers inside the network, are compensated for by random linear coding across packets at the source.

In short, the benefits of network coding can be viewed to arise from two basic and distinct reasons:

1. Resilience to losses and errors

2. Managing how flows share bottleneck links

Several solutions have been proposed that make use of coding across packets. Each solution has its own merits and demerits, and the optimal choice depends on the needs of the application. We compare below, three such approaches – digital fountain codes, random linear network coding and priority encoding transmission.

1. Digital fountain codes: The digital fountain codes ([19, 20, 21]) constitute a well-known approach to this problem. From a block of k transmit packets, the sender generates random linear combinations in such a way that the receiver can, with high probability, decode the block once it receives *any* set of slightly more than k linear combinations. This approach has low complexity and requires no feedback, except to signal successful decoding of the block. However, fountain codes are designed for a point-to-point erasure channel and in their original form, do not extend readily to a network setting. Consider a two-link tandem network. An end-to-end fountain code with simple forwarding at the middle node will result in throughput loss. If the middle node chooses to decode and re-encode an entire block, the scheme will be sub-optimal in terms of delay, as pointed out by [22]. In this sense, the fountain code approach is not composable across links. For the special case of tree networks, there has been some recent work on composing fountain codes across links by enabling the middle node to re-encode even before decoding the entire block [23].

2. Random linear network coding: Network coding was originally introduced for the case of error-free networks with specified link capacities ([1, 10]), and was extended to the case of erasure networks [15], [16], [17]. In contrast to fountain codes, the random linear network coding solution of [12] and [16] does not require decoding at intermediate nodes and can be applied in any network. Each node transmits a random linear combination of all coded packets it has received so far. This solution ensures that with high probability, the transmitted packet will have what we call the *innovation guarantee property*, *i.e.*, it will be *innovative*[1] to every receiver that receives it successfully, except if the receiver already knows as much as the sender. Thus, every successful reception will bring a unit of new information. In [16], this scheme is shown to achieve capacity for the case of a multicast session.

The work of Dana *et al.* [15] also studied a wireless erasure network with broadcast but no interference, and established the capacity region. Unlike [15] however, the scheme of [16] does not require the destination to be provided the location of all the erasures through-

[1] An innovative packet is a linear combination of packets which is linearly independent of previously received linear combinations, and thus conveys new information. See Section 2.3 for a formal definition.

out the network.

An important problem with both fountain codes and random linear network coding is that although they are rateless, the encoding operation is performed on a block (or generation) of packets. This means that in general, there is no guarantee that the receiver will be able to extract and pass on to higher layers, any of the original packets from the coded packets till the entire block has been received. This leads to a decoding delay.

Such a decoding delay is not a problem if the higher layers will anyway use a block only as a whole (*e.g.*, file download). This corresponds to traditional approaches in information theory where the message is assumed to be useful only as a whole. No incentive is placed on decoding "a part of the message" using a part of the codeword. However, many applications today involve broadcasting a continuous stream of packets in real-time (*e.g.*, video streaming). Sources generate a stream of messages which have an intrinsic temporal ordering. In such cases, playback is possible only till the point up to which all packets have been recovered, which we call *the front of contiguous knowledge*. Thus, there is incentive to decode the older messages earlier, as this will reduce the playback latency. The above schemes would segment the stream into blocks and process one block at a time. Block sizes will have to be large to ensure high throughput. However, if playback can begin only after receiving a full block, then large blocks will imply a large delay.

This raises an interesting question: can we code in such a way that playback can begin even before the full block is received? In other words, we are more interested in packet delay than block delay. These issues have been studied using various approaches by [24], [25] and [26] in a point-to-point setting. However, in a network setting, the problem is not well understood. Moreover, these works do not consider the queue management aspects of the problem. In related work, [27] and [28] address the question of how many original packets are revealed before the whole block is decoded in a fountain code setting. However, performance may depend on not only *how much data* reaches the receiver in a given time, but also *which part of the data*. For instance, playback delay depends on not just the number of original packets that are recovered, but also the order in which they are recovered. One option is to precode the packets using some form of multiple description code []. In that case, only the number of received coded packets would matter, and the order in which they

are received. However, in real-time streaming applications, this approach is likely to have high computational complexity. Therefore, it is better to design the network so that it is aware of the ordering of the data, and tries to deliver the earlier packets first.

3. Priority encoding transmission: The scheme proposed in [29], known as priority encoding transmission (PET), addresses this problem by proposing a code for the erasure channel that ensures that a receiver will receive the first (or highest priority) i messages using the first k_i coded packets, where k_i increases with decreasing priority. In [30], [31], this is extended to systems that perform network coding. A concatenated network coding scheme is proposed in [31], with a delay-mitigating pre-coding stage. This scheme guarantees that the k^{th} innovative reception will enable the receiver to decode the k^{th} message. In such schemes however, the ability to decode messages in order requires a reduction in throughput because of the pre-coding stage.

Even in the presence of coding, we still need feedback to implement various control mechanisms in the network. Especially, congestion control requires some mechanism to infer the build-up of congestion. The problem of decoding delay or queue management could also become simpler if we make use of feedback well. In short, both feedback and coding have their benefits and issues. A good communication system design will have to employ both concepts in a synergistic manner.

1.3 Problems addressed

This leads to the question – how to combine the benefits of ARQ and network coding? The goal is to extend ARQ's desirable properties in the point-to-point context, to systems that require coding across packets.

Throughout this thesis, we focus on linear codes. The exact mechanism for coding across packets in a practical system is described in Chapter 4. For now, we can visualize the coding operation as follows. Imagine the original data packets are unknown variables. The coded packets generated by the encoder can be viewed as linear equations in these unknown variables.

A variety of interesting problems arise when we try to incorporate a coding-based ap-

proach into an acknowledgment-based mechanism. The thesis focuses on the following problems:

1. *The problem of decoding delay:* One of the problems with applying ARQ to a coded system is that a new reception may not always reveal the next unknown packet to the receiver. Instead, it may bring in a linear equation involving the packets. In conventional ARQ, upon receiving an ACK, the sender drops the ACKed packet and transmits the next one. But in a coded system, upon receiving an ACK for a linear equation, it is not clear which linear combination the sender should pick for its next transmission to obtain the best system performance. This is important because, if the receiver has to collect many equations before it can decode the unknowns involved, this could lead to a large decoding delay.

2. *How does coding affect queue management?* A related question is: upon receiving the ACK for a linear equation, which packet can be excluded from future coding, *i.e.*, which packet can be dropped from the sender's queue? More generally, if the nodes perform coding across packets, what should be the policy for updating the queues at various nodes in the network?

   In the absence of coding, the conventional approach to queue management at the sender node is that once a packet has been delivered at the destination, the sender finds this out using the acknowledgments, and then drops the packet. As far as the intermediate nodes in the network are concerned, they usually store and forward the packets, and drop a packet once it has been forwarded.

   With coding however, this simple policy may not be the best. As pointed out in the example in the introduction to Chapter 2, the conventional approach of retaining in the sender's queue any packet that has not been delivered to the destination is not optimal. In fact, we will show in Chapter 2 that this policy leads to a much larger queue than necessary, especially as the traffic load on the network increases.

   Also, if the intermediate nodes want to perform coding for erasure correction as in the work of [16], they cannot drop a packet immediately after forwarding it. They will need to retain it and involve it in the coding for a while, in order to make sure

the receiver has sufficiently many redundant (coded) packets to ensure reliable data transfer. In such a case, we need to define an effective queue management policy for the intermediate nodes as well.

3. *Practical deployment of network coding:* In order to bring the ideas of network coding into practice, we need a protocol that brings out the benefits of network coding while requiring very little change in the protocol stack. Flow control and congestion control in today's internet are predominantly handled by the Transmission Control Protocol (TCP), which works using the idea of a sliding transmission window of packets, whose size is controlled based on feedback. On the theoretical side, Chen *et al.* [32] proposed distributed rate control algorithms for network coding, in a utility maximization framework, and pointed out its similarity to TCP. However, to implement such algorithms in practice, we need to create a clean interface between network coding and TCP.

   The main idea behind TCP is to use acknowledgments of packets as they arrive *in correct sequence order* in order to guarantee reliable transport and also as a feedback signal for the congestion control loop. Now, if we allow the network to perform network coding however, the notion of an ordered sequence of packets as used by TCP goes missing. What is received is a randomly chosen linear combination, which might not immediately reveal an original packet, even if it conveys new information. The current ACK mechanism in TCP does not allow the receiver to acknowledge a packet before it has been decoded. This essentially means that the decoding delay will enter the round-trip time measurement, thereby confusing the TCP source. For network coding, we need a modification of the standard TCP mechanism that allows the acknowledgment of every unit of information received, even if it does not cause a packet to get decoded immediately.

   In network coding, there have been several important advances in bridging the gap between theory and practice. The distributed random linear coding idea, introduced by Ho *et al.* [33], is a significant step towards a robust practical implementation of network coding. The work by Chou *et al.* [34] introduced the idea of embedding the

coefficients used in the linear combination in the packet header, and also the notion of generations (coding blocks). In other work, Katti *et al.* [35] used the idea of local opportunistic coding to present a practical implementation of a network coded system for unicast. The use of network coding in combination with opportunistic routing was presented in [36]. However, these works rely on a batch-based coding mechanism which is incompatible with TCP. Reference [37] proposed an on-the-fly coding scheme, but the packets are acknowledged only upon decoding. Thus, none of these works allows an ACK-based sliding-window network coding approach that is compatible with TCP. This is the problem we address in our current work.

These issues motivate the following questions – if we have feedback in a system with network coding, what is the best possible tradeoff between throughput, delay and queue size? In particular, how close can we get to the performance of ARQ for the point-to-point case? And finally, how to incorporate network coding in existing congestion control protocols such as TCP?

1.4 Main contributions

1. *Theoretical contributions:* The thesis introduces a new notion called the notion of a node "seeing a packet" (The reader is referred to Section 2.1 for the formal definition.). This notion enables the study of delay and queue occupancy in systems involving coding across packets, by mapping these problems to well-known problems within the framework of traditional queuing theory. Using this notion, we compute the expected queue size in a variety of scenarios with network coding. We also develop models to analyze the decoding and delivery delay in such systems. More specifically, our contributions include the following:

   (a) In a packet erasure broadcast scenario, we propose a novel queue management algorithm called 'drop-when-seen', which ensures that the queue size tracks the true information backlog between the sender and the receivers, without compromising on reliability or throughput. Our algorithm achieves the optimal heavy-

traffic asymptotic behavior of *the expected queue size at the sender* as a function of $1/(1-\rho)$, as the load factor ρ approaches its limiting value of 1.

(b) In the same scenario, we propose a new coding algorithm which is throughput-optimal and is conjectured to achieve the optimal heavy-traffic asymptotic growth of *the expected delay per packet*, as the system load approaches capacity. This algorithm is compatible with an appropriately modified drop-when-seen queue management policy as well, implying that the algorithm will also achieve the optimal asymptotic behavior of the queue size.

2. *Practical contributions:* Using the notion of seen packets, we develop practically useful queue management algorithms as well as a new congestion control protocol for coded networks. In particular, we present a new interpretation of acknowledgments (ACKs) where the nodes acknowledge a packet upon seeing it, without having to wait till it is decoded. This new type of ACK is expected to prove very useful in realizing the benefits of network coding in practice.

   In particular, it enables the incorporation of network coding into the current TCP/IP protocol stack. We propose the introduction of a new network coding layer between the TCP and IP layers. The network coding layer accepts packets from the sender TCP and transmits random linear combinations of these packets into the network. Nodes inside the network may further transform these coded packets by re-encoding them using random linear coding. We present a real-life implementation of our new protocol on a testbed and demonstrate its benefits. Our work is a step towards the implementation of TCP over lossy networks in conjunction with new approaches such as multipath and opportunistic routing, and potentially even multicast.

1.5 Outline of the thesis

The thesis is organized as follows. Chapter 2 studies the problem of queuing in coded networks and presents a generic approach to queue management in the presence of coding across packets. We also introduce a new type of acknowledgment in order to implement

this approach. Chapter 3 addresses the important problem of coding for delay-sensitive applications. In particular, it introduces a new algorithm for adaptive coding based on feedback, that ensures that the receiver does not experience much delay waiting to decode the original packets. Chapter 4 makes use of the new type of acknowledgment introduced in Chapter 2 in order to fit network coding into the TCP protocol. We propose the insertion of a new layer inside the protocol stack between TCP and IP that provides a clean interface between TCP and network coding. As a result, the error correction and multipath capabilities of coding are now made available to any application that expects a TCP interface. After laying the theoretical foundations of the new protocol in Chapter 4, we then present an experimental evaluation of the performance of this protocol in a real testbed. This is described in Chapter 5. Finally, Chapter 6 presents a summary of the thesis with pointers for potential extensions in the future.

Chapter 2

Queue management in coded networks

This chapter explores the option of using acknowledgments to manage effectively the queues at the nodes in a network that performs network coding. If packets arrive at the sender according to some stochastic process, (as in [38, 39]) and links are lossy (as in [16, 17]), then the queue management aspect of the problem becomes important. The main questions that we address in this chapter are – in a network that employs network coding, when can the sender drop packets from its queue? Also, which packets should intermediate nodes store, if any?

In the absence of coding, the conventional approach to queue management at the sender node is that once a packet has been delivered at the destination, the sender finds this out using the acknowledgments, and then drops the packet. As far as the intermediate nodes inside the network are concerned, they usually store and forward the packets, and drop a packet once it has been forwarded to the next hop.

With coding however, this simple policy may not be the best. As pointed out in the example below, the conventional approach of retaining in the sender's queue any packet that has not been delivered to the destination is not optimal. In fact, we shall show in this chapter that this policy leads to a much larger queue than necessary, especially as the traffic load on the network increases.

Also, if the intermediate nodes want to perform coding for erasure correction as in the work of [16], they cannot drop a packet immediately after forwarding it. They will need to retain it and involve it in the coded transmissions for a while, in order to make

sure the receiver has sufficiently many redundant (coded) packets to be able to decode the transmitted data. In such a case, we need to define an effective queue management policy for the intermediate nodes as well.

An example: Consider a packet erasure broadcast channel with one sender and many receivers. Assume that in each slot the sender transmits a linear combination of packets that have arrived thus far, and is immediately informed by each receiver whether the transmission succeeded or got erased. With such feedback, one option for the sender is to drop packets that every receiver has decoded, as this would not affect the reliability. However, storing all undecoded packets may be suboptimal. Consider a situation where the sender has n packets $\mathbf{p_1}, \mathbf{p_2} \ldots, \mathbf{p_n}$, and every receiver has received the following set of $(n-1)$ linear combinations: $(\mathbf{p_1}+\mathbf{p_2})$, $(\mathbf{p_2}+\mathbf{p_3})$, $\ldots$, $(\mathbf{p_{n-1}}+\mathbf{p_n})$. A drop-when-decoded scheme will not allow the sender to drop any packet, since no packet can be decoded by any receiver yet. However, the true backlog in terms of the amount of information to be conveyed has a size of just 1 packet. We ideally want queue size to correspond to the information backlog. Indeed, in this example, it would be sufficient if the sender stores any one $\mathbf{p_i}$ in order to ensure reliable delivery.

This example indicates that the ideal queuing policy will make sure that the physical queue occupancy tracks the backlog in degrees of freedom, which is also called the *virtual queue* ([38, 39]).

In this chapter, we propose a new queue management policy for coded networks. This policy allows a node to drop one packet for every degree of freedom that is delivered to the receiver. As a result, our policy allows the physical queue occupancy to track the virtual queue size. The chapter is organized as follows. Section 2.1 introduces the central idea of the thesis, namely, the notion of a node 'seeing' a packet. Section 2.2 explains our contribution in the light of the related earlier work. The problem setup is specified in Section 2.3. Section 2.4 presents and analyzes a baseline queue update algorithm called the drop-when-decoded algorithm. Section 2.5 presents our new queue update algorithm in a generic form. This is followed by Section 2.6 which proposes an easy-to-implement variant of the generic algorithm, called the drop-when-seen algorithm. Section 2.7 studies the overhead associated with implementing these algorithms. Finally, Section 2.8 presents

the conclusions.

2.1 'Seeing' a packet – a novel acknowledgment mechanism

In this work, we treat packets as vectors over a finite field. We restrict our attention to linear network coding. Therefore, the state of knowledge of a node can be viewed as a vector space over the field (see Section 2.3 for further details).

We propose a new acknowledgment mechanism that uses feedback to *acknowledge degrees of freedom*[1] *instead of original decoded packets*. Based on this new form of ACKs, we propose an online coding module that naturally generalizes ARQ to coded systems. The code implies a queue update algorithm that ensures that *the physical queue size at the sender will track the backlog in degrees of freedom*.

It is clear that packets that have been decoded by all receivers need not be retained at the sender. But, our proposal is more general than that. The key intuition is that we can ensure reliable transmission even if we restrict the sender's transmit packet to be chosen from a subspace that is independent[2] of the subspace representing the common knowledge available at all the receivers.

In other words, *the sender need not use for coding (and hence need not store) any information that has already been received by all the receivers*. Therefore, at any point in time, the queue simply needs to store a basis for a coset space with respect to the subspace of knowledge common to all the receivers. We define a specific way of computing this basis using the new notion of a node "seeing" a message packet, which is defined below.

Definition 1 (Index of a packet)**.** *For any positive integer k, the k^{th} packet that arrives at the sender is said to have an* index k.

[1] Here, *degree of freedom* refers to a new dimension in the appropriate vector space representing the sender's knowledge.

[2] A subspace S_1 is said to be *independent* of another subspace S_2 if $S_1 \cap S_2 = \{\mathbf{0}\}$. See [40] for more details.

Definition 2 (Seeing a packet)**.** *A node is said to have* seen *a message packet* $\mathbf{p}$ *if it has received enough information to compute a linear combination of the form* $(\mathbf{p} + \mathbf{q})$*, where* $\mathbf{q}$ *is itself a linear combination involving only packets with an index greater than that of* $\mathbf{p}$*. (Decoding implies seeing, as we can pick* $\mathbf{q} = \mathbf{0}$*.)*

In our scheme, the feedback is utilized as follows. In conventional ARQ, a receiver ACKs a packet upon decoding it successfully. However, in our scheme **a receiver ACKs a packet when it sees the packet**. Our new scheme is called the *drop-when-seen* algorithm because the sender *drops a packet if all receivers have seen (ACKed) it.*

Since decoding implies seeing, the sender's queue is expected to be shorter under our scheme compared to the drop-when-decoded scheme. However, we will need to show that in spite of dropping seen packets even before they are decoded, we can still ensure reliable delivery. To prove this, we present a deterministic coding scheme that uses only unseen packets and still guarantees that the coded packet will **simultaneously cause each receiver that receives it successfully, to see its next unseen packet**. We shall prove later that seeing a new packet translates to receiving a new degree of freedom. This means, the innovation guarantee property (Definition 7) is satisfied and therefore, reliability and 100% throughput can be achieved (see Algorithm 2 (b) and corresponding Theorems 6 and 8 in Section 2.6).

The intuition is that, if all receivers have seen $\mathbf{p}$, then their uncertainty can be resolved using only packets with index more than that of $\mathbf{p}$ because after decoding these packets, the receivers can compute $\mathbf{q}$ and hence obtain $\mathbf{p}$ as well. Therefore, even if the receivers have not decoded $\mathbf{p}$, no information is lost by dropping it, provided it has been seen by all receivers.

Next, we present an example that explains our algorithm for a simple two-receiver case. Section 2.6.3 extends this scheme to more receivers.

Example: Table 2.1 shows a sample of how the proposed idea works in a packet erasure broadcast channel with two receivers A and B. The sender's queue is shown after the arrival point and before the transmission point of a slot (see Section 2.3 for details on the setup). In each slot, based on the ACKs, the sender identifies the next unseen packet for A and

Time	Sender's queue	Transmitted packet	Channel state	A		B	
				Decoded	Seen but not decoded	Decoded	Seen but not decoded
1	$\mathbf{p_1}$	$\mathbf{p_1}$	$\rightarrow$ A, $\not\rightarrow$ B	$\mathbf{p_1}$	-	-	-
2	$\mathbf{p_1}, \mathbf{p_2}$	$\mathbf{p_1} \oplus \mathbf{p_2}$	$\rightarrow$ A, $\rightarrow$ B	$\mathbf{p_1}, \mathbf{p_2}$	-	-	$\mathbf{p_1}$
3	$\mathbf{p_2}, \mathbf{p_3}$	$\mathbf{p_2} \oplus \mathbf{p_3}$	$\not\rightarrow$ A, $\rightarrow$ B	$\mathbf{p_1}, \mathbf{p_2}$	-	-	$\mathbf{p_1}, \mathbf{p_2}$
4	$\mathbf{p_3}$	$\mathbf{p_3}$	$\not\rightarrow$ A, $\rightarrow$ B	$\mathbf{p_1}, \mathbf{p_2}$	-	$\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$	-
5	$\mathbf{p_3}, \mathbf{p_4}$	$\mathbf{p_3} \oplus \mathbf{p_4}$	$\rightarrow$ A, $\not\rightarrow$ B	$\mathbf{p_1}, \mathbf{p_2}$	$\mathbf{p_3}$	$\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$	-
6	$\mathbf{p_4}$	$\mathbf{p_4}$	$\rightarrow$ A, $\rightarrow$ B	$\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$	-	$\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p_4}$	-

Table 2.1: An example of the drop-when-seen algorithm

B. If they are the same packet, then that packet is sent. If not, their XOR is sent. It can be verified that, with this rule, every reception causes each receiver to see its next unseen packet.

In slot 1, $\mathbf{p_1}$ reaches A but not B. In slot 2, $(\mathbf{p_1} \oplus \mathbf{p_2})$ reaches A and B. Since A knows $\mathbf{p_1}$, it can also decode $\mathbf{p_2}$. As for B, it has now seen (but not decoded) $\mathbf{p_1}$. At this point, since A and B have seen $\mathbf{p_1}$, the sender drops it. This is acceptable even though B has not yet decoded $\mathbf{p_1}$, because B will eventually decode $\mathbf{p_2}$ (in slot 4), at which time it can obtain $\mathbf{p_1}$. Similarly, $\mathbf{p_2}$, $\mathbf{p_3}$ and $\mathbf{p_4}$ will be dropped in slots 3, 5 and 6 respectively. However, the drop-when-decoded policy will drop $\mathbf{p_1}$ and $\mathbf{p_2}$ in slot 4, and $\mathbf{p_3}$ and $\mathbf{p_4}$ in slot 6. Thus, our new strategy clearly keeps the queue shorter. This is formally proved in Theorem 1 and Theorem 6. The example also shows that it is fine to drop packets before they are decoded. Eventually, the future packets will arrive, thereby allowing the decoding of all the packets.

2.2 Background and our contribution

2.2.1 Related earlier work

In [41], Shrader and Ephremides study the queue stability and delay of block-based random linear coding versus uncoded ARQ for stochastic arrivals in a broadcast setting. However, this work does not consider the combination of coding and feedback in one scheme. In related work, [42] studies the case of load-dependent variable sized coding blocks with

ACKs at the end of a block, using a bulk-service queue model. The main difference in our work is that receivers ACK packets even before decoding them, and this enables the sender to perform online coding.

Sagduyu and Ephremides [43] consider online feedback-based adaptation of the code, and propose a coding scheme for the case of two receivers. This work focuses on the maximum possible stable throughput, and does not consider the use feedback to minimize queue size or decoding delay. In [44], the authors study the throughput of a block-based coding scheme, where receivers acknowledge the successful decoding of an entire block, allowing the sender to move to the next block. Next, they consider the option of adapting the code based on feedback for the multiple receiver case. They build on the two-receiver case of [43] and propose a greedy deterministic coding scheme that may not be throughput optimal, but picks a linear combination such that the number of receivers that immediately decode a packet is maximized. In contrast, in our work we consider throughput-optimal policies that aim to minimize queue size and delay.

In [37], Lacan and Lochin proposes an erasure coding algorithm called Tetrys to ensure reliability in spite of losses on the acknowledgment path. While this scheme also employs coding in the presence of feedback, their approach is to make minimal use of the feedback, in order to be robust to feedback losses. As opposed to such an approach, we investigate how best to use the available feedback to improve the coding scheme and other performance metrics. For instance, in the scheme in [37], packets are acknowledged (if at all) only when they are decoded, and these are then dropped from the coding window. However, we show in this work that, by dropping packets when they are seen, we can maintain a smaller coding window without compromising on reliability and throughput. A smaller coding window translates to lower encoding complexity and smaller queue size at the sender in the case of stochastic arrivals.

The use of ACKs and coded retransmissions in a packet erasure broadcast channel has been considered for multiple unicasts [45] and multicast ([46], [47], [48], [49]). The main goal of these works however, is to optimize the throughput. Other metrics such as queue management and decoding delay are not considered. In our work, we focus on using feedback to optimize these metrics as well, in addition to achieving 100% throughput in

a multicast setting. Our coding module (in Section 2.6.5) is closely related to the one proposed by Larsson in an independent work [48]. However, our algorithm is specified using the more general framework of seen packets, which allows us to derive the drop-when-seen queue management algorithm and bring out the connection between the physical queue and virtual queue sizes. Reference [48] does not consider the queue management problem. Moreover, using the notion of seen packets allows our algorithm to be compatible even with random coding. This in turn enables a simple ACK format and makes it suitable for practical implementation. (See Remark 2 for further discussion.)

2.2.2 Implications of our new scheme

The newly proposed scheme has many useful implications:

- **Queue size:** The physical queue size is upper-bounded by the sum of the backlogs in degrees of freedom between the sender and all the receivers. This fact implies that as the traffic load approaches capacity (as load factor $\rho \to 1$), the expected size of the physical queue at the sender is $O\left(\frac{1}{1-\rho}\right)$. This is the same order as for single-receiver ARQ, and hence, is order-optimal.

- **Queuing analysis:** Our scheme forms a natural bridge between the virtual and physical queue sizes. It can be used to extend results on the stability of virtual queues such as [38], [39] and [50] to physical queues. Earlier work has studied the backlog in degrees of freedom (virtual queue size) using traditional queuing theory techniques such as the transform based analysis for the queue size of M/G/1 queues, or even a Jackson network type approaches [16]. By connecting the degree-of-freedom occupancy to the physical queue size, we allow these results obtained for virtual queues, to be extended to the physical queue size of nodes in a network coded system.

- **Simple queue management:** Our approach based on *seen packets* ensures that the sender does not have to store linear combinations of the packets in the queue to represent the basis of the coset space. Instead, it can store the basis using the original uncoded packets themselves. Therefore, the queue follows a simple first-in-first-out service discipline.

- **Online encoding:** All receivers see packets in the same order in which they arrived at the sender. This gives a guarantee that the information deficit at the receiver is restricted to a set of packets that advances in a streaming manner and has a stable size (namely, the set of unseen packets). In this sense, the proposed encoding scheme is truly online.

- **Easy decoding:** Every transmitted linear combination is sparse – at most n packets are coded together for the n receiver case. This reduces the decoding complexity as well as the overhead for embedding the coding coefficients in the packet header.

- **Extensions:** We present our scheme for a single packet erasure broadcast channel. However, our algorithm is composable across links and can be applied to a tandem network of broadcast links. With suitable modifications, it can potentially be applied to a more general setup like the one in [17] provided we have feedback. Such extensions are discussed further in Chapter 6.

2.3 The setup

In this chapter, we consider a communication problem where a sender wants to broadcast a stream of data to n receivers. The data are organized into *packets*, which are essentially vectors of fixed size over a finite field $\mathbb{F}_q$. A packet erasure broadcast channel connects the sender to the receivers. Time is slotted. The details of the queuing model and its dynamics are described next.

The queuing model

Earlier work has studied the effect on throughput, of having a finite-sized buffer at the sender to store incoming packets [51], [52]. Fixing the queue size to a finite value makes the analysis more complicated, since we need to consider buffer overflow and its effects on the throughput. Instead, in this work, we take an approach where we assume that the sender has an infinite buffer, *i.e.*, a queue with no preset size constraints. We then study the behavior of the expected queue size in steady state in the limit of heavy traffic. This

analysis is more tractable, and will serve as a guideline for deciding the actual buffer sizes while designing the system in practice.

We assume that the sender is restricted to use linear codes. Thus, every transmission is a linear combination of packets from the incoming stream that are currently in the buffer. The vector of coefficients used in the linear combination summarizes the relation between the coded packet and the original stream. We assume that this coefficient vector is embedded in the packet header. A node can compute any linear combination whose coefficient vector is in the linear span of the coefficient vectors of previously received coded packets. In this context, the state of knowledge of a node can be defined as follows.

Definition 3 (Knowledge of a node)**.** *The* knowledge of a node *at some point in time is the set of all linear combinations of the original packets that the node can compute, based on the information it has received up to that point. The coefficient vectors of these linear combinations form a vector space called the* knowledge space *of the node.*

We use the notion of a virtual queue to represent the backlog between the sender and receiver in terms of linear degrees of freedom. This notion was also used in [38], [39] and [50]. There is one virtual queue for each receiver.

Definition 4 (Virtual queue)**.** *For $j = 1, 2, \ldots, n$, the size of the j^{th} virtual queue is defined to be the difference between the dimension of the knowledge space of the sender and that of the j^{th} receiver.*

We shall use the term *physical queue* to refer to the sender's actual buffer, in order to distinguish it from the virtual queues. Note that the virtual queues do not correspond to real storage.

Definition 5 (Degree of freedom)**.** *The term* degree of freedom *refers to one dimension in the knowledge space of a node. It corresponds to one packet worth of data.*

Definition 6 (Innovative packet)**.** *A coded packet with coefficient vector* $\mathbf{c}$ *is said to be* innovative *to a receiver with knowledge space V if $\mathbf{c} \notin V$. Such a packet, if successfully received, will increase the dimension of the receiver's knowledge space by one unit.*

Definition 7 (Innovation guarantee property)**.** *Let V denote the sender's knowledge space, and V_j denote the knowledge space of receiver j for $j = 1, 2, \ldots, n$. A coding scheme is said to have the* innovation guarantee property *if, in every slot, the coefficient vector of the transmitted linear combination is in $V \backslash V_j$ for every j such that $V_j \neq V$. In other words, the transmission is innovative to every receiver except when the receiver already knows everything that the sender knows.*

Arrivals

Packets arrive into the sender's physical queue according to a Bernoulli process[3] of rate λ. An arrival at the physical queue translates to an arrival at each virtual queue since the new packet is a new degree of freedom that the sender knows, but none of the receivers knows.

Service

The channel accepts one packet per slot. Each receiver either receives this packet with no errors (with probability μ) or an erasure occurs (with probability $(1 - \mu)$). Erasures occur independently across receivers and across slots. The receivers are assumed to be capable of detecting an erasure.

We only consider coding schemes that satisfy the innovation guarantee property. This property implies that if the virtual queue of a receiver is not empty, then a successful reception reveals a previously unknown degree of freedom to the receiver and the virtual queue size decreases by one unit. We can thus map a successful reception by some receiver to one unit of service of the corresponding virtual queue. This means, in every slot, each virtual queue is served independently of the others with probability μ.

The relation between the service of the virtual queues and the service of the physical queue depends on the queue update scheme used, and will be discussed separately under each update policy.

[3]We have assumed Bernoulli arrivals for ease of exposition. However, we expect the results to hold for more general arrival processes as well.

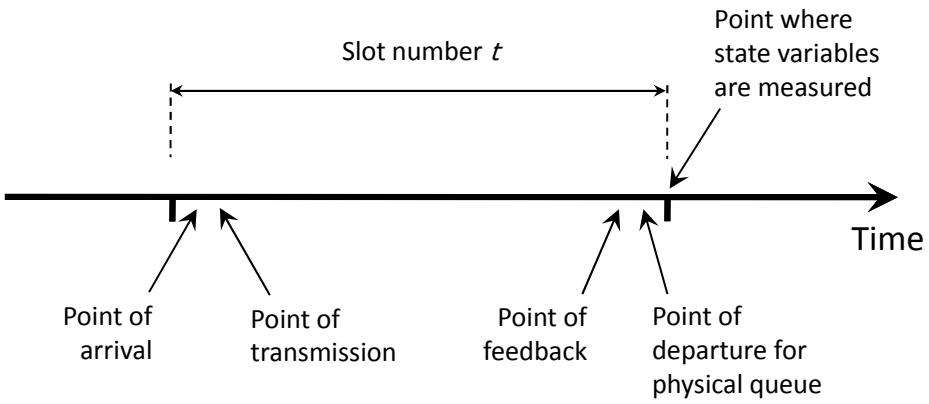


Figure 2-1: Relative timing of arrival, service and departure points within a slot

Feedback

We assume perfect delay-free feedback. In Algorithm 1 below, feedback is used to indicate successful decoding. For all the other algorithms, the feedback is needed in every slot to indicate the occurrence of an erasure.

Timing

Figure 2-1 shows the relative timing of various events within a slot. All arrivals are assumed to occur *just after the beginning* of the slot. The point of transmission is after the arrival point. For simplicity, we assume very small propagation time. Specifically, we assume that the transmission, unless erased by the channel, reaches the receivers before they send feedback for that slot and feedback from all receivers reaches the sender *before the end of the same slot*. Thus, the feedback incorporates the current slot's reception also. Based on this feedback, packets are dropped from the physical queue *just before the end of the slot*, according to the queue update rule. Queue sizes are measured at the end of the slot.

The load factor is denoted by $\rho \triangleq \lambda/\mu$. In what follows, we will study the asymptotic behavior of the expected queue size and decoding delay under various policies, in the heavy traffic limit, *i.e.*, as $\rho \to 1$ from below. For the asymptotics, we assume that either λ or μ is fixed, while the other varies causing ρ to increase to 1.

In the next section, we first present a baseline algorithm – retain packets in the queue until the feedback confirms that they have been decoded by all the receivers. Then, we

present a new queue update policy and a coding algorithm that is compatible with this rule. The new policy allows the physical queue size to track the virtual queue sizes.

2.4 Algorithm 1: Drop when decoded (baseline)

We first present the baseline scheme which we will call Algorithm 1. It combines a random coding strategy with a drop-when-decoded rule for queue update. The coding scheme is an online version of [16] with no preset generation size – a coded packet is formed by computing a random linear combination of all packets currently in the queue. With such a scheme, the innovation guarantee property will hold with high probability, provided the field size is large enough (We assume the field size is large enough to ignore the probability that the coded packet is not innovative. It can be incorporated into the model by assuming a slightly larger probability of erasure because a non-innovative packet is equivalent to an erasure.).

For any receiver, the packets at the sender are unknowns, and each received linear combination is an equation in these unknowns. Decoding becomes possible whenever the number of linearly independent equations catches up with the number of unknowns involved. The difference between the number of unknowns and number of equations is essentially the backlog in degrees of freedom, *i.e.,* the virtual queue size. Thus, *a virtual queue becoming empty translates to successful decoding at the corresponding receiver*. Whenever a receiver is able to decode in this manner, it informs the sender. Based on this, the sender tracks which receivers have decoded each packet, and drops a packet if it has been decoded by all receivers. From a reliability perspective, this is acceptable because there is no need to involve decoded packets in the linear combination.

Remark 1. *In general, it may be possible to solve for some of the unknowns even before the virtual queue becomes empty. For example, this could happen if a newly received linear combination cancels everything except one unknown in a previously known linear combination. It could also happen if some packets were involved in a subset of equations that can be solved among themselves locally. Then, even if the overall system has more unknowns than equations, the packets involved in the local system can be decoded. However, these*

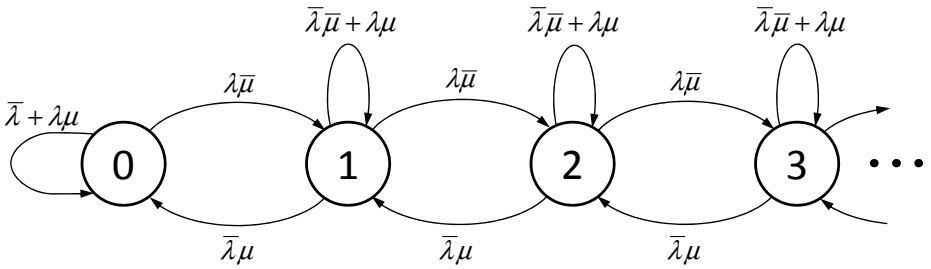


Figure 2-2: Markov chain representing the size of a virtual queue. Here $\bar{\lambda} := (1-\lambda)$ and $\bar{\mu} := (1-\mu)$.

are secondary effects and we ignore them in this analysis. Equivalently, we assume that if a packet is decoded before the virtual queue becomes empty, the sender ignores the occurrence of this event and waits for the next emptying of the virtual queue before dropping the packet. We believe this assumption will not change the asymptotic behavior of the queue size, since decoding before the virtual queue becoming empty is a rare event with random linear coding over a large field.

2.4.1 The virtual queue size in steady state

We will now study the behavior of the virtual queues in steady state. But first, we introduce some notation:

$Q(t)$:= Size of the sender's physical queue at the end of slot t,

$Q_j(t)$:= Size of the j^{th} virtual queue at the end of slot t.

Figure 2-2 shows the Markov chain for $Q_j(t)$. If $\lambda < \mu$, then the chain $\{Q_j(t)\}$ is positive recurrent and has a steady state distribution given by [53]:

$$\pi_k := \lim_{t\to\infty} \mathbb{P}[Q_j(t) = k] = (1-\alpha)\alpha^k, \qquad k \geq 0, \tag{2.1}$$

where $\alpha = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}$.

Thus, the expected size of any virtual queue in steady state is given by:

$$\lim_{t\to\infty} \mathbb{E}[Q_j(t)] = \sum_{j=0}^{\infty} j\pi_j = (1-\mu)\cdot\frac{\rho}{(1-\rho)}. \tag{2.2}$$

Next, we analyze the physical queue size under this scheme.

2.4.2 The physical queue size in steady state

The following theorem characterizes the asymptotic behavior of the queue size under Algorithm 1 in the heavy traffic limit, *i.e.*, as the load on the system approaches capacity ($\rho \to 1$). We assume that λ and μ are themselves away from 1, but only their ratio approaches 1 from below. Comparing with Equation (2.2), this result makes it clear that the physical queue size does not track the virtual queue size.

Theorem 1. *The expected size of the physical queue in steady state for Algorithm 1 is* $\Omega\left(\frac{1}{(1-\rho)^2}\right)$.

Proof. Let T be the time an arbitrary arrival in steady state spends in the physical queue before departure, excluding the slot in which the arrival occurs (Thus, if a packet departs immediately after it arrives, then T is 0.). A packet in the physical queue will depart when each virtual queue has become empty at least once since its arrival. Let D_j be the time starting from the new arrival, till the next emptying of the j^{th} virtual queue. Then, $T = \max_j D_j$ and so, $\mathbb{E}[T] \geq \mathbb{E}[D_j]$. Hence, we focus on $\mathbb{E}[D_j]$.

We condition on the event that the state seen by the new arrival just before it joins the queue, is some state k. There are two possibilities for the queue state at the end of the slot in which the packet arrives. If the channel is ON in that slot, then there is a departure and the state at the end of the slot is k. If the channel is OFF, then there is no departure and the state is $(k+1)$. Now, D_j is simply the first passage time from the state at the end of that slot to state 0, *i.e.*, the number of slots it takes for the system to reach state 0 for the first time, starting from the state at the end of the arrival slot. Let $\Gamma_{u,v}$ denote the expected first passage time from state u to state v. The expected first passage time from state u to state 0, for $u > 0$ is given by the following expression:

$$\Gamma_{u,0} = \frac{u}{\mu - \lambda}.$$

This can be derived as follows. Consider the Markov chain $\{Q_j(t)\}$ for the virtual queue size, shown in Figure 2-2. Assume that the Markov chain has an initial distribu-

tion equal to the steady state distribution (Equivalently, assume that the Markov chain has reached steady state.). We use the same notation as in Section 2.4.

Define $N_m := \inf\{t \geq 1 : Q_j(t) = m\}$. We are interested in deriving for $k \geq 1$, an expression for $\Gamma_{k,0}$, the expected first passage time from state k to 0, *i.e.*,

$$\Gamma_{k,0} = \mathbb{E}[N_0 | Q_j(0) = k].$$

Define for $i \geq 1$:

$$X_i \triangleq a(i) - d(i)$$

where, $a(i)$ is the indicator function for an arrival in slot i, and $d(i)$ is the indicator function for the channel being on in slot i. Let $S_t \triangleq \sum_{i=1}^{t} X_i$. If $Q_j(t) > 0$, then the channel being on in slot t implies that there is a departure in that slot. Thus the correspondence between the channel being on and a departure holds for all $0 \leq t \leq N_0$. This implies that:

$$\text{for } t \leq N_0, Q_j(t) = Q_j(0) + S_t.$$

Thus, N_0 can be redefined as the smallest $t \geq 1$ such that S_t reaches $-Q_j(0)$. Thus, N_0 is a valid stopping rule for the X_i's which are themselves IID, and have a mean $\mathbb{E}[X] = (\lambda - \mu)$. We can find $\mathbb{E}[N_0]$ using Wald's equality [54]:

$$\mathbb{E}[S_{N_0} | Q_j(0) = k] = \mathbb{E}[N_0 | Q_j(0) = k] \cdot \mathbb{E}[X]$$

$$\text{i.e., } -k = \mathbb{E}[N_0 | Q_j(0) = k] \cdot (\lambda - \mu),$$

which gives:

$$\Gamma_{k,0} = \mathbb{E}[N_0 | Q_j(0) = k] = \frac{k}{\mu - \lambda}.$$

Now, because of the property that Bernoulli arrivals see time averages (BASTA) [55], an arbitrary arrival sees the same distribution for the size of the virtual queues, as the steady state distribution given in Equation (2.1).

Using this fact, we can compute the expectation of D_j as follows:

$$\begin{aligned}
\mathbb{E}[D_j] &= \sum_{k=0}^{\infty} \mathbb{P}(\text{New arrival sees state } k)\mathbb{E}[D_j|\text{State } k] \\
&= \sum_{k=0}^{\infty} \pi_k[\mu\Gamma_{k,0} + (1-\mu)\Gamma_{k+1,0}] \\
&= \sum_{k=0}^{\infty} \pi_k \cdot \frac{\mu k + (1-\mu)(k+1)}{\mu - \lambda} \\
&= \frac{1-\mu}{\mu} \cdot \frac{\rho}{(1-\rho)^2}. \qquad (2.3)
\end{aligned}$$

Now, the expected time that an arbitrary arrival in steady state spends in the system is given by:

$$\mathbb{E}[T] = \mathbb{E}[\max_j D_j] \geq \mathbb{E}[D_j] = \Omega\left(\frac{1}{(1-\rho)^2}\right).$$

Since each virtual queue is positive recurrent (assuming $\lambda < \mu$), the physical queue will also become empty infinitely often. Then we can use Little's law to find the expected physical queue size.

The expected queue size of the physical queue in steady state if we use algorithm 1 is given by:

$$\lim_{t\to\infty} \mathbb{E}[Q(t)] = \lambda\mathbb{E}[T] = \Omega\left(\frac{1}{(1-\rho)^2}\right).$$

This completes the proof. □

2.5 Algorithm 2 (a): Drop common knowledge

In this section, we first present a generic algorithm that operates at the level of knowledge spaces and their bases, in order to ensure that the physical queue size tracks the virtual queue size. Later, we shall describe a simple-to-implement variant of this generic algorithm.

2.5.1 An intuitive description

The aim of this algorithm is to drop as much data as possible from the sender's buffer while still satisfying the reliability requirement and the innovation guarantee property. In other words, the sender should store just enough data so that it can always compute a linear combination which is simultaneously innovative to all receivers who have an information deficit. As we shall see, the innovation guarantee property is sufficient for good performance.

After each slot, every receiver informs the sender whether an erasure occurred, using perfect feedback. Thus, there is a slot-by-slot feedback requirement which means that the frequency of feedback messages is higher than in Algorithm 1. The main idea is to exclude from the queue any knowledge that is known to all the receivers. More specifically, the queue's contents must correspond to some basis of a vector space that is independent of the intersection of the knowledge spaces of all the receivers. We show in Lemma 2 that with this queuing rule, it is always possible to compute a linear combination of the current contents of the queue that will guarantee innovation, as long as the field size is more than n, the number of receivers.

The fact that the common knowledge is dropped suggests a modular or incremental approach to the sender's operations. Although the knowledge spaces of the receivers keep growing with time, the sender only needs to operate with the projection of these spaces on dimensions currently in the queue, since the coding module does not care about the remaining part of the knowledge spaces that is common to all receivers. Thus, the algorithm can be implemented in an incremental manner. It will be shown that this incremental approach is equivalent to the cumulative approach.

Table 2.2 shows the main correspondence between the notions used in the uncoded case and the coded case. We now present the queue update algorithm formally. Then we present theorems that prove that under this algorithm, the physical queue size at the sender tracks the virtual queue size.

All operations in the algorithm occur over a finite field of size $q > n$. The basis of a node's knowledge space is stored as the rows of a basis matrix. The representation and all

	Uncoded Networks	**Coded Networks**
Knowledge represented by	Set of received packets	Vector space spanned by the coefficient vectors of the received linear combinations
Amount of knowledge	Number of packets received	Number of linearly independent (innovative) linear combinations of packets received (*i.e.*, dimension of the knowledge space)
Queue stores	All undelivered packets	Linear combination of packets which form a basis for the *coset space* of the common knowledge at all receivers
Update rule after each transmission	If a packet has been received by all receivers drop it.	Recompute the common knowledge space V_Δ; Store a new set of linear combinations so that their span is independent of V_Δ

Table 2.2: The uncoded vs. coded case

operations are in terms of local coefficient vectors (*i.e.*, with respect to the current contents of the queue) and not global ones (*i.e.*, with respect to the original packets).

2.5.2 Formal description of the algorithm

Algorithm 2 (a)

1. Initialize basis matrices B, $B_1, \ldots, B_n$ to the empty matrix. These contain the bases of the incremental knowledge spaces of the sender and receivers in that order.

2. Initialize the vector g to the zero vector. This will hold the coefficients of the transmitted packet in each slot.
   In every time slot, do:

3. *Incorporate new arrivals:*

   Let a be the number of new packets that arrived at the beginning of the slot. Place these packets at the end of the queue. Let B have b rows. Set B to I_{a+b}. (I_m denotes the identity matrix of size m.) Note that B will always be an identity matrix. To make the number of columns of all matrices consistent (*i.e.*, equal to $a + b$), append a all-zero columns to each B_j.

4. *Transmission:*

   If B is not empty, update $\mathbf{g}$ to be any vector that is in $span(B)$, but not in $\cup_{\{j:B_j \subsetneq B\}} span(B_j)$. (Note: $span(B)$ denotes the row space of B.)

   Lemma 2 shows that such a $\mathbf{g}$ exists. Let $\mathbf{y_1}, \mathbf{y_2}, \ldots \mathbf{y_Q}$ represent the current contents of the queue, where the queue size $Q = (a + b)$. Compute the linear combination $\sum_{i=1}^{Q} g_i \mathbf{y_i}$ and transmit it on the packet erasure broadcast channel. If B is empty, set $\mathbf{g}$ to $\mathbf{0}$ and transmit nothing.

5. *Incorporate feedback:*

   Once the feedback arrives, for every receiver $j = 1$ to n, do:

   > If $\mathbf{g} \neq \mathbf{0}$ and the transmission was successfully received by receiver j in this slot, append $\mathbf{g}$ as a new row to B_j.

6. *Separate out the knowledge that is common to all receivers:*

   Compute the following (the set notation used here considers the matrices as a set of row vectors):

   B_Δ := Any basis of $\cap_{j=1}^{n} span(B_j)$.

   B' := Completion of B_Δ into a basis of $span(B)$.

   B'' := $B' \backslash B_\Delta$.

   B'_j := Completion of B_Δ into a basis of $span(B_j)$ in such a way that, if we define $B''_j := B'_j \backslash B_\Delta$, then the following holds: $B''_j \subseteq span(B'')$. Lemma 1 proves that this is possible.

7. *Update the queue contents:*

   Replace the contents of the queue with packets $\mathbf{y'_1}, \mathbf{y'_2}, \ldots \mathbf{y'_{Q'}}$ of the form $\sum_{i=1}^{Q} h_i \mathbf{y_i}$ for each $\mathbf{h} \in B''$. The new queue size Q' is thus equal to the number of rows in B''.

8. *Recompute local coefficient vectors with respect to the new queue contents:*

   Find a matrix C_j such that $B''_j = X_j B''$ (this is possible because $B''_j \subseteq span(B'')$). Call X_j the new B_j. Update the value of B to $I_{Q'}$.

9. Go back to step 3 for the next slot.

The above algorithm essentially removes, at the end of each slot, the common knowledge (represented by the basis B_Δ) and retains only the remainder, B''. The knowledge spaces of the receivers are also represented in an incremental manner in the form of B''_j, excluding the common knowledge. Since $B''_j \subseteq span(B'')$, the B''_j vectors can be completely described in terms of the vectors in B''. It is as if B_Δ has been completely removed from the entire setting, and the only goal remaining is to convey $span(B'')$ to the receivers. Hence, it is sufficient to store linear combinations corresponding to B'' in the queue. B'' and B''_j get mapped to the new B and B_j, and the process repeats in the next slot.

Lemma 1. *In step 5 of the algorithm above, it is possible to complete B_Δ into a basis B'_j of each $span(B_j)$ such that $B''_j \subseteq span(B'')$.*

Proof. We show that any completion of B_Δ into a basis of $span(B_j)$ can be changed to a basis with the required property.

Let $B_\Delta = \{\mathbf{b_1}, \mathbf{b_2}, \ldots, \mathbf{b_m}\}$. Suppose we complete this into a basis C_j of $span(B_j)$ such that:

$$C_j = B_\Delta \cup \{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_{|B_j|-m}}\}.$$

Now, we claim that at the beginning of step 6, $span(B_j) \subseteq span(B)$ for all j. This can be proved by induction on the slot number, using the way the algorithm updates B and the B_j's. Intuitively, any receiver knows a subset of what the sender knows.

Therefore, for each vector $\mathbf{c} \in C_j \backslash B_\Delta$, $\mathbf{c}$ must also be in $span(B)$. Now, since $B_\Delta \cup B''$ is a basis of $span(B)$, we can write $\mathbf{c}$ as $\sum_{i=1}^{m} \alpha_i \mathbf{b_i} + \mathbf{c'}$ with $\mathbf{c'} \in span(B'')$. In this manner, each $\mathbf{c_i}$ gives a distinct $\mathbf{c'_i}$. It is easily seen that $C'_j := B_\Delta \cup \{\mathbf{c'_1}, \mathbf{c'_2}, \ldots, \mathbf{c'_{|B_j|-m}}\}$ is also a basis of the same space that is spanned by C_j. Moreover, it satisfies the property that $C'_j \backslash B_\Delta \subseteq span(B'')$. □

Lemma 2. *Let $\mathcal{V}$ be a vector space with dimension k over a field of size q, and let $\mathcal{V}_1, \mathcal{V}_2, \ldots \mathcal{V}_n$, be subspaces of $\mathcal{V}$, of dimensions $k_1, k_2, \ldots, k_n$ respectively. Suppose that $k > k_i$ for all $i = 1, 2, \ldots, n$. Then, there exists a vector that is in $\mathcal{V}$ but is not in any of the $\mathcal{V}_i$'s, if $q > n$.*

Proof. The total number of vectors in $\mathcal{V}$ is q^n. The number of vectors in $\mathcal{V}_i$ is q^{n_i}. Hence, the number of vectors in $\cup_{i=1}^{k}\mathcal{V}_i$ is at most $\sum_{i=1}^{k} q^{n_i}$. Now,

$$\sum_{i=1}^{k} q^{n_i} \leq kq^{n_{max}} \leq kq^{n-1} < q^n$$

where, n_{max} is $\max_i n_i$, which is at most $(n-1)$. Thus, $\mathcal{V}$ has more vectors than $\cup_{i=1}^{k}\mathcal{V}_i$. This completes the proof. □

This lemma is also closely related to the result in [48], which derives the smallest field size needed to ensure innovation guarantee.

2.5.3 Connecting the physical and virtual queue sizes

In this subsection, we will prove the following result that relates the size of the physical queue at the sender and the virtual queues, which themselves correspond to the backlog in degrees of freedom.

Theorem 2. *For Algorithm 2 (a), the physical queue size at the sender is upper bounded by the sum of the backlog differences between the sender and each receiver in terms of the number of degrees of freedom.*

Let $a(t)$ denote the number of arrivals in slot t, and let $A(t)$ be the total number of arrivals up to and including slot t, *i.e.*, $A(t) = \sum_{t'=0}^{t} a(t')$. Let $B(t)$ (resp. $B_j(t)$) be the matrix B (resp. B_j) after incorporating the slot t arrivals, *i.e.*, at the end of step 3 in slot t. Let $H(t)$ be a matrix whose rows are the *global* coefficient vectors of the queue contents at the end of step 3 in time slot t, *i.e.*, the coefficient vectors in terms of the original packet stream. Note that each row of $H(t)$ is in $\mathbb{F}_q^{A(t)}$.

Let $\mathbf{g}(t)$ denote the vector $\mathbf{g}$ at the calculated in step 4 in time slot t, *i.e.*, the local coefficient vector of the packet transmitted in slot t. Also, let $B_\Delta(t)$ (resp. $B''(t)$, $B'_j(t)$ and $B''_j(t)$) denote the matrix B_Δ (resp. B'', B'_j and B''_j) at the end of step 6 in time slot t.

Lemma 3. *The rows of $H(t)$ are linearly independent for all t.*

Proof. The proof is by induction on t.

Basis step: In the beginning of time slot 1, $a(1)$ packets arrive. So, $H(1) = I_{a(1)}$ and hence the rows are linearly independent.

Induction hypothesis: Assume $H(t-1)$ has linearly independent rows.

Induction step: The queue is updated such that the linear combinations corresponding to local coefficient vectors in B'' are stored, and subsequently, the $a(t)$ new arrivals are appended. Thus, the relation between $H(t-1)$ and $H(t)$ is:

$$H(t) = \begin{bmatrix} B''(t-1)H(t-1) & 0 \\ 0 & I_{a(t)} \end{bmatrix}.$$

Now, $B''(t-1)$ has linearly independent rows, since the rows form a basis. The rows of $H(t-1)$ are also linearly independent by hypothesis. Hence, the rows of $B''(t-1)H(t-1)$ will also be linearly independent. Appending $a(t)$ zeros and then adding an identity matrix block in the right bottom corner does not affect the linear independence. Hence, $H(t)$ also has linearly independent rows. □

Define the following:

$$\begin{aligned}
U(t) &\triangleq \text{Row span of } H(t) \\
U_j(t) &\triangleq \text{Row span of } B_j(t)H(t) \\
U'_j(t) &\triangleq \text{Row span of } B'_j(t)H(t) \\
U'_\Delta(t) &\triangleq \cap_{j=1}^{n} U'_j(t) \\
U''(t) &\triangleq \text{Row span of } B''(t)H(t) \\
U''_j(t) &\triangleq \text{Row span of } B''_j(t)H(t).
\end{aligned}$$

All the vector spaces defined above are subspaces of $\mathbb{F}_q^{A(t)}$. Figure 2-3 shows the points at which these subspaces are defined in the slot.

The fact that $H(t)$ has full row rank (proved above in Lemma 3) implies that the operations performed by the algorithm in the domain of the local coefficient vectors can be mapped to the corresponding operations in the domain of the global coefficient vectors:

1. The intersection subspace $U'_\Delta(t)$ is indeed the row span of $B_\Delta(t)H(t)$.

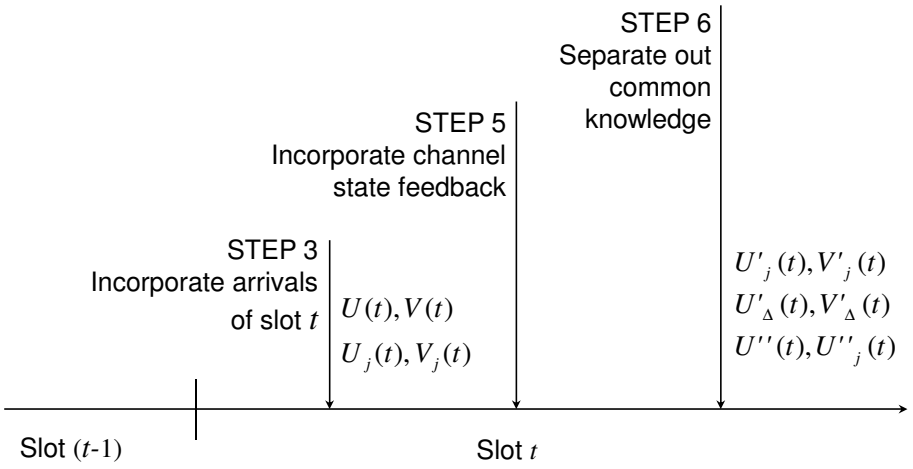


Figure 2-3: The main steps of the algorithm, along with the times at which the various $U(t)$'s are defined

2. Let $R_j(t)$ be an indicator (0-1) random variable which takes the value 1 if and only if the transmission in slot t is successfully received without erasure by receiver j and in addition, receiver j does not have all the information that the sender has. Let $\tilde{\mathbf{g}}_\mathbf{j}(t) := R_j(t)\mathbf{g}(t)H(t)$. Then,

$$U'_j(t) = U_j(t) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t)) \tag{2.4}$$

where $\oplus$ denotes direct sum of vector spaces. The way the algorithm chooses $\mathbf{g}(t)$ guarantees that if $R_j(t)$ is non-zero, then $\tilde{\mathbf{g}}_\mathbf{j}(t)$ will be outside the corresponding $U_j(t)$, *i.e.*, it will be innovative. This fact is emphasized by the direct sum in this equation.

3. Because of the way the algorithm performs the completion of the bases in the local domain in step 6, the following properties hold in the global domain:

$$U(t) = U'_\Delta(t) \oplus U''(t) \tag{2.5}$$

$$U'_j(t) = U'_\Delta(t) \oplus U''_j(t) \quad \text{and,} \tag{2.6}$$

$$U''_j(t) \subseteq U''(t), \quad \forall j = 1, 2, \ldots, n. \tag{2.7}$$

From the above properties, we can infer that $U''_1(t) + U''_2(t) + \ldots U''_n(t) \subseteq U''(t)$. After

incorporating the arrivals in slot $t+1$, this gives $U_1(t+1) + U_2(t+1) + \ldots U_n(t+1) \subseteq U(t+1)$. Since this is true for all t, we write it as:

$$U_1(t) + U_2(t) + \ldots U_n(t) \subseteq U(t). \tag{2.8}$$

Now, in order to relate the queue size to the backlog in number of degrees of freedom, we define the following vector spaces which represent the *cumulative* knowledge of the sender and receivers (See Figure 2-3 for the timing):

- $V(t) \triangleq$ Sender's knowledge space after incorporating the arrivals (at the end of step 3) in slot t. This is simply equal to $\mathbb{F}_q^{A(t)}$
- $V_j(t) \triangleq$ Receiver j's knowledge space at the end of step 3 in slot t
- $V_j'(t) \triangleq$ Receiver j's knowledge space in slot t, after incorporating the channel state feedback into $V_j(t)$, *i.e.*, $V_j'(t) = V_j(t) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t))$.
- $V_\Delta(t) \triangleq \cap_{j=1}^n V_j(t)$
- $V_\Delta'(t) \triangleq \cap_{j=1}^n V_j'(t)$.

For completeness, we now prove the following facts about direct sums of vector spaces that we shall use.

Lemma 4. *Let V be a vector space and let $V_\Delta, U_1, U_2, \ldots U_n$ be subspaces of V such that, V_Δ is independent of the span of all the U_j's,* i.e., *$dim[V_\Delta \cap (U_1 + U_2 + \ldots + U_n)] = 0$. Then,*

$$V_\Delta \oplus [\cap_{i=1}^n U_i] = \cap_{i=1}^n [V_\Delta \oplus U_i].$$

Proof. For any $z \in V_\Delta \oplus \cap_{i=1}^n U_i$, there is a $x \in V_\Delta$ and $y \in \cap_{i=1}^n U_i$ such that $z = x + y$. Now, for each i, $y \in U_i$. Thus, $z = x + y$ implies that $z \in \cap_{i=1}^n [V_\Delta \oplus U_i]$. Therefore, $V_\Delta \oplus \cap_{i=1}^n U_i \subseteq \cap_{i=1}^n [V_\Delta \oplus U_i]$.

Now, let $w \in \cap_{i=1}^n V_\Delta \oplus U_i$. Then for each i, there is a $x_i \in V_\Delta$ and $y_i \in U_i$ such that $w = x_i + y_i$. But, $w = x_i + y_i = x_j + y_j$ means that $x_i - x_j = y_i - y_j$. Now, $(x_i - x_j) \in V_\Delta$ and $(y_i - y_j) \in (U_1 + U_2 + \ldots + U_n)$. By hypothesis, these two vector spaces have only 0 in common. Thus, $x_i - x_j = y_i - y_j = 0$. All the x_i's are equal to a common $x \in V_\Delta$ and all the y_i's are equal to a common y which belongs to all the U_i's. This means, w

can be written as the sum of a vector in V_Δ and a vector in $\cap_{i=1}^n U_i$, thereby proving that $\cap_{i=1}^n [V_\Delta \oplus U_i] \subseteq V_\Delta \oplus \cap_{i=1}^n U_i$. □

Lemma 5. *Let $A, B,$ and C be three vector spaces such that B is independent of C and A is independent of $B \oplus C$. Then the following hold:*

1. *A is independent of B.*

2. *$A \oplus B$ is independent of C.*

3. *$A \oplus (B \oplus C) = (A \oplus B) \oplus C$.*

Proof. Statement 1 follows from the fact that B is a subset of $B \oplus C$. Hence, if $A \cap (B \oplus C)$ is empty, so is $A \cap B$.

For statement 2, we need to show that $(A \oplus B) \cap C = \{\mathbf{0}\}$. Consider any element $\mathbf{x} \in (A \oplus B) \cap C$. Since it is in $A \oplus B$, there exist unique $\mathbf{a} \in A$ and $\mathbf{b} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$. Now, since $\mathbf{b} \in B$ and $\mathbf{x} \in C$, it follows that $\mathbf{a} = \mathbf{x} - \mathbf{c}$ is in $B \oplus C$. It is also in A. Since A is independent of $B \oplus C$, $\mathbf{a}$ must be $\mathbf{0}$. Hence, $\mathbf{x} = \mathbf{b}$. But this means $\mathbf{x} \in B$. Since it is also in C, it must be $\mathbf{0}$, as B and C are independent. This shows that the only element in $(A \oplus B) \oplus C$ is $\mathbf{0}$.

Statement 3 can be proved as follows.

$$
\begin{aligned}
& \mathbf{x} \in A \oplus (B \oplus C) \\
\Leftrightarrow & \exists \text{ unique } \mathbf{a} \in A, \mathbf{d} \in B \oplus C \text{ s.t. } \mathbf{x} = \mathbf{a} + \mathbf{d} \\
\Leftrightarrow & \exists \text{ unique } \mathbf{a} \in A, \mathbf{b} \in B, \mathbf{c} \in C \text{ s.t. } \mathbf{x} = \mathbf{a} + \mathbf{b} + \mathbf{c} \\
\Leftrightarrow & \exists \text{ unique } \mathbf{e} \in A \oplus B, \mathbf{c} \in C \text{ s.t. } \mathbf{x} = \mathbf{e} + \mathbf{c} \\
\Leftrightarrow & \mathbf{x} \in (A \oplus B) \oplus C
\end{aligned}
$$

□

Theorem 3. *For all* $t \geq 0$,

$$\begin{aligned} V(t) &= V_\Delta(t) \oplus U(t) \\ V_j(t) &= V_\Delta(t) \oplus U_j(t) \quad \forall j = 1, 2, \ldots n \\ V'_\Delta(t) &= V_\Delta(t) \oplus U'_\Delta(t). \end{aligned}$$

Proof. The proof is by induction on t.

Basis step:

At $t = 0$, $V(0)$, $U(0)$ as well as all the $V_j(0)$'s and $U_j(0)$'s are initialized to $\{\mathbf{0}\}$. Consequently, $V_\Delta(0)$ is also $\{\mathbf{0}\}$. It is easily seen that these initial values satisfy the equations in the theorem statement.

Induction Hypothesis:

We assume the equations hold at t, *i.e.*,

$$V(t) = V_\Delta(t) \oplus U(t) \tag{2.9}$$

$$V_j(t) = V_\Delta(t) \oplus U_j(t), \forall j = 1, 2, \ldots n \tag{2.10}$$

$$V'_\Delta(t) = V_\Delta(t) \oplus U'_\Delta(t) \tag{2.11}$$

Induction Step: We now prove that they hold in slot $(t + 1)$. We have:

$$\begin{aligned} V(t) &= V_\Delta(t) \oplus U(t) && \text{(from (2.9))} \\ &= V_\Delta(t) \oplus [U'_\Delta(t) \oplus U''(t)] && \text{(from (2.5))} \\ &= [V_\Delta(t) \oplus U'_\Delta(t)] \oplus U''(t) && \text{(Lemma 5)} \\ &= V'_\Delta(t) \oplus U''(t) && \text{(from (2.11)).} \end{aligned}$$

Thus, we have proved:

$$V(t) = V'_\Delta(t) \oplus U''(t). \tag{2.12}$$

Now, we incorporate the arrivals in slot $(t+1)$. This converts $V'_\Delta(t)$ to $V_\Delta(t+1)$, $U''(t)$

to $U(t+1)$, and $V(t)$ to $V(t+1)$, owing to the following operations:

$$\begin{aligned}
\text{Basis of } V_\Delta(t+1) &= \begin{bmatrix} \text{Basis of } V'_\Delta(t) & 0 \end{bmatrix} \\
\text{Basis of } U(t+1) &= \begin{bmatrix} \text{Basis of } U''(t) & 0 \\ 0 & I_{a(t+1)} \end{bmatrix} \\
\text{Basis of } V(t+1) &= \begin{bmatrix} \text{Basis of } V(t) & 0 \\ 0 & I_{a(t+1)} \end{bmatrix}.
\end{aligned}$$

Incorporating these modifications into (2.12), we obtain:

$$V(t+1) = V_\Delta(t+1) \oplus U(t+1).$$

Now, consider each receiver $j = 1, 2, \ldots n$.

$V'_j(t)$

$$\begin{aligned}
&= V_j(t) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t)) \\
&= [V_\Delta(t) \oplus U_j(t)] \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t)) && \text{(from (2.10))} \\
&= V_\Delta(t) \oplus [U_j(t) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t))] && \text{(Lemma 5)} \\
&= V_\Delta(t) \oplus U'_j(t) && \text{(from (2.4))} \\
&= V_\Delta(t) \oplus [U'_\Delta(t) \oplus U''_j(t)] && \text{(from (2.6))} \\
&= [V_\Delta(t) \oplus U'_\Delta(t)] \oplus U''_j(t) && \text{(Lemma 5)} \\
&= V'_\Delta(t) \oplus U''_j(t) && \text{(from (2.11)).}
\end{aligned}$$

Incorporating the new arrivals into the subspaces involves adding $a(t+1)$ all-zero columns to the bases of $V'_j(t)$, $V'_\Delta(t)$, and $U''_j(t)$, thereby converting them into bases of $V_j(t+1), V_\Delta(t+1)$, and $U_j(t+1)$ respectively. These changes do not affect the above relation, and we get:

$$V_j(t+1) = V_\Delta(t+1) \oplus U_j(t+1), \quad \forall j = 1, 2, \ldots n.$$

Finally,

$$
\begin{aligned}
& V'_\Delta(t+1) \\
&= \cap_{j=1}^{n} V'_j(t+1) \\
&= \cap_{j=1}^{n} [V_j(t+1) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t+1))] \\
&= \cap_{j=1}^{n} [V_\Delta(t+1) \oplus U_j(t+1) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t+1))] \\
&\overset{(a)}{=} V_\Delta(t+1) \oplus \cap_{j=1}^{n} [U_j(t+1) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t+1))] \\
&= V_\Delta(t+1) \oplus U'_\Delta(t+1).
\end{aligned}
$$

Step (a) is justified as follows. Using equation (2.8) and the fact that $\tilde{\mathbf{g}}_\mathbf{j}(t+1)$ was chosen to be inside $U(t+1)$, we can show that the span of all the $[U_j(t+1) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t+1))]$'s is inside $U(t+1)$. Now, from the induction step above, $V_\Delta(t+1)$ is independent of $U(t+1)$. Therefore, $V_\Delta(t+1)$ is independent of the span of all the $[U_j(t+1) \oplus span(\tilde{\mathbf{g}}_\mathbf{j}(t+1))]$'s. We can therefore apply Lemma 4. □

Theorem 4. *Let $Q(t)$ denote the size of the queue after the arrivals in slot t have been appended to the queue.*

$$Q(t) = dim\ V(t) - dim\ V_\Delta(t).$$

Proof. $Q(t) = dim\ U(t) = dim\ U''(t-1) + a(t)$

$$
\begin{aligned}
&= dim\ U(t-1) - dim\ U'_\Delta(t-1) + a(t) \\
&\quad \text{(using (2.5))} \\
&= dim\ V(t-1) - dim\ V_\Delta(t-1) - dim\ U'_\Delta(t) + a(t) \\
&\quad \text{(from Theorem 3)} \\
&= dim\ V(t-1) - dim\ V'_\Delta(t) + a(t) \\
&\quad \text{(from Theorem 3)} \\
&= dim\ V(t) - dim\ V_\Delta(t).
\end{aligned}
$$

□

Lemma 6. *Let $V_1, V_2, \ldots, V_k$ be subspaces of a vector space V. Then, for $k \geq 1$,*

$$dim(V_1 \cap V_2 \cap \ldots \cap V_k) \geq \sum_{i=1}^{k} dim(V_i) - (k-1)dim(V).$$

Proof. For any two subspaces X and Y of V,

$$dim(X \cap Y) + dim(X + Y) = dim(X) + dim(Y)$$

where $X + Y$ denotes the span of subspaces X and Y.

Hence,

$$\begin{aligned} dim(X \cap Y) &= dim(X) + dim(Y) - dim(X + Y) \\ &\geq dim(X) + dim(Y) - dim(V) \qquad (2.13) \\ &\quad \text{(since } X + Y \text{ is also a subspace of } V\text{)}. \end{aligned}$$

Now, we prove the lemma by induction on k.

Basis step:

$k = 1$: LHS = $dim(V_1)$, RHS = $dim(V_1)$

$k = 2$: LHS = $dim(V_1 \cap V_2)$, RHS = $dim(V_1) + dim(V_2) - dim(V)$.

The claim follows from inequality (2.13).

Induction Hypothesis:

For some arbitrary k,

$$dim(\cap_{i=1}^{k-1} V_i) \geq \sum_{i=1}^{k-1} dim(V_i) - (k-2)dim(V).$$

Induction Step:

$$dim(\cap_{i=1}^{k} V_i) = dim(V_k \cap \cap_{i=1}^{k-1} V_i)$$

$$\begin{aligned} &\geq dim(V_k) + dim(\cap_{i=1}^{k-1} V_i) - dim(V) \quad \text{(using (2.13))} \\ &\geq dim(V_k) + \left[\sum_{i=1}^{k-1} dim(V_i) - (k-2)dim(V)\right] \\ &\quad -dim(V) \\ &= \sum_{i=1}^{k} dim(V_i) - (k-1)dim(V). \end{aligned}$$

The above result can be rewritten as:

$$dim(V) - dim(V_1 \cap V_2 \cap \ldots V_k) \leq \sum_{i=1}^{k} [dim(V) - dim(V_i)]. \tag{2.14}$$

□

Using this result, we can now prove Theorem 2.

Proof of Theorem 2: If we apply Lemma 6 to the vector spaces $V_j(t), j = 1, 2, \ldots, n$ and $V(t)$, then the left hand side of inequality (2.14) becomes the sender queue size (using Theorem 4), while the right hand side becomes the sum of the differences in backlog between the sender and the receivers, in terms of the number of degrees of freedom. Thus, we have proved Theorem 2.

2.6 Algorithm 2 (b): Drop when seen

The drop-when-seen algorithm can be viewed as a specialized variant of the generic Algorithm 2 (a) given above. It uses the notion of seen packets (defined in Section 2.1) to represent the bases of the knowledge spaces. This leads to a simple and easy-to-implement version of the algorithm which, besides ensuring that physical queue size tracks virtual queue size, also provides some practical benefits. For instance, the sender need not store linear combinations of packets in the queue like in Algorithm 2 (a). Instead only original packets need to be stored, and the queue can be operated in a simple first-in-first-out manner. We now present some mathematical preliminaries before describing the algorithm.

2.6.1 Some preliminaries

The newly proposed algorithm uses the notion of reduced row echelon form (RREF) of a matrix to represent the knowledge of a receiver. Hence, we first recapitulate the definition and some properties of the RREF from [40], and present the connection between the RREF and the notion of seeing packets.

Definition 8 (Reduced row echelon form (RREF))**.** *A matrix is said to be in reduced row echelon form if it satisfies the following conditions:*

1. *The first nonzero entry of every row is 1.*

2. *The first nonzero entry of any row is to the right of the first nonzero entry of the previous row.*

3. *The entries above the first nonzero row of any row are all zero.*

The RREF leads to a standard way to represent a vector space. Given a vector space, consider the following operation – arrange the basis vectors in any basis of the space as the rows of a matrix, and perform Gaussian elimination. This process essentially involves a sequence of elementary row transformations and it produces a unique matrix in RREF such that its row space is the given vector space. We call this the RREF basis matrix of the space. We will use this representation for the knowledge space of the receivers.

Let V be the knowledge space of some receiver. Suppose m packets have arrived at the sender so far. Then the receiver's knowledge consists of linear combinations of some collection of these m packets, *i.e.*, V is a subspace of $\mathbb{F}_q^m$. Using the procedure outlined above, we can compute the $dim(V) \times m$ RREF basis matrix of V over $\mathbb{F}_q$.

In the RREF basis, the first nonzero entry of any row is called a *pivot*. Any column with a pivot is called a *pivot column*. By definition, each pivot occurs in a different column. Hence, the number of pivot columns equals the number of nonzero rows, which is $dim[V]$. Let $\mathbf{p_k}$ denote the packet with index k. The columns are ordered so that column k maps to packet $\mathbf{p_k}$. The following theorem connects the notion of seeing packets to the RREF basis.

Theorem 5. *A node has seen a packet with index k if and only if the k^{th} column of the RREF basis B of the knowledge space V of the node is a pivot column.*

Proof. The 'if' part is clear. If column k of B is a pivot column, then the corresponding pivot row corresponds to a linear combination known to the node, of the form $\mathbf{p_k}+\mathbf{q}$, where $\mathbf{q}$ involves only packets with index more than k. Thus, the node has seen $\mathbf{p_k}$.

For the 'only if' part, suppose column k of B does not contain a pivot. Then, in any linear combination of the rows, rows with pivot after column k cannot contribute anything to column k. Rows with pivot before column k will result in a non-zero term in some column to the left of k. Since every vector in V is a linear combination of the rows of B, the first non-zero term of any vector in V cannot be in column k. Thus, $\mathbf{p_k}$ could not have been seen. □

Since the number of pivot columns is equal to the dimension of the vector space, we obtain the following corollary.

Corollary 1. *The number of packets seen by a receiver is equal to the dimension of its knowledge space.*

The next corollary introduces a useful concept.

Corollary 2. *If receiver j has seen packet $\mathbf{p_k}$, then it knows exactly one linear combination of the form $\mathbf{p_k} + \mathbf{q}$ such that $\mathbf{q}$ involves only* **unseen** *packets with index more than k.*

Proof. We use the same notation as above. The receiver has seen $\mathbf{p_k}$. Hence, column k in B is a pivot column. By definition of RREF, in the row containing the pivot in column k, the pivot value is 1 and subsequent nonzero terms occur only in non-pivot columns. Thus, the corresponding linear combination has the given form $\mathbf{p_k} + \mathbf{q}$, where $\mathbf{q}$ involves only *unseen* packets with index more than k.

We now prove uniqueness by contradiction. Suppose the receiver knows another such linear combination $\mathbf{p_k} + \mathbf{q}'$ where $\mathbf{q}'$ also involves only unseen packets. Then, the receiver must also know $(\mathbf{q} - \mathbf{q}')$. But this means the receiver has seen some packet involved in either $\mathbf{q}$ or $\mathbf{q}'$ – a contradiction. □

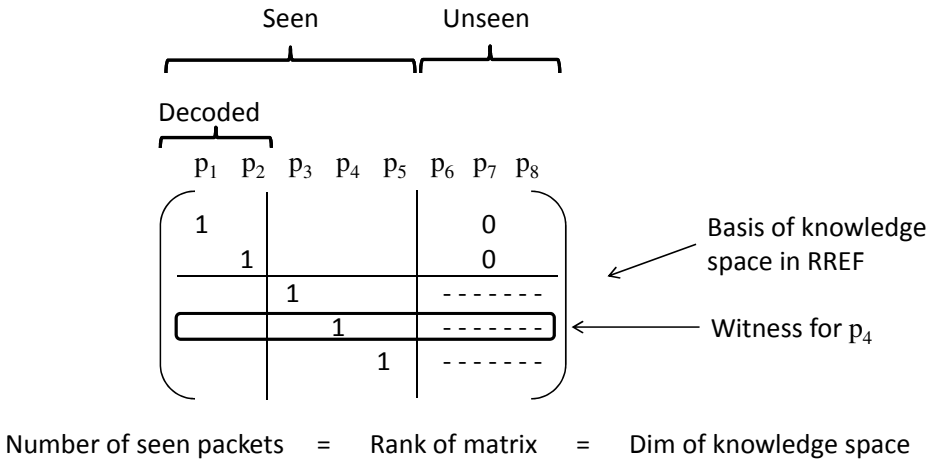


Figure 2-4: Seen packets and witnesses in terms of the basis matrix

Definition 9 (Witness)**.** *We denote the unique linear combination guaranteed by Corollary 2 as* $\mathbf{W_j(p_k)}$*, the* witness for receiver j seeing $\mathbf{p_k}$.

Figure 2-4 explains the notion of a seen packet and the notion of a witness in terms of the basis matrix.

Example: Suppose a node knows the following linear combinations: $\mathbf{x} = (\mathbf{p_1} + \mathbf{p_2})$ and $\mathbf{y} = (\mathbf{p_1} + \mathbf{p_3})$. Since these are linearly independent, the knowledge space has a dimension of 2. Hence, the number of seen packets must be 2. It is clear that packet $\mathbf{p_1}$ has been seen, since $\mathbf{x}$ satisfies the requirement of Definition 2. Now, the node can compute $\mathbf{z} \triangleq \mathbf{x} - \mathbf{y} = (\mathbf{p_2} - \mathbf{p_3})$. Thus, it has also seen $\mathbf{p_2}$. That means $\mathbf{p_3}$ is unseen. Hence, $\mathbf{y}$ is the witness for $\mathbf{p_1}$, and $\mathbf{z}$ is the witness for $\mathbf{p_2}$.

2.6.2 The main idea

The central idea of the algorithm is to keep track of seen packets instead of decoded packets. The two main parts of the algorithm are the coding and queue update modules.

In Section 2.6.5, we present the formal description of our coding module. The coding module computes a linear combination $\mathbf{g}$ that will cause any receiver that receives it, to see its next unseen packet. First, for each receiver, the sender computes its knowledge space using the feedback and picks out its next unseen packet. Only these packets will be involved in $\mathbf{g}$, and hence we call them the *transmit set*. Now, we need to select coefficients for each packet in this set. Clearly, the receiver(s) waiting to see the oldest packet in the

transmit set (say $\mathbf{p_1}$) will be able to see it as long as its coefficient is not zero. Consider a receiver that is waiting to see the second oldest packet in the transmit set (say $\mathbf{p_2}$). Since the receiver has already seen $\mathbf{p_1}$, it can subtract the witness for $\mathbf{p_1}$, thereby canceling it from $\mathbf{g}$. The coefficient of $\mathbf{p_2}$ must be picked such that after subtracting the witness for $\mathbf{p_1}$, the remaining coefficient of $\mathbf{p_2}$ in $\mathbf{g}$ is non-zero. The same idea extends to the other coefficients. The receiver can cancel packets involved in $\mathbf{g}$ that it has already seen by subtracting suitable multiples of the corresponding witnesses. Therefore, the coefficients for $\mathbf{g}$ should be picked such that for each receiver, after canceling the seen packets, the remaining coefficient of the next unseen packet is non-zero. Then, the receiver will be able to see its next unseen packet. Theorem 8 proves that this is possible if the field size is at least n, the number of receivers. With two receivers, the coding module is a simple XOR based scheme (see Table 2.1). Our coding scheme meets the innovation guarantee requirement because Theorem 5 implies that a linear combination that would cause a new packet to be seen brings in a previously unknown degree of freedom.

The fact that the coding module uses only the next unseen packet of all receivers readily implies the following queue update rule. **Drop a packet if all receivers have seen it.** This simple rule ensures that the physical queue size tracks the virtual queue size.

Remark 2. *In independent work, [48] proposes a coding algorithm which uses the idea of selecting those packets for coding, whose indices are one more than each receiver's rank. This corresponds to choosing the next unseen packets in the special case where packets are seen in order. Moreover, this algorithm picks coding coefficients in a deterministic manner, just like our coding module. Therefore, our module is closely related to the algorithm of [48].*

However, our algorithm is based on the framework of seen packets. This allows several benefits. First, it immediately leads to the drop-when-seen queue management algorithm, as described above. In contrast, [48] does not consider queuing aspects of the problem. Second, in this form, our algorithm readily generalizes to the case where the coding coefficients are picked randomly. The issue with random coding is that packets may be seen out of order. Our algorithm will guarantee innovation even in this case (provided the field is large), by selecting a random linear combination of the next unseen packets of the receivers

(the reasoning is similar to the arguments in [12]). However, the algorithm of [48] may not work well here, as it may pick packets that have already been seen, which could cause non-innovative transmissions.

The compatibility of our algorithm with random coding makes it particularly useful from an implementation perspective. With random coding, each receiver only needs to inform the sender the set of packets it has seen. There is no need to convey the exact knowledge space. This can be done simply by generating a TCP-like cumulative ACK upon seeing a packet. Thus, the ACK format is the same as in traditional ARQ-based schemes. Only its interpretation is different.

We next present the formal description and analysis of the queue update algorithm.

2.6.3 The queuing module

The algorithm works with the RREF bases of the receivers' knowledge spaces. The coefficient vectors are with respect to the current queue contents and not the original packet stream.

Algorithm 2 (b)

1. Initialize matrices $B_1, B_2, \ldots, B_n$ to the empty matrix. These matrices will hold the bases of the incremental knowledge spaces of the receivers.

2. *Incorporate new arrivals:* Suppose there are a new arrivals. Add the new packets to the end of the queue. Append a all-zero columns on the right to each B_j for the new packets.

3. *Transmission:* If the queue is empty, do nothing; else compute g using the coding module and transmit it.

4. *Incorporate channel state feedback:*

   For every receiver $j = 1$ to n, do:

   If receiver j received the transmission, include the coefficient vector of g in terms of the current queue contents, as a new row in B_j. Perform Gaussian elimination.

5. *Separate out packets that all receivers have seen:*

   Update the following sets and bases:

   S'_j := Set of packets corresponding to the pivot columns of B_j

   $S'_\Delta := \cap_{j=1}^{n} S'_j$

   New B_j := Sub-matrix of current B_j obtained by excluding columns in S'_Δ and corresponding pivot rows.

6. *Update the queue:* Drop the packets in S'_Δ.

7. Go back to step 2 for the next slot.

2.6.4 Connecting the physical and virtual queue sizes

The following theorem describes the heavy traffic asymptotic behavior of the expected physical queue size under our new queuing rule.

Theorem 6. *For Algorithm 2 (b), the physical queue size at the sender is upper-bounded by the sum of the virtual queue sizes,* i.e., *the sum of the degrees-of-freedom backlog between the sender and the receivers. Hence, the expected size of the physical queue in steady state for Algorithm 2 (b) is* $O\left(\frac{1}{1-\rho}\right)$.

In the rest of this section, we shall prove the above result. Now, in order to relate the queue size to the backlog in number of degrees of freedom, we shall need the following notation:

$S(t) \triangleq$ Set of packets arrived at sender till the end of slot t

$V(t) \triangleq$ Sender's knowledge space after incorporating the arrivals in slot t. This is simply equal to $\mathbb{F}_q^{|S(t)|}$

$V_j(t) \triangleq$ Receiver j's knowledge space at the end of slot t. It is a subspace of $V(t)$.

$S_j(t) \triangleq$ Set of packets receiver j has seen till end of slot t.

We shall now formally argue that Algorithm 2 (b) indeed implements the drop-when-seen rule in spite of the incremental implementation. In any slot, the columns of B_j are updated as follows. When new packets are appended to the queue, new columns are added to B_j on the right. When packets are dropped from the queue, corresponding columns are dropped from B_j. There is no rearrangement of columns at any point. This implies that a one-to-one correspondence is always maintained between the columns of B_j and the packets currently in the queue. Let $U_j(t)$ be the row space of B_j at time t. Thus, if $(u_1, u_2, \ldots, u_{Q(t)})$ is any vector in $U_j(t)$, it corresponds to a linear combination of the form $\sum_{i=1}^{Q(t)} u_i \mathbf{p_i}$, where $\mathbf{p_i}$ is the i^{th} packet in the queue at time t. The following theorem connects the incremental knowledge space $U_j(t)$ to the cumulative knowledge space $V_j(t)$.

Theorem 7. *In Algorithm 2 (b), for each receiver j, at the end of slot t, for any $\mathbf{u} \in U_j(t)$, the linear combination $\sum_{i=1}^{Q(t)} u_i \mathbf{p_i}$ is known to the receiver j, where $\mathbf{p_i}$ denotes the i^{th} packet in the queue at time t.*

Proof. We shall use induction on t. For $t = 0$, the system is completely empty and the statement is vacuously true. Let us now assume that the statement is true at time $(t-1)$. Consider the operations in slot t. A new row is added to B_j only if the corresponding linear combination has been successfully received by receiver j. Hence, the statement is still true. Row operations involved in Gaussian elimination do not alter the row space. Finally, when some of the pivot columns are dropped along with the corresponding pivot rows in step 5, this does not affect the linear combinations to which the remaining rows correspond because the pivot columns have a 0 in all rows except the pivot row. Hence, the three operations that are performed between slot $(t-1)$ and slot t do not affect the property that the vectors in the row space of B_j correspond to linear combinations that are known at receiver j. This proves the theorem. □

If a packet corresponds to a pivot column in B_j, the corresponding pivot row is a linear combination of the packet in question with packets that arrived after it. From the above theorem, receiver j knows this linear combination which means it has seen the packet. This leads to the following corollary.

Corollary 3. *If a packet corresponds to a pivot column in B_j, then it has been seen by receiver j.*

Thus, in step 5, $S'_\Delta(t)$ consists of those packets in the queue that all receivers have seen by the end of slot t. In other words, the algorithm retains only those packets that have not yet been seen by all receivers. Even though the algorithm works with an incremental version of the knowledge spaces, namely $U_j(t)$, it maintains the queue in the same way as if it was working with the cumulative version $V_j(t)$. Thus, the incremental approach is equivalent to the cumulative approach.

We require the following lemma to prove the main theorem.

Lemma 7. *Let $A_1, A_2, \ldots, A_k$ be subsets of a set A. Then, for $k \geq 1$,*

$$|A| - |\cap_{i=1}^{k} A_i| \leq \sum_{i=1}^{k} (|A| - |A_i|). \tag{2.15}$$

Proof.

$$\begin{aligned}
& |A| - |\cap_{i=1}^{k} A_i| \\
= \; & |A \cap (\cap_{i=1}^{k} A_i)^c| \text{ (since the } A_i\text{'s are subsets of } A) \\
= \; & |A \cap (\cup_{i=1}^{k} A_i^c)| \text{ (by De Morgan's law)} \\
= \; & |\cup_{i=1}^{k} (A \cap A_i^c)| \text{ (distributivity)} \\
\leq \; & \sum_{i=1}^{k} |A \cap A_i^c| \text{ (union bound)} \\
= \; & \sum_{i=1}^{k} (|A| - |A_i|).
\end{aligned}$$

□

Now, we are ready to prove Theorem 6.

Proof of Theorem 6: Since the only packets in the queue at any point are those that not all receivers have seen, we obtain the following expression for the physical queue size at

the sender at the end of slot t:

$$Q(t) = |S(t)| - |\cap_{j=1}^{n} S_j(t)|.$$

If we apply Lemma 7 to the sets $S(t)$ and $S_j(t), j = 1, 2, \ldots, n$ then the left hand side of inequality (2.15) becomes the sender queue size $Q(t)$ given above. Now, $|S_j(t)| = dim[V_j(t)]$, using Corollary 1. Hence the right hand side of inequality (2.15) can be rewritten as $\sum_{j=1}^{n} \left[dim[V(t)] - dim[V_j(t)]\right]$, which is the sum of the virtual queue sizes.

Finally, we can find the asymptotic behavior of the physical queue size in steady state under Algorithm 2 (b). Since the expected virtual queue sizes themselves are all $O\left(\frac{1}{1-\rho}\right)$ from Equation (2.2), we obtain the stated result.

2.6.5 The coding module

We now present a coding module that is compatible with the drop-when-seen queuing algorithm in the sense that it always forms a linear combination using packets that are currently in the queue maintained by the queuing module. In addition, we show that the coding module satisfies the innovation guarantee property.

Let $\{u_1, u_2, \ldots, u_m\}$ be the set of indices of the next unseen packets of the receivers, sorted in ascending order (in general, $m \leq n$, since the next unseen packet may be the same for some receivers). Exclude receivers whose next unseen packets have not yet arrived at the sender. Let $R(u_i)$ be the set of receivers whose next unseen packet is $\mathbf{p_{u_i}}$. We now present the coding module to select the linear combination for transmission.

1. *Loop over next unseen packets*

   For $j = 1$ to m, do:

   All receivers in $R(u_j)$ have seen packets $\mathbf{p_{u_i}}$ for $i < j$. Now, $\forall r \in R(u_j)$, find $\mathbf{y_r} := \sum_{i=1}^{j-1} \alpha_i \mathbf{W_r}(\mathbf{p_{u_i}})$, where $\mathbf{W_r}(\mathbf{p_{u_i}})$ is the witness for receiver r's seeing $\mathbf{p_{u_i}}$. Pick $\alpha_j \in \mathbb{F}_q$ such that α_j is different from the coefficient of $\mathbf{p_{u_j}}$ in $\mathbf{y_r}$ for each $r \in R(u_j)$.

2. *Compute the transmit packet:* $\mathbf{g} := \sum_{i=1}^{m} \alpha_i \mathbf{p_{u_i}}$.

It is easily seen that this coding module is compatible with the drop-when-seen algorithm. Indeed, it does not use any packet that has been seen by all receivers in the linear combination. It only uses packets that at least one receiver has not yet seen. The queue update module retains precisely such packets in the queue. The next theorem presents a useful property of the coding module.

Theorem 8. *If the field size is at least n, then the coding module picks a linear combination that will cause any receiver to see its next unseen packet upon successful reception.*

Proof. First we show that a suitable choice always exists for α_j that satisfies the requirement in step 1. For $r \in R(u_1)$, $\mathbf{y_r} = \mathbf{0}$. Hence, as long as $\alpha_1 \neq 0$, the condition is satisfied. So, pick $\alpha_1 = 1$. Since at least one receiver is in $R(u_1)$, we have that, for $j > 1$, $|R(u_j)| \leq (n-1)$. Even if each $\mathbf{y_r}$ for $r \in R(u_j)$ has a different coefficient for $\mathbf{p_{u_j}}$, that covers only $(n-1)$ different field elements. If $q \geq n$, then there is a choice left in $\mathbb{F}_q$ for α_j.

Now, we have to show that the condition given in step 1 implies that the receivers will be able to see their next unseen packet. Indeed, for all j from 1 to m, and for all $r \in R(u_j)$, receiver r knows $\mathbf{y_r}$, since it is a linear combination of witnesses of r. Hence, if r successfully receives $\mathbf{g}$, it can compute $(\mathbf{g} - \mathbf{y_r})$. Now, $\mathbf{g}$ and $\mathbf{y_r}$ have the same coefficient for all packets with index less than u_j, and a different coefficient for $\mathbf{p_{u_j}}$. Hence, $(\mathbf{g} - \mathbf{y_r})$ will involve $\mathbf{p_{u_j}}$ and only packets with index beyond u_j. This means r can see $\mathbf{p_{u_j}}$ and this completes the proof. □

Theorem 5 implies that seeing an unseen packet corresponds to receiving an unknown degree of freedom. Thus, Theorem 8 essentially says that the innovation guarantee property is satisfied and hence the scheme is throughput optimal.

This theorem is closely related to the result derived in [48] that computes the minimum field size needed to guarantee innovation. The difference is that our result uses the framework of seen packets to make a more general statement by specifying not only that innovation is guaranteed, but also that packets will be seen in order with this deterministic coding scheme. This means packets will be dropped in order at the sender.

2.7 Overhead

In this section, we comment on the overhead required for Algorithms 1 and 2 (b). There are several types of overhead.

2.7.1 Amount of feedback

Our scheme assumes that every receiver feeds back one bit after every slot, indicating whether an erasure occurred or not. In comparison, the drop-when-decoded scheme requires feedback only when packets get decoded. However, in that case, the feedback may be more than one bit – the receiver will have to specify the list of all packets that were decoded, since packets may get decoded in groups. In a practical implementation of the drop-when-seen algorithm, TCP-like cumulative acknowledgments can be used to inform the sender which packets have been seen.

2.7.2 Identifying the linear combination

Besides transmitting a linear combination of packets, the sender must also embed information that allows the receiver to identify what linear combination has been sent. This involves specifying which packets have been involved in the combination, and what coefficients were used for these packets.

Set of packets involved

The baseline algorithm uses all packets in the queue for the linear combination. The queue is updated in a first-in-first-out (FIFO) manner, *i.e.*, no packet departs before all earlier packets have departed. This is a consequence of the fact that the receiver signals successful decoding only when the virtual queue becomes empty[4]. The FIFO rule implies that specifying the current contents of the queue in terms of the original stream boils down to specifying the sequence number of the head-of-line packet and the last packet in the queue in every transmission.

[4]As mentioned earlier in Remark 1, we assume that the sender checks whether any packets have been newly decoded, only when the virtual queue becomes empty.

The drop-when-seen algorithm does not use all packets from the queue, but only at most n packets from the queue (the next unseen packet of each receiver). This set can be specified by listing the sequence number of these n packets.

Now, in both cases, the sequence number of the original stream cannot be used as it is, since it grows unboundedly with time. However, we can avoid this problem using the fact that the queue contents are updated in a FIFO manner (This is also true of our drop-when-seen scheme – the coding module guarantees that packets will be seen in order, thereby implying a FIFO rule for the sender's queue.). The solution is to express the sequence number relative to an origin that also advances with time, as follows. If the sender is certain that the receiver's estimate of the sender's queue starts at a particular point, then both the sender and receiver can reset their origin to that point, and then count from there.

For the baseline case, the origin can be reset to the current HOL packet, whenever the receiver sends feedback indicating successful decoding. The idea is that if the receiver decoded in a particular slot, that means it had a successful reception in that slot. Therefore, the sender can be certain that the receiver must have received the latest update about the queue contents and is therefore in sync with the sender. Thus, the sender and receiver can reset their origin. Note that since the decoding epochs of different receivers may not be synchronized, the sender will have to maintain a different origin for each receiver and send a different sequence number to each receiver, relative to that receiver's origin. This can be done simply by concatenating the sequence number for each receiver in the header.

To determine how many bits are needed to represent the sequence number, we need to find out what range of values it can take. In the baseline scheme, the sequence number range will be proportional to the busy period of the virtual queue, since this determines how often the origin is reset. Thus, the overhead in bits for each receiver will be proportional to the logarithm of the expected busy period, *i.e.*, $O\left(\log_2 \frac{1}{1-\rho}\right)$.

For the drop-when-seen scheme, the origin can be reset whenever the receiver sends feedback indicating successful reception. Thus, the origin advances a lot more frequently than in the baseline scheme.

Coefficients used

The baseline algorithm uses a random linear coding scheme. Here, potentially all packets in the queue get combined in a linear combination. So, in the worst case, the sender would have to send one coefficient for every packet in the queue. If the queue has m packets, this would require $m \log_2 q$ bits, where q is the field size. In expectation, this would be $O\left(\frac{\log_2 q}{(1-\rho)^2}\right)$ bits. If the receiver knows the pseudorandom number generator used by the sender, then it would be sufficient for the sender to send the current state of the generator and the size of the queue. Using this, the receiver can generate the coefficients used by the sender in the coding process. The new drop-when-seen algorithm uses a coding module which combines the next unseen packet of each receiver. Thus, the overhead for the coefficients is at most $n \log_2 q$ bits, where n is the number of receivers. It does not depend on the load factor ρ at all.

2.7.3 Overhead at sender

While Algorithm 2 (b) saves in buffer space, it requires the sender to store the basis matrix of each receiver, and update them in every slot based on feedback. However, storing a row of the basis matrix requires much less memory than storing a packet, especially for long packets. Thus, there is an overall saving in memory. The update of the basis matrix simply involves one step of the Gaussian elimination algorithm.

2.7.4 Overhead at receiver

The receiver will have to store the coded packets till they are decoded. It will also have to decode the packets. For this, the receiver can perform a Gaussian elimination after every successful reception. Thus, the computation for the matrix inversion associated with decoding can be spread over time.

2.8 Conclusions

Comparing Theorem 1 and Theorem 6, we see that the queue size for the new Algorithm 2 is significantly lower than Algorithm 1, especially at heavy traffic. If the memory at the sender is shared among many flows, then this reduction in queue size will prove useful in getting statistical multiplexing benefits. Algorithm 2 allows the physical queue size to track the virtual queue size. This extends stability and other queuing-theoretic results on virtual queues to physical queues. We believe the proposed scheme will be robust to delayed or lossy feedback, just like conventional ARQ. The scheme readily extends to a tree of broadcast links with no mergers, if intermediate nodes use witness packets in place of original packets. With suitable changes, we expect it to extend to other topologies as well. In summary, we propose in this chapter, a natural extension of ARQ for coded networks, and analyze it from a queue management perspective. This is the first step towards the goal of using feedback on degrees of freedom to control the network performance, by dynamically adjusting the extent to which packets are mixed in the network. In the next chapter, we focus on a different metric, the decoding delay. We study how the encoding process can be adapted dynamically based on the feedback, so as to ensure low delay.

Chapter 3

Adaptive online coding to reduce delay

In today's communication systems, the demand for supporting real-time applications is growing rapidly. In popular applications such as live video streaming and video conferencing, the user's experience is very sensitive to the per-packet delay. In pre-recorded video streaming (*i.e.*, not live), a low delay is still preferable because that would reduce the amount of buffering required for playback at the receiver.

Note that this notion of per-packet delay is very different from download delay [56]. While downloading a file, usually the main performance criterion is the time it takes to complete the download. From the system point of view, this goal essentially translates to a high throughput requirement. The implicit assumption in such a scenario is that the file is useful only as a whole.

From a throughput perspective, there are situations where coding across packets is very useful. One reason is that coding can help correct errors and erasures in the network. Another reason is, in certain network topologies such as the butterfly network from the network coding literature [1], coding is necessary to share bottleneck links across flows, in order to achieve the system capacity. Similarly, in broadcast-mode links, especially with erasures, coding across packets is critical for achieving a high throughput [16].

Now, any form of coding comes with an associated decoding delay. The receiver has to wait to collect sufficiently many coded packets before it can decode the original packets. Therefore, in delay-sensitive applications, it may be necessary to carefully design the coding scheme so that it not only satisfies the criteria needed to ensure high throughput, but

also achieves a low decoding delay.

Motivated by this goal, we explore in our work, the possibility of making use of feedback in order to adapt the coding scheme in an online manner. We focus on the single hop packet erasure broadcast channel with perfect immediate feedback. We propose and study a new coding module for any number of receivers. We show that it is throughput optimal and that it allows efficient queue management. We also study two different notions of delay. The first one is the decoding delay per packet. This is simply the average over all packets of the time between arrival and decoding at an arbitrary receiver. The second notion, known as delivery delay, is a much stronger notion of delay. It assumes that packets may be delivered to the receiver's application only in the order of their arrival at the sender. These notions were also studied in earlier work [26]. We conjecture that our scheme achieves the asymptotically optimal expected decoding delay and delivery delay in the limit of heavy traffic.

Note that with the coding module of Section 2.6.5 in Chapter 2, although a receiver can see the next unseen packet in every successful reception, this does not mean the packet will be decoded immediately. In general, the receiver will have to collect enough equations in the unknown packets before being able to decode them, resulting in a delay.

The rest of the chapter is organized as follows. In Section 3.1, we present the system model and the problem statement. Section 3.2 then motivates the problem in the context of related earlier work. We then study in Section 3.3.1, the delivery delay behavior of Algorithms 1 and 2(b) of Chapter 2, and provide an upper bound on the asymptotic expected delivery delay for any policy that satisfies the innovation guarantee property. This is followed by a generic lower bound on the expected decoding delay in Section 3.3.2. Section 3.4 presents the new generalized coding module for any number of receivers. The performance of this algorithm is described in Section 3.5. In Section 3.6, we present our simulation results. Finally, the conclusions and directions for future work are presented in Section 3.7.

3.1 The system model

The system model is identical to that in Chapter 2. The load factor ρ is defined to be $\rho := \lambda/\mu$ as before, where λ is the rate of the arrival process and μ is the probability of successful delivery with respect to a particular receiver. In this chapter, we again assume that the sender can only use linear codes. In other words, every transmission is a linear combination of the current contents of the buffer. The coefficient vector corresponding to a linear combination is conveyed to the receiver through the packet header.

Unlike Chapter 2 however, in this chapter we are interested not in the queue occupancy, but in the delay performance. We will use two notions of delay.

Definition 10 (Decoding Delay)**.** *The* decoding delay *of a packet with respect to a receiver is the time between the arrival of the packet at the sender and the decoding of the packet by the receiver under consideration.*

As discussed earlier, some applications can make use of a packet only if all prior packets have been decoded. In other words, the application will accept packets only up to *the front of contiguous knowledge*, defined as follows.

Definition 11 (Front of contiguous knowledge)**.** *In an application where the sender generates a stream of packets, the* front of contiguous knowledge *of a receiver is defined to be the largest packet index k such that the receiver has decoded all packets with index less than or equal to k.*

This motivates the following stronger notion of delay.

Definition 12 (Delivery Delay)**.** *The* delivery delay *of a packet with respect to a receiver is the time between the arrival of the packet at the sender and the delivery of the packet by the receiver to the application, with the constraint that packets may be delivered* **only in order**.

It is easily seen from these definitions that *the delivery delay is, in general, longer than the decoding delay*. Upon decoding the packets, the receiver will place them in a reordering buffer until they are delivered to the application.

It is well known that in this model, the queue can be stabilized as long as $\rho < 1$, by using linear network coding [16]. In this work, we are interested in the rate of growth of the decoding and delivery delay, in the heavy traffic regime of ρ approaching 1. We focus on the expectation of these delays for an arbitrary packet. It can be shown using ergodic theory that the long term average of the delay experienced by the packets in steady state converges to this expectation with high probability.

The problem we study in this chapter is the following: Is there an adaptive coding scheme that is throughput optimal and at the same time achieves the best possible rate of growth of the decoding and delivery delay, as a function of $1/(1-\rho)$?

3.2 Motivation and related earlier work

Coding for per-packet delay has been studied in earlier work by Martinian *et al.* [24]. However, that work considered a point-to-point setting unlike our broadcast scenario. The problem of the delay for recovering packets from a file has been studied in the rateless code framework with or without feedback, by [27] and [28]. Reference [25] also considered the problem of coding for delay using feedback. The setting there is in terms of a fixed delay model for point-to-point communication, where each packet has a deadline by which it needs to be delivered. A packet which does not meet its deadline is considered to be in error, and the corresponding error exponents are characterized.

In contrast, we consider the expected per-packet delay in a queuing theoretic framework, with no strict deadlines. Besides, our setting is a point-to-multipoint (broadcast) packet erasure channel.

For the case of packet erasure broadcast channel with two receivers, Durvy *et al.* [57] have proposed a feedback-based throughput-optimal coding scheme that ensures that every successful innovative reception at any receiver will cause it to decode an original packet. This property is called *instantaneous decodability*. However, the authors provided an example to show that for the three receiver case, instantaneous decodability cannot be achieved without losing throughput. In related work, Sadeghi *et al.* [58] formulated the instantaneous decodability problem as an integer linear program and proposed algorithms for

different scenarios.

Keller *et al.* [59] also studied this problem and proposed and compared several algorithms to reduce the decoding delay. This work did not consider the in-order delivery problem. Both [57] and [59] consider the transmission of a given finite set of packets. In contrast, [43] assumes that packets arrive at the source according to a stochastic process in a streaming manner and proposes a coding scheme for two receivers. The focus however, is to ensure stable throughput and not low delay. In [44], the authors propose a greedy coding scheme for the case of more than 2 receivers, which aims to maximize the number of receivers that can decode a packet instantaneously, at the expense of losing throughput.

Our current work considers stochastic packet arrivals. Whereas the earlier works did not consider the in-order delivery constraint, we study the delivery delay as well. We focus on throughput optimal schemes. Since instantaneously decodability cannot be guaranteed for more than 2 receivers, we consider the relaxed requirement of asymptotically optimal decoding and delivery delay, where the asymptotics are in the heavy traffic limit of the load factor $\rho \to 1$.

In Section 3.3.2, we present a lower bound on the asymptotic growth of the expected decoding delay of $O\left(\frac{1}{1-\rho}\right)$ by arguing that even the single receiver case has this linear rate of growth in terms of $\frac{1}{1-\rho}$. For the two receiver case, it can be proved that the algorithm of [57] indeed achieves this lower bound for decoding delay, and seems to achieve it for delivery delay as well, based on simulations.

In Chapter 2 (also in [60]), we presented a feedback-based coding scheme for any number of receivers. The main focus there, however, was to ensure efficient queue management. The queue size growth was shown to be $O\left(\frac{1}{1-\rho}\right)$. However, the decoding delay of the schemes proposed there, are seen to have a *quadratic* growth in $1/(1-\rho)$ based on simulations, as explained in Section 3.3.1 below. The section also shows that the delay of any policy that satisfies the innovation guarantee property is upper-bounded by a quadratic function of $1/(1-\rho)$.

Reference [61] proposed a coding scheme for the case of three receivers that was conjectured to achieve the asymptotic lower bound. However, it was not generalizable to multiple receivers. Reference [62] considers the case of heterogeneous channels to the receivers,

and proposes a systematic online encoding scheme that sends uncoded packets to enable frequent decoding at the receivers. However, no characterization of the asymptotic behavior of the decoding or delivery delay is provided. For more related work, the reader is referred to [62].

The contribution of our current work is to provide a new coding module for any number of receivers, that is at the same time throughput-optimal, allows asymptotically optimal queue sizes and is conjectured to achieve an asymptotically optimal $O\left(\frac{1}{1-\rho}\right)$ growth for both decoding and delivery delay in the heavy traffic limit. It can be shown that the two-receiver algorithm of [57] is a special case of our algorithm. The delay performance conjecture is verified through simulations.

Adaptive coding allows the sender's code to incorporate receivers' states of knowledge and thereby enables the sender to control the evolution of the front of contiguous knowledge. Our scheme may thus be viewed as a step towards feedback-based control of the tradeoff between throughput and decoding delay, along the lines suggested in [63].

3.3 Bounds on the delay

3.3.1 An upper bound on delivery delay

We now present the upper bound on delay for policies that satisfy the innovation guarantee property. The arguments leading to this bound are presented below.

Theorem 9. *The expected delivery delay of a packet for any coding module that satisfies the innovation guarantee property is* $O\left(\frac{1}{(1-\rho)^2}\right)$.

Proof. For any policy that satisfies the innovation guarantee property, the virtual queue size evolves according to the Markov chain in Figure 2-2. The analysis of Algorithm 1 in Section 2.4 therefore applies to any coding algorithm that guarantees innovation.

As explained in that section, the event of a virtual queue becoming empty translates to successful decoding at the corresponding receiver, since the number of equations now

matches the number of unknowns involved. Thus, an arbitrary packet that arrives at the sender will get decoded by receiver j at or before the next emptying of the j^{th} virtual queue. In fact, it will get delivered to the application at or before the next emptying of the virtual queue. This is because, when the virtual queue is empty, every packet that arrived at the sender gets decoded. Thus, the front of contiguous knowledge advances to the last packet that the sender knows.

The above discussion implies that Equation (2.3) gives an upper bound on the expected delivery delay of an arbitrary packet. We thus obtain the result stated above. □

We next study the decoding delay of Algorithm 2 (b). We define *the decoding event* to be the event that all seen packets get decoded. Since packets are always seen in order, the decoding event guarantees that the front of contiguous knowledge will advance to the front of seen packets.

We use the term *leader* to refer to the receiver which has seen the maximum number of packets at the given point in time. Note that there can be more than one leader at the same time. The following lemma characterizes sufficient conditions for the decoding event to occur.

Lemma 8. *The decoding event occurs in a slot at a particular receiver if in that slot:*

(a) The receiver has a successful reception which results in an empty virtual queue at the sender; OR

(b) The receiver has a successful reception and the receiver was a leader at the beginning of the slot.

Proof. Condition (a) implies that the receiver has seen all packets that have arrived at the sender up to that slot. Each packet at the sender is an unknown and each seen packet corresponds to a linearly independent equation. Thus, the receiver has received as many equations as the number of unknowns, and can decode all packets it has seen.

Suppose condition (b) holds. Let $\mathbf{p_k}$ be the next unseen packet of the receiver in question. The sender's transmitted linear combination will involve only the next unseen packets of all the receivers. Since the receiver was a leader at the beginning of the slot, the sender's

transmission will not involve any packet beyond $\mathbf{p_k}$, since the next unseen packet of all other receivers is either $\mathbf{p_k}$ or some earlier packet. After subtracting the suitably scaled witnesses of already seen packets from such a linear combination, the leading receiver will end up with a linear combination that involves only $\mathbf{p_k}$. Thus the leader not only sees $\mathbf{p_k}$, but also decodes it. In fact, none of the sender's transmissions so far would have involved any packet beyond $\mathbf{p_k}$. Hence, once $\mathbf{p_k}$ has been decoded, $\mathbf{p_{k-1}}$ can also be decoded. This procedure can be extended to all unseen packets, and by induction, we can show that all unseen packets will be decoded. □

The upper bound proved in Theorem 9 is based on the emptying of the virtual queues. This corresponds only to case (a) in Lemma 8. The existence of case (b) shows that in general, the decoding delay will be strictly smaller than the upper bound. A natural question is whether this difference is large enough to cause a different asymptotic behavior, *i.e.*, does Algorithm 2 (b) achieve a delay that asymptotically has a smaller exponent of growth than the upper bound as $\rho \to 1$? We conjecture that this is not the case, *i.e.*, that the decoding delay for Algorithm 2 (b) is also $\Omega\left(\frac{1}{(1-\rho)^2}\right)$, although the constant of proportionality will be smaller. For the two receiver case, based on our simulations, this conjecture seems to hold. Figure 3-1 shows the growth of the decoding delay averaged over a large number of packets, as a function of $\frac{1}{(1-\rho)}$. The resulting curve seems to be close to the curve $\frac{0.37}{(1-\rho)^2}$, implying a quadratic growth. The value of ρ ranges from 0.95 to 0.98, while μ is fixed to be 0.5. The figure also shows the upper bound based on busy period measurements. This curve agrees with the formula in Equation (2.3) as expected.

3.3.2 A lower bound on decoding delay

Lemma 9. *The expected per-packet decoding delay is* $\Omega\left(\frac{1}{1-\rho}\right)$

Proof. The expected per-packet delay for the single receiver case is clearly a lower bound for the corresponding quantity at one of the receivers in a multiple-receiver system. We will compute this lower bound in this section. Figure 2-2 shows the Markov chain for the queue size in the single receiver case. If $\rho = \frac{\lambda}{\mu} < 1$, then the chain is positive recurrent and the steady state expected queue size can be computed to be $\frac{\rho(1-\mu)}{(1-\rho)} = \Theta\left(\frac{1}{1-\rho}\right)$ (see

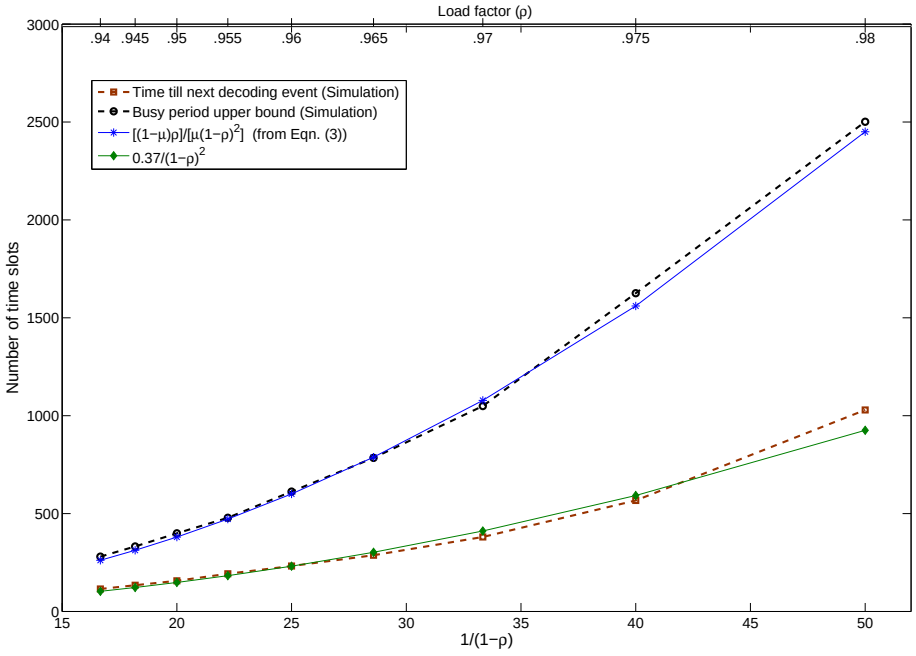


Figure 3-1: Delay to decoding event and upper bound for 2 receiver case, as a function of $\frac{1}{(1-\rho)}$. The corresponding values of ρ are shown on the top of the figure.

Equation (2.1)). Now, if $\rho < 1$, then the system is stable and Little's law can be applied to show that the expected per-packet delay in the single receiver system is also $\Theta\left(\frac{1}{1-\rho}\right)$. $\square$

3.4 The coding algorithm

We now present the new coding module for the general case of any number of receivers. First, we describe the main ideas behind the algorithm. Then, we present the detailed specification.

3.4.1 Intuitive description

The intuition behind the algorithm is first to identify for each receiver, the oldest packet that it has not yet decoded, which we will call the *request* of that receiver. The algorithm then transmits a linear combination that involves packets from only within this set.

The linear combination is constructed incrementally. The receivers are grouped accord-

ing to their request, and the groups are processed in descending order of their requested packet's index. First, the newest request (*i.e.*, the one with the largest index) is included in the linear combination, as otherwise, the corresponding receivers, having decoded everything older, will find the transmission non-innovative. Then, the algorithm checks whether the linear combination formed thus far is innovative to every receiver in the next group. If it is not innovative, then the coefficient of the next group's request is adjusted till it is simultaneously innovative to the whole group. The key idea is that, since the groups are processed in descending order of their requests, the choices made for the coefficient of subsequent groups' requests will not affect the innovation of earlier groups. This is because, the earlier groups have already decoded the subsequent groups' requests.

After processing all the groups in this order, the transmitted linear combination is thus chosen so that it satisfies the innovation guarantee property.

3.4.2 Representing knowledge

Before specifying the algorithm, we first propose a way to represent systematically the state of knowledge of the receivers. This is based on the representation used in Chapter 2, with a key difference described below.

The k^{th} packet to have arrived at the sender is said to have an *index* k and is denoted by $\mathbf{p_k}$. Suppose the total number of packets that have arrived at any time t is denoted by $A(t)$. Since we have a restriction that the coding must be linear, we can represent the state of knowledge of a node by a vector space consisting of all the linear combinations that a node can compute using what it has received thus far. We represent the state of knowledge using a basis of this vector space. The basis is represented as the rows of a matrix which is in the row-reduced echelon form (RREF). The matrix has $A(t)$ columns, one for each packet that has arrived thus far. While all this is identical to the representation in Chapter 2, the main difference is in the ordering of the columns of the basis matrix. We use the same framework, except that in our current work, the columns are ordered so that packet $\mathbf{p_k}$ maps to column $A(t) - k$. In other words, the columns are arranged in reverse order with respect to the order of arrival at the sender.

Throughout this chapter, we shall use the RREF representation of the basis matrix, with this reverse ordering of the packets. We also make use of the notion of seen packets that was introduced in Chapter 2. Note however that the definition becomes quite different from that in the previous chapter, if we use the reverse ordering on the packets.

Definition 13 (Seeing a packet)**.** *A node is said to have* seen *a packet with index k if and only if the k^{th} column from the right, of the RREF basis B of the knowledge space V of the node, is a pivot column. Alternatively, a node has seen a packet $\mathbf{p_k}$ if it has received enough information to compute a linear combination of the form $(\mathbf{p_k} + \mathbf{q})$, where $\mathbf{q}$ is itself a linear combination involving only packets with an index **less** than that of $\mathbf{p}$. (Decoding implies seeing, as we can pick $\mathbf{q} = \mathbf{0}$.)*

In contrast, the definition used in Chapter 2 had replaced the word "less" with the word "greater" in the above statement. We believe the reverse ordering is better suited to analyzing the delivery delay. We now make some observations about the new definition.

Observation 1: As with the forward ordering, the notion of seen with the reverse ordering also has connections to the dimension of the knowledge space. In particular, we can show that every innovative reception causes a new packet to be seen. In other words, the number of seen packets is equal to the dimension of the knowledge space.

Observation 2: Owing to the reverse ordering of the packets, we have an interesting property. For any $k > 0$, if all packets $\mathbf{p_1}$ to $\mathbf{p_k}$ have been seen, then they have also been decoded, and hence can be delivered to the application.

A more general definition that accommodates both the forward and reverse ordering is as follows. A packet is considered seen when the receiving node receives a linear combination including the packet and only subsequent packets, with respect to some fixed ordering on the original packet stream. Thus, a notion of seen exists for every order that is defined on the packet stream. For the remaining part of this chapter, we shall use the reverse ordering with respect to the order of arrival.

3.4.3 Algorithm specification

Now, we present the formal coding algorithm. Let us first define $\{u_1, u_2, \ldots, u_m\}$ to be the set of indices of the oldest undecoded packets of the n receivers, sorted in descending order ($m \leq n$, since the oldest undecoded packet may be the same for some receivers). Exclude receivers whose oldest undecoded packet has not yet arrived at the sender. We call this resulting set of packets the *transmit set*, since the coding module will use only these packets in computing the linear combination to be transmitted.

Let $R(u_i)$ be the group of receivers whose request is $\mathbf{p_{u_i}}$. We now present the coding module to select the linear combination for transmission.

Initialize the transmit coefficient vector $\mathbf{a}$ to an all zero vector of length Q, the current sender queue size.

for $j = 1$ to m **do** (*Loop over the transmit set*)

Initialize the veto list[1] to the empty set.

for all $r \in R(u_j)$ **do**

Zero out the coefficient of all packets seen by receiver r from the current transmission vector $\mathbf{a}$ by subtracting from $\mathbf{a}$, suitably scaled versions of the rows of the current RREF basis matrix, to get the vector $\mathbf{a}'$. (This is essentially the first step of Gaussian elimination.) Hence, find out which packet will be newly seen if the linear combination corresponding to $\mathbf{a}$ is transmitted. This is simply the index of the packet corresponding to the first non-zero entry in $\mathbf{a}'$.

if no packet is newly seen **then**

Append 0 to the veto list

else if the newly seen packet's index is u_j **then**

Append the additive inverse of the leading non-zero entry of $\mathbf{a}'$ to the veto list

else if the newly seen packet is anything else **then**

Do not add anything to the veto list

end if

end for

[1] This will hold the list of unacceptable coefficients of $\mathbf{p_{u_j}}$.

Arrange the elements of the finite field in any order, starting with 0. Choose a_{u_j} to be the first element in this order that is not in the veto list.

end for

Compute the transmit packet: $\mathbf{g} := \sum_{k=1}^{Q} a_k \mathbf{p_k}$

3.5 Properties of the algorithm

3.5.1 Throughput

To ensure correctness, the algorithm uses a finite field of size at least as large as the number of receivers. Theorem 10 shows that this is a sufficient condition to guarantee innovation.

Theorem 10. *If the field is at least as large as the number of receivers, then the above algorithm will always find values for the a_k's such that the resulting transmission satisfies the innovation guarantee property.*

Proof. We first show that the choices made by the algorithm guarantee innovation. For any $j > 0$, consider the j^{th} request group. Let $\mathbf{a}(j-1)$ be the value of the coefficient vector just before processing group j (Note, $\mathbf{a}(\mathbf{0}) = \mathbf{0}$.).

Any receiver in group j has not decoded $\mathbf{p_{u_j}}$ yet. Hence, it cannot know a linear combination of the form $\mathbf{a}(j-1) + \beta \mathbf{e_{u_j}}$ for more than one value of β, where $\mathbf{e_{u_j}}$ is the unit vector with a 1 in the u_j^{th} coordinate and 0 elsewhere. (If it knew two such combinations, it could subtract one from the other to find $\mathbf{p_{u_j}}$, a contradiction.)

Suppose the receiver knows exactly one such linear combination. Then, after the row reduction step, the vector $\mathbf{a}(j-1)$ will get transformed into $\mathbf{a}' = -\beta \mathbf{e_{u_j}}$. Hence, the leading non-zero coefficient of $\mathbf{a}'$ is $-\beta$, and its additive inverse gives β. (Note: the resulting value of β could be 0. This corresponds to the non-innovative case.) If the receiver does not know any linear combination of this form, then packet u_j is not seen, and nothing is added to the veto list.

In short, the values that are vetoed are those values of β for which some receiver knows a linear combination of the form $\mathbf{a}(j-1) + \beta \mathbf{e_{u_j}}$. Hence, by picking a value of a_{u_j} from

outside this list, we ensure innovation. Thus, the algorithm essentially checks for innovation by considering different coefficients β for including $\mathbf{p_{u_j}}$ into the transmission and eliminating the ones that do not work. Finally, processing subsequent groups will not affect the innovation of the previous groups because the subsequent groups will only change the coefficient of their requests, which have already been decoded by the previous groups.

We now show that the algorithm always has enough choices to pick such an a_{u_j} even after excluding the veto list. As argued above, at any point in the algorithm, each receiver adds at most one field element to the veto list. Hence, the veto list can never be longer than the number of receivers in the corresponding request group. Now, we consider two cases.

Case 1: If the group requesting the highest request u_1 does not include all the receivers, then none of the groups contain n receivers. Hence, the veto list for any group will always be strictly shorter than n, and hence if the field size is at least n, there is always a choice left for a_{u_j}.

Case 2: If all n receivers request the highest packet u_1, then it has to be the case that they have all decoded every packet before u_1. Hence, the only coefficient that any receiver would veto for $\mathbf{p_{u_1}}$ is 0, thus leaving other choices for a_{u_1}.

This completes the proof. □

3.5.2 Decoding and delivery delay

We conjecture that the coding module described above has good delay performance.

Conjecture 1. *For the coding module in Section 3.4.3, the expected decoding delay per packet, as well as the expected delivery delay per packet with respect to a particular receiver, grow as* $O\left(\frac{1}{1-\rho}\right)$ *as* $\rho \to 1$*, which is asymptotically optimal.*

The exact analysis of the delay and the proof of this conjecture are open problems. We believe that the notion of seen packets will be useful in this analysis. In particular, to analyze the delivery delay, we can make use of Observation 2 from Section 3.4.2. A packet is delivered if and only if this packet and all packets with a lower index have been seen. This condition is the same as what arises in problems involving a resequencing buffer. Thus, we can formulate our delivery delay problem in terms of traditional queuing problems.

In our formulation, we break down the delivery delay of a packet for a particular receiver into two parts, as though the packet has to traverse two queues in tandem. The first part is simply the time till the packet is seen. Once it is seen, the packet moves into a second queue which is essentially a resequencing buffer. The second part is the time spent in this buffer waiting for all older packets to be seen.

The expectation of the first part is easy to calculate, since every innovative reception causes a new packet to be seen. By Little's theorem, the delay is directly proportional to the size of the queue of unseen packets. This queue's behavior was studied in Chapter 2. Although that work used the older notion of seeing a packet, it can be shown that the analysis still holds even if we use the new notion of seen packets based on reverse ordering. Hence, we get a $O\left(\frac{1}{1-\rho}\right)$ bound on the first part of the delay. The analysis of the second part of the delay however, seems more complicated.

3.5.3 Queue management

The coding module described above makes use of only the oldest undecoded packet of each receiver in any given time-slot. Since our definition of seen packets uses reverse ordering of the packets (see Section 3.4.2), the oldest undecoded packet is always an unseen packet. In other words, the algorithm never uses packets that have been seen by all the receivers. This implies that **the algorithm is compatible with the drop-when-seen queuing algorithm that was proposed and analyzed in Chapter 2**, provided we use the new definition of seen. As pointed out in Observation 1 in Section 3.4.2, the new definition of seeing a packet has the same relation to the dimension of the knowledge space as the old definition of Chapter 2. Thus, we can recycle all the queue size guarantees that were obtained in Chapter 2. In other words, we can get a provable $O\left(\frac{1}{1-\rho}\right)$ growth of the expected queue size at the sender, in addition to the provable innovation guarantee property and the conjectured delay guarantees.

3.6 Simulation results

We now evaluate the performance of the newly proposed coding module through simulations. In particular, we study the behavior of the decoding delay and the delivery delay as a function of the load factor ρ, in the limit as ρ approaches 1, *i.e.*, as the loading on the system approaches capacity.

The probability of reception in any slot is $\mu = 0.5$. The packets arrive according to a Bernoulli process, whose arrival rate is calculated according to the load factor ρ. The load factor is varied through the following values: 0.8, 0.9, 0.92, 0.94, 0.96, 0.97, 0.98 and 0.99. The decoding delay and delivery delay are averaged across the packets over a large number of slots. The number of slots is set to 10^6 for the first four data points, 2×10^6 for the next two points, and at 5×10^6 for the last two points.

We consider two different cases. In the first case, there are three receivers. The entire operation is therefore performed over a $GF(3)$ (*i.e.*, integer operations modulo 3). In the second case, we consider the situation where there are five receivers. In this case, the operations are performed over a field of size 5.

Figure 3-2 shows the plot of the decoding and delivery delay as a function of $\frac{1}{1-\rho}$ for both the three and the five receiver cases. Figure 3-3 shows the same plot in a logarithmic scale. From both these figures, it is clearly seen that the algorithm achieves a linear growth of the delay in terms of $\frac{1}{1-\rho}$. We have thus verified Conjecture 1 for the case of 3 and 5 receivers, using simulations.

3.7 Conclusions

In this chapter, we have thus proposed a new coding module which not only achieves optimal throughput, but is conjectured to achieve asymptotically optimal decoding and in-order delivery delay as well in the heavy traffic limit. In addition, it also allows efficient queue management, leading to asymptotically optimal expected queue size. The algorithm applies to the case of any number of receivers. The conjecture on low delay is verified through simulations.

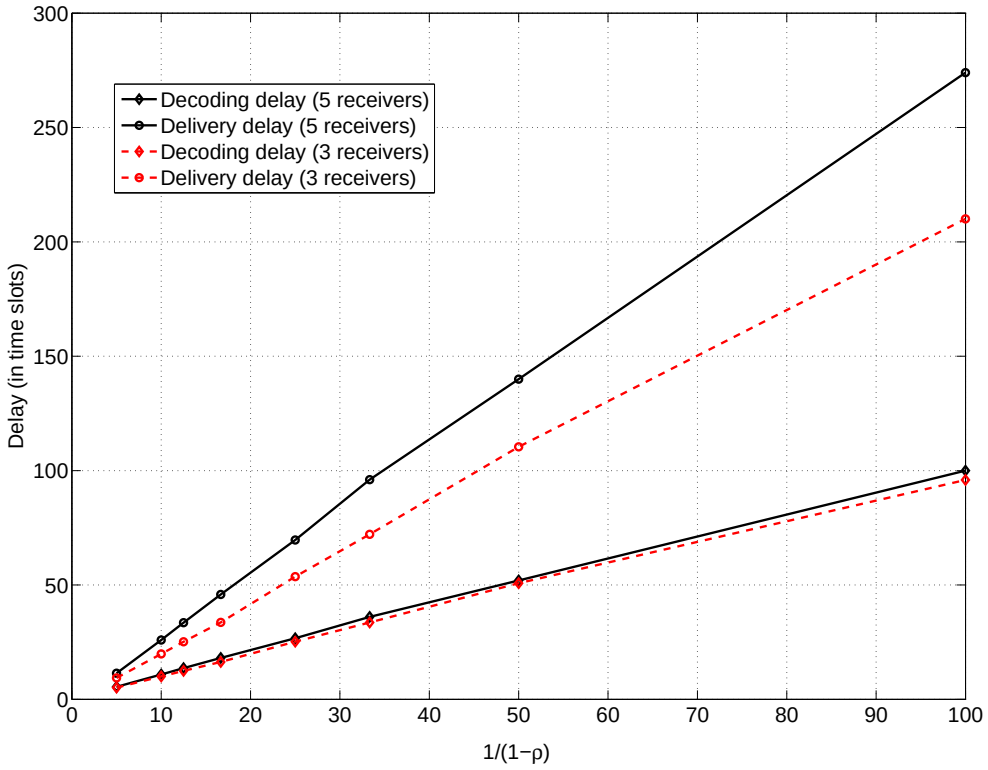


Figure 3-2: Linear plot of the decoding and delivery delay

Our work introduces a new way of adapting the encoding process based on feedback so as to ensure low delay in combination with high throughput. In the future, several extensions are possible. Of particular interest is the study of the effect of delayed or imperfect feedback on the code design. We believe that the main ideas of the coding algorithm will extend to the case where we have imperfections in the feedback link.

Also of interest for the future is the proof of Conjecture 1. The delivery delay is closely related to the problems concerning resequencing buffers, which have been studied in a different context in the literature [64], [65] (see also [66] and the references therein). The techniques used in those works might be useful in studying this problem as well.

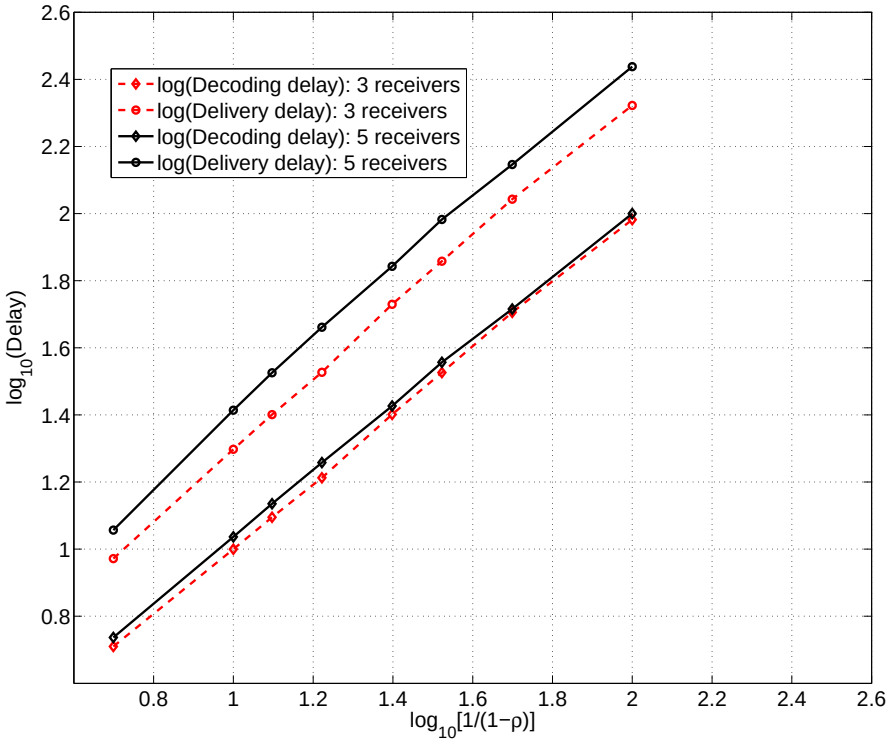


Figure 3-3: Log plot of the decoding and delivery delay

Chapter 4

Interfacing network coding with TCP/IP

Network coding has emerged as an important potential approach to the operation of communication networks, especially wireless networks. The major benefit of network coding stems from its ability to *mix* data, across time and across flows. This makes data transmission over lossy wireless networks robust and effective. Despite this potential of network coding, we still seem far from seeing widespread implementation of network coding across networks. We believe a major reason for this is that it is not clear how to naturally add network coding to current network systems (the incremental deployment problem) and how network coding will behave in a real network in the midst of the various other protocols and network dynamics.

In order to bring the ideas of network coding into practice, we need a protocol that brings out its benefits while requiring very little change in the protocol stack. Flow control and congestion control in today's internet are predominantly based on the Transmission Control Protocol (TCP), which works using the idea of a sliding transmission window of packets, whose size is controlled based on feedback. The TCP paradigm has clearly proven successful. We therefore see a need to find a sliding-window approach as similar as possible to TCP for network coding that makes use of acknowledgments for flow and congestion control. (This problem was initially proposed in [63].) Such an approach would necessarily differ from the generation-based approach more commonly considered for network coding [34], [36]. In this chapter, we show how to incorporate network coding into TCP, allowing its use with minimal changes to the protocol stack, and in such a way that incremental

deployment is possible.

The main idea behind TCP is to use acknowledgments of newly received packets as they arrive *in correct sequence order* in order to guarantee reliable transport and also as a feedback signal for the congestion control loop [7]. This mechanism requires some modification for systems using network coding. The key difference to be dealt with is that, under network coding, the receiver does not obtain original packets of the message, but linear combinations of the packets that are then decoded to obtain the original message once enough such combinations have arrived. Hence, the notion of an ordered sequence of packets as used by TCP is missing, and further, a linear combination may bring in new information to a receiver even though it may not reveal an original packet immediately. The current ACK mechanism does not allow the receiver to acknowledge a packet before it has been decoded. For network coding, we need a modification of the standard TCP mechanism that acknowledges every unit of information received. A new unit of information corresponds mathematically to a *degree of freedom*; essentially, once n degrees of freedom have been obtained, a message that would have required n uncoded packets can be decoded. We present a mechanism that performs the functions of TCP, namely reliable transport and congestion control, based on acknowledging every degree of freedom received, whether or not it reveals a new packet. In fact, whereas TCP is an end-to-end protocol, the proposed interface with network coding allows us to go beyond this and re-encode data inside the network for better erasure-correction, while still presenting the same TCP interface to the application layer above.

Our solution introduces a new network coding layer between the transport layer and the network layer of the protocol stack. Thus, we recycle the congestion control principle of TCP, namely that the number of packets involved in transmissions cannot exceed the number of acknowledgments received by more than the congestion window size. However, we introduce two main changes. First, whenever the source is allowed to transmit, it sends a random linear combination of all packets in the congestion window. Second, the receiver acknowledges degrees of freedom and not original packets. (This idea was previously introduced in Chapter 2 in the context of a single hop erasure broadcast link.) An appropriate interpretation of the degree of freedom allows us to order the receiver degrees of freedom

in a manner consistent with the packet order of the source. This allows us to use the standard TCP protocol with minimal change. We use the TCP-Vegas protocol [67] in the initial description, as it is more compatible with our modifications. However, in the next chapter, we shall demonstrate that our protocol is also compatible with the more commonly used TCP-Reno.

4.1 Implications for wireless networking

In considering the potential benefits of our TCP-compatible network coding solution, we focus on the area of wireless links. We now explain the implications of this new protocol for improving throughput in wireless networks.

TCP was originally developed for wired networks and was designed to interpret each packet loss as a congestion signal. Since wired networks have very little packet loss on the links and the predominant source of loss is buffer overflow due to congestion, TCP's approach works well. In contrast, wireless networks are characterized by packet loss on the link and intermittent connectivity due to fading. It is well known that TCP is not well suited for such lossy links. The primary reason is that it wrongly assumes the cause of link losses to be congestion, and reduces its transmission rate unnecessarily, leading to low throughput.

Adapting TCP for wireless scenarios is a very well-studied problem (see [68] and references therein for a survey). The general approach has been to mask losses from TCP using link layer retransmission [69]. However, it has been noted in the literature ([70], [71]) that the interaction between link layer retransmission and TCP's retransmission can be complicated and that performance may suffer due to independent retransmission protocols at different layers.

More importantly, if we want to exploit the broadcast nature of the wireless medium, conventional link layer retransmission may not be the best approach. For example, suppose a node with two neighbors transmits two packets A and B and suppose A is heard by only the first neighbor and B only by the second neighbor. Then, even though each neighbor has lost one packet, no retransmission may be necessary if they both opportunistically forward

the packets that they received, towards the final destination. Even if, for some reason, we want both neighbors to receive both packets, a coded transmission, namely A XOR B is more efficient than repeating either A or B, as it would simultaneously convey the missing packet to both neighbors.

There has been a growing interest in approaches that make active use of the intrinsic broadcast nature of the wireless medium. In the technique known as opportunistic routing [72], a node broadcasts its packet, and if any of its neighbors receives the packet, that node will forward the packet downstream, thereby obtaining a diversity benefit. In [73], the authors study the case of unicast traffic in a wireless erasure network with feedback, and present a capacity achieving flooding-based policy with no coding. Their algorithm requires each node to transmit a packet chosen randomly from its buffer. A node would drop a packet from its buffer upon hearing an ACK from the receiver. The feedback is thus not link-by-link, but of the form that every node throughout the network is immediately informed of a successful reception at the receiver. Moreover, this scheme could generate multiple copies of each packet. An alternate approach is that if more than one of the neighbors receive the packet, they would coordinate and decide who will forward the packet. A backpressure-based solution to this problem was proposed in [74]. However, in general, such coordination could require a lot of overhead.

The MORE protocol [36] proposed the use of intra-flow network coding in combination with opportunistic routing. The random linear mixing (coding) of incoming packets at a node before forwarding them downstream was shown to reduce the coordination overhead associated with opportunistic routing. Another advantage is that the coding operation can be easily tuned to add redundancy to the packet stream to combat erasures, even with limited feedback. Besides, such schemes can potentially achieve capacity, even for a multicast connection [12].

One issue with these approaches however, is that typical implementations use batches of packets instead of sliding windows, and are generally therefore not compatible with TCP. ExOR uses batching to reduce the coordination overhead, but as mentioned in [72], this interacts badly with TCP's window mechanism. MORE uses batching to perform the coding operation. In this case, the receiver cannot acknowledge the packets until an entire

batch has arrived and has been successfully decoded. Since TCP performance heavily relies on the timely return of ACKs, such a delay in the ACKs would affect the round-trip time calculation and thereby reduce the throughput.

Another issue with opportunistic routing is that it could lead to reordering of packets. Schemes such as [73], [72] or [74] that are based on forwarding of packets opportunistically, may not be able to deliver the packets in their original order at the sender. Such reordering is known to interact badly with TCP, as it can cause duplicate ACKs, and TCP interprets duplicate ACKs as a sign of congestion.

Our work addresses both these issues – batching and reordering. It proposes a TCP-compatible sliding window network coding scheme in combination with a new acknowledgment mechanism for running TCP over a network coded system. The sender would transmit a random linear combination of packets in the TCP congestion window. The new type of ACK allows the receiver to acknowledge every linear combination (degree of freedom) that is linearly independent from the previously received linear combinations. The receiver does not have to wait to decode a packet, but can send a TCP ACK for every degree of freedom received, thus eliminating the problems of using batchwise ACKs.

It is shown later (Lemma 10) that if the linear combination happens over a large enough finite field, then every incoming random linear combination will, with high probability, generate a TCP ACK for the very next unacknowledged packet in order, among the ones involved in the linear combination. This is because the random combinations do not have any inherent ordering. The argument holds true even when multiple paths deliver the random linear combinations. Hence the use of random linear coding with the acknowledgment of degrees of freedom can potentially *address the TCP reordering problem for multipath opportunistic routing schemes*.

Our scheme does not rely on the link layer for recovering losses. Instead, we use an erasure correction scheme based on random linear codes across packets. Coding across packets is a natural way to handle losses. A coding based approach is better suited for broadcast-mode opportunistic routing scenarios, as randomly chosen linear combinations of packets are more likely to convey new information, compared to retransmissions. Reference [18] proposed a scheme where intermediate nodes in the network have no buffers and

the resulting packet drops are compensated for by random linear coding at the sender. In contrast, our work uses random linear coding only to correct losses that occur on the link due to channel errors. We do not aim to mask buffer overflow losses, since these losses may be needed for TCP to measure the level of congestion.

The tradeoff between using coding and using retransmissions to correct errors has been studied at the physical layer from a theoretical point of view by [75]. A similar question arises at the TCP layer as well. In fact, the question is further complicated by the fact that error recovery using retransmission is directly linked with the congestion control mechanism of TCP. We need sufficient coding to mask network layer losses from TCP, but at the same time, we need to allow the buffer overflow losses to be recovered by the retransmission mechanism so that congestion may be correctly detected when it happens.

In summary, by providing an interface between TCP and a network coded system, we present a new approach to implementing TCP over wireless networks, and it is here where the benefits of our solution are most dramatic.

It is important to note that our scheme respects the end-to-end philosophy of TCP – it would work even if coding operations are performed only at the end hosts. Having said that, if some nodes inside the network also perform network coding, our solution naturally generalizes to such scenarios as well. The queuing analysis in Section 4.5.2 considers such a situation.

The rest of the chapter explains the details of our new protocol along with its theoretical basis, and analyzes its performance using simulations as well as an idealized theoretical analysis.

4.2 The new protocol

In this section, we present the logical description of our new protocol, followed by a way to implement these ideas with as little disturbance as possible to the existing protocol stack.

4.2.1 Logical description

The main aim of our algorithm is to mask losses from TCP using random linear coding. We make some important modifications in order to incorporate coding. First, instead of the original packets, we transmit random linear combinations of packets in the congestion window. While such coding helps with erasure correction, it also leads to a problem in acknowledging data. TCP operates with units of packets, which have a well-defined ordering. Thus, the packet sequence number can be used for acknowledging the received data. The unit in our protocol is a degree of freedom. However, when packets are coded together, there is no clear ordering of the degrees of freedom that can be used for ACKs. Our main contribution is the solution to this problem. The notion of seen packets defines an ordering of the degrees of freedom that is consistent with the packet sequence numbers, and can therefore be used to acknowledge degrees of freedom.

Upon receiving a linear combination, the sink finds out which packet, if any, has been newly seen because of the new arrival and acknowledges that packet. The sink thus pretends to have received the packet even if it cannot be decoded yet. We will show in Section 4.3 that in the end this is not a problem because if all the packets in a file have been seen, then they can all be decoded as well.

The idea of transmitting random linear combinations and acknowledging seen packets achieves our goal of masking losses from TCP as follows. With a large field size, every random linear combination is very likely to cause the next unseen packet to be seen (see Lemma 10). So, even if a transmitted linear combination is lost, the next successful reception will cause the next unseen packet to be seen. From TCP's perspective, this appears as though the degree of freedom waits in a fictitious queue until the channel stops erasing packets and allows it through. Thus, there will never be any duplicate ACKs. Every ACK will cause the congestion window to advance. In short, *the lossiness of the link is presented to TCP as an additional queuing delay that leads to a larger effective round-trip time*. The term round-trip time thus has a new interpretation. It is the effective time the network takes to deliver *reliably* a degree of freedom (including the delay for the coded redundancy, if necessary), followed by the return of the ACK. This is larger than the true network delay

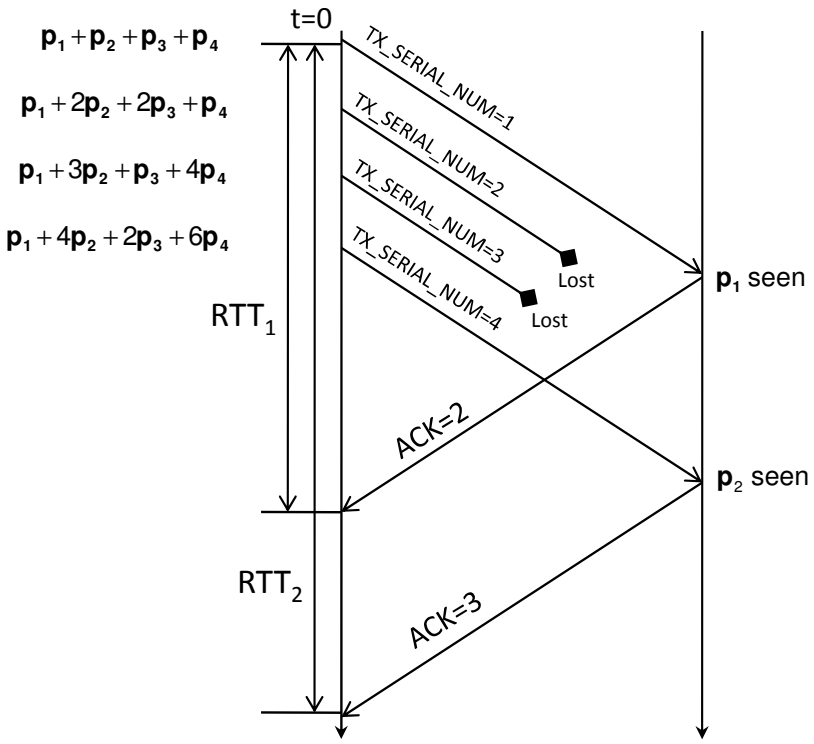


Figure 4-1: Example of coding and ACKs

needed for a lossless transmission and the return of the ACK. The more lossy the link is, the larger will be the effective RTT. Presenting TCP with a larger value for RTT may seem counterintuitive as TCP's rate is inversely related to RTT. However, if done correctly, it improves the rate by preventing loss-induced window closing, as it gives the network more time to deliver the data in spite of losses, before TCP times out. Therefore, losses are effectively masked.

Consider the example shown in Figure 4-1. Suppose the congestion window's length is 4. Assume TCP sends 4 packets to the network coding layer at $t = 0$. All 4 transmissions are linear combinations of these 4 packets. The 1^{st} transmission causes the 1^{st} packet to be seen. The 2^{nd} and 3^{rd} transmissions are lost, and the 4^{th} transmission causes the 2^{nd} packet to be seen (the discrepancy is because of losses). As far as the RTT estimation is concerned, transmissions 2, 3 and 4 are treated as attempts to convey the 2^{nd} degree of freedom. The RTT for the 2^{nd} packet must include the final attempt that successfully delivers the 2^{nd} degree of freedom, namely the 4^{th} transmission. In other words, the RTT is the time from $t = 0$ until the time of reception of ACK=3.

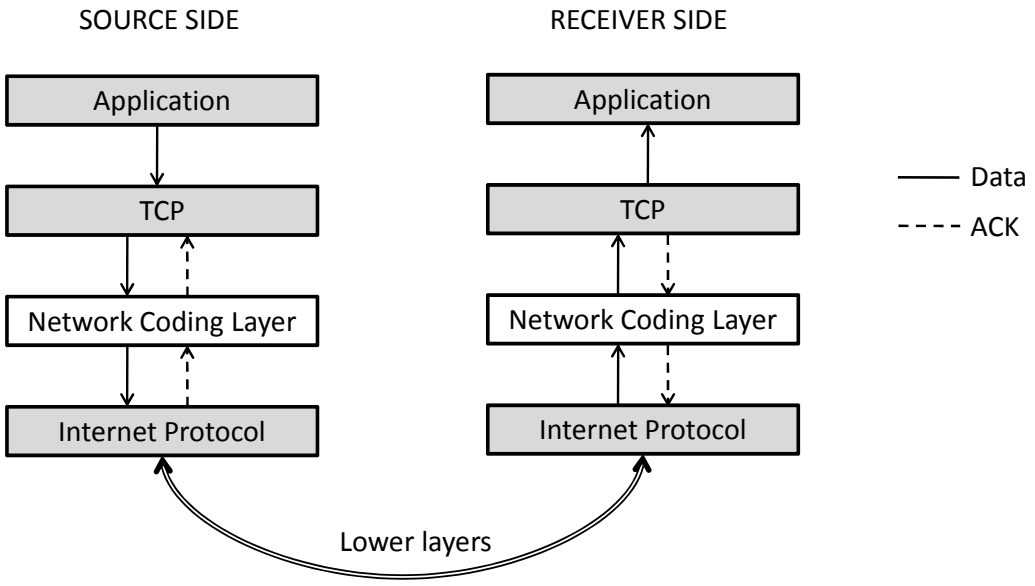


Figure 4-2: New network coding layer in the protocol stack

4.2.2 Implementation

The implementation of all these ideas in the existing protocol stack needs to be done in as non-intrusive a manner as possible. We present a solution which embeds the network coding operations in a separate layer below TCP and above IP on the source and receiver side, as shown in Figure 4-2. The exact operation of these modules is described next.

The sender module accepts packets from the TCP source and buffers them into an encoding buffer which represents the coding window[1], until they are ACKed by the receiver. The sender then generates and sends random linear combinations of the packets in the coding window. The coefficients used in the linear combination are also conveyed in the header.

For every packet that arrives from TCP, R linear combinations are sent to the IP layer on average, where R is the redundancy parameter. The average rate at which linear combinations are sent into the network is thus a constant factor more than the rate at which TCP's congestion window progresses. This is necessary in order to compensate for the loss rate

[1]Whenever a new packet enters the TCP congestion window, TCP transmits it to the network coding module, which then adds it to the coding window. Thus, the coding window is related to the TCP layer's congestion window but generally not identical to it. For example, the coding window will still hold packets that were transmitted earlier by TCP, but are no longer in the congestion window because of a reduction of the window size by TCP. However, this is not a problem because involving more packets in the linear combination will only increase its chances of being innovative.

of the channel and to match TCP's sending rate to the rate at which data is actually sent to the receiver. If there is too little redundancy, then the data rate reaching the receiver will not match the sending rate because of the losses. This leads to a situation where the losses are not effectively masked from the TCP layer. Hence, there are frequent timeouts leading to a low throughput. On the other extreme, too much redundancy is also bad, since then the transmission rate becomes limited by the rate of the code itself. Besides, sending too many linear combinations can congest the network. The ideal level of redundancy is to keep R equal to the reciprocal of the probability of successful reception. Thus, in practice the value of R should be dynamically adjusted by estimating the loss rate, possibly using the RTT estimates.

Upon receiving a linear combination, the receiver module first retrieves the coding coefficients from the header and appends it to the basis matrix of its knowledge space. Then, it performs a Gaussian elimination to find out which packet is newly seen so that this packet can be ACKed. The receiver module also maintains a buffer of linear combinations of packets that have not been decoded yet. Upon decoding the packets, the receiver module delivers them to the TCP sink.

The algorithm is specified below using pseudo-code. This specification assumes a one-way TCP flow.

Source side

The source side algorithm has to respond to two types of events – the arrival of a packet from the source TCP, and the arrival of an ACK from the receiver via IP.

1. Set NUM to 0.
2. *Wait state:* If any of the following events occurs, respond as follows; else, wait.
3. *Packet arrives from TCP sender:*
   (a) If the packet is a control packet used for connection management, deliver it to the IP layer and return to wait state.
   (b) If packet is not already in the coding window, add it to the coding window.

(c) Set NUM = NUM+R. (R=redundancy factor)

(d) Repeat the following $\lfloor NUM \rfloor$ times:

i) Generate a random linear combination of the packets in the coding window.

ii) Add the network coding header specifying the set of packets in the coding window and the coefficients used for the random linear combination.

iii) Deliver the packet to the IP layer.

(e) Set $NUM :=$ fractional part of NUM.

(f) Return to the wait state.

4. *ACK arrives from receiver:* Remove the ACKed packet from the coding buffer and hand over the ACK to the TCP sender.

Receiver side

On the receiver side, the algorithm again has to respond to two types of events: the arrival of a packet from the source, and the arrival of ACKs from the TCP sink.

1. *Wait state:* If any of the following events occurs, respond as follows; else, wait.

2. *ACK arrives from TCP sink:* If the ACK is a control packet for connection management, deliver it to the IP layer and return to the wait state; else, ignore the ACK.

3. *Packet arrives from source side:*

(a) Remove the network coding header and retrieve the coding vector.

(b) Add the coding vector as a new row to the existing coding coefficient matrix, and perform Gaussian elimination to update the set of seen packets.

(c) Add the payload to the decoding buffer. Perform the operations corresponding to the Gaussian elimination, on the buffer contents. If any packet gets decoded in the process, deliver it to the TCP sink and remove it from the buffer.

(d) Generate a new TCP ACK with sequence number equal to that of the oldest unseen packet.

4.3 Soundness of the protocol

We argue that our protocol guarantees reliable transfer of information. In other words, every packet in the packet stream generated by the application at the source will be delivered eventually to the application at the sink. We observe that the acknowledgment mechanism ensures that the coding module at the sender does not remove a packet from the coding window unless it has been ACKed, *i.e.*, unless it has been seen by the sink. Thus, we only need to argue that if all packets in a file have been seen, then the file can be decoded at the sink.

Theorem 11. *From a file of n packets, if every packet has been seen, then every packet can also be decoded.*

Proof. If the sender knows a file of n packets, then the sender's knowledge space is of dimension n. Every seen packet corresponds to a new dimension. Hence, if all n packets have been seen, then the receiver's knowledge space is also of dimension n, in which case it must be the same as the sender's and all packets can be decoded. □

In other words, seeing n different packets corresponds to having n linearly independent equations in n unknowns. Hence, the unknowns can be found by solving the system of equations. At this point, the file can be delivered to the TCP sink. In practice, one does not have to necessarily wait until the end of the file to decode all packets. Some of the unknowns can be found even along the way. In particular, whenever the number of equations received catches up with the number of unknowns involved, the unknowns can be found. Now, for every new equation received, the receiver sends an ACK. The congestion control algorithm uses the ACKs to control the injection of new unknowns into the coding window. Thus, the discrepancy between the number of equations and number of unknowns does not tend to grow with time, and therefore will hit zero often based on the channel conditions. As a consequence, the decoding buffer will tend to be stable.

An interesting observation is that the arguments used to show the soundness of our approach are quite general and can be extended to more general scenarios such as random linear coding based multicast over arbitrary topologies.

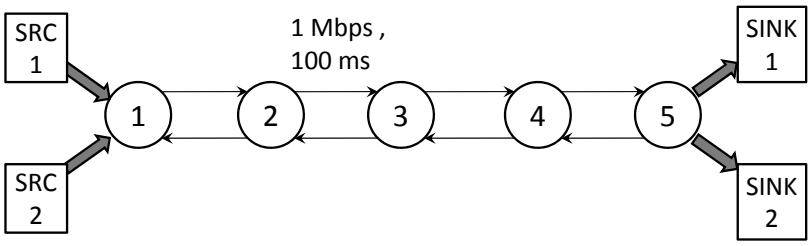


Figure 4-3: Simulation topology

4.4 Fairness of the protocol

Here, we study the fairness property of our algorithm through simulations.

4.4.1 Simulation setup

The protocol described above is simulated using the Network Simulator (ns-2) [76]. The topology for the simulations is a tandem network consisting of 4 hops (hence 5 nodes), shown in Figure 4-3. The source and sink nodes are at opposite ends of the chain. Two FTP applications want to communicate from the source to the sink. There is no limit on the file size. They emit packets continuously till the end of the simulation. They either use TCP without coding or TCP with network coding (denoted TCP/NC). In this simulation, intermediate nodes do not re-encode packets. All the links have a bandwidth of 1 Mbps, and a propagation delay of 100 *ms*. The buffer size on the links is set at 200. The TCP receive window size is set at 100 packets, and the packet size is 1000 bytes. The Vegas parameters are chosen to be $\alpha = 28, \beta = 30, \gamma = 2$ (see [67] for details of Vegas).

4.4.2 Fairness and compatibility – simulation results

By fairness, we mean that if two similar flows compete for the same link, they must receive an approximately equal share of the link bandwidth. In addition, this must not depend on the order in which the flows join the network. The fairness of TCP-Vegas is a well-studied problem. It is known that depending on the values chosen for the α and β parameters, TCP-Vegas could be unfair to an existing connection when a new connection enters the bottleneck link ([77], [78]). Several solutions have been presented to this problem in the literature (for example, see [79] and references therein). In our simulations, we first pick

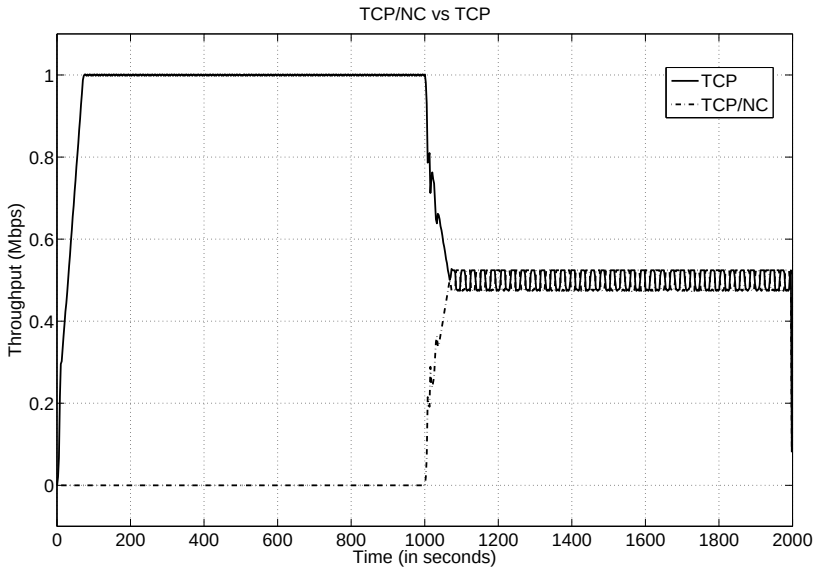


Figure 4-4: Fairness and compatibility - one TCP/NC and one TCP flow

values of α and β that allow fair sharing of bandwidth when two TCP flows without our modification compete with each other, in order to evaluate the effect of our modification on fairness. With the same α and β, we consider two cases:

Case 1: The situation where a network coded TCP flow competes with another flow running TCP without coding.

Case 2: The situation where two coded TCP flows compete with each other.

In both cases, the loss rate is set to 0% and the redundancy parameter is set to 1 for a fair comparison. In the first case, the TCP flow starts first at $t = 0.5s$ and the TCP/NC flow starts at $1000s$. The system is simulated for 2000 s. The current throughput is calculated at intervals of $2.5s$. The evolution of the throughput over time is shown in Figure 4-4. The figure shows that the effect of introducing the coding layer does not affect fairness. We see that, after the second flow starts, the bandwidth gets redistributed fairly.

For case 2, the experiment is repeated with the same starting times, but this time both flows are TCP/NC flows. The plot for this case is essentially identical to Figure 4-4 (and hence is not shown here) because in the absence of losses, TCP/NC behaves identically to TCP if we ignore the effects of field size. Thus, coding can coexist with TCP in the absence of losses, without affecting fairness.

4.5 Effectiveness of the protocol

We now show that the new protocol indeed achieves a high throughput, especially in the presence of losses. We first describe simulation results comparing the protocol's performance with that of TCP in Section 4.5.1. Next, in Section 4.5.2, we study the effectiveness of the random linear coding ideas in a theoretical model with idealized assumptions such as infinite buffer space, and known channel capacity. We show that, in such a scenario, our scheme stabilizes the queues for all rates below capacity.

4.5.1 Throughput of the new protocol – simulation results

The simulation setup is identical to that used in the fairness simulations (see Section 4.4.1).

We first study the effect of the redundancy parameter on the throughput of TCP/NC for a fixed loss rate of 5%. By loss rate, we mean the probability of a packet getting lost on each link. Both packets in the forward direction as well as ACKs in the reverse direction are subject to these losses. No re-encoding is allowed at the intermediate nodes. Hence, the overall probability of packet loss across 4 hops is given by $1-(1-0.05)^4$ which is roughly 19%. Hence the capacity is roughly 0.81 Mbps, which when split fairly gives 0.405 Mbps per flow. The simulation time is $10000s$.

We allow two TCP/NC flows to compete on this network, both starting at $0.5s$. Their redundancy parameter is varied between 1 and 1.5. The theoretically optimum value is approximately $1/(1-0.19) \simeq 1.23$. Figure 4-5 shows the plot of the throughput for the two flows, as a function of the redundancy parameter R. It is clear from the plot that R plays an important role in TCP/NC. We can see that the throughput peaks around $R = 1.25$. The peak throughput achieved is 0.397 Mbps, which is indeed close to the capacity that we calculated above. In the same situation, when two TCP flows compete for the network, the two flows see a throughput of 0.0062 and 0.0072 Mbps respectively. Thus, with the correct choice of R, the throughput for the flows in the TCP/NC case is very high compared to the TCP case. In fact, even with $R = 1$, TCP/NC achieves about 0.011 Mbps for each flow improving on TCP by almost a factor of 2.

Next, we study the variation of throughput with loss rate for both TCP and TCP/NC.

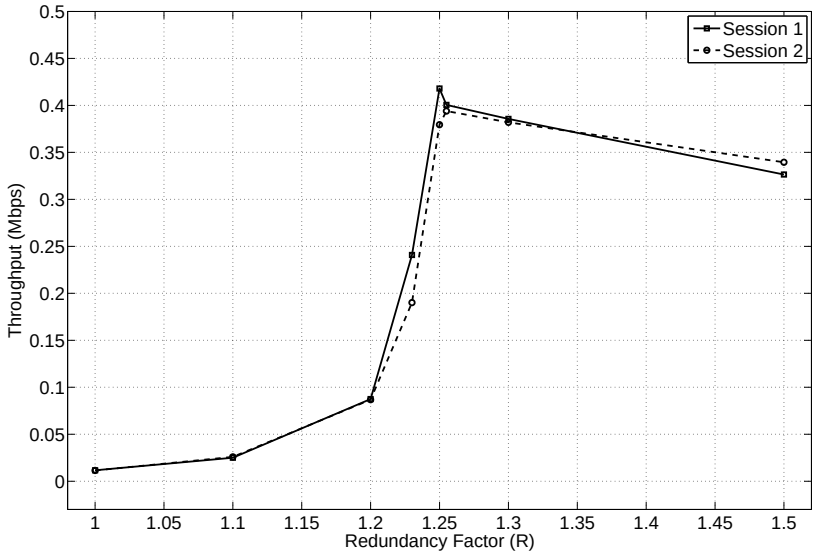


Figure 4-5: Throughput vs redundancy for TCP/NC

The simulation parameters are all the same as above. The loss rate of all links is kept at the same value, and this is varied from 0 to 20%. We compare two scenarios – two TCP flows competing with each other, and two TCP/NC flows competing with each other. For the TCP/NC case, we set the redundancy parameter at the optimum value corresponding to each loss rate. Figure 4-6 shows that TCP's throughput falls rapidly as losses increase. However, TCP/NC is very robust to losses and reaches a throughput that is close to capacity. (If p is the loss rate on each link, then capacity is $(1-p)^4$, which must then be split equally.)

Figure 4-7 shows the instantaneous throughput in a 642 second long simulation of a tandem network with 3 hops (*i.e.*, 4 nodes), where erasure probabilities vary with time in some specified manner. The third hop is, on average, the most erasure-prone link. The plots are shown for traditional TCP, TCP/NC with coding only at the source, and TCP/NC with re-encoding at node 3 (just before the worst link). The operation of the re-encoding node is very similar to that of the source – it collects incoming linear combinations in a buffer, and transmits, on average, R_{int} random linear combinations of the buffer contents for every incoming packet. The R of the sender is set at 1.8, and the R_{int} of node 3 is set at 1.5 for the case when it re-encodes. The average throughput is shown in the table. A considerable improvement is seen due to the coding, that is further enhanced by allowing intermediate node re-encoding. This plot thus shows that our scheme is also suited to systems with coding inside the network.

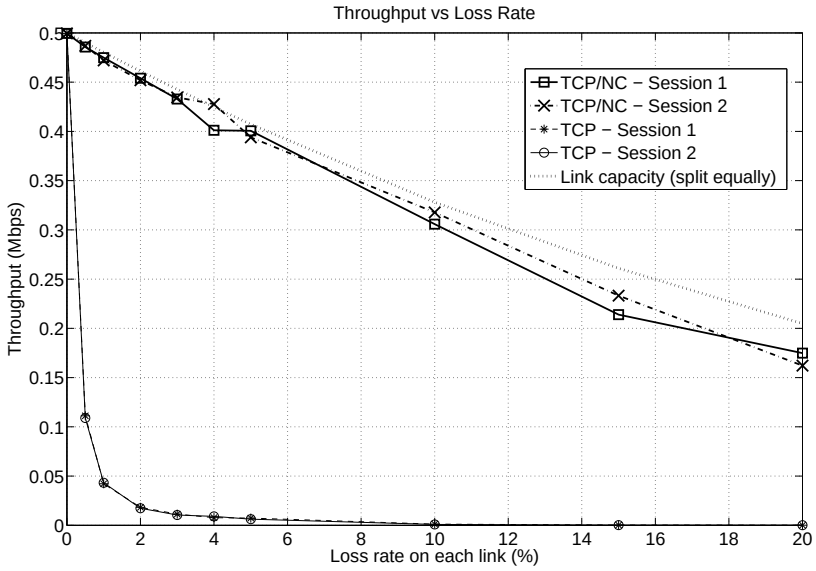
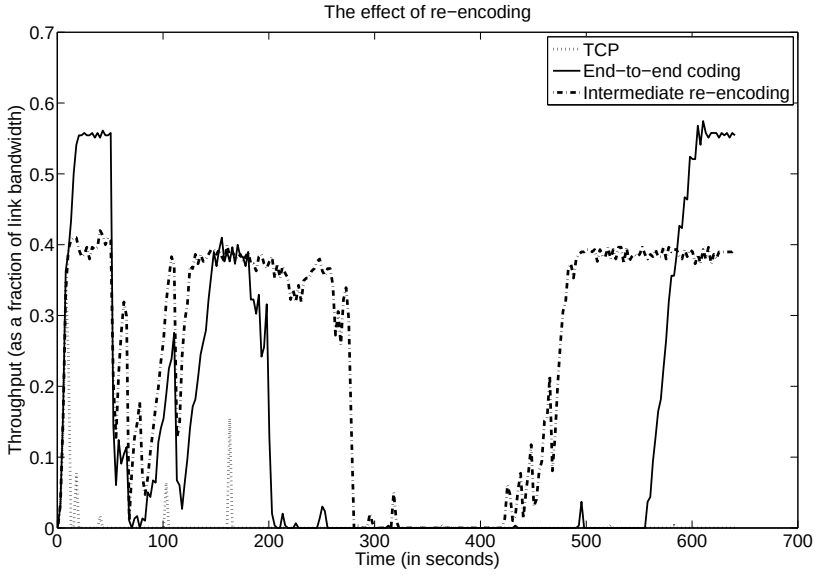


Figure 4-6: Throughput vs loss rate for TCP and TCP/NC



TCP	End-to-end coding	Re-encoding at node 3 only
0.0042 Mbps	0.1420 Mbps	0.2448 Mbps

Figure 4-7: Throughput with and without intermediate node re-encoding

Remark 3. *These simulations are meant to be a preliminary study of our algorithm's performance. Specifically, the following points must be noted:*

– Link layer retransmission is not considered for either TCP or TCP/NC. If allowed, this could improve the performance of TCP. However, as mentioned earlier, the retransmission approach does not extend to more general multipath routing solutions, whereas coding is better suited to such scenarios.

– The throughput values do not account for the overhead associated with the network coding headers. The main overhead is in conveying the coding coefficients and the contents of the coding window. If the source and sink share a pseudorandom number generator, then the coding coefficients can be conveyed succinctly by sending the current state of the generator. Similarly, the coding window contents can be conveyed in an incremental manner to reduce the overhead.

– The loss in throughput due to the finiteness of the field has not been modeled in the simulations. A small field might cause received linear combinations to be non-innovative, or might cause packets to be seen out of order, resulting in duplicate ACKs. However, the probability that such problems persist for a long time falls rapidly with the field size. We believe that for practical choices of field size, these issues will only cause transient effects that will not have a significant impact on performance. These effects remain to be quantified exactly.

– Finally, the decoding delay associated with the network coding operation has not been studied. We intend to focus on this aspect in experiments in the future. A thorough experimental evaluation of all these aspects of the algorithm, on a more general topology, is part of ongoing work.

4.5.2 The ideal case

In this section, we focus on an idealized scenario in order to provide a first order analysis of our new protocol. We aim to explain the key ideas of our protocol with emphasis on the interaction between the coding operation and the feedback. The model used in this section will also serve as a platform which we can build on to incorporate more practical situations.

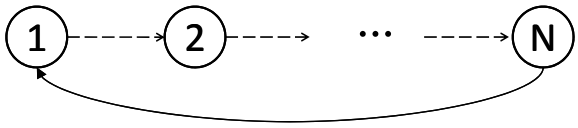


Figure 4-8: Topology: Daisy chain with perfect end-to-end feedback

We abstract out the congestion control aspect of the problem by assuming that the capacity of the system is fixed in time and known at the source, and hence the arrival rate is always maintained below the capacity. We also assume that nodes have infinite capacity buffers to store packets. We focus on a topology that consists of a chain of erasure-prone links in tandem, with perfect end-to-end feedback from the sink directly to the source. In such a system, we investigate the behavior of the queue sizes at various nodes.

System model

The network we study in this section is a daisy chain of N nodes, each node being connected to the next one by a packet erasure channel. We assume a slotted time system. The source generates packets according to a Bernoulli process of rate λ packets per slot. The point of transmission is at the very beginning of a slot. Just after this point, every node transmits one random linear combination of the packets in its queue. The relation between the transmitted linear combination and the original packet stream is conveyed in the packet header. We ignore this overhead for the analysis in this section. We ignore propagation delay. Thus, the transmission, if not erased by the channel, reaches the next node in the chain almost immediately. However, the node may use the newly received packet only in the next slot's transmission. We assume perfect, delay-free feedback from the sink to the source. In every slot, the sink generates the feedback signal after the instant of reception of the previous node's transmission. The erasure event happens with a probability $(1 - \mu_i)$ on the channel connecting node i and $(i + 1)$, and is assumed to be independent across different channels and over time. Thus, the system has a capacity $\min_i \mu_i$ packets per slot. We assume that $\lambda < \min_i \mu_i$, and define the load factor $\rho_i = \lambda/\mu_i$.

Queue update mechanism

Each node transmits a random linear combination of the current contents of its queue and hence, it is important to specify how the queue contents are updated at the different nodes. Queue updates at the source are relatively simple because in every slot, the sink is assumed to send an ACK directly to the source, containing the index of the oldest packet not yet seen by the sink. Upon receiving the ACK, the source simply drops all packets from its queue with an index lower than the sink's request.

Whenever an intermediate node receives an innovative packet, this causes the node to see a previously unseen packet. The node performs a Gaussian elimination to compute the witness of the newly seen packet, and adds this to the queue. Thus, intermediate nodes store the witnesses of the packets that they have seen. The idea behind the packet drop rule is similar to that at the source – an intermediate node may drop the witnesses of packets up to but excluding what it believes to be the sink's first unseen packet, based on its knowledge of the sink's status at that point of time.

However, the intermediate nodes, in general, may only know an outdated version of the sink's status because we assume that the intermediate nodes do not have direct feedback from the sink (see Figure 4-8). Instead, the source has to inform them about the sink's ACK through the same erasure channel used for the regular forward transmission. This feed-forward of the sink's status is modeled as follows. Whenever the channel entering an intermediate node is in the ON state (*i.e.*, no erasure), the node's version of the sink's status is updated to that of the previous node. In practice, the source need not transmit the sink's status explicitly. The intermediate nodes can infer it from the set of packets that have been involved in the linear combination – if a packet is no longer involved, that means the source must have dropped it, implying that the sink must have ACKed it already.

Remark 4. *This model and the following analysis also works for the case when not all intermediate nodes are involved in the network coding. If some node simply forwards the incoming packets, then we can incorporate this in the following way. An erasure event on either the link entering this node or the link leaving this node will cause a packet erasure. Hence, these two links can be replaced by a single link whose probability of being ON is*

simply the product of the ON probabilities of the two links being replaced. Thus, all non-coding nodes can be removed from the model, which brings us back to the same situation as in the above model.

Queuing analysis

We now analyze the size of the queues at the nodes under the queuing policy described above. The following theorem shows that if we allow coding at intermediate nodes, then it is possible to achieve the capacity of the network, namely $\min_k \mu_k$. In addition, it also shows that the expected queue size in the heavy-traffic limit ($\lambda \to \min_k \mu_k$) has an asymptotically optimal linear scaling in $1/(1-\rho_k)$.

Note that, if we only allow forwarding at some of the intermediate nodes, then we can still achieve the capacity of a new network derived by collapsing the links across the non-coding nodes, as described in Remark 4.

Theorem 12. *As long as $\lambda < \mu_k$ for all $0 \leq k < N$, the queues at all the nodes will be stable. The expected queue size in steady state at node k ($0 \leq k < N$) is given by:*

$$\mathbb{E}[Q_k] = \sum_{i=k}^{N-1} \frac{\rho_i(1-\mu_i)}{(1-\rho_i)} + \sum_{i=1}^{k-1} \rho_i$$

An implication: Consider a case where all the ρ_i's are equal to some ρ. Then, the above relation implies that in the limit of heavy traffic, *i.e.*, $\rho \to 1$, the queues are expected to be longer at nodes near the source than near the sink.

A useful lemma: The above theorem will be proved after the following lemma. The lemma shows that the random linear coding scheme has the property that every time there is a successful reception at a node, the node sees the next unseen packet with high probability, provided the field is large enough. This fact will prove useful while analyzing the evolution of the queues.

Lemma 10. *Let S_A and S_B be the set of packets seen by two nodes A and B respectively. Assume $S_A \backslash S_B$ is non-empty. Suppose A sends a random linear combination of its witnesses of packets in S_A and B receives it successfully. The probability that this transmission causes*

B to see the oldest packet in $S_A \backslash S_B$ is $\left(1 - \frac{1}{q}\right)$, where q is the field size.

Proof. Let M_A be the RREF basis matrix for A. Then, the coefficient vector of the linear combination sent by A is $\mathbf{t} = \mathbf{u}M_A$, where $\mathbf{u}$ is a vector of length $|S_A|$ whose entries are independent and uniformly distributed over the finite field $\mathbb{F}_q$. Let d^* denote the index of the oldest packet in $S_A \backslash S_B$.

Let M_B be the RREF basis matrix for B before the new reception. Suppose $\mathbf{t}$ is successfully received by B. Then, B will append $\mathbf{t}$ as a new row to M_B and perform Gaussian elimination. The first step involves subtracting from $\mathbf{t}$, suitably scaled versions of the pivot rows such that all entries of $\mathbf{t}$ corresponding to pivot columns of M_B become 0. We need to find the probability that after this step, the leading non-zero entry occurs in column d^*, which corresponds to the event that B sees packet d^*. Subsequent steps in the Gaussian elimination will not affect this event. Hence, we focus on the first step.

Let P_B denote the set of indices of pivot columns of M_B. In the first step, the entry in column d^* of $\mathbf{t}$ becomes

$$t'(d^*) = t(d^*) - \sum_{i \in P_B, i < d^*} t(i) \cdot M_B(r_B(i), d^*)$$

where $r_B(i)$ is the index of the pivot row corresponding to pivot column i in M_B. Now, due to the way RREF is defined, $t(d^*) = u(r_A(d^*))$, where $r_A(i)$ denotes the index of the pivot row corresponding to pivot column i in M_A. Thus, $t(d^*)$ is uniformly distributed. Also, for $i < d^*$, $t(i)$ is a function of only those $u(j)$'s such that $j < r_A(d^*)$. Hence, $t(d^*)$ is independent of $t(i)$ for $i < d^*$. From these observations and the above expression for $t'(d^*)$, it follows that for any given M_A and M_B, $t'(d^*)$ has a uniform distribution over $\mathbb{F}_q$, and the probability that it is not zero is therefore $\left(1 - \frac{1}{q}\right)$. □

Computing the expected queue size: For the queuing analysis, we assume that a successful reception always causes the receiver to see its next unseen packet, as long as the transmitter has already seen it. The above lemma argues that this assumption becomes increasingly valid as the field size increases. In reality, some packets may be seen out of order, resulting in larger queue sizes. However, we believe that this effect is minor and can

be neglected for a first order analysis.

With this assumption in place, the queue update policy described earlier implies that the size of the physical queue at each node is simply the difference between the number of packets the node has seen and the number of packets it believes the sink has seen.

To study the queue size, we define a virtual queue at each node that keeps track of the degrees of freedom backlog between that node and the next one in the chain. The arrival and departure of the virtual queues are defined as follows. A packet is said to arrive at a node's virtual queue when the node sees the packet for the first time. A packet is said to depart from the virtual queue when the next node in the chain sees the packet for the first time. A consequence of the assumption stated above is that the set of packets seen by a node is always a contiguous set. This allows us to view the virtual queue maintained by a node as though it were a first-in-first-out (FIFO) queue. The size of the virtual queue is simply the difference between the number of packets seen by the node and the number of packets seen by the next node downstream

We are now ready to prove Theorem 12. For each intermediate node, we study the expected time spent by an arbitrary packet in the physical queue at that node, as this is related to the expected physical queue size at the node, by Little's law.

Proof of Theorem 12: Consider the k^{th} node, for $1 \leq k < N$. The time a packet spends in this node's queue has two parts:

1) *Time until the packet is seen by the sink:*

The virtual queue at a node essentially behaves like a FIFO $Geom/Geom/1$ queue. The Markov chain governing its evolution is identical to that of the virtual queues studied in [60] and in Chapter 2 of this thesis (see Figure 2-2). Given that node k has just seen the packet in question, the additional time it takes for the next node to see that packet corresponds to the waiting time in the virtual queue at node k. For a load factor of ρ and a channel ON probability of μ, the expected waiting time can be derived using Little's theorem and Equation 2.2 to be $\frac{(1-\mu)}{\mu(1-\rho)}$. Now, the expected time until the sink sees the packet is the sum

of $(N-k)$ such terms, which gives

$$\sum_{i=k}^{N-1} \frac{(1-\mu_i)}{\mu_i(1-\rho_i)}.$$

2) *Time until sink's ACK reaches intermediate node:*

The ACK informs the source that the sink has seen the packet. This information needs to reach node k by the feed-forward mechanism. The expected time for this information to move from node i to node $i+1$ is the expected time until the next slot when the channel is ON, which is just $\frac{1}{\mu_i}$ (since the i^{th} channel is ON with probability μ_i). Thus, the time it takes for the sink's ACK to reach node k is given by

$$\sum_{i=1}^{k-1} \frac{1}{\mu_i}.$$

The total expected time T_k a packet spends in the queue at the k^{th} node ($1 \leq k < N$) can thus be computed by adding the above two terms. Now, assuming the system is stable (*i.e.*, $\lambda < \min_i \mu_i$), we can use Little's law to derive the expected queue size at the k^{th} node, by multiplying T_k by λ:

$$\mathbb{E}[Q_k] = \sum_{i=k}^{N-1} \frac{\rho_i(1-\mu_i)}{(1-\rho_i)} + \sum_{i=1}^{k-1} \rho_i$$

4.6 Conclusions

In this chapter, we have proposed a new approach to congestion control on lossy links based on the idea of random linear network coding. We have introduced a new acknowledgment mechanism that plays a key role in incorporating coding into the control algorithm. From an implementation perspective, we have introduced a new network coding layer between the transport and network layers on both the source and receiver sides. Thus, our changes can be easily deployed in an existing system.

A salient feature of our proposal is that it is simultaneously compatible with the case where only end hosts perform coding (thereby preserving the end-to-end philosophy of

TCP), as well as the case where intermediate nodes perform network coding. Theory suggests that a lot can be gained by allowing intermediate nodes to code as well. Our scheme naturally generalizes to such situations. Our simulations show that the proposed changes lead to large throughput gains over TCP in lossy links, even with coding only at the source. For instance, in a 4-hop tandem network with a 5% loss rate on each link, the throughput goes up from about 0.007 Mbps to about 0.39 Mbps for the correct redundancy factor. Intermediate node coding further increases the gains.

This chapter presents a new framework for combining coding with feedback-based rate-control mechanisms in a practical way. It is of interest to extend this approach to more general settings such as network coding based multicast over a general network. Even in the point-to-point case, these ideas can be used to implement a multipath-TCP based on network coding.

In the next chapter, we shall discuss some of the practical issues that arise in designing an implementation of the TCP/NC protocol compatible with real TCP/IP stacks. These issues were not considered in the idealized setting discussed up to this point. We shall explain how to implement a network-coding layer that provides a clean interface with TCP.

Chapter 5

Experimental evaluation

We now present a real-life network coding implementation based on the theoretical foundation presented in Chapter 4. The main contributions of this chapter are as follows:

1. We explain how to address the practical problems that arise in making the network coding and decoding operations compatible with TCP's window management system, such as variable packet length, buffer management, and network coding overhead.

2. We demonstrate the compatibility of our protocol with the widely used TCP Reno; the proposal of Chapter 4 considered only TCP Vegas.

3. We present experimental results on the throughput benefits of the new protocol for a TCP connection over a single-hop wireless link. Although currently our experiments only study behavior over a single hop, this restriction is not mandatory and the evaluation of the protocol over arbitrary topologies will be addressed elsewhere.

The rest of this chapter is organized as follows. Sections 5.1 and 5.2 describe the sender side and receiver side modules, respectively, in detail. In section 5.3, we discuss the parameters defined in the algorithm and how they affect the performance. Section 5.3.5 discusses the interface presented by the coding layer to TCP, on the sender as well as the receiver side. Section 5.4 presents the results obtained from the experiment.

In summary, we discuss the various measures needed in the actual system in order to ensure that the theoretical ideas of Chapter 4 can be implemented without violating the

primary requirement, which is, the correctness of the protocol. We show that it is possible to implement a TCP-aware network-coding layer that has the property of a clean interface with TCP.

5.1 Sender side module

The operation of the coding element at the sender is more involved than the sender side operations described in Chapter 4. Several complications arise, that need to be addressed before ensuring that the theoretical ideas carry over to the real system. We shall now describe these issues and the corresponding fixes.

5.1.1 Forming the coding buffer

The description of the protocol in Chapter 4 assumes a fixed packet length, which allows all coding and decoding operations to be performed symbol-wise on the whole packet. That is, in Chapter 4 an entire packet serves as the basic unit of data (*i.e.*, as a single unknown), with the implicit understanding that the exact same operation is being performed on every symbol within the packet. The main advantage of this view is that the decoding matrix operations (*i.e.*, Gaussian elimination) can be performed at the granularity of packets instead of individual symbols. Also, the ACKs are then able to be represented in terms of packet numbers. Finally, the coding vectors then have one coefficient for every packet, not every symbol. Note that the same protocol and analysis of Chapter 4 holds even if we fix the basic unit of data as a symbol instead of a packet. The problem is that the complexity will be very high as the size of the coding matrix will be related to the number of symbols in the coding buffer, which is much more than the number of packets (typically, a symbol is one byte long).

In actual practice, TCP is a byte-stream oriented protocol in which ACKs are in terms of byte sequence numbers. If all packets are of fixed length, we can still apply the packet-level approach, since we have a clear and consistent map between packet sequence numbers and byte sequence numbers. In reality, however, TCP might generate segments of different sizes. The choice of how many bytes to group into a segment is usually made based on

the Maximum Transmission Unit (MTU) of the network, which could vary with time. A more common occurrence is that applications may use the PUSH flag option asking TCP to packetize the currently outstanding bytes into a segment, even if it does not form a segment of the maximum allowed size. In short, it is important to ensure that our protocol works correctly in spite of variable packet sizes.

A closely related problem is that of repacketization. Repacketization, as described in Chapter 21 of [7], refers to the situation where a set of bytes that were assigned to two different segments earlier by TCP may later be reassigned to the same segment during retransmission. As a result, the grouping of bytes into packets under TCP may not be fixed over time.

Both variable packet lengths and repacketization need to be addressed when implementing the coding protocol. To solve the first problem, if we have packets of different lengths, we could elongate the shorter packets by appending sufficiently many dummy zero symbols until all packets have the same length. This will work correctly as long as the receiver is somehow informed how many zeros were appended to each packet. While transmitting these extra dummy symbols will decrease the throughput, generally this loss will not be significant, as packet lengths are usually consistent.

However, if we have repacketization, then we have another problem, namely it is no longer possible to view a packet as a single unknown. This is because we would not have a one-to-one mapping between packets sequence numbers and byte sequence numbers; the same bytes may now occur in more than one packet. Repacketization appears to destroy the convenience of performing coding and decoding at the packet level.

To counter these problems, we propose the following solution. The coding operation described in Chapter 4 involves the sender storing the packets generated by the TCP source in a *coding buffer*. We pre-process any incoming TCP segment before adding it to the coding buffer as follows:

1. First, any part of the incoming segment that is already in the buffer is removed from the segment.

2. Next, a separate TCP packet is created out of each remaining contiguous part of the

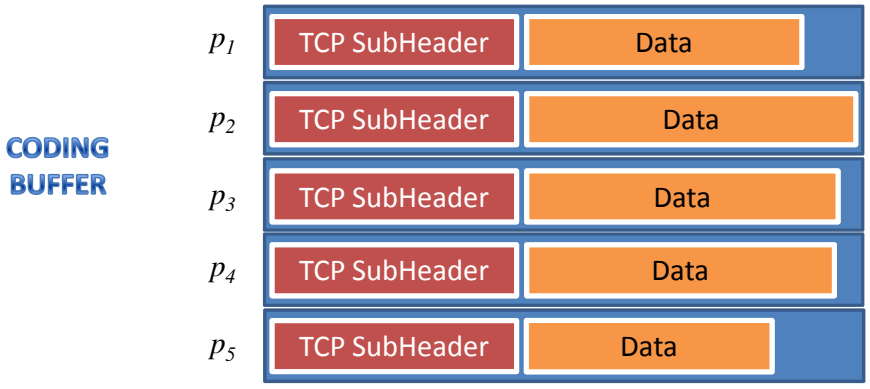


Figure 5-1: The coding buffer

segment.

3. The source and destination port information is removed. It will be added later in the network coding header.

4. The packets are appended with sufficiently many dummy zero bytes, to make them as long as the longest packet currently in the buffer.

Every resulting packet is then added to the buffer. This processing ensures that the packets in the buffer will correspond to disjoint and contiguous sets of bytes from the byte stream, thereby restoring the one-to-one correspondence between the packet numbers and the byte sequence numbers. The reason the port information is excluded from the coding is because port information is necessary for the receiver to identify which TCP connection a coded packet corresponds to. Hence, the port information should not be involved in the coding. We refer to the remaining part of the header as the TCP subheader.

Upon decoding the packet, the receiver can find out how many bytes are real and how many are dummy using the $Start_i$ and End_i header fields in the network coding header (described below). With these fixes in place, we are ready to use the packet-level algorithm of Chapter 4. All operations are performed on the packets in the coding buffer. Figure 5.1.1 shows a typical state of the buffer after this pre-processing. The gaps at the end of the packets correspond to the appended zeros. It is important to note that, as suggested in Chapter 4, the TCP control packets such as SYN packet and reset packet are allowed to bypass the coding buffer and are directly delivered to the receiver without any coding.

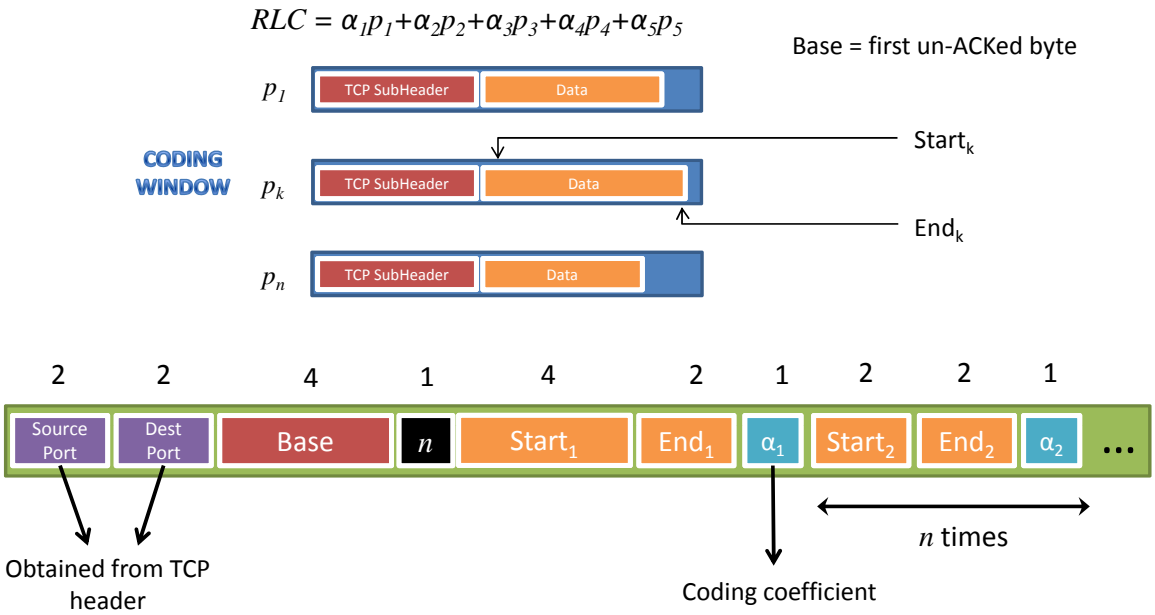


Figure 5-2: The network coding header

5.1.2 The coding header

A coded packet is created by forming a random linear combination of a subset of the packets in the coding buffer. The coding operations are done over a Galois field of size 256 in our implementation. Thus, one symbol corresponds to one byte, which is a natural choice for breaking up a packet into symbols. The header of a coded packet should contain information that the receiver can use to identify what is the linear combination corresponding to the packet. We now discuss the header structure in more detail.

We assume that the network coding header has the structure shown in Figure 5-2. The typical sizes (in bytes) of the various fields are written above them. The meaning of the various fields are described next:

- *Source and destination port:* The port information is needed for the receiver to identify the coded packet's session. It must not be included in the coding operation. It is taken out of the TCP header and included in the network coding header.

- *Base:* The TCP byte sequence number of the first byte that has not been ACKed. The field is used by intermediate nodes or the decoder to decide which packets can be safely dropped from their buffers without affecting reliability.

- *n:* The number of packets involved in the linear combination.
- $Start_i$*:* The starting byte of the i^{th} packet involved in the linear combination.
- End_i*:* The last byte of the i^{th} packet involved in the linear combination.
- α_i*:* The coefficient used for the i^{th} packet involved in the linear combination.

The $Start_i$ (except $Start_1$) and End_i are expressed relative to the previous packet's End and $Start$ respectively, to save header space. As shown in the figure, this header format will add $5n + 7$ bytes of overhead for the network coding header in addition to the TCP header, where n is the number of packets involved in a linear combination. (Note that the port information is not counted in this overhead, since it has been removed from the TCP header.) We believe it is possible to reduce this overhead by further optimizing the header structure.

5.1.3 The coding window

In the theoretical version of the algorithm, the sender transmits a random linear combination of all packets in the coding buffer. However, as noted above, the size of the header scales with the number of packets involved in the linear combination. Therefore, mixing all packets currently in the buffer will lead to a very large coding header.

To solve this problem, we propose mixing only a constant-sized subset of the packets chosen from within the coding buffer. We call this subset the *coding window*. The coding window evolves as follows. The algorithm uses a fixed parameter for the maximum coding window size W. The coding window contains the packet that arrived most recently from TCP (which could be a retransmission), and the $(W - 1)$ packets before it in sequence number, if possible. However, if some of the $(W - 1)$ preceding packets have already been dropped, then the window is allowed to extend beyond the most recently arrived packet until it includes W packets.

Note that this limit on the coding window implies that the code is now restricted in its power to correct erasures and to combat reordering-related issues. The choice of W will thus play an important role in the performance of the scheme. The correct value for W will

depend on the length of burst errors that the channel is expected to produce. Other factors to be considered while choosing W are discussed in Section 5.3. In our experiment, we fixed W based on trial and error.

5.1.4 Buffer management

A packet is removed from the coding buffer if a TCP ACK has arrived requesting a byte beyond the last byte of that packet. If a new TCP segment arrives when the coding buffer is full, then the segment with the newest set of bytes must be dropped. This may not always be the newly arrived segment, for instance, in the case of a TCP retransmission of a previously dropped segment.

5.2 Receiver side module

The decoder module's operations are outlined below. The main data structure involved is the decoding matrix, which stores the coefficient vectors corresponding to the linear combinations currently in the decoding buffer.

5.2.1 Acknowledgment

The receiver side module stores the incoming linear combination in the decoding buffer. Then it unwraps the coding header and appends the new coefficient vector to the decoding matrix. Gaussian elimination is performed and the packet is dropped if it is not innovative (i.e. if it is not linearly independent of previously received linear combinations). After Gaussian elimination, the oldest unseen packet is identified. Instead of acknowledging the packet number as in Chapter 4, the decoder acknowledges the last seen packet by *requesting the byte sequence number of the first byte of the first unseen packet*, using a regular TCP ACK. Note that this could happen before the packet is decoded and delivered to the receiver TCP. The port and IP address information for sending this ACK may be obtained from the SYN packet at the beginning of the connection. Any ACKs generated by the receiver TCP are not sent to the sender. They are instead used to update the receive window field that

is used in the TCP ACKs generated by the decoder (see subsection below). They are also used to keep track of which bytes have been delivered, for buffer management.

5.2.2 Decoding and delivery

The Gaussian elimination operations are performed not only on the decoding coefficient matrix, but correspondingly also on the coded packets themselves. When a new packet is decoded, any dummy zero symbols that were added by the encoder are pruned using the coding header information. A new TCP packet is created with the newly decoded data and the appropriate TCP header fields and this is then delivered to the receiver TCP.

5.2.3 Buffer management

The decoding buffer needs to store packets that have not yet been decoded and delivered to the TCP receiver. Delivery can be confirmed using the receiver TCP's ACKs. In addition, the buffer also needs to store those packets that have been delivered but have not yet been dropped by the encoder from the coding buffer. This is because, such packets may still be involved in incoming linear combinations. The $Base$ field in the coding header addresses this issue. $Base$ is the oldest byte in the coding buffer. Therefore, the decoder can drop a packet if its last byte is smaller than $Base$, and in addition, has been delivered to and ACKed by the receiver TCP. Whenever a new linear combination arrives, the value of $Base$ is updated from the header, and any packets that can be dropped are dropped.

The buffer management can be understood using Fig. 5-3. It shows the receiver side windows in a typical situation. In this case, $Base$ is less than the last delivered byte. Hence, some delivered packets have not yet been dropped. There could also be a case where $Base$ is beyond the last delivered byte, possibly because nothing has been decoded in a while.

5.2.4 Modifying the receive window

The TCP receive window header field is used by the receiver to inform the sender how many bytes it can accept. Since the receiver TCP's ACKs are suppressed, the decoder must copy this information in the ACKs that it sends to the sender. However, to ensure

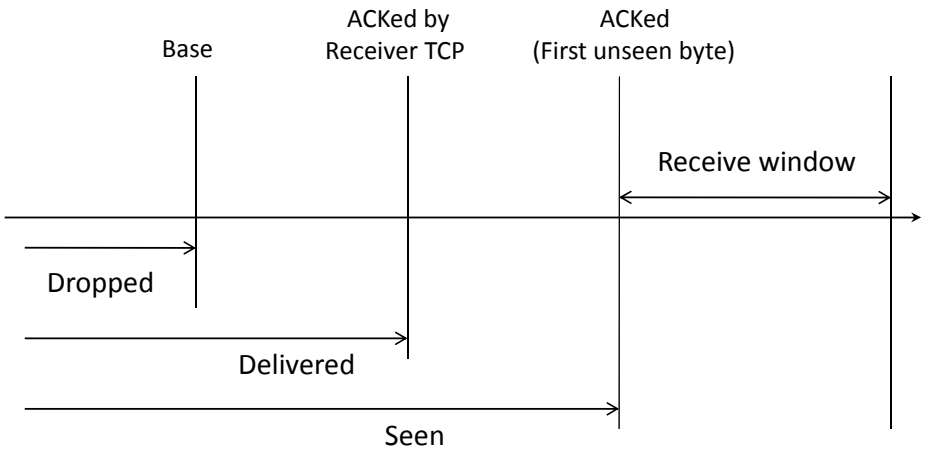


Figure 5-3: Receiver side window management

correctness, we may have to modify the value of the TCP receive window based on the decoding buffer size. The last acceptable byte should thus be the minimum of the receiver TCP's last acceptable byte and the last byte that the decoding buffer can accommodate. Note that while calculating the space left in the decoding buffer, we can include the space occupied by data that has already been delivered to the receiver because such data will get dropped when $Base$ is updated. If window scaling option is used by TCP, this needs to be noted from the SYN packet, so that the modified value of the receive window can be correctly reported. Ideally, we would like to choose a large enough decoding buffer size so that the decoding buffer would not be the bottleneck and this modification would never be needed.

5.3 Discussion of the practicalities

5.3.1 Redundancy factor

The choice of redundancy factor is based on the effective loss probability on the links. For a loss rate of p_e, with an infinite window W and using TCP Vegas, the theoretically optimal value of R is $1/(1-p_e)$, as shown in Chapter 4. The basic idea is that of the coded packets that are sent into the network, only a fraction $(1-p_e)$ of them are delivered on average. Hence, the value of R must be chosen so that in spite of these losses, the receiver is able to collect linear equations at the same rate as the rate at which the unknown packets are

mixed in them by the encoder. As discussed below, in practice, the value of R may depend on the coding window size W. As W decreases, the erasure correction capability of the code goes down. Hence, we may need a larger R to compensate and ensure that the losses are still masked from TCP. Another factor that affects the choice of R is the use of TCP Reno. The TCP Reno mechanism causes the transmission rate to fluctuate around the link capacity, and this leads to some additional losses over and above the link losses. Therefore, the optimal choice of R may be higher than $1/(1 - p_e)$.

5.3.2 Coding Window Size

There are several considerations to keep in mind while choosing W, the coding window size The main idea behind coding is to mask the losses on the channel from TCP. In other words, we wish to correct losses without relying on the ACKs. Consider a case where W is just 1. Then, this is a simple repetition code. Every packet is repeated R times on average. Now, such a repetition would be useful only for recovering one packet, if it was lost. Instead, if W was say 3, then every linear combination would be useful to recover any of the three packets involved. Ideally, the linear combinations generated should be able to correct the loss of any of the packets that have not yet been ACKed. For this, we need W to be large. This may be difficult, since a large W would lead to a large coding header. Another penalty of choosing a large value of W is related to the interaction with TCP Reno. This is discussed in the next subsection.

The penalty of keeping W small on the other hand, is that it reduces the error correction capability of the code. For a loss probability of 10%, the theoretical value of R is around 1.1. However, this assumes that all linear combinations are useful to correct any packet's loss. The restriction on W means that a coded packet can be used only for recovering those W packets that have been mixed to form that coded packet. In particular, if there is a contiguous burst of losses that result in a situation where the receiver has received no linear combination involving a particular original packet, then that packet will show up as a loss to TCP. This could happen even if the value of R is chosen according to the theoretical value. To compensate, we may have to choose a larger R.

The connection between W, R and the losses that are visible to TCP can be visualized as follows. Imagine a process in which whenever the receiver receives an innovative linear combination, one imaginary token is generated, and whenever the sender slides the coding window forward by one packet, one token is used up. If the sender slides the coding window forward when there are no tokens left, then this leads to a packet loss that will be visible to TCP. The reason is, when this happens, the decoder will not be able to see the very next unseen packet in order. Instead, it will skip one packet in the sequence. This will make the decoder generate duplicate ACKs requesting that lost (i.e., unseen) packet, thereby causing the sender to notice the loss.

In this process, W corresponds to the initial number of tokens available at the sender. Thus, when the difference between the number of redundant packets (linear equations) received and the number of original packets (unknowns) involved in the coding up to that point is less than W, the losses will be masked from TCP. However, if this difference exceeds W, the losses will no longer be masked. A theoretically optimal value of W is not known. However, we expect that the value should be a function of the loss probability of the link. For the experiment, we chose values of W based on trial and error.

5.3.3 Working with TCP Reno

By adding enough redundancy, the coding operation essentially converts the lossiness of the channel into an extension of the round-trip time (RTT). This is why Chapter 4 proposed the use of the idea with TCP Vegas, since TCP Vegas controls the congestion window in a smoother manner using RTT, compared to the more abrupt loss-based variations of TCP Reno. However, the coding mechanism is also compatible with TCP Reno. The choice of W plays an important role in ensuring this compatibility. The choice of W controls the power of the underlying code, and hence determines when losses are visible to TCP. As explained above, losses will be masked from TCP as long as the number of received equations is no more than W short of the number of unknowns involved in them. For compatibility with Reno, we need to make sure that whenever the sending rate exceeds the link capacity, the resulting queue drops are visible to TCP as losses. A very large value of

W is likely to mask even these congestion losses, thereby temporarily giving TCP a false estimate of capacity. This will eventually lead to a timeout, and will affect throughput. The value of W should therefore be large enough to mask the link losses and small enough to allow TCP to see the queue drops due to congestion.

5.3.4 Computational overhead

It is important to implement the encoding and decoding operations efficiently, since any time spent in these operations will affect the round-trip time perceived by TCP. The finite field operations over $GF(256)$ have been optimized using the approach of [80], which proposes the use of logarithms to multiply elements. Over $GF(256)$, each symbol is one byte long. Addition in $GF(256)$ can be implemented easily as a bitwise XOR of the two bytes.

The main computational overhead on the encoder side is the formation of the random linear combinations of the buffered packets. The management of the buffer also requires some computation, but this is small compared to the random linear coding, since the coding has to be done on every byte of the packets. Typically, packets have a length L of around 1500 bytes. For every linear combination that is created, the coding operation involves LW multiplications and $L(W-1)$ additions over $GF(256)$, where W is the coding window size. Note that this has to be done R times on average for every packet generated by TCP. Since the coded packets are newly created, allocating memory for them could also take time.

On the decoder side, the main operation is the Gaussian elimination. Note that, to identify whether an incoming linear combination is innovative or not, we need to perform Gaussian elimination only on the decoding matrix, and not on the coded packet. If it is innovative, then we perform the row transformation operations of Gaussian elimination on the coded packet as well. This requires $O(LW)$ multiplications and additions to zero out the pivot columns in the newly added row. The complexity of the next step of zeroing out the newly formed pivot column in the existing rows of the decoding matrix varies depending on the current size and structure of the matrix. Upon decoding a new packet, it needs to be packaged as a TCP packet and delivered to the receiver. Since this requires allocating

space for a new packet, this could also be expensive in terms of time.

As we will see in the next section, the benefits brought by the erasure correction begin to outweigh the overhead of the computation and coding header for loss rates of about 3%. This could be improved further by more efficient implementation of the encoding and decoding operations.

5.3.5 Interface with TCP

An important point to note is that the introduction of the new network coding layer does not require any change in the basic features of TCP. As described above, the network coding layer accepts TCP packets from the sender TCP and in return delivers regular TCP ACKs back to the sender TCP. On the receiver side, the decoder delivers regular TCP packets to the receiver TCP and accepts regular TCP ACKs. Therefore, neither the TCP sender nor the TCP receiver sees any difference looking downwards in the protocol stack. The main change introduced by the protocol is that the TCP packets from the sender are transformed by the encoder by the network coding process. This transformation is removed by the decoder, making it invisible to the TCP receiver. On the return path, the TCP receiver's ACKs are suppressed, and instead the decoder generates regular TCP ACKs that are delivered to the sender. This interface allows the possibility that regular TCP sender and receiver end hosts can communicate through a wireless network even if they are located beyond the wireless hosts.

While the basic features of the TCP protocol see no change, other special features of TCP that make use of the ACKs in ways other than to report the next required byte sequence number, will need to be handled carefully. For instance, implementing the timestamp option in the presence of network coding across packets may require some thought. With TCP/NC, the receiver may send an ACK for a packet even before it is decoded. Thus, the receiver may not have access to the timestamp of the packet when it sends the ACK. Similarly, the TCP checksum field has to be dealt with carefully. Since a TCP packet is ACKed even before it is decoded, its checksum cannot be tested before ACKing. One solution is to implement a separate checksum at the network coding layer to detect errors. In the same

way, the various other TCP options that are available have to be implemented with care to ensure that they are not affected by the premature ACKs.

5.4 Results

We test the protocol on a TCP flow running over a single-hop wireless link. The transmitter and receiver are Linux machines equipped with a wireless antenna. The experiment is performed over 802.11a with a bit-rate of 6 Mbps and a maximum of 5 link layer retransmission attempts. RTS-CTS is disabled.

Our implementation uses the Click modular router [81]. The Click code extracts the IP packets transmitted by the source TCP module using the $KernelTun$ element. These packets are then processed by an element that encapsulates the network coding layer at the sender. In order to control the parameters of the setup, we use the predefined elements of Click. Since the two machines are physically close to each other, there are very few losses on the wireless link. Instead, we artificially induce packet losses using the $RandomSample$ element. Note that these packet losses are introduced before the wireless link. Hence, they will not be recovered by the link layer retransmissions, and have to be corrected by the layer above IP. The round-trip delay is empirically observed to be in the range of a few tens of milliseconds. The encoder and decoder queue sizes are set to 100 packets, and the size of the bottleneck queue just in front of the wireless link is set to 5 packets. In our setup, the loss inducing element is placed before the bottleneck queue.

The quantity measured during the experiment is the goodput over a 20 second long TCP session. The goodput is measured using $iperf$ [82]. Each point in the plots shown is averaged over 4 or more iterations of such sessions, depending on the variability. Occasionally, when the iteration does not terminate and the connection times out, the corresponding iteration is neglected in the average, for both TCP and TCP/NC. This happens around 2 % of the time, and is observed to be because of an unusually long burst of losses in the forward or return path. In the comparison, neither TCP nor TCP/NC uses selective ACKs. TCP uses delayed ACKs. However, we have not implemented delayed ACKs in TCP/NC at this point.

Fig. 5-5 shows the variation of the goodput with the redundancy factor R for a loss rate of 10%, with a fixed coding window size of $W = 3$. The theoretically optimal value of R for this loss rate is 1.11 (=1/0.9) (see Chapter 4). However, from the experiment, we find that the best goodput is achieved for an R of around 1.25. The discrepancy is possibly because of the type of coding scheme employed. Our coding scheme transmits a linear combination of only the W most recent arrivals, in order to save packet header space. This restriction reduces the strength of the code for the same value of R. In general, the value of R and W must be carefully chosen to get the best benefit of the coding operation. As mentioned earlier, nother reason for the discrepancy could be the use of TCP Reno.

Fig. 5-6 plots the variation of goodput with the size of the coding window size W. The loss rate for this plot is 5%, with the redundancy factor fixed at 1.06. We see that the best coding window size is 2. Note that a coding window size of $W = 1$ corresponds to a repetition code that simply transmits every packet 1.06 times on average. In comparison, a simple sliding window code with $W = 2$ brings a big gain in throughput by making the added redundancy more useful. However, going beyond 2 reduces the goodput because a large value of W can mislead TCP into believing that the capacity is larger than it really is, which leads to timeouts. We find that the best value of W for our setup is usually 2 for a loss rate up to around 5 %, and is 3 for higher loss rates up to 25%. Besides the loss rate, the value of W could also depend on other factors such as the round-trip time of the path.

Fig. 5-4 shows the goodput as a function of the packet loss rate. For each loss rate, the values of R and W have been chosen by trial and error, to be the one that maximizes the goodput. We see that in the lossless case, TCP performs better than TCP/NC. This could be because of the computational overhead that is introduced by the coding and decoding operations, and also the coding header overhead. However, as the loss rate increases, the benefits of coding begin to outweigh the overhead. The goodput of TCP/NC is therefore higher than TCP. Coding allows losses to be masked from TCP, and hence the fall in goodput is more gradual with coding than without. The performance can be improved further by improving the efficiency of the computation.

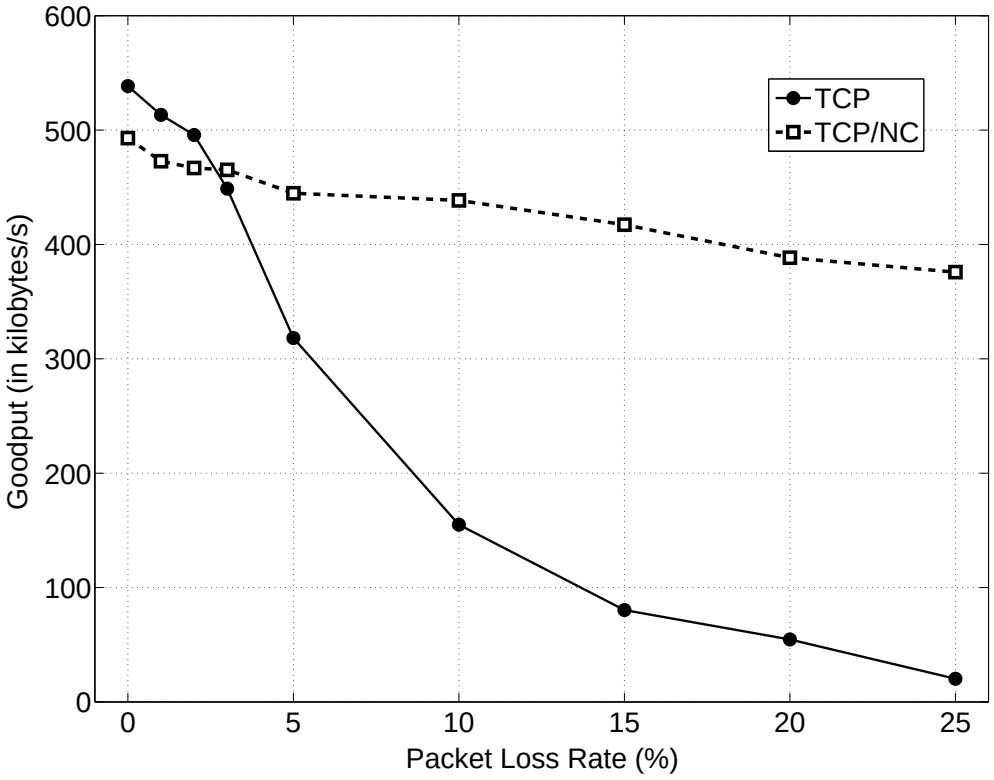


Figure 5-4: Goodput versus loss rate

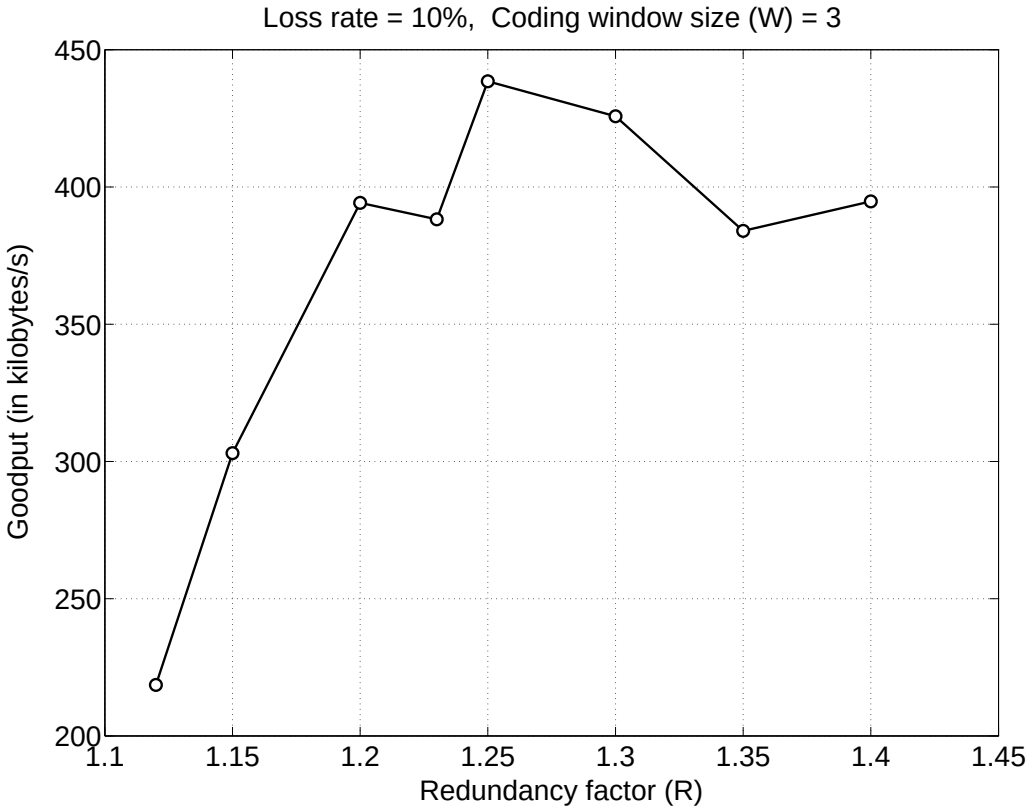


Figure 5-5: Goodput versus redundancy factor for a 10% loss rate and W=3

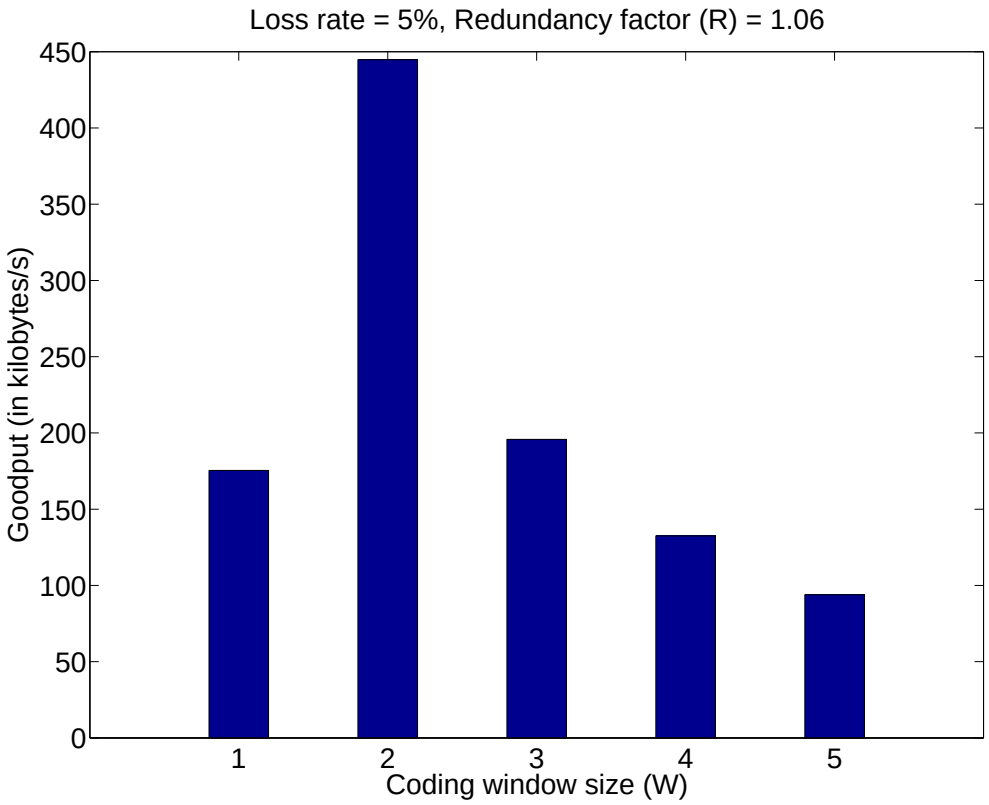


Figure 5-6: Goodput versus coding window size for a 5% loss rate and R=1.06

Chapter 6

Conclusions and future work

In today's world, wireless communication networks are becoming increasingly important. Their relative ease of deployment and convenience for the end-user have been primarily responsible for the rapid growth of wireless communications that we have seen and continue to see today. Hence, understanding the fundamental limits of communication over the wireless medium, and designing effective protocols that approach these limits, are the main goals of many communication engineers today.

In the applications side, the end-users are increasingly demanding multimedia data, usually with the added requirement of low delay, due to the real-time nature of the communication. Moreover, the demand for multicast traffic is also on the rise. These demands place some difficult requirements on the network infrastructure, and we need some new approaches to meet these demands.

Network coding provides a fundamentally new way to operate networks, by generalizing the routing operation at nodes to one involving coding across packets. The literature in the field of network coding has shown that the traditional way of operating networks, namely using store-and-forward (routing) strategies, is not best-suited for meeting such requirements over lossy environments such as wireless. Instead, the option of intermediate nodes in the network being able to code across packets adds a new dimension to the solution space, and more often than not, brings with it a benefit in terms of either an improvement in the efficiency, or a reduction in the complexity of the implementation, or both.

In order to realize these theoretical promises in practice however, it is important to

address the interfacing problem – when incorporating a new technological idea into an existing infrastructure, it is crucial to ensure that the idea is compatible with current protocols. In this thesis, we have addressed this question from the point of view of interfacing acknowledgment-based protocols and the network coding operations.

The thesis is an effort to understand how to make use of feedback in the form of acknowledgments, in the context of a network that employs network coding. We have studied three different uses of feedback – efficient queue management, coding for low decoding delay, and finally the congestion control problem. In each of these three cases, we have provided a theoretical framework to analyze the problem, as well as designed algorithms that, we believe, are simple enough to be easily deployed in practice to improve the network performance.

The notion of *seeing a packet*, introduced in Chapter 2, is the central idea of the thesis, and it leads to a completely online feedback-based network coding scheme, which is readily compatible with the widely used sliding-window congestion control protocol, namely TCP. Based on this idea, we propose new queue management strategies that can be used in any scenario with coding across packets. The notion of seen packets is generalized in Chapter 3, where we study feedback-based adaptation of the code itself, in order to optimize the performance from a delay perspective.

We believe that there are several extensions to the problems studied in this thesis. The notion of seen packets also plays a crucial role in mapping the delay and queue management problems in network coded systems, to traditional problems from queuing theory. Moreover, it gives rise to several new and interesting queuing problems that are well motivated from the network coding framework. A striking example is the decoding delay problem, discussed in Chapter 3, and its relation to the resequencing buffer problem from the queuing literature. We believe the techniques developed to study resequencing buffers will prove useful in studying the delay performance of various schemes in the network coding framework.

Another extension that is possible using the framework developed in this thesis is the development of a TCP-like protocol for multicast sessions, even if we have multiple senders. From the network coding theory, it is known that the presence of multiple re-

ceivers does not significantly complicate the problem, as long as all of them have identical demands. We believe that this understanding can be combined with a suitably defined notion of seen packets, to develop an acknowledgment-based protocol for multiple-sender multicast sessions with intra-session coding.

To summarize, we believe that the work done in this thesis is a step towards realizing the benefits of network coding in a practical setting, by enabling a simple implementation that is compatible with the existing infrastructure.

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