X. GASEOUS ELECTRONICS*

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A. OSCILLATIONS OF AN INHOMOGENEOUS PLASMA

This report indicates the usefulness of operator methods for dealing with problems involving Vlasov's equation in its more general linearized form, that is, when applied to inhomogeneous bounded anisotropic plasmas. Although these operator methods have been successfully applied by Laplace (in working with diffusion theory), their use seems almost to have been restricted only to quantum theory where they are used in full detail. The first application of operational techniques to plasma physics is probably the work of Buchsbaum and Hasegawa (B. H.).[1] In the present report we shall derive some generalizations of their work and indicate how to generalize their theory to one that is correct to all orders of temperature. First, we shall indicate how to derive the fundamental equation; second, we shall use this equation to explain the structure of the Tonks-Dattner resonances and those recently observed in microwave emission by Mitani, Kubo, and Tanaka[2] and in microwave absorption by Buchsbaum and Hasegawa[1]; and third, we shall explain the rules for obtaining partial differential equations that are correct to all orders of temperature. Important extensions such as damping and coupling to outside waves are in progress.

1. Derivation of the Fundamental Equation

We start with the Vlasov equation $\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial \vec{x}}-\frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}}=0$ which we linearize according to the usual Landau prescription $f(\vec{x},\vec{v},t) = g(\vec{x})\, f_o(\vec{v}) + f_1(\vec{x},\vec{v},t)$, in which we have assumed that the unperturbed distribution function separates into a spatial part $g(\vec{x})$, and a Maxwell-Boltzmann distribution function $f_o(\vec{v})$. The particle density of the inhomogeneous plasma is given by $N(\vec{x}) = N_o g(\vec{x})$. The fields are given by

$$\vec{E}(\vec{x},t) = 0 + \vec{E}_1(\vec{x},t); \qquad \vec{B}(\vec{x},t) = \vec{B}_o + \vec{B}_1(\vec{x},t). \tag{1}$$

Setting the zero-order electric field equal to zero has been discussed but physically meaningful results appear to be obtained; probably the best justification is that we arrive at correct results by so doing. We choose the z axis along the constant external magnetic field $\vec{B}_o$ and cylindrical coordinates in velocity space $\vec{v} = (v_\perp \cos\phi, v_\perp \sin\phi, v_\parallel)$. If we consider one component in the Fourier time spectrum we find that, using the above simplifications, we can write the Vlasov equation in the linearized form

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$$\left(\frac{\partial}{\partial\phi} + i\alpha - \beta\cos\phi\frac{\partial}{\partial x} - \beta\sin\phi\frac{\partial}{\partial y}\right) f_1 = Av_\perp\hat{f}g(\vec{x})\,(\cos\phi\, E_x + \sin\phi\, E_y), \tag{2}$$

where

$$\alpha = -\frac{\omega}{\omega_b}, \qquad \beta = -\frac{v_\perp}{\omega_b}; \qquad A = \frac{(1/2\pi)^{3/2}\ \omega_p/\omega_b}{4\pi e L_D V_T^4}; \qquad \hat{f} = \exp - \frac{v_\perp^2 + v_\parallel^2}{2v_T^2}.$$

Here, ω_b is the cyclotron frequency; L_D, ω_p, and V_T are the Debye length, plasma frequency, and thermal velocity; $L_D\omega_p = V_T$; $\omega_p^2 = 4\pi N_o e^2/m$, etc. Equation 2 is formally identical to an equation found in Allis, Buchsbaum and Bers[4,3] and can also be considered a kind of "inhomogeneous Schrödinger equation" with the Hamiltonian

$$H = i\alpha - \beta\cos\phi\frac{\partial}{\partial x} - \beta\sin\phi\frac{\partial}{\partial y}. \tag{3}$$

Then we can formally integrate Eq. 2 and (after imposing the condition that the perturbed distribution function be axially symmetric in velocity space) obtain the expression

$$f_1(\vec{x},\vec{v},\omega) = Av_\perp\hat{f}\exp\left(\beta\sin\phi\frac{\partial}{\partial x} - \beta\cos\phi\frac{\partial}{\partial y}\right)\sum_{n,m\,-\infty}^{+\infty} i^{m-1}\, J_n\left(i\beta\frac{\partial}{\partial x}\right) J_m\left(\frac{\beta}{i}\frac{\partial}{\partial y}\right)$$

$$\cdot\left\{\frac{G(\vec{x})\, e^{i(m+n+1)\phi}}{m+n+\alpha+1} + \frac{F(\vec{x})\, e^{i(m+n+1)\phi}}{m+n-1+\alpha}\right\} \tag{4}$$

where J_k stands for Bessel functions of first kind and order k and

$$F(\vec{x}) = g(\vec{x})\, E_+(\vec{x}); \qquad G(\vec{x}) = g(\vec{x})\, E_-(\vec{x}); \qquad E_\pm(\vec{x}) = \frac{1}{2}\left(E_x(\vec{x}) \pm iE_y(\vec{x})\right). \tag{5}$$

Since $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are commuting operators, we see that Glauber's lemma allows us to write

$$\exp\left(\beta\sin\phi\frac{\partial}{\partial x} - \beta\cos\phi\frac{\partial}{\partial y}\right) J_n\left(i\beta\frac{\partial}{\partial x}\right) J_m\left(\frac{\beta}{i}\frac{\partial}{\partial y}\right)$$

$$= J_n\left(i\beta\frac{\partial}{\partial x}\right) J_m\left(\frac{\beta}{i}\frac{\partial}{\partial y}\right)\exp\left(\beta\sin\phi\frac{\partial}{\partial x}\right)\exp\left(-\beta\cos\phi\frac{\partial}{\partial y}\right). \tag{6}$$

If we recall the property of the displacement operator $\exp(\vec{x}_o\cdot\vec{\nabla})\, M(\vec{x}) = M(\vec{x}+\vec{x}_o)$, we can write the exact integral of the linearized Vlasov equation in the form

$$f_1 = Av_\perp\hat{f}\sum_{n,m\,-\infty}^{+\infty} i^{m-1}\, J_n\left(i\beta\frac{\partial}{\partial x}\right) J_m\left(\frac{\beta}{i}\frac{\partial}{\partial y}\right)\left\{\frac{e^{i(m+n+1)\phi}}{m+n+\alpha+1}\, G(x+\beta\sin\phi, y-\beta\cos\phi)\right.$$

$$\left. + \frac{e^{i(m+n+1)\phi}}{m+n+\alpha-1}\, F(x+\beta\sin\phi, y-\beta\cos\phi)\right\}. \tag{7}$$

Heretofore no approximations have been made: Eq. 7 is the exact integral of the linearized Vlasov equation. Since we have an infinite series, we have to limit the number of terms to be kept in the expansion and we choose to keep those terms that are linear in temperature. This greatly simplifies the algebra and also, from the structure of the differential equations so obtained, permits their extension when all powers of temperature are taken into account. Keeping terms linear in temperature means that only terms in β^0, β^1, β^2, and β^3 need be kept in the expansion. When we do this the equations obtained have denominators of the form $\omega \pm n\omega_b$, with only n = 0, 1, 2. The expansion is valid when the Larmor radius is small compared with the wavelength, or the scale length of the density gradient, and it is clear that dropping terms above β^3 makes the solution invalid in the neighborhood of cyclotron harmonics above the second. The solution is found to be correct for $\omega_b = 0$. The exact criterion of validity in terms of the frequency is not yet understood, but in terms of temperature the equations are valid up to linear terms only.

To calculate the charge density $-e \int d^3 v f_1$, we have to perform integrals of the perturbed distribution function in velocity space.

The ϕ integrals are easily performed because of well-known orthogonality relations. The $v_{\|}$ integral is also easily done and amount to multiplication by $\sqrt{2\pi}\, V_T$ and changing $\hat{f}$ into $\exp\left(-v_\perp^2/2v_T^2\right)$. The $v_\perp$ integrals amount to evaluating integrals of the sort[5]

$$\int_0^\infty (dx)\, x^{s-1}\, e^{a'x^2-\gamma x} = (2a')^{-3/2}\, \Gamma(s) \exp\left(\frac{\gamma^2}{8a'}\right) D_{-s}\left[\gamma(2a')^{-1/2a'}\right],$$

where Γ is the gamma function, and D_{-s} is a parabolic cylinder function of order $-s$.

If we perform all of the tediously lengthy algebra of the expansions, we arrive at an expression for Poisson's equation div E = $4\pi\rho$ in the form

$$\text{div}\,\vec{E} = \left(\epsilon_1+\lambda_1^2\nabla^2\right) \text{div}\, g\vec{E} + i\left(\epsilon_2+\lambda_2^2\nabla^2\right) (\text{rot}\, g\vec{E}) \cdot \vec{e}_3, \tag{8}$$

where $\vec{e}_3$ is a unit vector along $\vec{B}_o$, div, ∇^2, and rot are the usual divergence, Laplacian, and curl operators and also

$$\epsilon_2 = \frac{\omega_b}{\omega}\epsilon_1 = \frac{\omega_b}{\omega}\frac{\omega_p^2}{\omega^2-\omega_b^2}$$

$$\lambda_2^2 = 2\frac{\omega_b}{\omega}\lambda_1^2 = 2\frac{\omega_b}{\omega}\frac{3V_T^2\omega_p^2}{\left(\omega^2-\omega_b^2\right)\left(4\omega_b^2-\omega^2\right)}, \qquad V_T = L_D\omega_p. \tag{9}$$

This equation has already been obtained by Buchsbaum[6] and Hasegawa; in their paper they were restricted to one dimension in which the curl term was absent. For the case

$\omega_b \to 0$, Eq. 8 becomes

$$\text{div}\,\vec{E} = \left(\epsilon_o - \lambda_o^2 \nabla^2\right) \text{div}\, g\vec{E}, \qquad \lambda_o^2 = 3L_D^2\left(\frac{\omega_p}{\omega}\right)^4, \qquad \epsilon_o = \left(\frac{\omega_p}{\omega}\right)^2. \tag{10}$$

Equation 8 will be called the fundamental second-order equation. It is simply a kinetic interpretation of Poisson's equation. The calculation of the conductivity current and use of the full set of Maxwell's equation will be carried out. We shall now be concerned with some consequences and generalizations of Eq. 8.

2. Consequences of the Fundamental Second-Order Equation

We now show that Eq. 8 duplicates the results of other existing theories whenever these are based on Vlasov's equation, but find some disagreement when they are based on the method of moments, with their chain broken with some ad hoc assumption.

We compare therefore our fundamental equation with results from the theory of the cold, homogeneous, and unbounded plasma as derived by Allis, Buchsbaum, and Bers[4]; with Bohm-Gross[7] dispersion relation which applies to the hot homogeneous and unbounded plasma (which is also correct to order T); and with some of Bernstein's[8] result. The last comparison involves the eigenfrequency spectrum that is still to be derived.

To compare with Allis, Buchsbaum, and Bers we first set T = 0 and also g = 1. Equation 8 becomes

$$\text{div}\, E = \epsilon_1\, \text{div}\, gE + i\epsilon_2(\text{rot}\, gE) \cdot \vec{e}_3 \tag{11}$$

which can be written

$$\frac{\partial}{\partial x}\left[(1-\epsilon_1 g)E_x - i\epsilon_2 g E_y\right] + \frac{\partial}{\partial y}\left[(1-\epsilon_1 g)E_y + i\epsilon_2 g E_x\right] = 0. \tag{12}$$

This can be written div $\vec{D} = 0$, where $D_i = \epsilon_{ij}E_j$, $\epsilon_{xx} = \epsilon_{yy} = 1 - \epsilon_1 g$, and $\epsilon_{xy} = -\epsilon_{yx} = -i\epsilon_2 g$. These according to Eq. 9 can be written

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \frac{\omega_p^2 g(\vec{x})}{\omega^2 - \omega_b^2}; \qquad \epsilon_{xy} = -\epsilon_{yx} = -i\left(\frac{\omega_c}{\omega}\right) \frac{\omega_p^2 g(\vec{x})}{\omega^2 - \omega_b^2}, \tag{13}$$

which, apart from notation, are the same as Allis' Eqs. 2.20 when we set g(x) = 1; and as might be expected $g(\vec{x})$ multiplies ω_p^2 when the plasma is inhomogeneous. We also confirm Allis, Buchsbaum, and Bers' statement[10] concerning the validity of cold-plasma theory to the first cyclotron frequency interval only: to obtain the second harmonic we have to give up the cold-plasma assumption.

Bohm and Gross' well-known dispersion relation is $\omega^2 = \omega_p^2 + 3k^2V_T^2$. This formula is a consequence of magnetohydrodynamic theory. If we evaluate the singular integral $k^2 = \omega_p^2 \int_{-\infty}^{+\infty} dv \frac{G(v)}{v - \omega/k}$, $G(v) = -(2\pi)^{-1/2}\, v_T^{-3/2} \exp\left(-v^2/2v_T^2\right)$ in the sense of a principal value integral (If G(x) does not vanish at $x = x_o$ and x_o is in the range of integration then $P\int_a^b dx\, G(x)\,(x-x_o)^{-1} = \lim_{\epsilon\to 0}\left(\int_a^{x_o-\xi} + \int_{x_o+\xi}^{b}\right) dx\, G(x)\,(x-x_o)^{-1}$, where $\epsilon > 0$,) $k^2 = \omega_p^2 P\int_{-\infty}^{-\infty} dv\, G(v)\,(v-\omega/k)^{-1}$, then it can be seen that the correct Bohm-Gross formula is $\omega^2 = \omega_p^2 + 3k^2V_T^2(\omega_p/\omega)^2$. This result is easily obtained by Fourier-analyzing the electric field in Eq. 10 which then becomes simply

$$1 = \epsilon_o + \lambda_o^2 k^2 \tag{14}$$

which agrees with the correct Bohm and Gross formula.

We consider next the solutions of Eq. 8 when applied to the slab geometry $-L \leq x \leq L$ and for the infinite cylinder geometry under the situations $\omega_c = 0$ and $\omega_c \neq 0$ and for the homogeneous plasma $(g(\vec{x})=1)$ and the inhomogeneous plasma, where $g^{-1}(\vec{x}) = 1 + \gamma\vec{x}^2/\vec{x}_o^2$. This profile is in reasonable agreement with experiment and leads to analytical solutions in closed form. The partial differential equations for the homogeneous plasma are quite simple and only the results will be stated. (The eigenfrequencies are obtained by setting the electric field (current) equal to zero at the boundary wall.)

For $g(x) = 1$ and $\omega_c = 0$ no eigenfrequency exists when $\omega < \omega_p$. The eigenfunctions are sines and cosines of argument kx, $k^2 = (1-\epsilon_o)/\lambda_o^2$. For $\omega > \omega_p$ these are given by

$$\omega^2 = \frac{\omega_p^2}{2}\left\{1 + \sqrt{1 + 12\left(\frac{L_D}{L}\right)^2 (2n+1)^2 \frac{\pi^2}{4}}\right\}, \qquad n = 0, \pm 1, \pm 2 \ldots \tag{15}$$

or by the same expression with $(2n+1)\,\pi/2$ replaced by $n\pi$. For small L_D/L Eq. 15 reduces to $\omega^2 = \omega_p^2 + 3(2n+1)^2 \frac{\pi^2}{4} \frac{V_T^2}{L^2}$ which apart from the factor of 3 on the right-hand side is a formula previously obtained by Weissglas. The case $\omega_b \neq 0$ does not present further complications. Then it is seen that the resonances exist only when $\omega^2 > \omega_p^2 + \omega_c^2$. These are extraordinary waves.[4] The eigenfunctions are sines and cosines of argument kx, $k^2 = \frac{L^2}{\lambda_1^2}(\epsilon_1 - 1)$. Their resonances are given by

$$\omega^2 = \frac{1}{2}\left\{\omega_p^2 + 5\omega_b^2 + \sqrt{\left(\omega_p^2 - 3\omega_b^2\right)^2 + 12\left(\frac{L_D}{L}\right)^2 \omega_p^4 (2n+1)^2 \frac{\pi^2}{4}}\right\}, \tag{16}$$

where n = 0, ±1, etc. and we could also have $n\pi$ instead of $(2n+1)\,\pi/2$; apart from notation, Eq. 16 agrees with an equation previously derived by Bernstein.[8] It is important to notice that for $\omega_b = 0$ the resonances exist only for $\omega > \omega_p$, whereas for $\omega_b \neq 0$ these waves exist only for $\omega^2 > \omega_p^2 + \omega_b^2$ which corresponds to the experimentally observed fact that the Tonks-Dattner ($\omega_b=0$) waves are trapped between the wall and the high-density regions, whereas the Buchsbaum-Hasegawa waves exist in the other region, that is, they are trapped in the high-density regions.

The inhomogeneous case $g^{-1}(x) = 1 + \gamma x^2/L^2$ has been treated by Buchsbaum and Hasegawa,[1] but only for $\omega_c < \omega < 2\omega_c$. The eigenfunctions are parabolic cylinder functions:

$$\frac{d^2G}{dZ^2} + \left(n+\frac{1}{2}-\frac{1}{4}Z^2\right)G = 0,$$

where $G(Z) = g(Z)\ E(Z)$, and

$$n = \frac{1}{2}\left[\frac{L(\epsilon_1-1)}{\lambda_1\sqrt{\gamma}} - 1\right], \qquad x = \left(\frac{L^2\lambda_1^2}{4\gamma}\right)^{1/4} Z. \tag{17}$$

Therefore

$$G = A[D_n(Z)+D_n(-Z)], \tag{18}$$

Here, D_n is a parabolic cylinder function of order n. The condition that the total electrostatic energy in the plasma be finite restricts the solution to a parabolic cylinder function of integral order, in which case a dispersion relation of the form of Eq. 16 can be derived and is given by

$$\omega^2 = \frac{1}{2}\left(\omega_p^2 + 5\omega_b^2 + \sqrt{\left(\omega_p^2-3\omega_b^2\right)^2 + 36\gamma\omega_p^4\left(\frac{L_D}{L}\right)^2 (2n+1)^2}\right). \tag{19}$$

The two formulas would give the same results if γ were equal to $\pi^2/12 = 0.82$; however, γ is by far smaller, approximately 0.02 for a cylinder of 2.5-cm radius, T = 4.9 ev, and this alters the spacing of the resonances significantly, thereby making the agreement with experiment much better.

The proper treatment of the cylinder problems with the curl term included is very complicated and we shall limit ourselves to cases in which $g(\vec{x})$ depends only on r and the electric field is purely radial. In this case the curl term drops out and we can write

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} + \left(\frac{\epsilon_1 - 1}{\lambda_1^2} - \frac{1}{r^2} - \frac{\gamma r^2}{\lambda_1^2 r_o^2}\right)G = 0,$$

where $G(r) = g(r)\ E_r(r)$. By setting $r^2 = \lambda_1 r_o \gamma^{-1/2} Z$ and $G = Z^{-1/2} v$, Eq. 18 reduces to a Whittaker equation; proper behavior at the origin gives the solution in terms of a transformed Whittaker function:

$$\vec{E}(r) = \vec{e}_1 \left(1 + \gamma \frac{r^2}{r_o^2}\right) \frac{\gamma^{1/2}}{\lambda_1 r_o} Ar \exp\left(-\frac{\gamma^{1/2} r^2}{2\lambda_1 r_o}\right) {}_1F_1\left(1-k, 2; \frac{\gamma^{1/2} r^2}{\lambda_1 r_o}\right), \tag{21}$$

where ${}_1F_1$ is a confluent hypergeometric function, $k = \frac{r_o}{\lambda_1} \frac{\epsilon_1 - 1}{4\sqrt{\gamma}}$. Equation 19 was carefully compared with experimental measurements by Gruber and Bekefi.[9] They actually measured the electric fields inside the plasma and the agreement between theory and

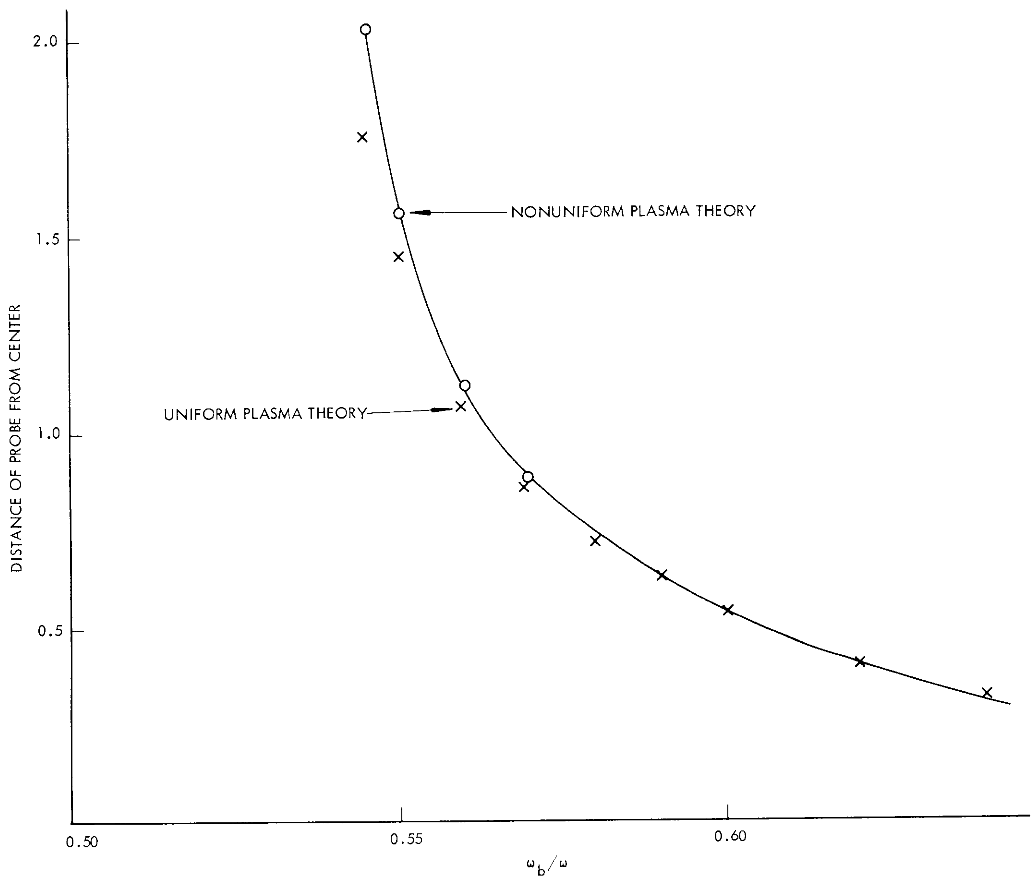

Fig. X-1. Solid line shows the locus of the first zero of the electric field (after Gruber and Bekefi[9]) oscillations as function of ω_b/ω. Open circles represent the first zero of the hypergeometric function; crosses are computed under the assumption of a uniform plasma $\gamma = 0$, $g(x) \equiv 1$. Agreement with experiment is very good even for the uniform plasma. It can be seen that the uniform plasma predictions deviate from the experimental results when the zero falls near the wall.

experiment is good. Their results will be published elsewhere[9] (see Fig. X-1 for the result of one of their measurements). In some limiting cases asymptotic expansions of

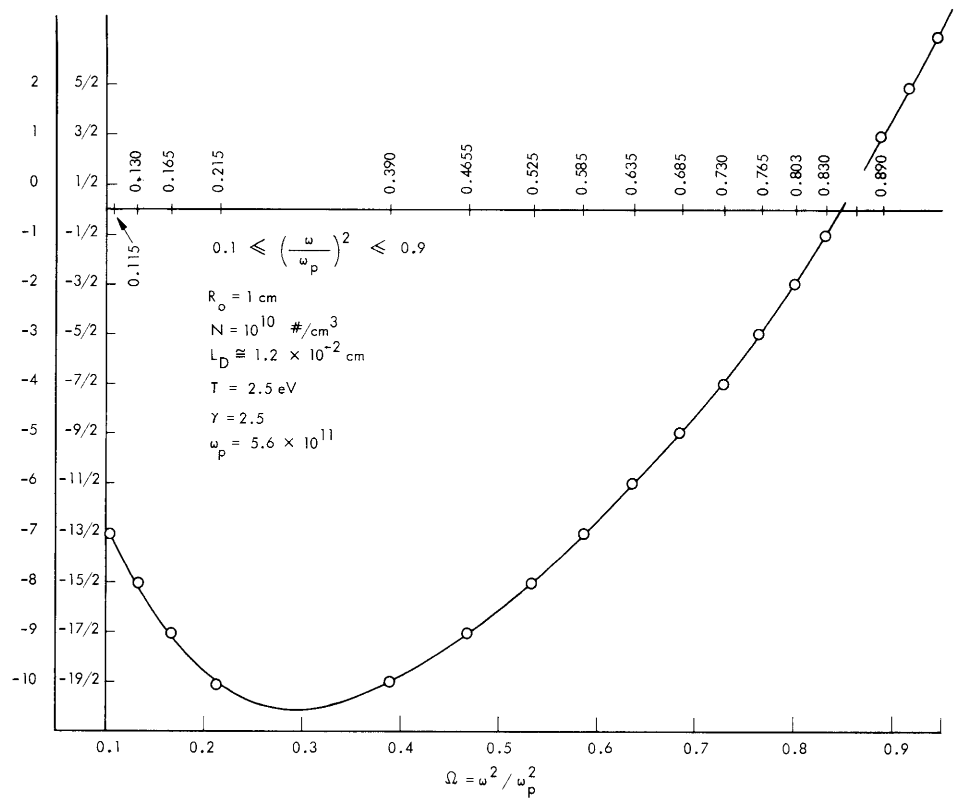

Fig. X-2. Frequency spectrum of the Tonks-Dattner resonance.

transformed Whitaker functions can be used; if we perform this expansion and match the conditions at the wall, we obtain an expression similar to Eq. 16, except that $(2n+1)\ \pi/2$ is replaced by $1/12 + (n+1)\pi$.

The Tonks-Dattner case is also solved along similar lines. In this case we set $G = gE$ in $\text{div}\ \vec{E} = \left(\epsilon_o - \lambda_o^2\nabla^2\right)\ \text{div}\ g\vec{E}$, set $r^2 = x$ and $G = x^{-1/2}R$ and obtain

$$\frac{d^2R}{dx^2} + \left(\frac{\gamma}{4r_o^2\lambda_o^2} - \frac{(1-\epsilon_o)/4\lambda_o^2}{x}\right) R = 0 \tag{22}$$

which is the equation for the Rutherford scattering of S-waves whose solutions are known.

$$R = A\,\frac{\sqrt{\gamma}}{\Gamma_o\lambda_o}\,(ix)\,\Gamma\left(1 + i\left(\frac{r_o}{\lambda_o}\right)\frac{1-\epsilon_o}{4\sqrt{\gamma}}\right)\exp\left(-i\,\frac{\sqrt{\gamma}\,r^2}{2r_o\lambda_o}\right)\,{}_1F_1\left(1 + i\left(\frac{r_o}{\lambda_o}\right)\frac{1-\epsilon_o}{4\sqrt{\gamma}},\,2,\,\frac{i\sqrt{\gamma}\,r^2}{r_o\lambda_o}\right) \tag{23}$$

which also has an asymptotic expansion

$$\lim_{\frac{\sqrt{\gamma}\, r^2}{2r_o\lambda_o}\to\infty} R = \frac{2r_o\lambda_o}{\sqrt{\gamma}\, r^2}\sin\left(\frac{r^2\sqrt{\gamma}}{2r_o\lambda_o}+\frac{r_o}{\lambda_o}\frac{1-\epsilon_o}{4\sqrt{\gamma}}\ln\frac{r^2\sqrt{\gamma}}{r_o\lambda_o}-\Omega_{ko}\right),$$

where $\Omega_{ko} = \arg\Gamma\left(1+i\frac{r_o}{\lambda_o}\frac{1-\epsilon_o}{4\sqrt{\gamma}}\right)$. We evaluate Ω_{ko} and set $R(r_o) = 0$ to obtain

$$\frac{1}{2\pi}\left(\frac{\omega}{\omega_p}\right)^2\frac{r_o}{L_D}\left(\frac{\gamma}{3}\right)^{1/2}+\frac{r_o}{\pi L_D}\frac{(\omega/\omega_p)^2-1}{4\sqrt{3\gamma}}\ln\left[\frac{r_o}{L_D}\left(\frac{\gamma}{3}\right)^{1/2}\frac{\omega^2}{\omega_p^2}\right] = n+\frac{1}{2}, \qquad n = 0, \pm1, \pm2, \dots \tag{24}$$

This equation can be solved numerically by plotting the left-hand side versus ω^2/ω_p^2. The abscissa of the intersection of this curve with the straight lines $n + 1/2$ are the resonances looked for. Part of this plot for $0.1 < (\omega/\omega_p)^2 < 0.9$ is found in Fig. X-2.

3. Partial Differential Equations Correct to All Orders of Temperature

Our theory has been hampered by two restrictions, one which we tacitly imposed when we restricted to two dimensions having propagation across B_o: $k_Z = 0$. The other consisted in keeping only terms of order less than T^2. It is important to get rid of these restrictions, principally the first one. Both of them, however, will be lifted simultaneously. To achieve this we rewrite Eq. 8 in the form

$$\frac{\partial}{\partial x}\left\{\left(\frac{1}{g}-\epsilon_1-\lambda_1^2\nabla^2\right)gE_x - i\left(\epsilon_2+\lambda_2^2\nabla^2\right)gE_y\right\} + \frac{\partial}{\partial y}\left\{\left(\frac{1}{g}-\epsilon_1-\lambda_1^2\nabla^2\right)gE_y + i\left(\epsilon_2+\lambda_2^2\nabla^2\right)gE_x\right\} = 0. \tag{25}$$

If in this equation we set $g(\vec{x}) = 1$ and assume that the plasma is infinite, we can set according to Fourier analysis $\nabla^2 = -k^2 = -\left(k_x^2+k_y^2\right)$. So that Eq. 25 really reduces to the proper components of the dielectric tensor of the homogeneous unbounded medium like

$$\epsilon_{xx} = 1 - \frac{\omega_p^2}{\omega^2-\omega_b^2} + \frac{3L_D^2\omega_p^4k^2}{\left(\omega^2-\omega_b^2\right)\left(4\omega_b^2-\omega^2\right)}. \tag{26}$$

These are correct to order T. Equation 26 and other similar equations for ϵ_{yx}, ϵ_{xy}, etc., are an important check on the validity of the theory developed thus far; but this means much more. For, if we keep in mind the operator procedures used above, we see that the physical interpretation of the terms $g^{-1} - \epsilon_1 - \lambda_1^2\nabla^2, \pm i\left(\epsilon_2+\lambda_2^2\nabla^2\right)$ is that they are the components of the dielectric tensor operators of the bounded plasma, that is,

$$\hat{\epsilon}_{xx} = \frac{1}{g(\vec{x})} - \frac{\omega_p^2}{\omega^2 - \omega_b^2} - \frac{3L_D^2\omega_p^4\nabla^2}{\left(\omega^2-\omega_b^2\right)\left(4\omega_b^2-\omega^2\right)}, \text{ etc.} \tag{27}$$

Therefore we propose the following rules:

The solution of the linearized Vlasov equation and substitution of the distribution function in Poisson's equation gives rise to a partial differential equation which when correct to order T^n is of order $2^n + 1$. This equation can be written down as well as the conductivity current by use of the following four rules.

RULE ONE: Take the dielectric tensor derived for the infinite homogeneous medium and change the unit term into $1/g(\vec{x})$; at the same time change the wave vector $\vec{k}$ into $-i\vec{\nabla}$; to obtain the dielectric tensor operator

$$\epsilon^{ij}\left(1, \frac{\omega_b}{\omega}, \frac{\omega_p}{\omega}, L_D\vec{k}\right) \rightarrow \hat{\epsilon}^{ij}\left(\frac{1}{g}, \frac{\omega_b}{\omega}, \frac{\omega_p}{\omega}, -iL_D\vec{\nabla}\right).$$

RULE TWO: Construct the vector $T_j(\vec{x}) = g(\vec{x})\, E_j(\vec{x})$, where $E_j(\vec{x})$ is the j component of the electric field at point $\vec{x}$ and $g(\vec{x}) = N(\vec{x})/N_o$.

RULE THREE: Contract both tensors obtained in one and two to obtain the displacement vector $D^i(\vec{x}) = \epsilon^{ij}g(\vec{x})\, E_j(\vec{x})$. The equation $D^i, i = 0$ or $\text{div}\,\vec{D} = 0$ corresponds to Poisson's equation.

In order to obtain the electromagnetic fields inside the plasma, we still have to calculate $\vec{J}(\vec{x})$, the conductivity current. We can infer from the structure of the kinetic equations, as well as Maxwell's, that all that we have to do is to take the dielectric tensor operator and construct from it the conductivity tensor operator $\hat{\sigma}^{ij}$ as given by

RULE FOUR: $\hat{\sigma}^{ij} = \frac{i\omega}{4\pi}(\delta^{ij}-\hat{\epsilon}^{ij})$ from which the conductivity current is given by

$$J^l(\vec{x}) = \hat{\sigma}^{lm}g(\vec{x})\, E_m(\vec{x}).$$

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B. RADIOFREQUENCY DIPOLE RESONANCE PROBE

In Quarterly Progress Report No. 74 (pages 91-98) Bekefi and Smith presented a theory for the dipole resonance probe. They derived an expression for the complex susceptance, B (= j×complex admittance), for a probe-plasma system in which the sheath region around the conducting dipole sphere was taken to be a vacuum and the surrounding plasma extended to infinity. By using a computer, their solution for the susceptance has been evaluated for several values of the parameters f and ν_c, f is the ratio of the radius of the sheath cavity to the probe radius, and ν_c is the collision frequency normalized to plasma frequency. Figure X-3 shows the first two multipole contributions to the probe admittance for f = 1.2 and $\nu_c = 0.1$. The octopole contribution (ℓ=3) is ~0.2 of the dipole contribution (ℓ=1). Also, the octopole resonance occurs at a higher frequency than the dipole resonance and the octopole antiresonance occurs at a lower frequency than the dipole antiresonance. The half-width of the conductance peak is approximately equal to ν_c for the dipole mode. Contributions from the monopole mode and the octopole mode will tend to broaden the observed half-width because their resonant frequencies differ from that of the dipole. It is found that, by increasing the value of f, the magnitude of the admittance decreases and the susceptance curve moves upward and remains positive except at the resonant frequency where it dips down to touch the zero axis. This happens because of the assumption of a vacuum sheath around the probe. As the sheath size increases (f increases), the probe response will approach the free-space value of a capacitor.

Measurements are being made with the apparatus shown in Fig. X-4 to determine the actual behavior of the complex admittance of spherical dipole probes. By allowing gas to flow into the side arms while the pumping mechanism is left open to the sphere, the gas pressure in the sphere can be maintained a factor of 30-40 below the pressure in the side arms. The gas is broken down in the side arms at a pressure of ~30 microns

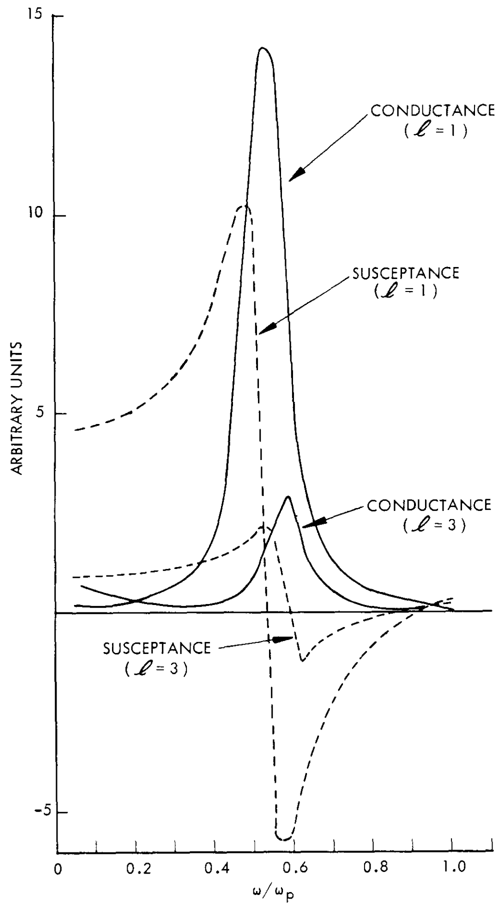

Fig. X-3. Real and imaginary parts of the admittance of a spherical dipole as a function of frequency for f = 1.2 and $\nu_c = 0.1$.

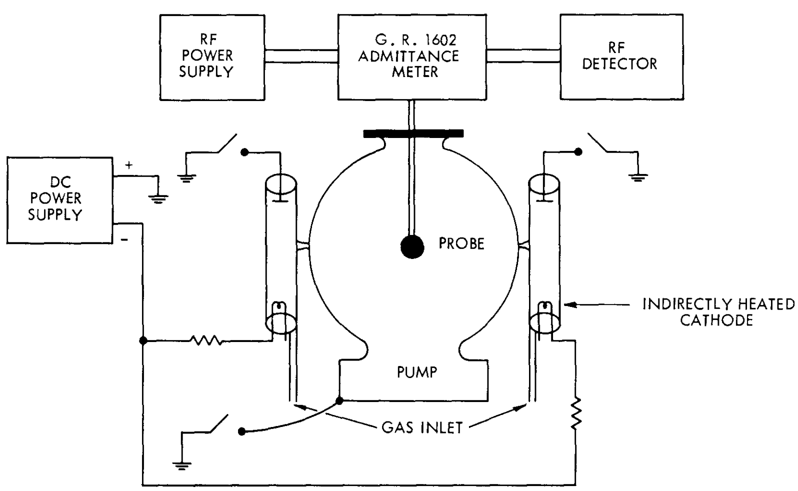

Fig. X-4. Schematic diagram of the experiment.

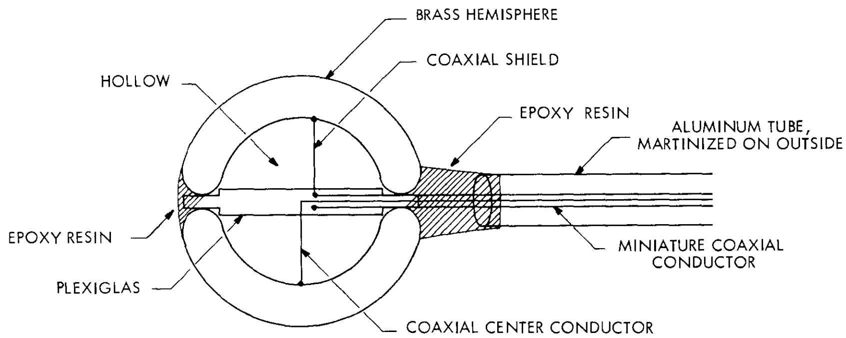

Fig. X-5. Schematic diagram of the probe construction.

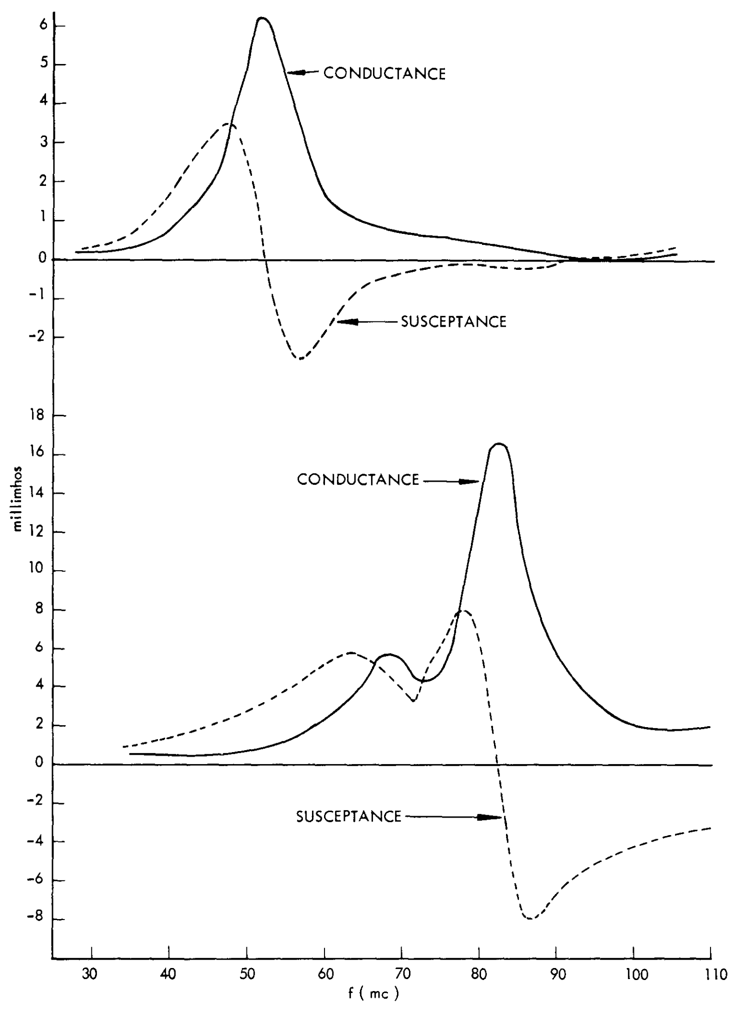

Fig. X-6. Measured real and imaginary parts of the admittance for a 1.75-inch probe.

and the resulting plasma is made to enter the low-pressure region of the sphere by making the pump the anode. Plasma densities of 10^{+8}/cc with electron temperatures of ~4 ev at a background gas pressure of a fraction of a micron are typical using argon.

The probes used (Fig. X-5) consist of two brass hemispheres separated by a plexiglas wafer and vacuum-sealed with an epoxy resin. The edge of each hemisphere has been rounded to reduce the effect of fringing fields. The probes are coated with a thin coating of Insl-X to prevent DC current from flowing.

Two representative plots of the conductance and susceptance for a 1.75-inch diameter probe are shown in Fig. X-6. For Fig. X-6a the plasma density and electron temperature as measured by a guard-ring Langmuir probe are

$$n = 1.38 \times 10^{+8}/\text{cc} \text{ and } T_e = 3.41 \text{ ev}.$$

The admittance plot shows a resonance at 57.5 Mc and an antiresonance at 96 Mc. Assuming that the dipole mode is the dominant mode in this case, we can solve the equations given by Bekefi and Smith[1] for the resonant and antiresonant frequencies and arrive at values for ω_p and f. The result is $\omega_p = 2\pi \times 104$ Mc and f = 1.22; ω_p as measured by the Langmuir probe is $2\pi \times 105$ Mc. Using the measured value of 3.41 ev for the electron temperature, one finds that the sheath thickness in this case is 4.2 λ_D.

For Fig. X-6b the plasma density and electron temperature are $3.28 \times 10^{+8}$/cc and 4.21 ev, respectively. Figure X-6b has a double peak which indicates that in this case a monopole mode in addition to the dipole mode was probably excited. Using the Langmuir probe value for the density and assuming that the resonance at 82.5 Mc is due to the dipole mode, one gets f = 1.18. The sheath thickness in this case is 4.78 λ_D.

More data are being taken to determine the relationship between sheath thickness, probe size, and Debye length. The resonance probe can then be used to determine the electron density and temperature and the collision frequency in low-density plasmas.

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