Energy-Loss Calculation of Gain in a Plane Sinusoidal Free-Electron Laser

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Abstract—The gain of a free-electron laser (FEL) made with a plane sinusoidal undulator is calculated by the electron beam energy loss.

THE free-electron laser (FEL) consists of an undulator (periodic transverse magnetic field) where an EM wave coming in the same direction as the ultrarelativistic ($\gamma^2 \gg 1$) electrons stimulates the emission of radiation.

The gain of an FEL can be calculated classically by the energy lost by the electron beam in the combined field of the undulator and of the input wave [1]–[4]. All of these calculations have only considered the case of the helical undulator.

It has also been remarked that there is a general relationship between gain and spontaneous spectrum for a free-electron device, independent of the particular undulator structure and allowing the gain of an arbitrary FEL to be calculated without describing the dynamics of the electrons in the device. In this way, the gain for a plane sinusoidal undulator has also been obtained [6]–[9].

As most of the experiments now in progress (e.g., on ACO and ADONE [10]) use plane sinusoidal undulators, it is of interest to show that the same result can be obtained by an energy-loss calculation, which better illustrates the physical origin of the differences between the two cases, and shows the more complex dynamics of the electrons in this case, which is not described by a simple pendulum potential.

We are considering an undulator

$$\vec{B} = \hat{y} B_0 \sin \frac{2\pi}{\lambda_0} x \tag{1}$$

of length $L = N \lambda_0$, with an electron beam of current I, electron energy γmc^2, and effective cross-sectional area σ (including filling factor).

Let us first summarize the derivation of the gain from the gain-spectrum relationship

$$G = \frac{2\gamma^2}{mc^2} \frac{d}{d\gamma} \frac{dW(\theta = 0)}{d\Omega \, d\gamma \, d\sigma} \tag{2}$$

where $dW/d\Omega \, d\nu$ is the spontaneous power per unit solid angle and bandwidth (a function of $\nu = c/\lambda$ and depending on γ).

The spectrum of incoherent emission from a plane sinusoidal undulator at $\theta = 0$ (forward direction) has been calculated by

Alferov, Bashmakov, and Bessonov [11]: for the nth harmonic we have

$$\frac{dW(\theta = 0)}{d\Omega \, d\nu} = \frac{2\pi e}{c} I\gamma^2 N^2 \frac{b^2 n^2}{1 + \frac{1}{2} b^2}$$
$$\cdot [J_{(n-1)/2}(n\epsilon) - J_{(n+1)/2}(n\epsilon)]^2 \tag{3}$$

where $b = (eB_0 \lambda_0 / 2\pi mc) = \alpha\gamma$ is the "deflection parameter" (MKS units, α maximum deflection, $\alpha^2 \ll 1$)

$$\xi = \frac{\pi}{2} \frac{\delta\nu}{\Delta\nu},$$

where $\delta\nu$ is the detuning from peak of spontaneous spectrum and $\Delta\nu$ is the spontaneous bandwidth $\sim 1/N$,

$$\epsilon = \frac{\frac{1}{4} b^2}{1 + \frac{1}{2} b^2}$$

and J are Bessel functions.

As $(d/d\gamma) = (\pi N/\gamma)(d/d\xi)$, from (1) and (2) we find, immediately, the single-pass gain on the nth harmonic ($n = 1, 3, 5, 7 \cdots$):

$$G_n = \pi^2 \frac{I}{\sigma I_A} f(\xi) \frac{N^3 \lambda_0^2}{\gamma^3} b^2 M_n(b) = 4\pi^2 \frac{I}{\sigma I_A}$$
$$\cdot f(\xi) \gamma N^3 \lambda^2 \frac{b^2 n^2}{(1 + \frac{1}{2} b^2)^2} M_n(b)$$
$$= G_H(b/\sqrt{2}) M_n(b) \tag{4}$$

where

$$M_n(b) = [J_{(n-1)/2}(n\epsilon) - J_{(n+1)/2}(n\epsilon)]^2$$

and G_H is the gain of a helical FEL, $I_A = mc^3/e \simeq 17$ kA, and

$$f(\xi) = \frac{d}{d\xi} \left(\frac{\sin \xi}{\xi} \right)^2 \tag{5}$$

which has a maximum value $\simeq 0.54$; more generally, $f(\xi)$ is the derivative of the modulus square of the Fourier transform of the undulator.

Coming now to the energy-loss calculation, we have to find the maximum relative energy gain per unit length $\delta\gamma_s^{max}/\gamma_s$ for a "synchronous" electron (i.e., an electron which "sees" the undulator and the input wave E_i as having the same frequency, or for which the input wave is tuned at the peak of the spontaneous spectrum). Considering a general mono-energetic beam and averaging the energy lost in the distance

L over the initial positions of the electrons, the gain per pass will be

$$G = \frac{\gamma m c^2 \langle \delta\gamma/\gamma \rangle}{(c/4\pi) \langle E_i^2 \rangle \sigma} \frac{I}{e} = \frac{2\pi^2 mc}{e\sigma \langle E_i^2 \rangle} \left(\frac{\delta\gamma_s^{\max}}{\gamma_s} \right)^2 INL^2 \gamma f(\xi) \quad (6)$$

as

$$\left\langle \frac{\delta\gamma}{\gamma} \right\rangle = \frac{\pi}{2} \left(\frac{\delta\gamma_s^{\max}}{\gamma_s} \right)^2 L^2 N f(\xi), \quad \text{with } \xi = \pi N \frac{\gamma - \gamma_s}{\gamma_s} = \frac{\pi}{2} \frac{\delta\gamma}{\Delta\gamma}.$$

Let us find $\delta\gamma_s^{\max}/\gamma_s$ in our case [(1)].

For an electron with velocity $v \simeq c(1 - \frac{1}{2}\gamma^2)(\gamma^2 \gg 1)$, with suitable initial conditions

$$\frac{dz}{dt} = \frac{cb}{\gamma} \cos \omega_0 t \quad (\omega_0 = 2\pi c/\lambda_0) \quad (7)$$

$$F_z(t) = eE_i \cos \omega \left(t - \frac{x(t)}{c} \right) \quad (8)$$

$$x(t) = \left(1 - \frac{1 + \frac{1}{2}b^2}{2\gamma^2} \right) ct - \frac{\lambda_0}{16} \frac{b^2}{\gamma^2} \sin 2 \, \omega_0 t. \quad (9)$$

Let

$$\omega_1 = \frac{\omega}{2\gamma^2} \left(1 - \frac{1}{2}b^2 \right).$$

We remark that the longitudinal oscillation amplitude is comparable with the output wavelength $\lambda = 2\pi c/\omega$:

$$\frac{\delta\gamma_s^{\max}}{\gamma_s} = \frac{1}{\gamma m c^3} \left\langle F_z \frac{dz}{dt} \right\rangle = \frac{eE_i b}{\gamma^2 m c^2}$$

$$\cdot \langle \cos \omega_0 t \cos (\omega_1 t + \eta \sin 2 \, \omega_0 t) \rangle \quad (10)$$

where

$$\eta = \frac{1}{8} \frac{\lambda_0}{\lambda} \frac{b^2}{\gamma^2}$$

where $\langle \ \rangle$ represents time average for a given electron. Remembering that

$$\cos (\omega_1 t + \eta \sin 2 \, \omega_0 t) = \cos \omega_1 t \cos (\eta \sin 2 \, \omega_0 t)$$

$$- \sin \omega_1 t \sin (\eta \sin 2 \, \omega_0 t)$$

$$= \cos \omega_1 t \left[J_0(\eta) + 2 \sum_1^\infty {}_k J_{2k}(\eta) \cos 4k \, \omega_0 t \right]$$

$$- \sin \omega_1 t \left[2 \sum_0^\infty {}_j J_{2j+1}(\eta) \sin 2 (2j+1) \omega_0 t \right] \quad (11)$$

and that the average value of the products of trigonometric functions is $\frac{1}{4}$ ($\frac{1}{2}$ for the J_0 term) if $\omega_i = n\omega_0$, with $n = 4k \pm 1$ or $n = 4j + 2 \pm 1$, that is, with n odd positive integer, and $= 0$ otherwise. We can write $\eta = n\epsilon$ with $\epsilon = [\frac{1}{4} b^2/(1 + \frac{1}{2}b^2)]$. The single-pass gain G_n on the nth harmonic ($n = 1, 3, 5, 7, \cdots$), substituting (10) and (11) into (6), is then identical to (4).

This is for an energy spread $\Delta\gamma \ll \gamma/2N$ and angular spread $\Delta\theta \ll [(1 + \frac{1}{2}b^2)/\sqrt{N}\gamma]$: for a beam with nonnegligible $\Delta\gamma$ and $\Delta\theta$, the result has to be convoluted with a longitudinal velocity spread distribution corresponding to the energy and angular distributions.

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