QMUL-PH-07-22
CERN-PH-TH/2007-157

On Perturbative Field Theory and Twistor String Theory[1]

James Bedford[a,b]

[a] Centre for Research in String Theory, Department of Physics
Queen Mary, University of London
Mile End Road, London E1 4NS, UK

[b] Department of Physics, CERN - Theory Division
1211 Geneva 23, Switzerland

ABSTRACT

It is well-known that perturbative calculations in field theory can lead to far simpler answers than the Feynman diagram approach might suggest. In some cases scattering amplitudes can be constructed for processes with any desired number of external legs yielding compact expressions which are inaccessible from the point of view of conventional perturbation theory. In this thesis we discuss some attempts to address the nature of this underlying simplicity and then use the results to calculate some previously unknown amplitudes of interest. Witten's twistor string theory is introduced and the CSW rules at tree-level and one-loop are described. We use these techniques to calculate the one-loop gluonic MHV amplitudes in $\mathcal{N}=1$ super-Yang-Mills as a verification of their validity and then proceed to evaluate the general MHV amplitudes in pure Yang-Mills with a scalar running in the loop. This latter amplitude is a new result in QCD. In addition to this, we review some recent on-shell recursion relations for tree-level amplitudes in gauge theory and apply them to gravity. As a result we present a new compact form for the n-graviton MHV amplitudes in general relativity. The techniques and results discussed are relevant to the understanding of the structure of field theory and gravity and the non-supersymmetric Yang-Mills amplitudes in-particular are pertinent to background processes at the LHC. The gravitational recursion relations provide new techniques for perturbative gravity and have some bearing on the ultraviolet properties of Einstein gravity.

[1] PhD thesis presented in June 2007.

To my parents and in loving memory of my grandfather Leonard Rogers.

Acknowledgements

First and foremost I would like to thank my supervisor Andreas Brandhuber for all his guidance and support throughout this work and for doing so much to mature my approach to physics. Similarly, I am especially grateful to Gabriele Travaglini and Bill Spence who have also acted as advisors and to Lance Dixon for being an inspirational mentor.

I am greatly indebted to David Mulryne, Constantinos Papageorgakis, John Ward and Konstantinos Zoubos for their unquestioning friendship and support and without whom I would not be the person I am today. I would also like to express my gratitude to my other collaborators Narit Pidokrajt and Diego Rodríguez-Gómez for putting up with me. It is likewise a pleasure to thank all these people for greatly helping the development of my physical understanding.

For interesting discussions and help with the subject-matter I am extremely grateful to Luis Alvarez-Gaumé, Charalampos Anastasiou, Pascal Anastasopoulos, Lilya Anguelova, Marcos Marino Beiras, David Berman, David Birtles, Emil Bjerrum-Bohr, Vincent Bouchard, Jacob Bourjaily, Ruth Britto, Tom Brown, John Charap, Joe Conlon, Cedric Delaunay, Sara Dobbins, Ellie Dobson, Vittorio Del Duca, David Dunbar, David Dunstan, Dario Duo, John Ellis, Kazem Bitaghsir Fadafan, Malcolm Fairbairn, Bo Feng, Joshua Friess, Michael Green, Umut Gursoy, Stefan Hohenegger, Harald Ita, Bob Jones, Chris Hull, Valya Khoze, Johanna Knapp, David Kosower, Theo Kreouzis, Duc Ninh Le, Wolfgang Lerche, Simon McNamara, Alex Morisse, Michele Della Morte, Adele Nasti, Georgios Papageorgiou, Alice Perrett, Roger Penrose, Brian Powell, Are Raklev, Sanjaye Ramgoolam, Christian Romelsberger, Rodolfo Rousso, Ricardo Schiappa, Shima Shams, Jonathan Shock, Laura Tadrowski, Alireza Tavanfar, Steve Thomas, David Tong, Nicolaos Toumbas, Massimiliano Vincon, Daniel Waldram, Marlene Weiss, Arne Wiebalck and Giulia Zanderighi.

The hospitality of both Queen Mary, University of London & CERN and their support via a Queen Mary studentship and a Marie Curie early stage training grant are gratefully acknowledged, as are the many staff including Terry Arter, Kathy Boydon, Denise Paige, Nanie Perrin, Jeanne Rostant, Kate Shine, Diana De Toth and Suzy Vascotto who make these places truly special.

For help with the manuscript I am especially grateful to Andi Brandhuber, Johanna Knapp, Costis Papageorgakis, Jon Shock, Gabriele Travaglini, Marlene Weiss and Costas Zoubos.

The support of many friends and family members, the number of whom is too great to list is also gratefully acknowledged. Especially, these include Sam Chamberlain, Jeremy and Sine Pickles, Barbara and Leonard Rogers and Christiane and Varos Shahbazian. Last, but by no means least I owe a great debt to my parents Andrew and Linda, my brother Simon and to Claire.

Table of contents

List of Figures

Introduction

In the realm of high energy physics, the standard model of particle physics is our crowning achievement to-date.[2] It describes the fundamental forces of nature – excluding gravity – as a quantum (gauge) field theory with gauge symmetry group $SU(3) \otimes SU(2) \otimes U(1)$. In this description, the strong force – described by a gauge theory known as quantum chromodynamics with gauge group $SU(3)$ – is adjoined to electro-weak theory which is itself a unification of quantum electrodynamics and the weak interaction. The standard model is well-verified experimentally and will soon be put to even greater tests by the large hadron collider at CERN which will start running later in 2007.

However, there are a number of features of the standard model (SM) which are not fully understood. Most prominent of these is perhaps that it predicts the existence of a scalar particle called the Higgs boson of mass $M_H > 114.4$ GeV [6] which is responsible for the generation of mass in electro-weak symmetry breaking and which has not yet been observed, though few doubt that it will not be found. Indeed one of the central goals of the large hadron collider (LHC) is to find such a particle. There is also evidence that neutrinos should have (tiny) masses and mixings and the SM should be extended to accomodate this.

On the other hand there are also theoretical issues that lead physicists to believe that the standard model is not the final story. For a start, (quantum) gravity is not incorporated into the theory. In addition, the SM suffers from a problem known as the hierarchy problem. This problem asks why there is such a large hierarchy of scales for the interaction strengths of the different forces present. It seems natural to theorists that just as the electromagnetic and weak forces are unified into the electro-weak (EW) force at scales $M_{EW} \sim 100$ GeV, so should EW theory be unified with quantum chromodynamics (QCD) at some (higher) scale. As such it is generally believed that SM particles are coming from a grand unified theory (GUT) that spontaneously broke to $SU(3) \otimes SU(2) \otimes U(1)$ at energies $M_{GUT} \sim 10^{16}$ GeV. Popular gauge groups that might unify those of the SM include $SU(5)$ and $SO(10)$.

[2] See e.g. [1] for an introductory text on the standard model and e.g. [2, 3, 4, 5] for treatises on quantum field theory in general.

Several ideas which try to deal with the hierarchy problem exist. One of these is a theory called technicolour [7, 8] which considers all scalar fields in the SM to be bound states of fermions joined by a new set of interactions. Another idea is that a new symmetry may exist such as supersymmetry – see e.g. [9, 10, 11, 12] for an introduction. Supersymmetry (SUSY) relates bosons and fermions and predicts that many more particles exist than are currently observed as each boson/fermion is associated with a partner fermion/boson. It can, however, unify the gauge couplings of the various component theories of the standard model and thus solve the hierarchy problem. As such the SM would be replaced by some supersymmetric version, the minimal realisation of which is usually termed the minimally supersymmetric standard model (MSSM).[3] The LHC is also geared towards searching for physics beyond the standard model such as technicolour and supersymmetry.

The case for unification with gravity is very much more speculative at present. This is not least because its tiny interaction strength compared with the other forces of nature makes experimental tests of gravity on small length-scales difficult to perform with existing technology. As such there is no accepted quantum theory of gravity at present let alone a unification of quantum gravity with the SM. Currently studied theories that address the issue of the quantisation of gravity include causal set theory [15, 16], loop quantum gravity [17, 18] and string theory [19, 20, 21, 22]. Of these, string theory has also emerged as a possible framework for providing a complete unified theory of all the forces of nature or a theory of everything (TOE) as it is sometimes called.

For string theory, the starting point is best understood as a generalisation of the world-line approach to particle physics as opposed to the spacetime approach of quantum field theory. In this approach one considers particles from the point of view of their world-volume or world-line (as their trajectories are lines in spacetime) and describes this trajectory using an action of the form

$$S_{\text{particle}} = \frac{1}{2}\int d\tau \left(e^{-1}\, \partial_\tau X^\mu \partial_\tau X_\mu - e m^2 \right);$$

where τ is a parameter along the world-line which can naturally be taken to be the proper time. $e(\tau)$ is a function[4] introduced to make the action valid for zero particle mass ($m = 0$) as well as $m \neq 0$ and $X^\mu(\tau)$ represents the position vector of the particle in the 'target' space in which it lives. For the sake of generality we may consider the target space to be d-dimensional though of course four dimensions is what we're aiming for. While S_{particle} describes a free particle, interactions may be included by adding terms such as $\int dX^\mu A_\mu(X)$ for a coupling to the electromagnetic field.

[3] See [13, 14] for an overview containing the action and Feynman rules.

[4] Actually $e(\tau)$ is an einbein.

To go from point-particles to strings we simply replace S_{particle} by an action appropriate for describing the world-sheet of a string embedded in spacetime. An action which naturally incorporates both massive and massless strings is the Brink-Di Vecchia-Howe or Polyakov action

$$S_{\text{string}} = -\frac{T}{2}\int d^2\sigma \sqrt{-\det|\gamma|}\,\left(\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right).$$

Here the parameters of the world-sheet are $\sigma^\alpha = \tau, \sigma$, the tension of the string is T and $\gamma_{\alpha\beta}$ can be thought of as a metric on the world-sheet.[5] Consistently quantizing S_{string} leads (eventually) to the many interesting consequences that string theory predicts, not least of these being that gravity is quantized and the demand that the dimension of the target space be 26-dimensional for the bosonic string (the action of which is the one given by S_{string} above) or 10-dimensional for any of its supersymmetric extensions.

There are 5 of these consistent supersymmetric string theories that are known as type I, type IIA, type IIB, heterotic SO(32) and heterotic $E_8 \times E_8$ respectively, each of which has its use in describing the physics of this 10-dimensional universe in different scenarios. They are, however, intrinsically perturbative constructions and as such it has been proposed that each of these theories is just a different limit of a unique 11-dimensional theory which describes the full non-perturbative range of physics and is known as M-theory [23].

The intrinsically higher-dimensional nature of these theories is clearly in contrast with current experimental results, although such results do not extend down to the Planck scale $M_P \sim 10^{19}$ GeV where it is believed that the effects of quantum gravity will be most prevalent. Nonetheless it is hoped by many that a compactification down to four dimensions or a realisation of string theory on a 4-dimensional submanifold such as a brane [24] may provide a unified description of the standard model plus gravity in 3 + 1 dimensions.

Aside from quantizing gravity or being a possible TOE, string theory has many other facets. Not least among these is the capacity to provide alternative or 'dual' descriptions of many well-known 4-dimensional quantum field theories. In particular these quantum field theories include highly symmetric gauge theories such as maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills, but also extend to certain aspects of QCD for example.

It has long been thought that gauge theories may be described by string theories and the idea goes back at least till 't Hooft's diagrammatic proposal [25]. However, it wasn't until much more recently that this proposal was realised in a concrete way by

[5] Note that the string tension is usually written as $T = 1/(2\pi\alpha')$ where $\alpha' = l_s^2$ with l_s the string length.

Maldacena [26] who discovered a duality between type IIB string theory with target space $AdS_5 \times S^5$ – the product of 5-dimensional anti-de-Sitter space and a 5-sphere – and a certain conformal field theory (CFT), namely $\mathcal{N} = 4$ super-Yang-Mills theory in Minkowski space with gauge group $SU(N)$.[6] The duality is a 'weak-strong' one in the sense that weakly coupled strings are describing the strong coupling regime of a gauge theory and as such this provides a fascinating perturbative window into non-perturbative 4-dimensional physics.

In addition to this, the duality provides a concrete realisation of the so-called holographic principle [27, 28] which asserts that physics in d-dimensional spacetimes that include gravity may be describable by degrees of freedom in $d-1$ dimensions. One of the key ideas in this is that the Bekenstein-Hawking entropy of a black hole (a system whose dominant force is gravity) is given by $S_{BH} = A/4$ in 'natural' units where A is the area of the event horizon. This is in contrast with the fact that entropy is an extensive variable and thus usually scales with the volume of the system concerned. In the case of the Maldacena conjecture (also known as the AdS/CFT correspondence), the 5-sphere essentially scales to a point and we are left with gravity (i.e. closed strings) in 5 dimensions being described by Yang-Mills (i.e. open strings) in 4 dimensions.

In any case, it is not only the non-perturbative aspects of four-dimensional gauge theory that we would like to understand better. Although weak-coupling perturbation theory is in-principle well understood for such theories, the complexity is so great as to make many calculations intractable. The asymptotic freedom of QCD [29, 30] means additionally that perturbative results become more important as the energy of interaction is increased, and many of these will be necessary input for the discovery of new physics at colliders such as the LHC. As such it would be very interesting from both a theoretical and a phenomenological perspective if a duality existed that might describe a 4-dimensional gauge theory at weak coupling.

In fact a key step was taken in this direction by Witten at the end of 2003 [31]. He discovered a remarkable new duality between weakly-coupled $\mathcal{N} = 4$ super-Yang-Mills theory in Minkowski space and a weakly-coupled topological string theory (known as the B-model) whose target space is the Calabi-Yau super-manifold $\mathbb{CP}^{3|4}$. This manifold has 6 real bosonic dimensions which are related to the usual 4-dimensional spacetime of the quantum field theory by the twistor construction of Penrose [32].

In [31], it was observed that tree-level gluon-scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills[7] localise on holomorphically embedded algebraic curves in twistor space and proposed that they could be calculated from a string theory by integrating over the moduli space of D1-brane instantons in the B-model on (super)-twistor space. The

[6] Note that in order to treat the strings perturbatively we must actually take $N \to \infty$.

[7] The same applies to QCD at tree-level due to an effective supersymmetry – see §1.3.

localisation properties of these amplitudes helped to explain the unexpectedly simple structure that often arises in their calculation from Feynman diagrams despite the large degree of complexity at intermediate stages in the computation. In the simplest case the maximally helicity violating (MHV) amplitudes, which describe the scattering of 2 gluons of negative helicity with $n-2$ gluons of positive helicity, are localised on simple straight lines in twistor space. Similarly, amplitudes which are known to vanish such as those involving n gluons of positive helicity or 1 gluon of negative helicity and $n-1$ gluons of positive helicity are explained in this scheme.

The beautifully simple localisation properties of the MHV amplitudes led Cachazo, Svrcek and Witten [33] to propose a new diagrammatic way of calculating tree amplitudes in gauge theory using MHV amplitudes as effective vertices. These are taken off-shell and glued together with simple scalar propagators to give amplitudes with successively greater numbers of negative-helicity particles. These rules turn out to be just the Feynman rules for light-cone Yang-Mills theory with a particular non-local change of variables and have more recently been put on a firmer theoretical footing [34, 35].

The situation at loop-level is not as clear. In [36] it was shown that states of conformal supergravity are present which do not decouple at one-loop and the procedure for calculating loop amplitudes in Yang-Mills from a twistor string theory is not clear. Despite this, it is a remarkable result of Brandhuber, Spence and Travaglini [37] that the so-called CSW rules can also be applied at loop-level. In [37] it was shown that the one-loop MHV amplitudes originally found by Bern, Dixon, Dunbar and Kosower (BDDK) in [38] could be calculated using MHV amplitudes as effective vertices in the same spirit as [33]. This strongly hints at the existence of a full quantum duality between maximally supersymmetric Yang-Mills and a twistor string theory, though the situation is unresolved at present.[8]

A natural question now arises: Can the MHV rules be applied at loop level in any gauge theory? The answer to this is not a priori clear as the duality in [31] applies to $\mathcal{N}=4$ Yang-Mills which is known to be very special due to its high degree of symmetry. Without the existence of a formal proof of the MHV rules at loop level, one way to proceed is certainly to try a similar method to that in [37] in other theories. To that end, the present author and the authors of [37] used the MHV rules to calculate the one-loop MHV amplitudes in $\mathcal{N}=1$ super-Yang-Mills [40] (see also Chapter 2 of this thesis). This was independently confirmed by Quigley and Rosali in [41] and both results found complete agreement with the amplitudes first presented by BDDK [42].

[8] An interesting possibility has recently arisen in [39] where a number of new dualities were constructed between field theories involving gravity and twistor string theory, One of which is a duality between $\mathcal{N}=4$ Yang-Mills coupled to Einstein supergravity and a twistor string theory. An interesting feature of this appears to be the existence of a decoupling limit giving pure Yang-Mills which might open the prospect of a twistor string formulation of Yang-Mills at loop-level.

Following this, the authors of [40] tackled the MHV amplitudes in pure Yang-Mills with a scalar running in the loop [43]. There the amplitudes for arbitrary positions of the negative-helicity gluons were derived for the first time and complete agreement was found with the existing special cases [42, 44]. It was discovered, however, that the MHV-vertex formalism calculates only the so-called 'cut-constructible' part – that is, the part containing branch cuts – of the amplitudes and thus misses possible rational terms. These rational terms are also present in the cases of the supersymmetric amplitudes, but it turns out that they are intrinsically linked to the cut-constructible parts [38, 42] and thus it is enough to know the cuts to fully determine the amplitudes. More recently, and building on the results in [43], the rational terms for the MHV amplitudes in pure Yang-Mills have been found [45] and due to a supersymmetric decomposition of one-loop amplitudes described in §1.3, this means that the complete n-gluon MHV amplitudes in QCD are now known. The calculation of the cut-constructible part of the amplitudes in pure Yang-Mills will be the subject of Chapter 3.

In a different direction, various results emerging from twistor string theory [46, 47] inspired Britto, Cachazo and Feng to propose certain on-shell recursion relations for tree-level amplitudes in gauge theory [48] which were later proved more rigorously in a paper with Witten [49]. These represent tree amplitudes as sums over amplitudes containing smaller numbers of external particles connected by scalar propagators. Starting from amplitudes with 3 particles one can thus build up all n-point tree-level amplitudes recursively.

Subsequently the present author, together with Brandhuber, Spence and Travaglini showed that similar on-shell recursion relations for tree-level amplitudes in gravity could be constructed [50], where a new form for the n-graviton MHV amplitudes was also proposed. Such recursion relations for gravity were independently found by Cachazo and Svrcek in [51] which has some overlap with [50]. One striking feature of these recursion relations is that they require a certain behaviour of the amplitudes ($\mathcal{M}$) as a function of momenta in the ultraviolet (UV) such that when thought of as a function of a complex parameter z, $\lim_{z\to\infty}\mathcal{M}(z) = 0$. For Yang-Mills amplitudes this was proved to be the case in [49], but it is a priori less clear how gravity might behave. In [50, 51] the particular amplitudes in question were shown to have this behaviour and more recently it was established for all tree-level gravity amplitudes in [52]. This unambiguously establishes the validity of the recursion relation in gravity, the construction of which is the subject of Chapter 4, and also lends support to the recent conjectures that gravity as a field theory may not be as divergent as previously thought [53, 54, 55, 56, 57, 58, 59, 60].

This thesis will be concerned with a few [40, 43, 50] of the many developments arising from twistor string theory [31]. These include the use of MHV vertices to calculate many tree-level (and some one-loop) processes [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71] as well as the use of the so-called holomorphic anomaly [72] (which arose to solve a discrepancy between the twistor space picture of one-loop amplitudes presented in [73] with the derivation in [37]) to evaluate one-loop amplitudes [74, 75, 76]. MHV vertices have also been found at tree-level in gravity [77] (after understanding how to deal with the non-holomorphicity which stalled initial progress [78]) and the CSW rules in gauge theory at loop level have been more rigorously proved in [79] together with recent advances at elucidating the loop structure in pure Yang-Mills [80, 81, 82, 83].

Recent improvements [47, 84, 85] to the unitarity method pioneered in [38, 42, 86, 87, 88, 89, 90] use complex momenta (in similarity with the on-shell recursion relations presented in [48, 49, 50, 51]) which allows, for example, a simple and purely algebraic determination of integral coefficients [47, 91]. In [92] Britto, Buchbinder, Cachazo and Feng developed efficient techniques for evaluating generic one-loop unitarity cuts which have since been applied in [93] and further developed in [94, 95, 96].

Stemming from the on-shell recursion relations written down at tree-level by Britto, Cachazo and Feng [48] (which have been successfully exploited in [97, 98, 99] and understood in terms of twistor-diagram theory in [100, 101, 102]) is the application of on-shell recursion to one-loop amplitudes which allows for a practical and systematic construction of their rational parts. These have been pioneered in [103, 104, 105, 106, 107, 108, 109, 110], leading to the full expression for the rational terms of the one-loop MHV amplitudes in QCD in [45]. Some success has also been had with such on-shell recursion in one-loop gravity [111].

Progress on the string theory side has been somewhat more limited after some promising initial work. Alternative twistor string theories to that introduced by Witten [31] to describe perturbative $\mathcal{N} = 4$ Yang-Mills have been put forward, though these have generally seemed to be more formal and less practical than the original proposal. Most notably there is that of Berkovits (and Motl) [112, 113] which was also addressed at loop level in [36], and which has been recently used to calculate loop amplitudes in Yang-Mills coupled to conformal supergravity [114]. Other proposals include those of [115, 116, 117, 118].

Similarly, dual twistor string theories have been constructed for other field theories including marginal deformations of $\mathcal{N} = 4$ (and non-supersymmetric theories) [119, 120], orbifolds of Witten's original proposal to include theories with less supersymmetry and product gauge groups [121, 122] as well as twistor string descriptions of supergravity theories. This latter section of work includes twistor descriptions of $\mathcal{N} = 1, 2$ conformal supergravity [123, 124], as well as a more recent construction for Einstein supergravity [39, 125] following initial observations of the special properties of graviton amplitudes [31, 77, 78, 126]. Additionally, twistor string dual constructions have been presented for truncations of self-dual $\mathcal{N} = 4$ super-Yang-Mills [127], lower dimensional theories [128, 129, 130, 131, 132] and $\mathcal{N} = 4$ SYM with a chiral mass term [133].

Directly following from [31], it was shown how to construct amplitudes that are more complex than the MHV amplitudes from an integral over a suitable moduli space of curves in twistor string theory. Some simple 5-point next-to-MHV (NMHV) amplitudes were addressed in [134] as well as all n-gluon $\overline{\text{MHV}}$ amplitudes[9] in [135] and all 6-gluon amplitudes in [136].

Another avenue that has proved illuminating is the study of gauge and gravity theories in twistor space. This includes [137] where the partition function of $\mathcal{N} = 4$ Yang-Mills was examined in twistor space, [138] where the CSW rules were treated from a purely gauge theoretic perspective in twistor space and [139] where loops have been studied and other related work including [140, 141]. Furthermore, self-dual supergravity theories have been investigated from a twistor space perspective in [142, 143], relations between twistors, hidden symmetries and integrability elucidated in [144, 145], and the connection with string field theory developed in [146]. Finally, twistor string theory has inspired a great deal of work in understanding supermanifolds and their connections with string theory and gauge theory such as that of [147, 148, 149, 150] and references therein.

[9] i.e. the amplitudes which are MHV amplitudes when the helicities of all particles are reversed. They thus describe the scattering of 2 gluons of positive helicity with $n-2$ gluons of negative helicity.

Summary

This thesis is organised as follows:

In Chapter 1 we discuss perturbative gauge theory and the unexpectedly simple results that it can produce despite the huge number of Feynman diagrams that have to be summed. We introduce various techniques for explaining this simplicity including colour ordering, the spinor helicity formalism, supersymmetric decompositions, supersymmetric ward identities and the use of twistor space. We go on to review the twistor string theory introduced in [31] and show how it can be used to calculate tree-level scattering amplitudes of gluons. Finally we describe some key ideas in perturbative gauge theory that were inspired by the twistor string theory. In particular we present an overview of the CSW rules and their application at tree- and loop-level in $\mathcal{N} = 4$ super-Yang-Mills.

Chapter 2 is devoted to elucidating the calculation of MHV loop amplitudes in $\mathcal{N} = 1$ Yang-Mills using a perturbative expansion in terms of MHV amplitudes as vertices as was introduced for $\mathcal{N} = 4$ Yang-Mills in [37]. We follow [40] where the calculation was originally performed and use the decomposition of the integration measure advocated in [37, 79] to reconstruct the n-gluon MHV amplitudes in $\mathcal{N} = 1$ Yang-Mills first given in [42]. This provides strong evidence that the MHV diagram method is valid in general supersymmetric field theories at loop level. Some technical details are relegated to Appendix E.

In much the same spirit, Chapter 3 describes the calculation of the MHV amplitudes in pure Yang-Mills with a scalar running in the loop. We take the same approach as in Chapter 2 and closely follow [43]. This produces the first results for the (cut-constructible part of the) n-gluon MHV amplitudes with arbitrary positions for the negative-helicity particles in pure Yang-Mills. The results obtained are in complete agreement with the previously known special cases in [42, 44] and as with Chapter 2, many technical details to do with the evaluation of integrals are omitted and provided in Appendix G.

In Chapter 4 we describe some tree-level on-shell recursion relations in gravity as constructed in [50] and highlight some of their similarities with the on-shell recursion relations proposed for gauge theory in [48, 49]. The format followed is that of [50] and as such we describe a new compact form for the n-graviton MHV amplitudes arising from the recursion relation. We also comment on the existence of recursion relations in other field theories such as ϕ^4 theory and mention the connection between the CSW rules at tree-level and these on-shell recursion relations.

We conclude and discuss future directions in Chapter 5. Additionally, there are appendices describing the spinor helicity formalism and Feynman rules for massless gauge theory in such a formalism, d-dimensional Lorentz-invariant phase space, unitarity and the Kawai-Lewellen-Tye (KLT) relations in gravity which relate tree amplitudes in gravity to (products of) tree amplitudes in Yang-Mills.

CHAPTER 1

PERTURBATIVE GAUGE THEORY

In the traditional approach to quantum field theory, one writes down a classical Lagrangian and can quantise the theory by defining the Feynman path integral. Perturbative physics can then be studied by drawing Feynman diagrams and using the Feynman rules generated by the path integral to calculate scattering amplitudes. For a non-Abelian gauge theory the classical theory is well-described by the Yang-Mills Lagrangian [151]:

$$
\begin{aligned}
\mathcal{L} = {} & \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + g\bar{\psi}A^a_\mu\gamma^\mu T^a\psi \\
& - gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} - \frac{1}{4}g^2(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^{\mu c}A^{\nu d}) \,; \qquad (1.0.1)
\end{aligned}
$$

where ψ is a fermion field, A_μ the gauge boson field and g is the coupling. Greek indices are associated with spacetime, while Roman indices describe the structure in gauge group space. This can then be used to construct the Feynman rules in the usual way.

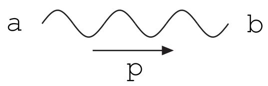
Although this construction is somewhat technical it is easy so see what these interactions will be from a heuristic standpoint. The first two terms in (1.0.1) will give the fermion and gauge boson propagators respectively. The third term involves two ψs and an A and thus represents a vertex where two fermions interact with a gauge boson. The fourth term involves 3 As and represents a 3-boson vertex while the fifth term gives a 4-boson vertex.

If we work everything out properly then we find that, in Feynman gauge for example where we have set $\xi = 1$ in a more general gauge boson propagator of the form

$$
\frac{-i}{p^2 + i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)\delta_{ab} \,; \qquad (1.0.2)
$$

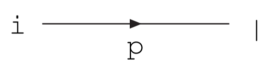
the Feynman rules for an $SU(N_c)$ gauge theory are:

Gauge Boson Propagator:



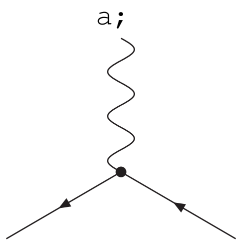
$$= \frac{ig}{p^2 + i\epsilon}\ {}^{ab}$$

Fermion Propagator:



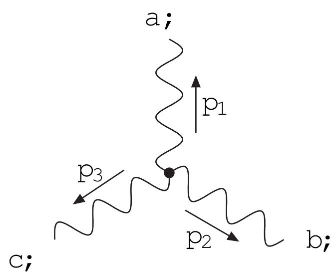
$$= \frac{i(\not p + m)}{p^2 \quad m^2 + i\epsilon}\ {}_{i}{}^{l}$$

Fermion Vertex:



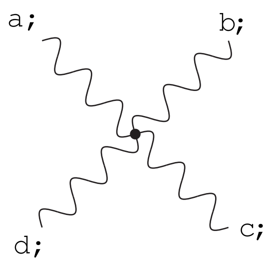
$$= ig \quad T^a$$

3-Boson Vertex:



$$\begin{aligned} = \quad & gf^{abc}[g \quad (p_1 \quad p_2) \\ & + g \quad (p_2 \quad p_3) \\ & + g \quad (p_3 \quad p_1)\ ] \end{aligned}$$

4-Boson Vertex:



$$\begin{aligned} = \quad & 2ig^2[f^{abe}f^{ecd}g^{[} \ g^{]} \\ & + f^{dae}f^{ebc}g^{[} \ g^{]} \\ & + f^{cae}f^{ebd}g^{[} \ g^{]}\ ] \end{aligned}$$

Figure 1.1: Feynman rules for $SU(N_c)$ Yang-Mills theory in Feynman gauge.

In the above rules we have taken all particles to be outgoing and we use the convention that $C^{[\mu\nu]} = (C^{\mu\nu} - C^{\nu\mu})/2$ for some 2-index object $C^{\mu\nu}$. We have also ignored the contributions due to ghost fields and will stick to these choices in what follows unless otherwise specified. Amplitudes for physical processes are obtained by drawing all the ways that the process can occur using the above rules and associating each of these with a specific mathematical expression. They are then evaluated and added up to produce the desired result. Classical results are obtained from diagrams without any closed loops while quantum corrections involve an increasing number of loops. For more details see e.g. [2].

Even though gauge theories present many technical challenges, the way to proceed (at least perturbatively) is in-principle well understood. In practice, however, the calculational complexity grows rapidly with the number of external particles (legs) and the number of loops. For example, even at tree-level where there are no loops to consider, the number of Feynman diagrams describing n-particle scattering of external gluons in QCD grows faster than factorially with n [152, 153]:[1]

n	4	5	6	7	8	9	10
# diagrams	4	25	220	2,485	34,300	559,405	10,525,900

Figure 1.2: The number of Feynman diagrams required for tree-level n-gluon scattering.

Despite this, the final result is often simple and elegant. A prime example is the so-called Maximally Helicity Violating (MHV) amplitude describing the scattering of 2 gluons (i and j) of negative helicity with $n-2$ gluons of positive helicity. At tree-level the amplitude is given by:

$$A_n^{\text{tree}} = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1\, n\rangle\langle n1\rangle}\,, \tag{1.0.3}$$

for any n. We will leave the explanation of the meaning of this expression to later in the chapter, but the reader is nonetheless able to appreciate its simplicity compared with the ever increasing number of Feynman diagrams needed to produce it.

The question then arises of: Why is there this simplicity underlying the apparently more complex perturbative expansion and how does it arise. The rest of this chapter is devoted to setting up a framework in which these questions may be addressed.

[1] Note that the following numbers are relevant for the case where one is considering a single colour structure only. The total number of diagrams after summing over all possible colour structures is even greater still. For more on this see §1.1

1.1 Colour ordering

One prominent complication experienced by gauge theories is the extra structure inherent in their gauge invariance. This means that fields of the theory do not just carry spacetime indices but also indices relating to their transformation under the gauge group. In the standard model it has been found that $SU(N_c)$ groups are the most appropriate ones for describing the gauge symmetry and so unless otherwise specified we will consider gauge groups of this type.

As is well-known, gluons carry an adjoint colour index $a = 1,2,\ldots,N_c^2-1$, while quarks and antiquarks carry fundamental (N_c) or anti-fundamental ($\bar{N}_c$) indices $i,\bar{\jmath} = 1,2,\ldots,N_c$. The $SU(N_c)$ generators in the fundamental representation are traceless Hermitian $N_c \times N_c$ matrices, $(T^a)_i^{\ \bar{\jmath}}$ which we normalise to $\mathrm{tr}(T^aT^b) = \delta^{ab}$.[2] The Lie-algebra is defined by $[T^a,T^b] = i f^{abc}T^c$, where the structure constants f^{abc} satisfy the Jacobi Identity:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0\,. \tag{1.1.1}$$

Let us begin by considering a generic tree-level scattering amplitude. It is apparent from the Feynman rules given in Figure 1 that each quark-gluon vertex contributes a group theory factor of $(T^a)_i^{\ \bar{\jmath}}$ and each tri-boson vertex a factor of f^{abc}, while four-boson vertices contribute more complicated contractions involving pairs of structure constants such as $f^{abe}f^{cde}$. The quark and gluon propagators will then contract many of the indices together using their group theory factors of δ_{ab} and $\delta_i^{\ \bar{\jmath}}$. We can now start to illuminate the general colour structure of the amplitudes if we first use the definition of the Lie-algebra to re-write the structure constants as

$$f^{abc} = -i\,\mathrm{tr}(T^a[T^b,T^c])\,. \tag{1.1.2}$$

Doing this means that all colour factors in the Feynman rules can be replaced by linear combinations of strings of T^as, e.g. $\sum \mathrm{tr}(\ldots T^aT^b\ldots)\,\mathrm{tr}(\ldots T^bT^c\ldots)\ldots\mathrm{tr}(\ldots T^d\ldots)$ if we only have external gluons, or $\ldots(T^a\ldots T^b)_i^{\ \bar{\jmath}}\,\mathrm{tr}(T^b\ldots T^c)(T^cT^d\ldots)_k^{\ \bar{l}}\ldots$ – where the strings are terminated by (anti)-fundamental indices – if external quarks are present.

In order to reduce the number of traces we make use of the identity

$$\sum_{a=1}^{N_c^2-1}(T^a)_i^{\ \bar{\jmath}}(T^a)_k^{\ \bar{l}} = \delta_i^{\ \bar{l}}\delta_k^{\ \bar{\jmath}} - \frac{1}{N_c}\delta_i^{\ \bar{\jmath}}\delta_k^{\ \bar{l}}\,; \tag{1.1.3}$$

[2] This is different from the more familiar $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$, but is purely a convention used to avoid the proliferation of factors of 2. Note that the Feynman rules written down at the beginning of the chapter use $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$. To rewrite the diagrams in a way that is consistent with these 'more natural' colour ordering conventions one simply has to replace $T^a \to T^a/\sqrt{2}$ and $f^{abc} \to f^{abc}/\sqrt{2}$. See also Appendix B.

which is just an algebraic statement of the fact that the generators T^a form a complete set of traceless Hermitian matrices. This in turn gives rise to simplifications such as

$$\begin{aligned}\sum_a \mathrm{tr}(T^{a_1}\ldots T^{a_k}T^a)\,\mathrm{tr}(T^aT^{a_{k+1}}\ldots T^{a_n}) &= \mathrm{tr}(T^{a_1}\ldots T^{a_k}T^{a_{k+1}}\ldots T^{a_n}) \\ &\quad - \frac{1}{N_c}\mathrm{tr}(T^{a_1}\ldots T^{a_k})\,\mathrm{tr}(T^{a_{k+1}}\ldots T^{a_n}) \qquad (1.1.4)\end{aligned}$$

and

$$\begin{aligned}\sum_a \mathrm{tr}(T^{a_1}\ldots T^{a_k}T^a)(T^aT^{a_{k+1}}\ldots T^{a_n})_i^{\bar{\jmath}} &= (T^{a_1}\ldots T^{a_k}T^{a_{k+1}}\ldots T^{a_n})_i^{\bar{\jmath}} \\ &\quad - \frac{1}{N_c}\mathrm{tr}(T^{a_1}\ldots T^{a_k})(T^{a_{k+1}}\ldots T^{a_n})_i^{\bar{\jmath}}. \qquad (1.1.5)\end{aligned}$$

In Eq. (1.1.3) the $1/N_c$ term corresponds to the subtraction of the trace of the $U(N_c)$ group in which $SU(N_c)$ is embedded and thus ensures tracelessness of the T^a. This trace couples directly only to quarks and commutes with $SU(N_c)$. As such the terms involving it disappear after one sums over all the permutations present – a fact which is easy to check directly. We can thus see that we are ultimately left with either sums of single traces of generators if we only have external gluons as in Eq. (1.1.4) or sums of strings of generators terminated by fundamental indices as in Eq. (1.1.5) if we also have external quarks [153, 154].[3] In most of what we do we will only be concerned with gluon scattering and can therefore write the colour decomposition of amplitudes as

$$A_n^{\mathrm{tree}}(a_i) = g^{n-2}\sum_{\sigma\in S_n/Z_n}\mathrm{tr}(T^{a_{\sigma(1)}}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(n)}})A_n^{\mathrm{tree}}(\sigma(1),\sigma(2),\ldots,\sigma(n)), \qquad (1.1.6)$$

where S_n is the set of permutations of n objects and Z_n is the subset of cyclic permutations. g is the coupling constant of the theory. The A_n^{tree} sub-amplitudes are colour-stripped and depend only on one ordering of the external particles. It is therefore sufficient to consider $A_n^{\mathrm{tree}}(1,2,\ldots,n)$ – the 'reduced colour-ordered amplitude' – and sum over all $(n-1)!$ non-cyclic permutations at the end.

It is interesting to note that the same conclusion can be arrived at from string theory in a somewhat more natural way [155, 156]. This arises because of the observation that in an open string theory the full on-shell amplitude for the scattering of n vector mesons can be written as a sum over non-cyclic permutations of external legs carrying Chan-Paton factors [157] multiplied by Koba-Nielsen partial amplitudes [158]. For

[3] Note that Eq. (1.1.5) is appropriate for the case where we have just one $q\bar{q}$ pair. With more pairs there will be products of strings with each string terminated by fundamental and anti-fundamental indices giving terms like $(T^a\ldots T^b)_i^{\bar{\jmath}}\ldots(T^c\ldots T^d)_k^{\bar{l}}$. In the final expression, each generator will appear only once in any given term of course.

the scattering of external gluons that we are interested in we need not worry about fundamental matter because at tree-level the Feynman rules forbid it from appearing as internal lines. In the infinite-tension limit ($T \to \infty$, $\alpha' \to 0$) the $U(N_c)$ string theory reduces to a $U(N_c)$ gauge theory and the trace part of this decouples as we have seen. We can thus immediately conclude that the gauge theory scattering amplitudes decompose as Eq. (1.1.6).

For one-loop amplitudes a similar colour decomposition exists [156]. In this case, however, there are up to two traces over $SU(N_c)$ generators and one must sum over the spins of the different particles that can circulate in the loop. In an expansion in N_c, the leading (as $N_c \to \infty$) contributions to the amplitudes are planar and the colour structure is simply a single trace – in fact it is N_c times the tree-level colour factor when there are no particles in the fundamental representation propagating in the loop. In this case an almost identical formula to (1.1.6) can be written down for a decomposition of one-loop amplitudes of external gluons [156]:

$$
\begin{aligned}
\mathcal{A}_n^{\text{1-loop}}(a_i) = g^n \Bigg[& \sum_{\sigma \in S_n/Z_n} N_c \,\mathrm{tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_{n;1}^{\text{1-loop}}(\sigma(1), \ldots, \sigma(n)) \\
& + \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \mathrm{tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(c-1)}})\, \mathrm{tr}(T^{a_{\sigma(c)}} \cdots T^{a_{\sigma(n)}}) \\
& \qquad A_{n;c}^{\text{1-loop}}(\sigma(1), \ldots, \sigma(n)) \Bigg], \qquad (1.1.7)
\end{aligned}
$$

and we have left the sum over spins as being implicit in the definitons of the colour-ordered partial amplitudes $A_{n;1}^{\text{1-loop}}$ and $A_{n;c}^{\text{1-loop}}$. $\lfloor r \rfloor$ is the largest integer less than or equal to r and $S_{n;c}$ is the subset of permutations of n objects leaving the double trace structure invariant.

It is a remarkable result of Bern, Dixon, Dunbar and Kosower that at one-loop, non-planar (multi-trace) amplitudes are simply obtained as a sum over permutations of the planar (single-trace) ones. This is discussed in Section 7 of [38] where it was also noted that this applies to a generic $SU(N_c)$ theory (both supersymmetric and non-supersymmetric) with external particles and those running in the loop both in the adjoint representation. As far as loop amplitudes go we will only be concerned with particles that are in the adjoint, so it will be enough for us to consider only one cyclic ordering (i.e. only $A_{n;1}^{\text{1-loop}}$, which we will generally abbreviate to $A_n^{\text{1-loop}}$) and then sum over all the relevant permutations at the very end. We will not actually perform this summation in what follows but leave it as something which can easily be implemented to obtain the full amplitude.

The colour-ordered sub-amplitudes obey a number of identities such as gauge invariance, cyclicity, order-reversal up to a sign, factorization properties and more. This means that there isn't a huge proliferation in the number of partial amplitudes that have to be computed. For 5-point gluon scattering for example, there are only 4 independent tree-level sub-amplitudes and it turns out that 2 of these vanish identically because of a 'hidden' supersymmetry (see §1.4). For a more complete list of identities see [153].

1.2 Spinor helicity formalism

So far we have seen that we can reduce some of the complexity of our task by removing the colour structure and considering only colour-ordered amplitudes. We'll also only consider massless particles and this restricts us further, though there are still a large number of things that A_n can depend on. For spinless particles (scalars), the situation is clear and $A_n = A_n(p_i)\ ^{(4)}(\sum_{i=1}^{n} p_i)$, where the p_i are the momenta of the external particles obeying $p^2 = p\ p = 0$ and we have written the delta function of momentum conservation explicitly [31, 159]. In fact the momentum dependence only appears in terms of Lorentz-invariant quantities such as $p_i\ p_j$.

For massless particles with spin the situation is more complicated and we have to consider their wavefunctions $_i$, giving $A_n = A_n(p_i;\ _i)\ ^{(4)}(\sum_{i=1}^{n} p_i)$. Textbook definitions have the $_i$ being different depending on the spin being considered. For example in the case of spin $1/2$ electrons and positrons in QED the wavefunctions are usually taken to be the familiar $u(p)$ and $v(p)$ and their conjugates (see e.g. Section (3.3) of [2]), while in the case of spin 1 gauge bosons the polarisation vectors in a suitably chosen basis are common. A more unifying description would be highly desirable and in fact one can be found using the so-called spinor helicity formalism [160].

1.2.1 Spinors

We start with the fact that, when complexified, the Lorentz group is locally isomorphic to

$$SO(1;3;\mathbb{C}) = SL(2;\mathbb{C})\ \ SL(2;\mathbb{C})\,; \tag{1.2.1}$$

and thus the finite-dimensional representations are classified as $(p;q)$, where p and q are integers or half-integers.[4] Negative- and positive-chirality spinors transform in the $(1/2;0)$ and $(0;1/2)$ representations respectively. For a generic negative-chirality spinor we write with $= 1;2$ and for a generic positive-chirality spinor we write $\tilde{}_{\dot{}}$ with $\dot{} = 1;2$.

The spinor indices introduced here are raised and lowered with the antisymmetric tensors and as $=$ and $=$ with $^{12} = 1$ and $=$

[4] Note that this section is based largely on the spinor helicity reviews of [31, 159, 161]. See also Appendix A for more details and identities and [162] for another good review covering many aspects of this chapter.

(and likewise for dotted indices). Given two spinors λ and χ of negative chirality we can then form a Lorentz-invariant scalar product as

$$\langle \lambda, \chi \rangle = \epsilon_{ab}\lambda^a \chi^b \,, \tag{1.2.2}$$

from which it follows that $\langle \lambda, \chi \rangle = -\langle \chi, \lambda \rangle$. Similar formulae apply for positive-chirality spinors except that we use square brackets to distinguish the two: $[\tilde\lambda, \tilde\chi] = \epsilon_{\dot a \dot b}\tilde\lambda^{\dot a}\tilde\chi^{\dot b}$. It is worth noting in-particular that $\langle \lambda, \chi \rangle = 0$ implies $\lambda = c\chi$ where c is a complex number and similarly for $\tilde\lambda$ and $\tilde\chi$. We will often use even more compact notation for these scalar products and write $\langle \lambda_1, \lambda_2 \rangle = \langle \lambda_1 \lambda_2 \rangle = \langle 12 \rangle$ etc.

The vector representation of $SO(1,3,C)$ is the $(1/2,1/2)$ and as such we can represent a momentum vector p_μ as a bi-spinor $p_{a\dot a}$. We can go to such a representation by using the chiral representation of the Dirac γ-matrices – a process that is well-known in supersymmetric field theories, see e.g. [9, 10]. In signature $+---$ the Dirac matrices can then be represented as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} , \tag{1.2.3}$$

where $(\sigma^\mu)_{a\dot a} = (1, \vec\sigma)$ and $(\bar\sigma^\mu)^{\dot a a} = \epsilon^{ab}\epsilon^{\dot a \dot b}(\sigma^\mu)_{b\dot b} = (1, -\vec\sigma)$ and $\vec\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices as given in Equation (A.1.2). For a given vector p we then have

$$\begin{aligned} p_{a\dot a} &= p_\mu \sigma^\mu_{a\dot a} \\ &= p_0 \mathbb{1} + \vec p \cdot \vec\sigma \qquad (1.2.4) \\ &= \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix} , \qquad (1.2.5) \end{aligned}$$

from which it follows that $p_\mu p^\mu = \det(p_{a\dot a})$. Hence p is light-like ($p^2 = 0$) if $\det(p_{a\dot a}) = 0$, which in turn means that massless vectors are those for which

$$p_{a\dot a} = \lambda_a \tilde\lambda_{\dot a} \,, \tag{1.2.6}$$

for some spinors λ_a and $\tilde\lambda_{\dot a}$. These spinors are unique up to the scaling $(\lambda, \tilde\lambda) \to (c\lambda, c^{-1}\tilde\lambda)$ for a complex number c.

If we wish p_μ to be real in Lorentz signature (in which case $p_{a\dot a}$ is hermitian) then we must take $\tilde\lambda = \pm\bar\lambda$ where $\bar\lambda$ is the complex conjugate of λ. The sign determines whether p has positive or negative energy. It is also possible (and sometimes useful) to consider other signatures. In signature $++--$ λ and $\tilde\lambda$ are real and independent while in Euclidean signature $(++++)$ the spinor representations are pseudoreal. Light-like vectors cannot be real with Euclidean signature.

The formula for $p_\mu p^\mu = \det(p_{a\dot a})$ generalises for any two momenta p and q and using the fact that $(\ \ )_{\dot{}}(\ \ )^{-} = 2$ we can write the scalar product for two light-like vectors $p_{\dot{}} = \ \tilde{}_{\dot{}}$ and $q_{\dot{}} = \ \tilde{}_{\dot{}}$ as

$$2(p \quad q) = h \quad i[\tilde{}\ ]: \tag{1.2.7}$$

This is the standard convention in the perturbative field theory literature and differs from the conventions in [31,161] by a sign that is related to the choice of how to contract indices using .

1.2.2 Wavefunctions

Once p is given, the additional information involved in specifying (and hence $\tilde{}$ in complexified Minkowski space with real $p_{\dot{}}$) is equivalent to a choice of wavefunction for a spin $1/2$ particle of momentum p. To see this, we can write the massless Dirac equation for a negative-chirality spinor as

$$i(\ \ )_{\dot{}}@ \quad = 0: \tag{1.2.8}$$

A plane wave $= \ e^{ip\ x}$ with constant obeys this equation $i\ p_{\dot{}} \quad = 0$ which implies that $= c$. Similar considerations apply for positive-chirality spinors and thus we can write fermion wavefunctions of helicity[5] $1/2$ as

$$_{-} = \tilde{}_{-}e^{ix\ \ \tilde{}\ }; \qquad = \quad e^{ix\ \ \tilde{}\ } \tag{1.2.9}$$

respectively.

For massless particles of spin 1 the usual method is to specify a polarization vector (which we should be careful not to confuse with) in addition to their momentum and together with the constraint $p \quad = 0$. This constraint is equivalent to the Lorentz gauge condition and deals with fixing the gauge invariance inherent in gauge field theories. It is clear that if we add any multiple of p to then this condition is still satisfied and we have the gauge invariance

$$\hat{} = \quad + !\,p : \tag{1.2.10}$$

If one now has a decomposition of a light-like vector particle with momentum $p_{\dot{}} = \ \tilde{}_{\dot{}}$ then one can take the polarisation vectors to be [31] (see also [153, 163] and references therein):

$$^{+}_{-} = \frac{\ \tilde{}\ }{h \quad i}; \qquad _{-} = \frac{\ \tilde{}\ }{[\tilde{}\ \tilde{}\ ]}; \tag{1.2.11}$$

[5] We will often use the terms chirality and helicity interchangeably.

for positive- and negative-helicity particles respectively. η and $\tilde{\eta}$ are arbitrary negative- and positive-chirality spinors (not proportional to λ or $\tilde{\lambda}$) respectively and it is worth noting that the positive-helicity polarization vector is proportional to the positive-helicity spinor ($\tilde{\lambda}_{\dot{a}}$) associated with the momentum vector p while the negative-helicity polarization vector is proportional to the negative-helicity one (λ_a). These polarization vectors clearly obey the constraint $0 = p \cdot \epsilon^{+} = p \cdot \epsilon^{-}$ since $\langle \lambda \lambda \rangle = [\tilde{\lambda} \tilde{\lambda}] = 0$ and are independent of η and $\tilde{\eta}$ up to a gauge transformation [31, 161]. The wavefunctions for positive and negative-helicity massless vector bosons can thus be written as [161]

$$A^{+}_{a\dot{a}} = \epsilon^{+}_{a\dot{a}} e^{i x_{b\dot{b}} \lambda^{b} \tilde{\lambda}^{\dot{b}}} ; \quad A^{-}_{a\dot{a}} = \epsilon^{-}_{a\dot{a}} e^{i x_{b\dot{b}} \lambda^{b} \tilde{\lambda}^{\dot{b}}} : \tag{1.2.12}$$

Spinless particles have wavefunction $\phi = e^{i x_{a\dot{a}} \lambda^{a} \tilde{\lambda}^{\dot{a}}}$ as usual.

1.2.3 Variable reduction

One of the central motivations for all this song and dance is that we can use the results to homogenise our description of scattering amplitudes. The plethora of variables that we had before can simply be traded for the bi-spinors λ and $\tilde{\lambda}$ to yield the compact form of a general scattering amplitude as

$$A_n = A_n(\lambda_i, \tilde{\lambda}_i, h_i)\, \delta^{(4)}\left(\sum_{i=1}^{n} \lambda_i^{a} \tilde{\lambda}_i^{\dot{a}} \right) ; \tag{1.2.13}$$

where h_i is the helicity of the ith particle. In this scheme we can therefore calculate amplitudes for the scattering of specific helicity configurations of specific colour orderings of massless particles. The full amplitude is obtained by summing over all helicity configurations and all appropriate colour orderings.

As a final remark in this section it is useful to note (and easy to show - see [31, 161]) that under the scaling-invariance inherent in the decomposition of Eq. (1.2.6), the wavefunction of a massless particle of helicity h scales as c^{-2h} and thus obeys the condition

$$\left(\lambda^{a} \frac{\partial}{\partial \lambda^{a}} - \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \right) \phi(\lambda, \tilde{\lambda}) = -2h\, \phi(\lambda, \tilde{\lambda}) : \tag{1.2.14}$$

Similarly, the amplitude in Eq. (1.2.13) obeys

$$\left(\lambda_i^{a} \frac{\partial}{\partial \lambda_i^{a}} - \tilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} \right) A_n(\lambda_i, \tilde{\lambda}_i, h_i) = -2h_i A_n(\lambda_i, \tilde{\lambda}_i, h_i) \tag{1.2.15}$$

for each i separately.[6]

[6] The full expression $A_n(\lambda_i, \tilde{\lambda}_i, h_i)\, \delta^{(4)}\left(\sum_{i=1}^{n} \lambda_i^{a} \tilde{\lambda}_i^{\dot{a}} \right)$ also obeys (1.2.15) [31].

The interested reader can find the Feynman rules for massless $SU(N_c)$ Yang-Mills gauge theory in the spinor helicity formalism in Appendix B.

1.3 Supersymmetric decomposition

Supersymmetric field theories are in many ways very similar to the usual Yang-Mills theories whose Feynman rules we wrote down at the start of the chapter. The presence of this extra symmetry – supersymmetry – means that the particles of the theories arrange themselves into supersymmetric multiplets containing equal numbers of bosonic and fermionic degrees of freedom and this can often give rise to great simplifications.

Maximally supersymmetric ($N = 4$) Yang-Mills for example, which has the maximum amount of supersymmetry consistent with a gauge theory (i.e. particles with spin less than or equal to 1) in four dimensions, contains only 1 multiplet consisting of 1 vector boson A_μ (2 real degrees of freedom (d.o.f.)), 6 real scalars ϕ^I (6 real d.o.f.) and 4 Weyl (i.e. chiral) fermions λ (8 real d.o.f.) which lives in the adjoint of the gauge group. This multiplet is often written in a helicity-basis (the helicities of the particles here are $h = (-1; -1/2; 0; 1/2; 1)$) as $(A^-; \lambda^-; \phi; \lambda^+; A^+) = (1;4;6;4;1)$ and is often referred to as the adjoint multiplet of $N = 4$. The meaning of this notation is that one of the degrees of freedom of the vector boson is associated with a negative-helicity (-1) state and the other with a positive-helicity $(+1)$ state. Similarly, the chiral fermions are split into two, with 4 degrees of freedom being associated with helicity $-1/2$ and 4 with helicity $+1/2$. The scalars are of course spinless and thus associated with helicity 0. Other common multiplets in four dimensions include the vector multiplet of $N = 2$ $(1;2;2;2;1)$ – which consists of 1 vector, 2 fermions and 2 scalars – the hypermultiplet of $N = 2$ $(0;2;4;2;0)$ and the vector $(1;1;0;1;1)$ and chiral $(0;1;2;1;0)$ multiplets of $N = 1$ supersymmetry.

The existence of these supersymmetric multiplets generally leads to a better control of the field theory in question, and most-importantly for us a greater control of its perturbative expansion. Heuristically, fermions propagating in loops give terms which have the opposite sign to bosons and the exact matching of the bosonic and fermionic degrees of freedom leads to cancellations in the ultraviolet divergences that plague non-supersymmetric field theories. In particular, $N = 4$ super-Yang-Mills is believed to be completely finite in four dimensions as well as having quantum-mechanical conformal invariance. Massless QCD on the other hand is classically conformally-invariant, although this is broken by quantum effects as is well-known from the existence of its one-loop (and higher) β-function. QCD is also UV divergent at loop-level and thus must be renormalised order-by-order in perturbation theory.

$N = 4$ super-Yang-Mills has the most striking features of these four-dimensional supersymmetric gauge theories and we will concern ourselves with this theory as well as $\mathcal{N}=1$ super-Yang-Mills. In fact, the results for $\mathcal{N}=1$ amplitudes in Chapter 2 also apply to certain $\mathcal{N}=2$ amplitudes by virtue of the fact that the $\mathcal{N}=2$ hypermultiplet is twice the $\mathcal{N}=1$ chiral multiplet and the $\mathcal{N}=2$ vector multiplet is equal to an $\mathcal{N}=1$ vector multiplet plus an $\mathcal{N}=1$ chiral multiplet.

As we have already mentioned, we will mostly be concerned with gluon scattering in $SU(N_c)$ Yang-Mills theories (including QCD) and thus will only consider this case here. At tree-level it is easy to see that gluon scattering amplitudes are the same in QCD as they are in $\mathcal{N}=4$ super-Yang-Mills theory. This is because vertices connecting gluons to fermions or scalars in these theories couple gluons to pairs of these particles. Thus one cannot create fermions or scalars internally without also creating a loop [164]. These QCD scattering amplitudes therefore have a 'hidden' $\mathcal{N}=4$ supersymmetry:

$$A^{\rm tree}_{\rm QCD} = A^{\rm tree}_{\mathcal{N}=4}\,. \tag{1.3.1}$$

The same can of course be said about any supersymmetric field theory with adjoint fields when one is concerned with the scattering of external gluons at tree-level. We thus have the more general result that

$$A^{\rm tree}_{\rm QCD} = A^{\rm tree}_{\mathcal{N}=4} = A^{\rm tree}_{\mathcal{N}=2} = A^{\rm tree}_{\mathcal{N}=1}\,. \tag{1.3.2}$$

At one-loop we can of course have other particles propagating in the loop, but where gluon-scattering only is concerned we can still find a supersymmetric decomposition. It is:

$$A^{\rm one\text{-}loop}_{\rm QCD} = A^{\rm one\text{-}loop}_{\mathcal{N}=4} - 4A^{\rm one\text{-}loop}_{\mathcal{N}=1,\,{\rm chiral}} + 2A^{\rm one\text{-}loop}_{\rm scalar}\,. \tag{1.3.3}$$

In words this says that an all-gluon scattering amplitude in QCD at one loop can be decomposed into 3 terms: Firstly a term where an $\mathcal{N}=4$ multiplet propagates in the loop. Secondly a term where an $\mathcal{N}=1$ chiral multiplet propagates in the loop and lastly a term in pure Yang-Mills where we only have 2 real scalars (or one complex scalar) in the loop. This is easily seen due to the multiplicities of the various multiplets in question: $(1,0,0,0,1) = (1,4,6,4,1) - 4(0,1,2,1,0) + 2(0,0,1,0,0)$.

As we have already discussed many times, the LHS of (1.3.3) is extremely complicated to evaluate. However, the 3 pieces on the RHS are relatively much easier to deal with. The first two pieces are contributions coming from supersymmetric field theories and these extra (super)-symmetries greatly help to reduce the complexity of the calculations there. Much of the difficulty is thus pushed into the last term which is the most complex of the three, but is still far easier to evaluate than the LHS.

It is therefore clear that supersymmetric field theories are not only simpler toy models with which to try to understand the gauge theories of the standard model, but relevant theories in themselves which contribute parts (and sometimes the entireity in the case of certain tree-level amplitudes (1.3.2)) of the answer to calculations in theories such as QCD. These supersymmetric decompositions will be of great assistance to us in our quest to understand the hidden simplicity of scattering amplitudes and in order to perform actual calculations.

For more information on supersymmetric field theories see any one of a multitude of books, papers and reviews including [9, 10, 11, 12].

1.4 Supersymmetric Ward identities

As we can now see, for a large number of scattering amplitudes in gauge theories we can reduce the complexity of our problem by considering an appropriate colour-ordered sub-amplitude that only depends on the positive- and negative-helicity spinors associated with the external momenta (we usually drop the h_i dependence of (1.2.13) and leave it as being implicit in the definition of the amplitude being considered). Using our 'hidden' (or not, depending on the theory in question) supersymmetry we are now in a position to learn something about the scattering amplitudes in question. The following is also nicely reviewed in a number of places including [153, 154] and was first considered in [164, 165, 166, 167]. See also e.g. [168] for a recent application of supersymmetric Ward identities to loop amplitudes.

1.4.1 $N = 1$ SUSY constraints

Let us consider what is in some ways the simplest possible setup, an adjoint (vector) multiplet in an $N = 1$ supersymmetric field theory where the SUSY is unbroken. This $N = 1$ theory has only one supercharge $Q(\eta)$ that generates the supersymmetry with η being the fermionic parameter of the transformation [9]. Because supersymmetry is unbroken we know that Q must annihilate the vacuum: $Q(\eta)|0\rangle = 0$. This in turn gives rise to the following supersymmetric Ward identity (SWI)

$$0 = \langle 0|[Q(\eta), \Phi_1 \ldots \Phi_n]|0\rangle = \sum_{i=1}^{n} \langle 0|\Phi_1 \ldots [Q(\eta), \Phi_i] \ldots \Phi_n|0\rangle\,, \tag{1.4.1}$$

for some fields Φ_i. In addition, if we use a suitable helicity basis in which we have a massless vector $A^\pm$ and a massless spin 1/2 fermion $\Lambda^\pm$, then $Q(\eta)$ acts on the doublet $(A^\pm, \Lambda^\pm)$ (i.e. $(A^-, \Lambda^-, 0, \Lambda^+, A^+)$ in the notation of the previous subsection) as [166, 167]:

$$\begin{aligned} [Q(\eta), A^\pm(p)] &= \mp\Gamma^\pm(p;\eta)\Lambda^\pm\,, \\ [Q(\eta), \Lambda^\pm(p)] &= \mp\Gamma^\mp(p;\eta)A^\pm\,, \end{aligned} \tag{1.4.2}$$

for some momentum p associated with these states. $\Gamma(\eta;p)$ is linear in η and can be constructed by using the Jacobi identity

$$[[Q(\eta),Q(\bar\eta)],\Lambda(p)]+[[Q(\eta),\Lambda(p)],Q(\bar\eta)]+[[\Lambda(p),Q(\bar\eta)],Q(\eta)]=0 \qquad (1.4.3)$$

and the SUSY algebra relation $[Q(\eta),Q(\bar\eta)]=-2i\,\bar\eta\not P\eta$, where $\not P=\sigma^\mu P_\mu$ as usual. By considering (1.4.4) for any of the chiral fields ($A^+(p)$ for example), we can readily deduce that

$$\Gamma^+(p;\eta)\,\Gamma^-(p;\bar\eta)-\Gamma^+(p;\bar\eta)\,\Gamma^-(p;\eta)=-2i\,\bar\eta\not p\eta\,; \qquad (1.4.4)$$

which can be solved to give (in the notation of §1.2) [153,154,166,167]:

$$\Gamma^+(p;q;\theta)=\theta[pq]\,;\qquad \Gamma^-(p;q;\theta)=\theta\langle pq\rangle\,. \qquad (1.4.5)$$

In this expression we have written $p=\lambda_p\tilde\lambda_p$ and our parameter η in terms of a Grassmann parameter θ and an arbitrary reference momentum $q=\lambda_q\tilde\lambda_q$. We have also used the shorthand notation $\langle\lambda_p\lambda_q\rangle=\langle pq\rangle$ and $[\tilde\lambda_p\tilde\lambda_q]=[pq]$ which will often be employed henceforth.

Now consider (1.4.1) with $\Lambda_1=\Lambda_1^+$ and $\Lambda_i=A_i^+$ for $i\neq 1$:

$$\begin{aligned}
0 &= \langle 0|[Q(\eta(q;\theta)),\Lambda_1^+(p_1)A_2^+(p_2)\dots A_n^+(p_n)]|0\rangle\\
&= -\Gamma(p_1;q;\theta)\langle 0|A_1^+(p_1)A_2^+(p_2)\dots A_n^+(p_n)|0\rangle\\
&\quad+\Gamma(p_2;q;\theta)\langle 0|\Lambda_1^+(p_1)\Lambda_2^+(p_2)\dots A_n^+(p_n)|0\rangle\\
&\quad\ \vdots\\
&\quad+\Gamma(p_n;q;\theta)\langle 0|\Lambda_1^+(p_1)A_2^+(p_2)\dots\Lambda_n^+(p_n)|0\rangle\\
&= -\Gamma(p_1;q;\theta)A_n(A_1^+,A_2^+,\dots,A_n^+)\\
&\quad+\Gamma^+(p_2;q;\theta)A_n(\Lambda_1^+,\Lambda_2^+,\dots,A_n^+)\\
&\quad\ \vdots\\
&\quad+\Gamma^+(p_n;q;\theta)A_n(\Lambda_1^+,A_2^+,\dots,\Lambda_n^+)\,.
\end{aligned} \qquad (1.4.6)$$

As all of the couplings of fermions to vectors conserve helicity (you always get one fermion of each helicity coupling to a vector), the $n-1$ terms involving two fermions and $n-2$ gluons must vanish and thus the first term involving only gluons of positive helicity must vanish too $A_n(A_1^+,A_2^+,\dots,A_n^+)=0$. Since supersymmetry commutes with colour we can write our amplitudes as colour-ordered ones straight away and then the relations apply to each colour-ordered amplitude separately.

If we consider the case where we have one negative-helicity in our SWI so that $\Lambda_1=\Lambda_1^+$, $\Lambda_2=A_2^-$ and $\Lambda_i=A_i^+$ for $i\neq 1,2$ for example, then we can also show that all amplitudes with one negative-helicity particle and $n-1$ positive-helicity particles vanish. This is so both for the case of all gluon scattering and the case of $n-2$ gluons and two fermions of opposite helicities. With more than one negative-helicity (such as $\Phi_1 = \Lambda_1^+$, $\Phi_2 = A_2^-$, $\Phi_3 = A_3^-$ and $\Phi_i = A_i^+$ for $i \neq 1,2,3$) we can start to relate non-zero amplitudes to each other. In all of these cases it is useful to remember that the reference momentum q is arbitrary and can thus be taken to be one of the external momenta ($q = p_i$) for example at any given stage in order to simplify the calculations and deduce useful results.

1.4.2 Amplitude relations

Some of the useful relations that we can obtain are:

$$A_n^{SUSY}(1^\pm, 2^+, \ldots, n^+) = 0, \tag{1.4.7}$$
$$A_n^{SUSY}(1^\mp, 2^-, \ldots, n^-) = 0, \tag{1.4.8}$$

for any spins of the particles involved and [38]

$$A_n^{SUSY}(A_i^-, \ldots, \Lambda_r^-, \ldots, \Lambda_s^+, \ldots) = \frac{\langle is\rangle}{\langle ir\rangle} A_n^{SUSY}(A_i^-, \ldots, A_r^-, \ldots, A_s^+, \ldots), \tag{1.4.9}$$
$$A_n^{SUSY}(A_i^-, \ldots, \phi_r^-, \ldots, \phi_s^+, \ldots) = \frac{\langle is\rangle^2}{\langle ir\rangle^2} A_n^{SUSY}(A_i^-, \ldots, A_r^-, \ldots, A_s^+, \ldots), \tag{1.4.10}$$

where we have also played the same game with an $N = 2$ Vector multiplet in order to include scalars ϕ. These relations hold order-by-order in the loop expansion of supersymmetric field theories as no perturbative approximations were made in deriving them, and by virtue of (1.3.2) they apply directly to tree-level QCD amplitudes involving gluons. It turns out that tree-level QCD amplitudes involving fundamental quarks can also be obtained from (1.4.9) because of relations between sub-amplitudes involving gluinos (i.e. fermionic superpartners of gluons in an adjoint multiplet such as the Λ above) and those involving fundamental quarks [153, 169].

Equations (1.4.7) and (1.4.8) amount to the statement that for any supersymmetric theory with only adjoint fields, the 'all-plus' and 'all-minus' helicity amplitudes must vanish and the amplitudes with one minus and $n-1$ plusses (or vice-versa) must also vanish. The same statement holds for the tree-level gluon scattering amplitudes of QCD. As a result of this, the first non-vanishing set of amplitudes in a supersymmetric theory are the ones with two negative helicities and $n-2$ positive helicities. These are thus termed the Maximally Helicity Violating (MHV) amplitudes. Their parity conjugates, the amplitudes with two positive helicities and $n-2$ negative helicities are similarly non-vanishing and are sometimes termed googly MHV (or $\overline{\text{MHV}}$) amplitudes [31]. Similarly, amplitudes with three negative helicities and $n-3$ positive helicities are termed next-to-MHV (NMHV) amplitudes. The next ones are thus called next-to-next-to-MHV

(NNMHV) and so on.

The tree-level MHV amplitudes for gluon scattering, proposed at n-point in [170] and then proved in [171], are given by (1.0.3) or by

$$A_n(1^+;\ldots;i^-;\ldots;j^-;\ldots;n^+) = \frac{\langle ij\rangle^4}{\prod_{k=1}^{n}\langle k\,k+1\rangle}\,; \qquad (1.4.11)$$

up to a factor. i and j are the gluons of negative helicity and the amplitude obeys (1.2.15). The amplitude is cyclic in the ordering of the gluons and so the $n+1^{\text{th}}$ spinor appearing in the denominator of (1.4.11) just denotes the spinor of the 1^{st} gluon. Note in particular that this function is entirely 'holomorphic' in the negative-helicity spinors – i.e. it does not depend on any of the $\tilde{\lambda}$s – and this will be important to us presently. We will not discuss NMHV and other amplitudes yet except to mention that they do depend on the $\tilde{\lambda}$s.

1.5 Twistor space

There is a way in which we can understand some of the properties of amplitudes that we have discussed above such as the vanishing of certain helicity configurations and the simple structure of the MHV amplitudes and that is by going to twistor space [31]. This has two primary motivations. One is that the conformal symmetry group has a rather exotic representation in terms of the λ and $\tilde{\lambda}$ variables and the other is that the scaling-invariance mentioned under equation (1.2.6) has an opposite action on the holomorphic spinors λ compared with the anti-holomorphic spinors $\tilde{\lambda}$. It would be nice to put the conformal group[7] into a more standard representation and it may also be nice to have the same scaling for the negative and positive-helicity spinors.

In terms of the spinors we have already introduced in §1.2, the conformal generators

[7] The gluon amplitudes at tree-level are invariant under the full conformal group rather than just the Poincaré group. This is because of the classical conformal invariance of both massless QCD and any of the other supersymmetric field theories that we have been considering. Amplitudes in some of these supersymmetric theories (especially $\mathcal{N}=4$ Yang-Mills) also have quantum conformal invariance.

are [31]

$$
\begin{aligned}
P_{\alpha\dot\alpha} &= \lambda_\alpha \tilde\lambda_{\dot\alpha} ; && (1.5.1)\\
J_{\alpha\beta} &= \frac{i}{2}\left(\lambda_\alpha \frac{\partial}{\partial\lambda^\beta} + \lambda_\beta \frac{\partial}{\partial\lambda^\alpha}\right) ; && (1.5.2)\\
\tilde J_{\dot\alpha\dot\beta} &= \frac{i}{2}\left(\tilde\lambda_{\dot\alpha} \frac{\partial}{\partial\tilde\lambda^{\dot\beta}} + \tilde\lambda_{\dot\beta} \frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\right) ; && (1.5.3)\\
D &= \frac{i}{2}\left(\lambda^\alpha \frac{\partial}{\partial\lambda^\alpha} + \tilde\lambda^{\dot\alpha} \frac{\partial}{\partial\tilde\lambda^{\dot\alpha}} + 2\right) ; && (1.5.4)\\
K_{\alpha\dot\alpha} &= \frac{\partial^2}{\partial\lambda^\alpha\,\partial\tilde\lambda^{\dot\alpha}} ; && (1.5.5)
\end{aligned}
$$

where $P_{\alpha\dot\alpha}$ is the momentum operator, $J_{\alpha\beta}$ and $\tilde J_{\dot\alpha\dot\beta}$ the Lorentz generators, D the dilatation operator and $K_{\alpha\dot\alpha}$ the generator of special conformal transformations. These give rise to the algebra of the conformal group as

$$
\begin{aligned}
[J_{\alpha\beta}, P_{\gamma\dot\gamma}] &= \frac{i}{2}(\epsilon_{\alpha\gamma} P_{\beta\dot\gamma} + \epsilon_{\beta\gamma} P_{\alpha\dot\gamma}) ;\\
[\tilde J_{\dot\alpha\dot\beta}, P_{\gamma\dot\gamma}] &= \frac{i}{2}\left(\epsilon_{\dot\alpha\dot\gamma} P_{\gamma\dot\beta} + \epsilon_{\dot\beta\dot\gamma} P_{\gamma\dot\alpha}\right) ;\\
[J_{\alpha\beta}, J_{\gamma\delta}] &= \frac{1}{4}(\epsilon_{\alpha\gamma} J_{\beta\delta} + \epsilon_{\alpha\delta} J_{\beta\gamma} + \epsilon_{\beta\gamma} J_{\alpha\delta} + \epsilon_{\beta\delta} J_{\alpha\gamma}) ;\\
[\tilde J_{\dot\alpha\dot\beta}, \tilde J_{\dot\gamma\dot\delta}] &= \frac{1}{4}\left(\epsilon_{\dot\alpha\dot\gamma} \tilde J_{\dot\beta\dot\delta} + \epsilon_{\dot\alpha\dot\delta} \tilde J_{\dot\beta\dot\gamma} + \epsilon_{\dot\beta\dot\gamma} \tilde J_{\dot\alpha\dot\delta} + \epsilon_{\dot\beta\dot\delta} \tilde J_{\dot\alpha\dot\gamma}\right) ;\\
[D, P_{\alpha\dot\alpha}] &= \frac{i}{2} P_{\alpha\dot\alpha} ;\\
[K_{\alpha\dot\alpha}, D] &= 2K_{\alpha\dot\alpha} ;\\
[J_{\alpha\beta}, K_{\gamma\dot\gamma}] &= \frac{i}{2}(\epsilon_{\alpha\gamma} K_{\beta\dot\gamma} + \epsilon_{\beta\gamma} K_{\alpha\dot\gamma}) ;\\
[\tilde J_{\dot\alpha\dot\beta}, K_{\gamma\dot\gamma}] &= \frac{i}{2}\left(\epsilon_{\dot\alpha\dot\gamma} K_{\gamma\dot\beta} + \epsilon_{\dot\beta\dot\gamma} K_{\gamma\dot\alpha}\right) ;\\
[K_{\alpha\dot\alpha}, P_{\beta\dot\beta}] &= i\left(\epsilon_{\alpha\beta} \tilde J_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} J_{\alpha\beta} + \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} D\right) ; \qquad (1.5.6)
\end{aligned}
$$

with all other commutators being zero. However, as can be seen from (1.5.1)–(1.5.5), the momentum operator is a multiplication operator, the Lorentz generators are first order homogeneous differential operators, the dilatation operator an inhomogeneous first order differential operator and the special conformal generator a degree two differential operator. We have quite a mix.

We can in fact reduce these to a more standard representation by performing a the transformation [31,32]

$$
\begin{aligned}
\tilde\lambda_{\dot\alpha} &\to i\frac{\partial}{\partial\mu^{\dot\alpha}} ;\\
\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}} &\to i\mu_{\dot\alpha} . \qquad (1.5.7)
\end{aligned}
$$

This breaks the symmetry between λ and $\tilde{\lambda}$ as we have chosen to transform one rather than the other, but giving the advantage that all the generators become first order differential operators:

$$P_{\alpha\dot{\alpha}} = i\lambda_\alpha \frac{\partial}{\partial \mu^{\dot{\alpha}}}; \tag{1.5.8}$$

$$K_{\alpha\dot{\alpha}} = i\mu_{\dot{\alpha}} \frac{\partial}{\partial \lambda^{\alpha}}; \tag{1.5.9}$$

$$J_{\alpha\beta} = \frac{i}{2}\left(\lambda_\alpha \frac{\partial}{\partial \lambda^\beta} + \lambda_\beta \frac{\partial}{\partial \lambda^\alpha}\right); \tag{1.5.10}$$

$$\tilde{J}_{\dot{\alpha}\dot{\beta}} = \frac{i}{2}\left(\mu_{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\beta}}} + \mu_{\dot{\beta}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}\right); \tag{1.5.11}$$

$$D = \frac{i}{2}\left(\lambda^\alpha \frac{\partial}{\partial \lambda^\alpha} - \mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}\right). \tag{1.5.12}$$

The scaling properties of λ and μ are also changed such that there is an invariance under

$$(\lambda;\mu) \to (c\lambda; c\mu); \tag{1.5.13}$$

for a complex number c, and the amplitude scalings (1.2.15) become

$$\left(\lambda_i \frac{\partial}{\partial \lambda_i} + \mu_i^{\dot{\alpha}} \frac{\partial}{\partial \mu_i^{\dot{\alpha}}}\right)\tilde{A}_n(\lambda_i;\mu_i;h_i) = -(2h_i + 2)\tilde{A}_n(\lambda_i;\mu_i;h_i); \tag{1.5.14}$$

where $\tilde{A}_n$ is the appropriately transformed amplitude.

This transformation is perhaps easiest to understand in signature $++--$. In this case one can consider λ and μ to be real and independent and thus they parametrise a copy of $\mathbb{R}^4$. The scaling (1.5.13) is then a real scaling and reduces the space to real-projective three-space $\mathbb{RP}^3$ and the transform (1.5.7) is implemented by a '1/2-Fourier' transform analagous to that encountered in quantum mechanics [31]:

$$\tilde{f}(\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} e^{i\mu^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}} f(\tilde{\lambda}). \tag{1.5.15}$$

In other signatures (such as Minkowski space) it may be more natural to regard λ and μ as being complex and independent. They thus parametrise a copy of $\mathbb{C}^4$ which reduces to $\mathbb{CP}^3$ under the scaling (1.5.13). These spaces – $\mathbb{RP}^3$ and $\mathbb{CP}^3$ – were called twistor spaces by Penrose [32] and we will often use coordinates Z^I with $I = 1 \ldots 4$ on them thus combining λ and μ together. One should really refer to 'real/complex projective twistor space' respectively, but we will denote them all as being twistor space ($\mathbb{T}$) and let the context dictate what we mean by that.

In the complex cases, the choice of a contour for the transformation as given by (1.5.15) is not necessarily clear and it seems necessary to take the more sophisticated approach of Penrose and use Dolbeault- or sheaf-cohomology [32]. Naïvely, this interprets the integrand and measure of (1.5.15) as a $(0,2)$-form on twistor space, while equation (1.5.14) suggests that the amplitudes are best thought of not as functions, but sections of a line bundle L_h of degree $-2h-2$, $L_h = O(-2h-2)$ for each h. The amplitudes are thus elements of $H^{(0,2)}(CP^{3'}, O(-2h-2))$ [31].[8]

The transformation of wavefunctions to twistor space is in some ways more complex. One cannot perform such a naïve '1/2-Fourier' transform in essence because the wavefunctions are defined by being solutions to the massless free wave equations and so one must see how one can solve these in twistor space. It turns out that these solutions can be written as integrals of functions of degree $2h-2$ and the wavefunctions are then described by elements of the $\bar{\partial}$-cohomology group $H^{(0,1)}(CP^{3'}, O(2h-2))$ – see e.g. [31, 172, 173, 174] for details.

In particular these descriptions mean that scattering amplitudes with specific external states make sense in twistor space. In a usual field theory construction one would multiply a momentum-space scattering amplitude with its momentum-space wavefunctions and integrate over all momenta to create a scattering amplitude with specific external states in the position-space representation. If the wavefunctions in position-space satisfying the appropriate free wave equations are given by $\varphi_i(x) = \int d^4p_i\, \delta(p_i^2) e^{ip_i \cdot x} \phi_i(p_i)$, then we have schematically $A(\varphi_i) = \int \prod (d^4p_i\, \delta(p_i^2) e^{ip_i \cdot x} \phi_i(p_i)) \tilde{A}(p_i)$.

In twistor space, multiplying an amplitude in $H^{(0,2)}(CP^{3'}, O(-2h-2))$ with a wavefunction which is in $H^{(0,1)}(CP^{3'}, O(2h-2))$ gives an element of $H^{(0,3)}(CP^{3'}, O(-4))$. The natural measure on CP^3 is a $(3,0)$-form of degree 4 (it is in fact the Ω' of (1.6.12)), and so the final integral will be of a $(3,3)$-form of degree 0 which makes sense (i.e. the integrand is a top-form on twistor space invariant under (1.5.13)) as an integral over $CP^{3'}$. Doing this for each external particle gives the required scattering amplitude in position-space.

Following the original suggestions of Nair [175], there is a similar construction which is particularly apt for amplitudes in $\mathcal{N} = 4$ Yang-Mills. In this case, particles are described by λ, $\tilde{\lambda}$ and an additional spinless fermionic variable η_A with $A = 1, \ldots, 4$ in the 4 representation of the R-symmetry group $SU(4)_R$ of $\mathcal{N} = 4$ Yang-Mills. The spacetime symmetry group in this case is no-longer the usual conformal group, but the superconformal group $PSU(2,2|4)$ and one can write down generators in terms of λ, $\tilde{\lambda}$ and η which are again in a somewhat exotic form. After a Penrose transform to

[8] Here we follow [31] and write $CP^{3'}$ instead of CP^3 because $H^{(0,2)}(CP^3, O(-2h-2)) = 0$ and we should really work with a suitable open set of CP^3 (which we denote with a prime) rather then all of twistor space.

super-twistor space, which just consists of (1.5.7) plus

$$\eta_A \to i\frac{\partial}{\partial \eta^A}$$
$$\frac{\partial}{\partial \eta_A} \to i\eta^A; \qquad (1.5.16)$$

all superconformal generators similarly become first order differential operators and the space spanned by λ, $\tilde{\mu}$ and η^A is $RP^{3|4}$ or $CP^{3|4}$. The scaling invariance of super-twistor space is:

$$(Z^I;\eta^A) \to (cZ^I; c\eta^A): \qquad (1.5.17)$$

In this case, the helicity operator

$$h = 1 - \frac{1}{2}\eta^A\frac{\partial}{\partial \eta^A} \qquad (1.5.18)$$

modifies the scaling relation (1.5.14) so that it becomes

$$\left(Z_i^I\frac{\partial}{\partial Z_i^I} + \eta_i^A\frac{\partial}{\partial \eta_i^A}\right)\tilde{A}_n(\lambda_i;\mu_i;\eta_A;h_i) = 0; \qquad (1.5.19)$$

and so the scattering amplitudes are elements of $H^{(0,2)}(CP^{3|4'};\mathcal{O}(0))$.

On super-twistor space, the wavefunctions are now elements of $H^{(0,1)}(CP^{3|4'};\mathcal{O}(0))$ and can be given explicitly for a particle of helicity h by [31, 36, 120, 161]

$$\Phi(\lambda;\mu;h) = \bar{\delta}(\langle\lambda;\lambda_i\rangle)\left(\frac{\xi}{\lambda_i}\right)^{2h-1}\exp\left(i[\tilde{\mu};\tilde{\lambda}_i]\right) g_h(\eta); \qquad (1.5.20)$$

where $g_h(\eta)$ is simply a factor of $2-2h$ ηs.[9] For example, for a positive-helicity gluon g_h is 1 while for a negative-helicity gluon it is $\eta^1\eta^2\eta^3\eta^4$. (In fact it is just the factor of η that the associated state multiplies in the expansion of the superfield $\mathcal{A}$ in (1.6.10).) $\bar{\delta}$ is a 'holomorphic' delta function which is a $(0;1)$-form given by $\bar{\delta}(f) = \delta^{(2)}(f)d\bar{f}$ for any holomorphic function f – see Appendix A for a more detailed discussion.

In this case, the multiplication of scattering amplitude and wavefunction leads to an element of $H^{(0,3)}(CP^{3|4'};\mathcal{O}(0))$ and the volume form is a $(3;0)$-form of degree 0 (explicitly given by (1.6.11)), so the result makes sense (again as a scaling invariant top-form) to be integrated over $CP^{3|4'}$ and gives the scattering amplitude in position-space.

For our treatment of amplitudes, we will generally use the definition (1.5.15) and signature $++--$ and interpret our results in other signatures when necessary. It is

---

[9] This factor of η^I is precisely what converts the wavefunctions from being of degree $2h-2$ to being of degree 0. One might also wonder why the power of ξ/λ is only $2h-1$ and not $2h-2$ given that the wavefunctions on CP^3 (i.e. with the factor of g_h omitted) are of degree $2h-2$. This is because the holomorphic delta function is of degree -1 and thus gives the correct scaling properties overall.

also worth mentioning that we have glossed over many subtleties in the considerations above such as the real nature of momenta already alluded to in §1.2, and the exclusion of the 'point at infinity' in twistor space (i.e. the use of $\mathbb{T}'$ rather than $\mathbb{T}$). For more details on all these and more detailed discussions of twistor Theory we refer the reader to [31,32,172,173,174,176,177,178] and related references.

1.5.1 Amplitude localisation

Interpreting (1.5.15) as the way to transform amplitudes into twistor space, we are now ready to see what the tree-level MHV amplitudes look like there. If we recall that these amplitudes depend only on the negative-helicity spinors λ_i, the transformed amplitudes are [31]:

$$
\begin{aligned}
\tilde{A}^{MHV}_n(\lambda_i;\mu_i) &= \int \prod_{j=1}^{n} \frac{d^2\tilde\lambda_j}{(2\pi)^2} e^{i\mu_j^{\dot a}\tilde\lambda_{j\dot a}}\, \delta^{(4)}\!\left(\sum_{k=1}^{n}\lambda_k\tilde\lambda_k\right) A^{MHV}_n(\lambda_i;\tilde\lambda_i) \\
&= \int d^4x \int \prod_{j=1}^{n} \frac{d^2\tilde\lambda_j}{(2\pi)^2} \exp\!\left(i\sum_{k=1}^{n}\mu_k^{\dot a}\tilde\lambda_{k\dot a}\right) \exp\!\left(ix_{a\dot a}\sum_{k=1}^{n}\lambda_k^{a}\tilde\lambda_k^{\dot a}\right) A^{MHV}_n(\lambda_i) \\
&= \int d^4x \prod_{j=1}^{n} \delta^{(2)}\!\left(\mu_{j\dot a} + x_{a\dot a}\lambda_j^{a}\right) A^{MHV}_n(\lambda_i)\,.
\end{aligned}
\tag{1.5.21}
$$

In the second line we have used a standard position-space representation for the delta function of momentum conservation and then in the third we have similarly interpreted the $\tilde\lambda$ integrals as delta functions. The MHV amplitudes are thus supported only when $\mu_{j\dot a} + x_{a\dot a}\lambda_j^a = 0$ for all j and for $\dot a = 1,2$. For each $x_{a\dot a}$ these equations define a curve of degree one and genus zero in $\mathbb{RP}^3$ or $\mathbb{CP}^3$ (depending on whether the variables are real or complex) which is in fact an $\mathbb{RP}^1$ or a $\mathbb{CP}^1$ [31]. $x_{a\dot a}$ is the parameter or modulus describing any one of these curves and (1.5.21) is thus an integral over the moduli space of degree one genus zero curves in $\mathbb{T}$. As there is a delta function for every external particle, the integral is only non-zero when all n-points $(\lambda_i;\mu_{i\dot a})$ lie on one of these curves in twistor space.[10] Thus the MHV amplitudes are localised on simple algebraic curves in twistor space, which are (projective) straight lines in the real case and spheres[11] in the complex case.

In the maximally supersymmetric case we have an additional localisation from transforming the fermionic variables to twistor space. As well as the delta function of momen-

[10] Techincally the space is really n copies of twistor space.

[11] Recall that $S^2 = \mathbb{CP}^1$.

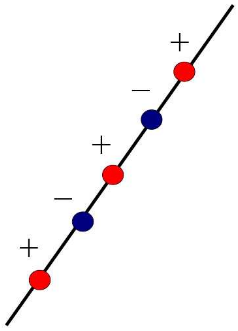


Figure 1.3: The MHV amplitudes localise on simple straight lines in twistor space. Here the 5-point MHV amplitude is depicted as an example.

tum conservation coming with the amplitudes, we also have a fermionic delta function

$$\delta^{(8)}(\eta) = \delta^{(8)}\left(\sum_{i=1}^{n} \lambda_i \eta_i\right) = \int d^8\theta \, \exp\left(i\theta^A \sum_{i=1}^{n} \lambda_i \eta_{iA}\right); \qquad (1.5.22)$$

and the MHV amplitudes for $N = 4$ Yang-Mills are given by [31, 175]

$$A_n^{MHV}(\lambda_i, \tilde{\lambda}_i, \eta_i) = \delta^{(4)}(P)\,\delta^{(8)}(\eta)\,\frac{1}{\prod_{i=1}^{n}\langle i\, i+1\rangle}. \qquad (1.5.23)$$

The transform to super-twistor space is a straightforward generalisation of (1.5.21) and the result is [31]

$$\tilde{A}_n^{MHV}(\lambda_i, \mu_i, \psi_i) = \int d^4x\, d^8\theta \prod_{j=1}^{n} \delta^{(2)}\left(\mu_{j\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda_j^{\alpha}\right)\delta^{(4)}\left(\psi_j^A + \theta^A_{\alpha}\lambda_j^{\alpha}\right)\frac{1}{\prod_{i=1}^{n}\langle i\, i+1\rangle}. \qquad (1.5.24)$$

The equations $\mu_{j\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda_j^{\alpha} = 0$ and $\psi_j^A + \theta^A_{\alpha}\lambda_j^{\alpha} = 0$ then define (for each j) a $CP^{1|0}$ or an $RP^{1|0}$ in $CP^{3|4}$ or $RP^{3|4}$ respectively on which the amplitudes lie.

The equation $\mu_{\dot{\alpha}} + x_{\alpha\dot{\alpha}}\lambda^{\alpha} = 0$ is in fact of central importance in twistor theory and is traditionally taken to be the definition of a twistor. For a given x (as in our case above), it can be regarded as an equation for λ and μ which as we have seen defines a degree one genus zero curve that is topologically an S^2. A point in complexified Minkowski space is thus represented by a sphere in twistor space and hence complexified Minkowski space is the moduli space of such curves. Alternatively, if λ and μ (i.e. a point in twistor space) are given, it can be regarded as an equation for x. The set of solutions is a two complex-dimensional subspace of complexified Minkowski space that is null and self-dual called an α-plane. The null condition means that any tangent vector to the plane is null, and the self-duality means that the tangent bi-vector is self-dual in a certain sense. These α-planes can essentially be regarded as being light-rays and twistor space is the moduli space of α-planes.

Other amplitudes involving more and more negative helicities can also be treated, though in these cases performing the Penrose transform (1.5.15) explicitly becomes harder. In these cases it has been found that certain differential operators can be constructed which help to elucidate their localisation properties in twistor space [31,73]. In particular, given three points $P_i, P_j, P_k \in CP^3$ with coordinates Z_i^I, Z_j^I and Z_k^I, the condition that they lie on a 'line' (i.e. a linearly-embedded copy of CP^1 as discussed above) is that $F_{ijkL} = 0$ where

$$F_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K \, . \tag{1.5.25}$$

Similarly, the condition that four points in twistor space are 'coplanar' (i.e. lie on a linearly embedded $CP^2 \subset CP^3$ is given by $K_{ijkl} = 0$ where

$$K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L \, . \tag{1.5.26}$$

When these are explicitly used, μ is substituted for $\partial/\partial\tilde{\lambda}$ and then they act on amplitudes as differential operators.

The localisation properties of many amplitudes have been checked [31,43,47,53,72,73,76,91,179,180,181,182,183,184], and it has been found that amplitudes with more and more negative helicities localise on curves of higher and higher degree. For tree-level amplitudes in particular this means that an amplitude with q negative-helicity gluons localises on a curve of degree $q-1$. In general, the twistor version of an n-particle scattering amplitude is supported on an algebraic curve in twistor space whose degree is given by [31]

$$d = q - 1 + l \, ; \tag{1.5.27}$$

where q is the number of negative-helicity gluons and l the number of loops. The curve is not necessarily connected and its genus g is bounded by $g \le l$.

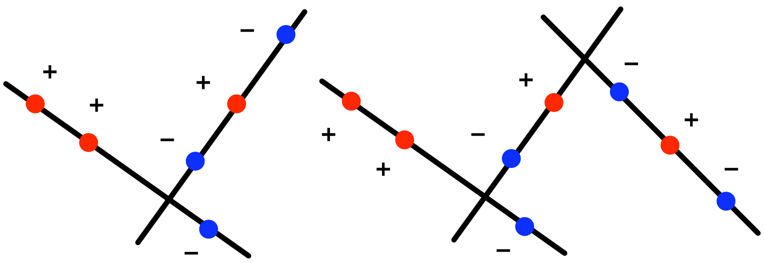


Figure 1.4: Twistor space localisation of tree amplitudes with $q = 3$ and $q = 4$

Tree-level next-to-MHV amplitudes for example are supported on curves of degree 2, while NNMHV amplitudes are supported on curves of degree 3 as shown in Figure 1.4 above. We can also get a geometrical understanding of the vanishing of the all-plus amplitude and the amplitude with one minus and $n-1$ plusses at tree-level. By (1.5.27) these would be supported on curves of degree $d = -1$ and $d = 0$ in twistor space. In the first case, there are no algebraic curves of degree -1, so these amplitudes must trivially vanish. In the second, a curve of degree 0 is simply a point and so amplitudes of this type are supported by configurations where all the gluons are attached to the same point $(\lambda_i; \mu_i) = (\lambda; \mu)\ \forall\, i$ in twistor space. Recalling from equation (1.2.7) that $p_i \cdot p_j \propto \langle\lambda_i\, \lambda_j\rangle[\tilde{\lambda}_j\, \tilde{\lambda}_i]$, all these invariants must be zero for these amplitudes. This on the other hand is impossible for non-trivial scattering amplitudes with $n \geq 4$ particles and thus these must vanish at tree-level.

For $n = 3$ things are a bit more subtle because on-shellness, $p_i^2 = 0$ and momentum conservation, $p_1 + p_2 + p_3 = 0$, guarantee that for real momenta in Lorentz signature $p_i \cdot p_j = 0$. However, for complex momenta and/or other signatures the 3-point amplitude makes more sense. As $0 = 2p_i \cdot p_j = \langle\lambda_i\, \lambda_j\rangle[\tilde{\lambda}_j\, \tilde{\lambda}_i]$, the independence of λ_i and $\tilde{\lambda}_i$ implies that either $\langle\lambda_i\, \lambda_j\rangle = 0$ or $[\tilde{\lambda}_j\, \tilde{\lambda}_i] = 0$. Thus all λ_i are proportional or all $\tilde{\lambda}_i$ are proportional. As can be read-off from the Yang-Mills Lagrangian (or seen as a special case of the googly MHV amplitudes), the $-++$ amplitude is given by

$$A = \frac{[\tilde{\lambda}_1, \tilde{\lambda}_2]^4}{[\tilde{\lambda}_1, \tilde{\lambda}_2][\tilde{\lambda}_2\, \tilde{\lambda}_3][\tilde{\lambda}_3\, \tilde{\lambda}_1]}. \tag{1.5.28}$$

This would vanish identically if all the $\tilde{\lambda}_i$ are proportional, so we should pick all the λ_i to be proportional to ensure momentum conservation. However, $SL(4,\mathbb{R})$ invariance in twistor space[12] then implies that the $(\lambda_i; \mu_i)$ all coincide and thus the gluons are supported at a single point in twistor space as predicted by (1.5.27) [31].

1.6 Twistor string theory

In this section we will give a very brief overview of a string theory that provides a natural framework for understanding the properties of scattering amplitudes discussed in the previous sections. We will only describe the original approach (which has also been the one most computationally useful to date) taken by Witten [31] though other approaches, notably by Berkovits [112, 113, 114], have been considered. Further proposals include [115, 116, 117, 118], though these have not so far been used to calculate any amplitudes. A good introduction to the material presented in this section can again be found in [161].

It is well known that the usual type I, type II and heterotic string theories live in the critical dimension of $d = 10$, which is where they really make sense quantum mechanically. However, there are other string theories known as topological string theories which are typically simpler than ordinary string theories and can make sense

[12] $SO(3,3) = SL(4,\mathbb{R})$ is the conformal group in signature $++--$.

in other dimensions. They are called topological because they can be obtained from certain topological field theories which are field theories whose correlation functions only depend on the topological information of their target space and in-particular do not depend on the local information such as the metric of the space. Witten introduced topological string theory in [185, 186] as a simplified model of string theory, and it has been extensively studied since then. We will only give a 'lightning' review here and refer the reader to the original papers and such excellent introductions as [187] for more details.

1.6.1 Topological field theory

One starts with a field theory in 2-dimensions with $N = 2$ supersymmetry. The supersymmetry generators usually transform as spin $1/2$ fermions under the Lorentz group, but in 2-d this is $SO(2) = U(1)$ locally and the spin $1/2$ representation is reducible into two representations which have opposite charge under the $U(1)$. Things living in these representations are often termed left-movers and right-movers, and the supersymmetry is usually written as being $N = (2,2)$ with 2 left-moving supercharges and 2 right-moving supercharges.

The symmetries of the theory consist of both the usual Poincare algebra as well as the $N = 2$ supersymmetry algebra and the R-symmetry of the theory associated with the supersymmetry. We will not write all of these down here, but in-particular the supersymmetry generators and their complex conjugates obey the non-zero anti-commutation relations (in the language of [187]):

$$\begin{aligned} \{Q_\pm, \bar{Q}_\pm\} &= P \pm H \\ \{D_\pm, \bar{D}_\pm\} &= -(P \pm H)\,; \end{aligned} \tag{1.6.1}$$

where $H \equiv d/d\sigma^0$ and $P \equiv d/d\sigma^1$ are the Hamiltonian and momentum operators of the 2-d space with coordinates σ^α.

One thing that we can now do is to define new operators Q_A and Q_B which are linear combinations of supercharges as

$$\begin{aligned} Q_A &= \bar{Q}_+ + Q_- \\ Q_B &= \bar{Q}_+ + \bar{Q}_-\,; \end{aligned} \tag{1.6.2}$$

and then it follows from (1.6.1) that

$$Q_A^2 = Q_B^2 = 0 \tag{1.6.3}$$

and Q_A and Q_B look like BRST operators. However, $Q_{A/B}$ are not scalars, so we would violate Lorentz invariance by interpreting them as BRST operators straight away. In fact what we can do is to make an additional modification to the Lorentz generator of the 2-d space by making linear combinations of it and the R-symmetry generators in such a way that the $Q_{A/B}$ are scalars under the new generators. This procedure is called twisting and produces two different topological field theories labelled by A and B.

Now that we have a BRST operator, we can use the usual definitions for the physical states of our theory in terms of BRST cohomology (see for example Chapter 16 of [2] or Chapter 15 of [4] for an introduction). Physical states $|\psi\rangle$ are given by the condition $Q_{A/B}|\psi\rangle = 0$ with states being equivalent if they differ by something which is BRST exact such as $Q_{A/B}|\chi\rangle$ for some $|\chi\rangle$. Similarly, physical operators are taken to be those which commute with the BRST operator modulo those which can be written as an anti-commutator of $Q_{A/B}$ with some other operator. In particular one can show that that the stress-tensors of the twisted theories are BRST exact as they can be written in the form $T_{A/B} = \{Q_{A/B}, \Lambda\}$ for some Λ. This is a general property of topological field theories.

1.6.2 Topological string theory

What we have so far constructed are two 2-dimensional topological field theories. However, we can promote these to string theories by considering the theories to be living on the worldsheet of a string and ensuring that we integrate over all metrics of the 2-dimensional space in the path integral as well as the other fields appearing in the action (see e.g. Chapter 3 of [19] for how this works in the usual string theory settings). The Euclidean path integral

$$Z_E = \int Dh(\sigma) D\phi(\sigma) e^{-S_{2d}[h;\phi]}, \tag{1.6.4}$$

where $h_{\alpha\beta}$ is the world-sheet metric, ϕ are the fields of our 2-d field theory and σ are the coordinates of the 2-d space then defines our topological string theory. If we have re-defined our Lorentz generators to make Q_A a scalar then the string theory is known as the A-model, while if we choose to make Q_B a scalar we arrive at the B-model [185, 186].

We can also say something about the target spaces of these topological string theories. In 'normal' string theory settings these target spaces – the spaces in which the strings live – are known to be 10-dimensional (or 26-dimensional for the purely bosonic string) in order for them to be quantum-mechanically anomaly-free. The $\mathcal{N} = (2,2)$ field theories discussed above, however, naturally give rise to target spaces which are special types of complex manifolds known as Kähler manifolds – even before we perform the topological twisting. These spaces are complex manifolds that are endowed with a Hermitian metric (i.e. a real metric – real in the sense that $g_{\bar\imath\bar\jmath} = (g_{ij})^*$ and $g_{\bar\imath j} = (g_{i\bar\jmath})^*$)

– with $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$) and which we can write locally as the second derivative of some function termed the Kähler potential $K(z;\bar{z})$:

$$g_{i\bar{\jmath}} = \frac{\partial^2 K(z;\bar{z})}{\partial z^i \partial \bar{z}^{\bar{\jmath}}} \, . \tag{1.6.5}$$

Here z^i and $\bar{z}^{\bar{\jmath}}$ are appropriate complex coordinates on the target space. When we do the twisting described by (1.6.2) it turns out that the A-model twist can be performed for any Kähler target space, while the B-model twist requires the space to be of a yet more specialised form known as a Calabi-Yau manifold.

There are many different ways to define a Calabi-Yau manifold, but one way that is good for our purposes is that it is a Kähler manifold that is also Ricci-flat, $R_{i\bar{\jmath}} = 0$. The moduli (essentially the parameters) describing the variety of such spaces are of two types which are termed the Kähler moduli and the complex-structure moduli. It can be shown that the space of Kähler moduli is locally $H^{(1,1)}(M_{CY})$ – that is to say it is locally given by the Dolbeault cohomology class of $(1,1)$-forms – while the complex-structure moduli space is locally the cohomology class of $(2,1)$-forms $H^{(2,1)}(M_{CY})$. Because Calabi-Yau manifolds are automatically Kähler manifolds to begin with and because of their high degree of symmetry, the A-model is often also considered on a Calabi-Yau. Finally, it can be shown that the central charge of the Virasoro algebra of the A- and B-models vanishes identically in any number of dimensions [187], so topological strings are well-defined in target spaces of any dimension. For more comprehensive discussions of complex, Kähler and Calabi-Yau manifolds see e.g. [187, 188, 189, 190, 191].

As for the physical operators in these models, we briefly state without proof that in the A-model, Q_A can be viewed as being $Q_A \simeq d$ – the de Rham exterior derivative – and the local physical operators are in one-to-one correspondence with de Rham cohomology elements on the target space:

$$O_A \leftrightarrow A_{i_1 \ldots i_p \bar{\jmath}_1 \ldots \bar{\jmath}_q}(\phi)\, d\phi^{i_1} \wedge \ldots \wedge d\phi^{i_p} \wedge d\phi^{\bar{\jmath}_1} \wedge \ldots \wedge d\phi^{\bar{\jmath}_q} \, . \tag{1.6.6}$$

For the B-model on the other hand one can show that $Q_B \simeq \bar{\partial}$ – the Dolbeault exterior derivative – and the local physical operators are now just $(0,p)$-forms with values in the antisymmetrized product of q holomorphic tangent spaces – which we denote by $\bigwedge^q T^{(1,0)}(M_{CY})$:

$$O_B \leftrightarrow B_{\bar{\imath}_1 \ldots \bar{\imath}_p}{}^{j_1 \ldots j_q}(\phi)\, d\phi^{\bar{\imath}_1} \wedge \ldots \wedge d\phi^{\bar{\imath}_p} \frac{\partial}{\partial \phi^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial \phi^{j_q}} \, . \tag{1.6.7}$$

These theories also have the intruiging property of mirror symmetry [192, 193] – see e.g. [194] and references therein for a comprehensive review – that the A-model on one Calabi-Yau is equivalent to the B-model on a different Calabi-Yau which is known as its Mirror. In the mirror map, the hodge numbers $h^{1,1}$ and $h^{2,1}$ are swapped which pertains to the exchange of Kähler and complex-structure moduli. This is especially useful as the B-model is generally easier to compute-with than the A-model, while the A-model is more physically interesting in many scenarios. Hard computations in the A-model can often be mapped to easier ones in the B-model.

1.6.3 The B-model on super-twistor space

In his original construction [31], Witten considered the B-model and we will do the same here. The target space on which we will want it to live will be $\mathbb{CP}^{3|4}$, which is a Calabi-Yau super-manifold (with bosonic and fermionic degrees of freedom) rather than a bosonic manifold as is more common. This is fortunate because $\mathbb{CP}^3$ is not Calabi-Yau, while $\mathbb{CP}^{3|4}$ is.[13] In addition, if we recall that the closed-string sector is where gravity states arise, we would like to consider the open-string B-model on twistor space in order that we may end up with degrees of freedom with spin 1 or less. In the simplest case this consists of adding N bosonic-space-filling D5-branes (thus spanning all 6 bosonic directions of $\mathbb{CP}^{3|4}$ in analogy with the purely bosonic case of [195]. In addition (as Witten did), we take the D5s to wrap the fermionic directions ψ^I and $\bar{\psi}^J$ in such a way that we can set $\bar{\psi}^J$ to zero. It is not entirely clear how this should be interpreted, but one might say that the branes wrap the ψ directions while being localised in the $\bar{\psi}$ directions. The presence of N branes gives rise to a $U(N)$ gauge symmetry as usual due to the Chan-Paton factors of the open strings ending on them.

So far we have been considering things from a worldsheet perspective. However, for open strings we also have the spacetime perspective of open-string field theory [196]. This has a multiplication law $\star$, an operator Q obeying $Q^2 = 0$ and Lagrangian

$$\mathscr{L} = \frac{1}{2}\int \left(\mathscr{A} \star Q\mathscr{A} + \frac{2}{3}\mathscr{A} \star \mathscr{A} \star \mathscr{A}\right) , \tag{1.6.8}$$

where $\mathscr{A}$ is the string field. In the presence of D5-branes on a 6-dimensional (bosonic) manifold this has been shown to reduce to holomorphic Chern-Simons theory [195], where the D5-D5 modes of the string field $\mathscr{A}$ give a $(0,1)$-form gauge field $A = A_{\bar{\imath}}(z,\bar{z})d\bar{z}^{\bar{\imath}}$ on the branes. On the other hand, when the target space is the super-manifold $\mathbb{CP}^{3|4}$, $\mathscr{A}$ reduces to the $(0,1)$-form gauge superfield $A = A_{\bar{I}}(Z,\bar{Z};\psi,\bar{\psi})d\bar{Z}^{\bar{I}}$, while Q becomes the $\bar{\partial}$ operator and $\star$ the usual wedge product operation $\wedge$. The action descends to

$$S = \frac{1}{2}\int_{\mathbb{CP}^{3|4}} \Omega \wedge \mathrm{tr}\left(A \wedge \bar{\partial}A + \frac{2}{3}A \wedge A \wedge A\right) , \tag{1.6.9}$$

[13] In fact $\mathbb{CP}^{3|N}$ is Calabi-Yau iff $N = 4$.

and with $\bar{\psi} = 0$ the superfield $\mathcal{A}$ can be expanded as

$$\mathcal{A}(Z,\bar{Z},\psi) = A + \psi^I \lambda_I + \frac{1}{2}\psi^I\psi^J\phi_{IJ} + \frac{1}{3!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\tilde{\lambda}^L + \frac{1}{4!}\epsilon_{IJKL}\psi^I\psi^J\psi^K\psi^L G, \tag{1.6.10}$$

where $A, \lambda_I, \phi_{IJ}, \tilde{\lambda}^I, G$ are all functions of Z and $\bar{Z}$ and we have suppressed the $(0,1)$-form structure. Ω in (1.6.9) is a $(3,0)$-form and is the holomorphic volume-form on $CP^{3|4}$

$$\Omega = \frac{1}{4!^2}\epsilon_{IJKL}\epsilon_{MNPQ}Z^I dZ^J dZ^K dZ^L d\psi^M d\psi^N d\psi^P d\psi^Q . \tag{1.6.11}$$

Because dZ^I and $d\psi^I$ scale oppositely – as follows from (1.5.17) and the fermionic nature of ψ^I ($d\psi^I \to c^{-1}d\psi^I$ under (1.5.17)) – it is clear that (1.6.11) is invariant under this scaling and thus the action (1.6.9) is only invariant if $\mathcal{A}$ is of degree zero, $\mathcal{A} \in H^{(0,1)}(CP^{3|4}, \mathcal{O}(0))$. This means that each component field in the ψ expansion (1.6.10) must be of degree $2h-2$ and thus describes a field of helicity h in spacetime – c.f. the twistor description of wavefunctions for particles of helicity h of Eq. (1.5.20) and surrounding paragraphs. In addition, the fermionic nature of the ψ^I restricts the number of degrees of freedom of the component fields and it can quickly be seen that (1.6.10) describes the $\mathcal{N}=4$ multiplet,[14] which in the notation of §1.3 can be written as $(A_-, \lambda_-, \phi, \lambda_+, A_+) \leftrightarrow (G, \tilde{\lambda}^I, \phi_{IJ}, \lambda_I, A)$, while the action in component form can be written as

$$\begin{aligned} S = \int_{CP^3}\Omega' \wedge \mathrm{tr}\Big(& G\wedge(\bar{\partial}A + A\wedge A) - \tilde{\lambda}^I\wedge(\bar{\partial}\lambda_I + [A,\lambda_I]) \\ & + \frac{1}{4}\epsilon^{IJKL}\phi_{IJ}\wedge(\bar{\partial}\phi_{KL} + A\wedge\phi_{KL}) - \frac{1}{2}\epsilon^{IJKL}\lambda_I\wedge\lambda_J\wedge\phi_{KL}\Big), \end{aligned} \tag{1.6.12}$$

where $\Omega' = \epsilon_{IJKL}Z^I dZ^J dZ^K dZ^L/4!$ is the bosonic reduction of Ω obtained after integrating out the ψ^I[15] and $[A,\lambda_I] = A\wedge\lambda_I + \lambda_I\wedge A$. The equations of motion following from (1.6.9) are $\bar{\partial}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$ and the gauge invariance is $\delta\mathcal{A} = \bar{\partial}\omega + [\mathcal{A},\omega]$.

What we have arrived at is 'half' of $\mathcal{N}=4$ super-Yang-Mills. We have all the fields as is apparent from (1.6.10), but it turns out that not all the interactions are present. One of the easiest ways to see this is to note that the symmetries of the B-model generally leave Ω invariant [31]. However there are also interesting transformations of the target space that leave the complex structure invariant but transform Ω non-trivially. One such transformation is a $U(1)_R$ part of the R-symmetry group $U(4)_R = SU(4)_R \times U(1)_R$ that acts as[16]

$$S: \; Z^I \to Z^I, \quad \psi^I \to e^{i\alpha}\psi^I \tag{1.6.13}$$

[14] To be more precise it is the twistor transform of the $\mathcal{N}=4$ multiplet [32, 197].

[15] Recall that for Grassmann variables, $\int d\psi \equiv \partial/\partial\psi$ with $\int d\psi^I\psi^J = \delta^{IJ}$ and $\delta(\psi) = \psi$.

[16] Note that the I indices on the component fields in (1.6.12) are fundamental indices of this $SU(4)_R$.

with $d\eta^I \to e^{-i\alpha} d\eta^I$ because of their fermionic nature. $\Omega \to e^{4i\alpha}\Omega$ thus has $S = -4$ and hence so does the action (1.6.9) as the transformation of the η^I inside $\mathcal{A}$ are compensated by equal and opposite transformations of the component fields: A has $S = 0$, ψ_I has $S = -1$, ϕ_{IJ} has $S = -2$, $\tilde{\chi}^I$ has $S = -3$ and G has $S = -4$. In fact the component action (1.6.12) is made up entirely of terms with $S = -4$. However, the usual $\mathcal{N} = 4$ Yang-Mills action in component form consists of terms which have $S = -4$ and $S = -8$. For example the scalar kinetic terms $(\partial\phi)^2$ have $S = -4$ while the scalar potential ϕ^4 has $S = -8$. The holomorphic Chern-Simons action (1.6.9) thus captures all the fields of maximally supersymmetric Yang-Mills, but not all the interactions. Although we will not discuss it here, the theory described by (1.6.9) is in fact self-dual $\mathcal{N} = 4$ super-Yang-Mills [198] – that is, (super)-Yang-Mills theory for a gauge field A^0 whose field strength appearing in the action is self-dual. A^0 is the spacetime field corresponding to the homogeneity 0 field (A) in (1.6.10) and the spacetime action of this theory is $S = \int G^0 \wedge * F^0 \equiv \int G^0 \wedge F^0_{SD}$. Here G^0 is a self-dual 2-form whose twistor transform is the homogeneity -4 field (G) in (1.6.10), F^0_{SD} is the self-dual part[17] of $F^0 = dA^0 + A^0 \wedge A^0$ and $*$ is the Hodge duality operation.

1.6.4 D1-brane instantons

Witten's solution to the aforementioned problem of the absence of the entire set of interactions was to enrich the B-model on $\mathbb{CP}^{3|4}$ with instantons. The ones in question are Euclidean D1-branes which wrap holomorphic curves in super-twistor space and on which the open strings can end. These holomorphic curves are precisely the ones that we met earlier on which the scattering amplitudes were found to localise. We won't go into much detail here (more can be found in [31]), but the basic idea is that these instantons have S-charge $-4(d + 1 - g)$ for the connected degree d and genus g case. Thus for the 'classical' tree-level MHV case[18] these instantons provide the terms with $S = -8$ as we had hoped.

We can now consider other types of strings apart from D5-D5s. We also have D1-D1s, D1-D5s and D5-D1s. The D1-D1 strings give rise to a $U(1)$ gauge field on the world-volume of an instanton which describes the motion of the instanton in $\mathbb{T}$. We will thus ignore the D1-D1 strings from now on. Of course we do want to involve the D1-instantons, so we'll focus on the D1-D5 and D5-D1 strings. Witten argued that these strings give rise to fermionic (0;0)-form fields living on the world-volume of the instanton. The D1-D5 modes give rise to a fermion α^i and the D5-D1 modes give a

[17] Note that formally we can write the self-dual and anti-self-dual parts of F^0 as $F^0_{SD} = (F^0 + *F^0)/2$ and $F^0_{ASD} = (F^0 - *F^0)/2$. Here we have taken $**F^0 = F^0$.

[18] We refer to the tree-level MHV amplitudes as being the 'classical' case as it turns out that we can re-formulate perturbation theory in terms of 'MHV-vertices' – see §1.7 – and they are thus appropriate for consideration of S-charge violation at the level of the action.

fermion $_\imath$, with $\imath$ and i (anti)-fundamental $U(N)$ indices respectively. The effective action for the low-energy modes is then

$$S_{eff} = \int_{D1} dz \quad (\partial_{\bar z} + A_{\bar z} d\bar z) \quad ; \tag{1.6.14}$$

where z and $\bar z$ are local complex coordinates on the $D1$ and $A_{\bar z}$ (which is a background field on the $D1$) is the component of the superfield generated by the $D5$-$D5$ strings lying along the $D1$. The first term is the kinetic term of these modes (with $\partial_{\bar z}$ the $\bar\partial$ operator restricted to the $D1$), while the second describes their interaction with the gauge field A and can be written as

$$S_{int} = \int_{D1} J \wedge (A_{\bar z} d\bar z) \; ; \tag{1.6.15}$$

where we define J to be the current $J_\imath{}^{j} = \quad _\imath \, {}^{j} dz$.

Any particular external state will contribute just one component of this superfield A and therefore its coupling will be

$$V_s = \int_{D1} J_s \wedge \quad _s \; ; \tag{1.6.16}$$

where $_s$ is the wavefunction of the state in twistor space and thus a $(0;1)$-form there.[19] Then if the curve which the $D1$ wraps were to have no moduli (i.e. there were only one possibility for it), one would be able to compute scattering amplitudes by evaluating the correlator $\langle V_{s_1} \ldots V_{s_n} \rangle$. However, we know from the discussion in §1.5.1 that these curves do have moduli and thus we should integrate this correlator over their moduli space. Our prescription for computing n-point scattering amplitudes whose external particles have wave functions $_{s_i}$ will then be

$$A_n = \int dM_d \, \langle V_{s_1} \ldots V_{s_n} \rangle \; ; \tag{1.6.17}$$

where dM_d is an appropriate measure on the moduli space of holomorphic curves of degree d (and genus zero for our current purposes).

1.6.5 The MHV amplitudes

As an example of how (1.6.17) is implemented let us calculate the MHV amplitudes using this prescription. From §1.5.1 we saw that the MHV amplitudes lie on holomorphic

[19] We use subscripts s_i etc. to denote the ith particle for the rest of this section in order to avoid confusion with the gauge indices.

curves that are embedded in $CP^{3|4}$ via the equations

$$
\begin{aligned}
\mu^{s_k}_{\dot\alpha} + x_{\alpha\dot\alpha}\lambda^{\alpha}_{s_k} &= 0 \\
\psi^{A}_{s_k} + \theta^{A}_{\alpha}\lambda^{\alpha}_{s_k} &= 0\,.
\end{aligned}
\tag{1.6.18}
$$

λ_{s_k} are the homogeneous coordinates on the curves (with $s_k = 1\ldots n$ denoting the k^{th} particle) and their moduli are $x_{\alpha\dot\alpha}$ and θ^{A}_{α}. These are thus the curves that we will take the D1-instantons to be wrapping. $x_{\alpha\dot\alpha}$ has 4 (bosonic) degrees of freedom while θ^{A}_{α} has 8 (fermionic) ones and a natural measure on the moduli space is then $dM_1 = d^4x\,d^8\theta$.

For clarity let us specialise to the case of 4-particle (gluon) scattering where particles 1 and 3 have negative-helicity and particles 2 and 4 positive-helicity. The n-particle case is an easy generalisation of this. Formally we have

$$
\begin{aligned}
A_4 &= \int dM_1 \langle V_{s_1} V_{s_2} V_{s_3} V_{s_4} \rangle \\
&= \int d^4x\,d^8\theta \int_{CP^{1|0}} J_{s_1} \wedge \phi_{s_1} \ldots \int_{CP^{1|0}} J_{s_4} \wedge \phi_{s_4}\;;
\end{aligned}
\tag{1.6.19}
$$

where we assume that the wavefunctions ϕ_{s_k} take values in the Lie-algebra of $U(N)$ and thus contain a generator T^{a_k} in addition to (1.5.20). $(J_{s_k})_{\imath}^{\;j} = \beta_{\imath}(z_k)\gamma^{j}(z_k)dz_k$ then gives

$$
A_4 = \int d^4x\,d^8\theta \int dz_1 \ldots dz_4\, \beta_{\imath_1}(z_1)\gamma^{j_1}(z_1)\phi_{s_1} \ldots \beta_{\imath_4}(z_4)\gamma^{j_4}(z_4)\phi_{s_4}
\tag{1.6.20}
$$

up to a factor. Separating-out the Lie-algebra generators from the rest of the wavefunctions ($\phi_{s_k} = \phi^{0}_{s_k} T^{a_k}$) we can re-write the correlator as

$$
\int d^4x\,d^8\theta \int dz_1 \ldots dz_4\, \phi^{0}_{s_1} \ldots \phi^{0}_{s_4} \left\langle \beta_{\imath_1}(T^{a_1})^{i_1}_{\;\jmath_1}\gamma^{j_1} \ldots \beta_{\imath_4}(T^{a_4})^{i_4}_{\;\jmath_4}\gamma^{j_4} \right\rangle\,.
\tag{1.6.21}
$$

This correlator has many different contributions (105 in total) coming from the possible ways of Wick contracting the fermions β and γ. Let us consider the cyclic one where we contract $\gamma(z_1)$ with $\beta(z_2)$, $\gamma(z_2)$ with $\beta(z_3)$ and so on (with $\gamma(z_4)$ contracted with $\beta(z_1)$). Because β and γ are fermions living on (in this case) CP^1, their propagator is the usual one for free fermions on the complex plane

$$
\langle \gamma^{j}(z_k)\beta_{\imath}(z_l) \rangle = \frac{\delta^{j}_{\imath}}{z_k - z_l}
\tag{1.6.22}
$$

and the relevant Wick contraction is

$$\begin{aligned} W_{\text{cyclic}} &= (T^{a_1})^{i_1}{}_{l_1}\cdots(T^{a_4})^{i_4}{}_{l_4}\langle\psi^{j_1}(z_1)\bar\psi_{l_2}(z_2)\rangle\cdots\langle\psi^{j_4}(z_4)\bar\psi_{l_1}(z_1)\rangle \\ &= (T^{a_1})^{i_1}{}_{l_1}\cdots(T^{a_4})^{i_4}{}_{l_4}\frac{\delta^{j_1}_{l_2}}{z_1-z_2}\cdots\frac{\delta^{j_4}_{l_1}}{z_4-z_1} \\ &= \frac{\mathrm{tr}(T^{a_1}\cdots T^{a_4})}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_1)}\,. \end{aligned} \tag{1.6.23}$$

Dropping the single-trace colour factor for now, (1.6.21) is

$$\begin{aligned} A_4 &= \int d^4x d^8\theta \int dz_1\cdots dz_4 \frac{\phi'_{s_1}\cdots\phi'_{s_4}}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_1)} \\ &= \int d^4x d^8\theta \int \langle\sigma_{s_1} d\sigma_{s_1}\rangle\cdots\langle\sigma_{s_4} d\sigma_{s_4}\rangle\frac{\phi'_{s_1}\cdots\phi'_{s_4}}{\langle\sigma_{s_1}\sigma_{s_2}\rangle\cdots\langle\sigma_{s_4}\sigma_{s_1}\rangle}\,; \end{aligned} \tag{1.6.24}$$

where we have changed to homogeneous coordinates σ_{s_k} on the CP^1s by setting $z_k = \sigma^2_{s_k}/\sigma^1_{s_k}$ with 1 and 2 indicating spinor 'α' indices here.

Now we must introduce the explicit form for the wavefunctions and integrate over the σ_k. For this it is useful to note that with $z = \sigma^2/\sigma^1$ and making the more specific choices of $\lambda = (1;z)$ and $\pi = (1;b)$, (A.2.9) becomes (A.2.10):

$$\int \langle\lambda\, d\lambda\rangle\,\delta(\langle\lambda\,\pi\rangle)F(\lambda) = iF(\pi)\,: \tag{1.6.25}$$

Omitting the integral over moduli, (1.6.24) thus gives

$$\begin{aligned} A_4 &= \int_{CP^{1|0}} \langle\sigma_{s_1} d\sigma_{s_1}\rangle\,\delta(\langle\sigma_{s_1}\pi_{s_1}\rangle)\left(\frac{\lambda_{s_1}}{\sigma_{s_1}}\right)^{2h_1-1} e^{i[\tilde\pi_{s_1}\mu_{s_1}](\lambda_{s_1}=\sigma_{s_1})}g_{h_1}(\psi_{s_1}) \\ &\quad\vdots \\ &\quad\int_{CP^{1|0}} \langle\sigma_{s_4} d\sigma_{s_4}\rangle\,\delta(\langle\sigma_{s_4}\pi_{s_4}\rangle)\left(\frac{\lambda_{s_4}}{\sigma_{s_4}}\right)^{2h_4-1} e^{i[\tilde\pi_{s_4}\mu_{s_4}](\lambda_{s_4}=\sigma_{s_4})}g_{h_4}(\psi_{s_4})H(\sigma_{s_i}) \\ &= {}^{1}_{s_1}{}^{2}_{s_1}{}^{3}_{s_1}{}^{4}_{s_1}{}^{1}_{s_3}{}^{2}_{s_3}{}^{3}_{s_3}{}^{4}_{s_3}H(\sigma_{s_i})e^{i\sum_{s_k=1}^4[\tilde\pi_{s_k}\mu_{s_k}]}\,; \end{aligned} \tag{1.6.26}$$

where H is the denominator in (1.6.24).

We must now perform the integral over the moduli. For this purpose we can recall the equations describing the embedding (1.6.18) and substitute $\mu_{s_k\dot\alpha} = x_{\alpha\dot\alpha}\lambda^\alpha_{s_k}$ and $\psi^A_{s_k} = \theta^A_\alpha\lambda^\alpha_{s_k}$ [20] whereupon the integral over x gives

$$\int d^4x \exp\left(i x_{\alpha\dot\alpha}\sum_{s_k=1}^4 \lambda^\alpha_{s_k}\tilde\pi^{\dot\alpha}_{s_k}\right) = \delta^{(4)}\left(\sum_{s_k=1}^4 \lambda_{s_k}\tilde\pi_{s_k}\right)\,; \tag{1.6.27}$$

[20] Recall that the delta functions of (1.6.26) have set $\lambda_{s_k} = \pi_{s_k}$.

which is just the delta function of momentum conservation. For the fermionic moduli we have (for example):

$$\begin{aligned}
\eta^1_{s_1}\eta^1_{s_3} &= (\theta^1_1\pi^1_{s_1} + \theta^1_2\pi^2_{s_1})(\theta^1_1\pi^1_{s_3} + \theta^1_2\pi^2_{s_3}) \\
&= \theta^1_2\theta^1_1\pi^2_{s_1}\pi^1_{s_3} + \theta^1_1\theta^1_2\pi^1_{s_1}\pi^2_{s_3} \\
&= \theta^1_1\theta^1_2(\pi^1_{s_1}\pi^2_{s_3} - \pi^2_{s_1}\pi^1_{s_3}) \\
&= \theta^1_1\theta^1_2\langle\pi_{s_1}\pi_{s_3}\rangle\,. \qquad (1.6.28)
\end{aligned}$$

After dealing with all the θs in a similar way and then integrating over the eight θ variables gives $\langle\lambda_{s_1}\lambda_{s_3}\rangle^4$. Putting all the pieces together we get

$$\begin{aligned}
A_4(1^-,2^+,3^-,4^+) &= \mathrm{tr}(T^{a_1}\ldots T^{a_4})\frac{\langle\lambda_{s_1}\lambda_{s_3}\rangle^4}{\langle\lambda_{s_1}\lambda_{s_2}\rangle\ldots\langle\lambda_{s_4}\lambda_{s_1}\rangle}\delta^{(4)}\left(\sum_{s_k=1}^{4}\lambda_{s_k}\tilde\lambda_{s_k}\right) \\
&= \mathrm{tr}(T^{a_1}\ldots T^{a_4})\frac{\langle 13\rangle^4}{\langle 12\rangle\ldots\langle 41\rangle}\delta^{(4)}\left(\sum_{i=1}^{4}\lambda_i\tilde\lambda_i\right), \qquad (1.6.29)
\end{aligned}$$

which is precisely the formula for the MHV amplitudes that we wrote down before, though we have kept the colour structure explicit here.

We should be careful to note that we have simply picked the particular Wick contraction that we needed in order to get a cyclic colour ordering. All the terms with non-cyclic colour orderings but a single trace are also present as well as multi-trace terms which in [31,36] were suggested to be a sign of the presence of closed-string (and thus gravitational) states.

We have explicitly described the construction of the MHV amplitudes from the B-model in twistor space. Other amplitudes can be calculated in this way too, though the complexity is greater so we will not go into any detail on this. The NMHV amplitudes for example require one to integrate over the moduli space of degree 2 curves in $\mathbb{T}$ and some simple cases were calculated this way in [134]. Other cases such as the n-point googly MHV amplitudes (with 2 positive-helicity gluons and $n-2$ of negative helicity) were worked out in [135] and all 6-point amplitudes in [136]. For these integrals over curves of degree $d > 1$, one encounters the possibility of describing these as connected curves of degree d, or disconnected curves of degree d_i with $\sum d_i = d$. In [134,135,136] it was found that the connected prescription alone reproduces the entire amplitudes in the cases considered (at least up to a factor). However, there is also strong evidence that the same amplitudes can be computed using the purely disconnected prescription [33]. Indeed, this disconnected prescription led directly to the proposal of new rules for doing perturbative gauge theory which we will describe in the next section. The authors of [199] argued that the integrals involved in the connected prescription localised on the subspace where a connected curve of degree d degenerates to the intersection of curves of degree d_i with $\sum d_i = d$ and thus provided strong evidence that there are ultimately d different prescriptions which are all equivalent. The extreme possibilities are that we have just one degree d curve to consider, or alternatively d degree one curves. This latter case was the inspiration for [33].

We have also not said anything about loop diagrams here except for the formal statement that they localise on curves of degree $d = q - 1 + l$ with $g \leq l$. The structure of many loop diagrams of $\mathcal{N} = 4$ super-Yang-Mills was elucidated in [31, 72, 73, 180, 181, 183], though the situation with their calculation from the B-model is far less clear than that for trees unfortunately. In [36] it was shown that closed string modes give rise to states of $\mathcal{N} = 4$ conformal supergravity describing deformations of the target twistor space as well as the expected $\mathcal{N} = 4$ Yang-Mills states. Conformal supergravity[21] in 4 dimensions has a Lagrangian which is the Weyl tensor of gravity squared, $S = \int d^4x \sqrt{\det|g|}\, |W|^2$, and is usually considered to be a somewhat unsavoury theory as it gives rise to fourth order differential equations which are generally held to lead to a lack of unitarity (see e.g. [201]). One might still hope to decouple these states, but because the coupling constant is the same in both sectors the amplitudes mix and one ends up with a theory of $\mathcal{N} = 4$ conformal supergravity coupled to $\mathcal{N} = 4$ super-Yang-Mills, some amplitudes of which were computed in [36] at tree-level and more recently in [114] at loop-level (see also [202]). Despite all this, it was shown by Brandhuber, Spence and Travaglini that the proposals of [33] can be extended to loop-level and provide a new perturbative expansion for field theory which is valid in the quantum regime as well as the classical one. This discovery will be a central theme in the following chapters of this thesis.

As a final remark in this section we point out that twistor string theories have also been constructed to describe other theories with less supersymmetry and/or product gauge groups [119, 120, 121, 122, 124] as well as more recently to describe Einstein supergravity [39]. Indeed the proposals in [39] include a twistor description of $\mathcal{N} = 4$ SYM coupled to Einstein supergravity which may lead to a resolution of the problem of loops if they can be consistently decoupled.

1.7 CSW rules (tree-level)

Motivated by the findings we have so far discussed, Cachazo, Svrcek and Witten proposed a set of alternative graphs for tree-level amplitudes in Yang-Mills theory based on the MHV vertices [33]. The essential idea is the observation that one can seemingly compute tree-level amplitudes from the totally disconnected prescription alluded to above by gluing d disconnected lines together (on each of which there is an MHV amplitude localised) for an amplitude involving $d + 1$ negative-helicity gluons. The gluing

[21] For a review see e.g. [200].

procedure is made concrete by connecting the lines with twistor space propagators. In field theory terms this corresponds to the use of MHV amplitudes as the fundamental building blocks – because their localisation properties in twistor space translates to a point-like interaction in Minkowski space – and gluing these together with simple scalar propagators $1/P^2$. The two ends of any propagator must have opposite helicity labels because an incoming gluon of one helicity is equivalent to an outgoing gluon of the opposite helicity.

1.7.1 Off-shell continuation

In order to glue MHV vertices together we must continue them off-shell since one or more of the legs must be connected to the off-shell propagator $1/P^2$. For this purpose, consider a generic off-shell momentum vector, L. On general grounds, it can be decomposed as [65, 179]

$$L = l + z\,\eta\ ; \tag{1.7.1}$$

where $l^2 = 0$, and η is a fixed and arbitrary null vector, $\eta^2 = 0$; z is a real number. Equation (1.7.1) determines z as a function of L:

$$z = \frac{L^2}{2(L\cdot\eta)}\ . \tag{1.7.2}$$

Using spinor notation, we can write l and η as $l_{\alpha\dot\alpha} = l_\alpha \tilde{l}_{\dot\alpha}$, $\eta_{\alpha\dot\alpha} = \eta_\alpha \tilde{\eta}_{\dot\alpha}$. It then follows that

$$l_\alpha = \frac{L_{\alpha\dot\alpha}\tilde{\eta}^{\dot\alpha}}{[\tilde{l}\,\tilde{\eta}]}\ ; \tag{1.7.3}$$

$$\tilde{l}_{\dot\alpha} = \frac{L_{\alpha\dot\alpha}\eta^{\alpha}}{\langle l\,\eta\rangle}\ . \tag{1.7.4}$$

We notice that (1.7.3) and (1.7.4) coincide with the CSW prescription proposed in [33] to determine the spinor variables l and $\tilde{l}$ associated with the non-null, off-shell four-vector L defined in (1.7.1). The denominators on the right hand sides of (1.7.3) and (1.7.4) turn out to be irrelevant for our applications since the expressions we will be dealing with are homogeneous in the spinor variables l; hence we will usually discard them. This defines our off-shell continuation.

1.7.2 The procedure: An example

The CSW rules for joining these MHV amplitudes together are probably best illustrated with an example. It is clear that a tree diagram with v MHV vertices has $2v$ negative-helicity legs, $v-1$ of which are connected together with propagators. As mentioned above, each propagator must subsume precisely one negative-helicity leg and thus we are left with $v+1$ external negative helicities. To put it another way, if we wish to compute a scattering amplitude with q negative-helicity gluons we will need $v = q \quad 1$ MHV vertices. As such let us consider the simplest case of the 4-point NMHV amplitude $A_4(1^+; 2 \; ; 3 \; ; 4 \;)$. We know from our discussions in x1.4.2 that this must vanish and we would thus like our calculations here to support that. Even though we end up computing something trivial, it is a good illustration of the procedure to follow.

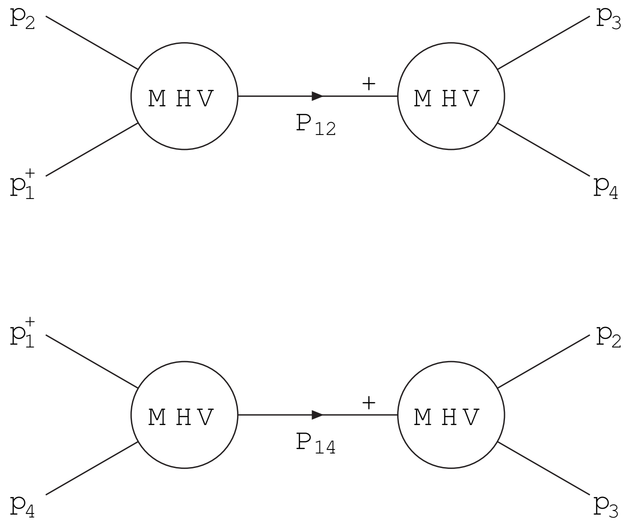


Figure 1.5: The two MHV diagrams contributing to the + amplitude. All external momenta are taken to be outgoing.

As shown in Figure 1.5, there are two diagrams to consider. For each diagram we should write down the MHV amplitudes corresponding to each vertex and join them together with the relevant scalar propagator, remembering to use the o -shell continuation of (1.7.3) and (1.7.4) to deal with the spinors associated with the internal particles. The rst (uppermost) diagram gives

$$C_1 = \frac{\langle \;_2 \; _{P_{12}}\rangle^3}{\langle \;_{P_{12}} \; _1\rangle\langle \;_1 \; _2\rangle}\frac{1}{P_{12}^2}\frac{\langle \;_3 \; _4\rangle^3}{\langle \;_4 \; _{P_{12}}\rangle\langle \;_{P_{12}} \; _3\rangle}\,; \tag{1.7.5}$$

where the momentum of the propagator is $P_{12} = \;(p_1 + p_2) = \;(p_3 + p_4)$. As the external momenta are massless, $P_{12}^2 = 2(p_1 \quad p_2) = \langle 12\rangle[21]$ and the o -shell continuation tells us that

$$\begin{aligned} {}_{P_{12}} &= \frac{P_{12}\tilde{\;}}{[\tilde{\;}_{P_{12}}\tilde{\;}]} \\ &= \frac{(\;_1\tilde{\;}_1 + \;_2\tilde{\;}_2)\tilde{\;}}{[\tilde{\;}_{P_{12}}\tilde{\;}]} \\ &= \;_1\frac{a_1}{b} \quad \;_2\frac{a_2}{b}\,; \end{aligned} \tag{1.7.6}$$

where we have written $\langle i\,\eta\rangle = a_i$ and $[\tilde\lambda_{P_{12}}\,\tilde\eta] = b$ for clarity and have kept the denominators of the off-shell continuation explicit in order to demonstrate that they will drop out of the expressions. Similarly, an appropriate form for eliminating $\lambda_{P_{12}}$ from the MHV vertex on the right is

$$\lambda_{P_{12}} = \lambda_3\frac{a_3}{b} + \lambda_4\frac{a_4}{b} : \qquad (1.7.7)$$

On substituting for $\lambda_{P_{12}}$ and P_{12}^2 (1.7.5) then becomes

$$\begin{aligned} C_1 &= \frac{ba_1^3}{a_2b^3}\frac{\langle 2\,1\rangle^3}{\langle 2\,1\rangle\langle 1\,2\rangle}\frac{1}{\langle 1\,2\rangle[2\,1]}\frac{\langle 3\,4\rangle^3}{\langle 4\,3\rangle\langle 4\,3\rangle}\frac{b^2}{a_3a_4} \\ &= -\frac{a_1^3}{a_2a_3a_4}\frac{\langle 3\,4\rangle}{[1\,2]} : \end{aligned} \qquad (1.7.8)$$

Going through the same procedure for the second contribution in Figure 1.5 gives

$$C_2 = \frac{a_1^3}{a_2a_3a_4}\frac{\langle 2\,3\rangle}{[4\,1]} ; \qquad (1.7.9)$$

and the final answer is $A_4 = C_1 + C_2$. Momentum conservation is $\sum_{i=1}^{4}\lambda_i\tilde\lambda_i = 0$, which can be applied to an expression of the form $\langle 3\,i\rangle[i\,1]$ to give $\sum_{i=1}^{4}\langle 3\,i\rangle[i\,1] = \langle 3\,2\rangle[2\,1] + \langle 3\,4\rangle[4\,1] = 0$ and means that $\langle 3\,4\rangle/[1\,2] = -\langle 2\,3\rangle/[4\,1]$. Thus $C_1 = -C_2$ and we get $A_4 = 0$ as expected.

There are two essential points to note here. The first is that when we performed the off-shell continuation all the denominators of (1.7.3) cancelled out. This is in fact generally true for the amplitudes we will be interested in and thus we will discard them from now on. The second point is that in C_1 and C_2, the arbitrary null momentum η of the off-shell continuation was still present, lurking as an $\tilde\eta$ in the λ_i. The contributions cancelled in the end so we didn't care too much about this, but we might worry about the presence of this arbitrary momentum in the calculation of amplitudes that don't vanish. In fact it tends to crop up frequently and the expressions that one arrives at seem to depend on η at first sight. However, it can be shown that the amplitudes are η-independent and it can therefore sometimes be of use to set η to be one of the external momenta in the problem.

This procedure has been implemented both for amplitudes with more external gluons and amplitudes with more negative helicities. In both cases the complexity grows, but the number of diagrams grows for large n at most as n^2 [33] which is a marked improvement on the factorial growth of the number of Feynman diagrams needed to compute the same processes. Further evidence for the procedure and a heuristic proof from twistor string theory can be found in [33], while a proof based on recursive techniques was given by Risager in [34] which was then used to give an MHV-vertex approach to gravity amplitudes [77]. Evidence for the validity of the procedure for tree and loop amplitudes was given in [79].[22]

On the other hand Mansfield found a transformation which takes the usual Yang-Mills Lagrangian and maps it to one where the vertices are explicitly MHV vertices [35] (see also [203]). This involves formulating pure Yang-Mills theory in light-cone coordinates and performing a non-local change of variables which maps the usual 3- and 4-point vertices that arise in Feynman diagram perturbation theory into an infinite sequence of MHV vertices starting with the 3-point $--+$ vertex. The procedure also clarifies the origin of the null vector η that we have used to define the off-shell continuation. It is just the same null vector as is used to define the light-cone formulation of the theory [35, 81]. For further work related to understanding the CSW rules from a Lagrangian approach see [80, 81, 82, 137, 138, 139, 204].

1.8 Loop diagrams from MHV vertices

The CSW rules at tree-level provide a new and effective way of re-organising perturbation theory and thus lead to more efficient methods for calculating tree-level amplitudes which often yield simpler results than more traditional approaches. Naturally we would like to be able to extend this method beyond tree-level and consider quantum corrections which are often a substantial contribution to the overall result. However as already mentioned the picture from twistor string theory is not as clear at loop-level and one might expect the CSW procedure to fail there due to the presence of conformal supergravity.

Nonetheless Brandhuber, Spence and Travaglini showed that the CSW rules are still valid at one-loop and provided a concrete procedure to follow from which they re-derived the one-loop n-point MHV gluon scattering amplitudes in $\mathcal{N} = 4$ super-Yang-Mills [37]. The answers they obtained are in complete agreement with the original results derived at 4-point by Green, Schwarz and Brink from the low energy limit of a string theory [205] and then at n-point by BDDK [38]. We will briefly review the method proposed in [37] and outline how it can be used to derive the $\mathcal{N} = 4$ amplitudes. Chapters 2 and 3 will then be devoted to applying the same method to the $\mathcal{N} = 1$ amplitudes and those in pure Yang-Mills with a scalar running in the loop respectively, thus calculating all cut-constructible[23] contributions to the n-point MHV gluon scattering amplitudes in QCD (1.3.3).

[22] We will say more about loop amplitudes shortly.

[23] It turns out that the CSW approach at loop-level only calculates the cut-containing terms, thus mirroring the cut-constructibility approach of BDDK. The rational terms are inextricably linked to these in supersymmetric theories but must be obtained in other ways in non-supersymmetric ones. See also Appendix D.

1.8.1 BST rules

The procedure proposed in [37] can be summarised as follows [40]:

1. Consider only the colour-stripped or partial amplitudes introduced in §1.1. As already mentioned there, the remarkable results discussed in Section 7 of [38] mean that this is sufficient to re-construct the entire colour-dependent amplitude.
2. Lift the MHV tree-level scattering amplitudes to vertices, by continuing the internal lines off-shell using the prescription described in §1.7.1. Internal lines are then connected by scalar propagators which join particles of the same spin but opposite helicity.
3. Build MHV diagrams with the required external particles at loop level using the MHV tree-level vertices and sum over all independent diagrams obtained in this fashion for a fixed ordering of external helicity states.
4. Re-express the loop integration measure in terms of the off-shell parametrisation employed for the loop momenta.
5. Analytically continue to $4-2\epsilon$ dimensions in order to deal with infrared divergences and perform all loop integrations.

1.8.2 Integration measure

The loop legs that we must integrate over are off-shell and in order to proceed we must work out the integration measure used in [37]. The details of the measure were more concretely worked-out in [79] using the Feynman tree theorem [206, 207, 208] and we use certain results from there as well as from the original construction of [37] while following the review of Section 3 of [40].

We need to re-express the usual integration measure d^4L over the loop momentum L in terms of the new variables l and z introduced previously. After a short calculation we find that[24] [37, 79]

$$\frac{d^4L}{L^2+i\varepsilon} = dN(l)\,\frac{dz}{z+i\varepsilon}\,; \qquad (1.8.1)$$

where we define $d^4L := \prod_{i=0}^{3} dL_i$ and have introduced the Nair measure [175]

$$dN(l) := \frac{1}{4i}\left(\langle l\, dl\rangle d^2\tilde{l} - [\tilde{l}\, d\tilde{l}]\, d^2 l\right) = \frac{d^3 l}{2 l_0}\,. \qquad (1.8.2)$$

[24] The $i\varepsilon$ prescription in the left- and right-hand sides of (1.8.1) was understood in [37], and, as stressed in [79, 179, 209] it is essential in order to correctly perform loop integrations.

Eq. (1.8.1) is key to the procedure. It is important to notice that the product of the measure factor with a scalar propagator $d^4L/(L^2+i\varepsilon)$ in (1.8.1) is independent of the reference vector η. In [175], it was noticed that the Lorentz-invariant phase space measure for a massless particle can be expressed precisely in terms of the Nair measure:

$$d^4 l\, \delta^{(+)}(l^2) = dN(l)\,, \tag{1.8.3}$$

where, as before, we write the null vector l as $l_{\alpha\dot\alpha} = l_\alpha \tilde{l}_{\dot\alpha}$, and in Minkowski space we identify $\tilde{l} = \pm\bar{l}$ depending on whether l_0 is positive or negative.

Next, we observe that in computing one-loop MHV scattering amplitudes from MHV diagrams (shown in Figure 1.6)[25], the four-dimensional integration measure which appears is [37, 79]

$$d\mathcal{M} := \frac{d^4L_1}{L_1^2+i\varepsilon}\,\frac{d^4L_2}{L_2^2+i\varepsilon}\,\delta^{(4)}(L_2 - L_1 + P_L)\,, \tag{1.8.4}$$

where L_1 and L_2 are loop momenta, and P_L is the external momentum flowing outside the loop[26] so that $L_2 - L_1 + P_L = 0$.

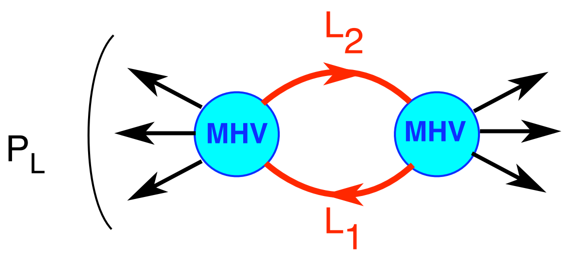


Figure 1.6: A generic MHV diagram contributing to a one-loop MHV scattering amplitude.

Now we express L_1 and L_2 as in (1.7.1),

$$L_{i;\alpha\dot\alpha} = l_{i\alpha}\tilde{l}_{i\dot\alpha} + z_i\,\eta_{\alpha\dot\alpha}\,, \qquad i = 1, 2\,. \tag{1.8.5}$$

Using (1.8.5), we re-write the argument of the delta function as

$$L_2 - L_1 + P_L = l_2 - l_1 + P_{L;z}\,, \tag{1.8.6}$$

where we have defined

$$P_{L;z} := P_L - z\eta\,, \tag{1.8.7}$$

[25] We thank the authors of [79] for allowing the re-production of Figure 17 of that paper.

[26] In our conventions all external momenta are outgoing.

and

$$z := z_1 - z_2 \, . \tag{1.8.8}$$

Notice that we use the same η for both the momenta L_1 and L_2. Using (1.8.5), we can then recast (1.8.4) as [37, 79]

$$dM = \frac{dz_1}{z_1 + i\varepsilon_1}\frac{dz_2}{z_2 + i\varepsilon_2}\frac{d^3 l_1}{2l_{10}}\frac{d^3 l_2}{2l_{20}}\,\delta^{(4)}(l_2 - l_1 + P_{L;z}) \, , \tag{1.8.9}$$

where $\varepsilon_i := \mathrm{sgn}(\eta_0 l_{i0})\varepsilon = \mathrm{sgn}(l_{i0})\varepsilon$, $i = 1, 2$ (the last equality holds since we are assuming $\eta_0 > 0$).

We now convert the integration over z_1 and z_2 into an integration over z and $z' := z_1 + z_2$ and with a careful treatment of the integrals [79] we can integrate out z'. We also make the replacement

$$\frac{d^3 l_1}{2l_{10}}\frac{d^3 l_2}{2l_{20}}\,\delta^{(4)}(l_2 - l_1 + P_{L;z}) \;\to\; d\mathrm{LIPS}(l_2^-, -l_1^+; P_{L;z}) \, , \tag{1.8.10}$$

where

$$d\mathrm{LIPS}(l_2^-, -l_1^+; P_{L;z}) := d^4 l_1\,\delta^{(+)}(l_1^2)\,d^4 l_2\,\delta^{(-)}(l_2^2)\,\delta^{(4)}(l_2 - l_1 + P_{L;z}) \tag{1.8.11}$$

is the two-particle Lorentz-invariant phase space (LIPS) measure and we recall that $\delta^{(\pm)}(l^2) := \theta(\pm l_0)\,\delta(l^2)$. Trading the final integral over z for an integration over $P^2_{L;z}$, the integration measure finally becomes [37, 79]

$$dM = 2\pi i\,\theta(P^2_{L;z})\,\frac{dP^2_{L;z}}{P^2_{L;z} - P^2_L - i\varepsilon}\,d\mathrm{LIPS}(l_2^-, -l_1^+; P_{L;z}) \, . \tag{1.8.12}$$

This can now be immediately dimensionally regularised, which is accomplished by simply replacing the four-dimensional LIPS measure by its continuation to $D = 4 - 2\epsilon$ dimensions:

$$d^D\mathrm{LIPS}(l_2^-, -l_1^+; P_{L;z}) := d^D l_1\,\delta^{(+)}(l_1^2)\,d^D l_2\,\delta^{(-)}(l_2^2)\,\delta^{(D)}(l_2 - l_1 + P_{L;z}) \, . \tag{1.8.13}$$

Eq. (1.8.12) was one of the key results of [37]. It gives a decomposition of the original integration measure into a D-dimensional phase space measure and a dispersive measure. According to Cutkosky's cutting rules [210], the LIPS measure computes the discontinuity of a Feynman diagram across its branch cuts. Which discontinuity is evaluated is determined by the argument of the delta function appearing in the LIPS measure; in (1.8.12) this is $P_{L;z}$ (defined in (1.8.7)). Notice that $P_{L;z}$ always contains a term proportional to the reference vector η, as prescribed by (1.8.7). Finally, discontinuities are integrated using the dispersive measure in (1.8.12), thereby reconstructing the full amplitude.

As a last remark, notice that in contradistinction with the cut-constructibility approach of BDDK, here we sum over all the cuts – each of which is integrated with the appropriate dispersive measure.

1.9 MHV amplitudes in N = 4 super-Yang-Mills

In this section we will briefly review the one-loop MHV $\mathcal{N} = 4$ super-Yang-Mills amplitudes and their derivation using MHV vertices. Many more details can be found in [37, 38].

1.9.1 General integral basis

It is known that, at one-loop, all amplitudes in massless gauge field theories can be written in terms of a certain basis of integral functions termed boxes, triangles and bubbles as well as possible rational contributions (i.e. contributions which do not contain any branch cuts) [38, 42]. These functions may involve some number of loop momenta in the numerator of their integrand, in which case they are termed tensor boxes, triangles or bubbles, though the basic scalar integrals remain the same and at 4-, 3- and 2-point respectively are the basic integrals arising at one-loop in scalar ϕ^3 theory.

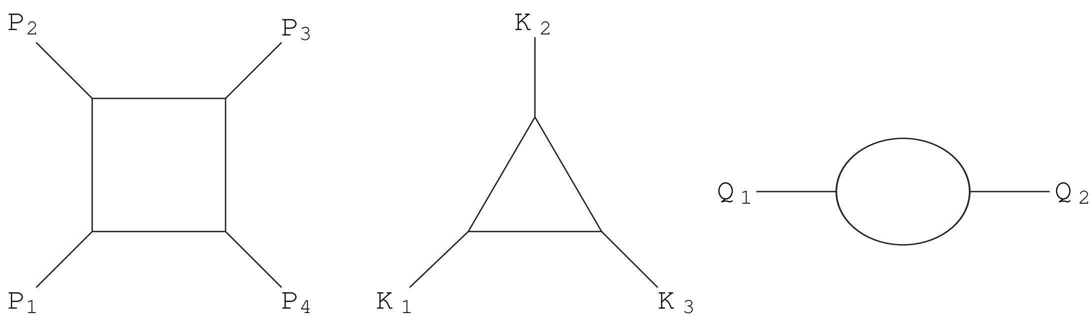


Figure 1.7: Boxes, Triangles and Bubbles. Here P_i, K_i and Q_i are generic momenta representing the contribution of one or more external particles. The different functions discussed below (1-mass, 2-mass etc.) are all special cases of these.

A box integral is characterised by having 4 vertices, a triangle integral by having 3 vertices while a bubble has 2. The specific functions that occur are then characterised not-only by possible powers of loop momenta arising in the numerator, but by the number of vertices with more than one external leg. If a vertex has only one external leg it is called a massless vertex (as the external momentum is massless in the theories we are considering), whilst if it has more than one external leg it is termed a massive vertex as the external momentum emanating from it does not square to zero.

There are thus 4 generic types of box integrals: 4-mass boxes where all 4 vertices are massive; 3-mass boxes; 2-mass 'easy' boxes where the massive vertices are opposite each other; 2-mass 'hard' boxes where the massive vertices are adjacent and 1-mass boxes. At 4-point the only possible box integral is a massless box. Similarly one can have 3-mass triangles, 2-mass triangles, 1-mass triangles, 2-mass bubbles and 1-mass bubbles (as well as massless triangles and massless bubbles at 3- and 2-point respectively).[27] Explicit forms for all these functions can be found in Appendix I of [42].

1.9.2 The N = 4 MHV one-loop amplitudes

Concerning the above decomposition, maximally supersymmetric Yang-Mills theory is special in that its high degree of symmetry prescribes that its one-loop amplitudes only contain scalar box integral functions (up to finite order in the dimensional regularisation parameter ϵ) [38, 42]. In particular, the MHV amplitudes only depend on the 2-mass easy (2me) box functions. The full one-loop n-point MHV amplitudes are proportional to the tree-level MHV amplitudes and are given by [38]

$$A_{n;1}^{N=4\,MHV} = A_n^{tree} V_n^g \, ; \tag{1.9.1}$$

where [38, 73]

$$V_n^g = \sum_{i=1}^{n} \sum_{r=1}^{[\frac{n}{2}]-1} \left(1 - \frac{1}{2}\delta_{\frac{n}{2}-1,r}\right) F_{n:r;i}^{2m\,e} \, : \tag{1.9.2}$$

The basic scalar box integral I_4 is defined by

$$I_4 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-P_1)^2(p-P_1-P_2)^2(p+P_4)^2} \, ; \tag{1.9.3}$$

where dimensional regularisation is used to take care of infrared divergences. The relevant integrals arising in (1.9.2) are related to I_4 for different choices of the external momenta at each vertex P_i $(i = 1 \ldots 4)$. These are denoted by $I_{4:r;i}^{2m\,e}$ – see Figure 1.8 – and are given in terms of the $F_{n:r;i}^{2m\,e}$ by

$$I_{4:r;i}^{2m\,e} = -\frac{2F_{n:r;i}^{2m\,e}}{t_i^{[r]} t_{i+r+1}^{[n-r-2]} - t_{i-1}^{[r+1]} t_i^{[r+1]}} \, ; \tag{1.9.4}$$

[27] Note that 1-mass and zero-mass bubbles are usually taken to vanish in dimensional regularization which is interpreted as a cancellation of infrared and ultraviolet divergences [42, 211].

with

$$t_i^{[r]} = (k_i + k_{i+1} + \cdots + k_{i+r-1})^2 \; ; \; r > 0$$
$$t_i^{[r]} = t_i^{[n-r]} \; ; \; r < 0 \; ; \qquad (1.9.5)$$

where the k_i are the external momenta. The explicit form of $F^{2me}_{n:r;i}$ is given by [38]

$$\begin{aligned} F^{2me}_{n:r;i} = & -\frac{1}{\epsilon^2}\Big[(-t_{i-1}^{[r+1]})^{-\epsilon} + (-t_i^{[r+1]})^{-\epsilon} - (-t_i^{[r]})^{-\epsilon} - (-t_{i+r+1}^{[n-r-2]})^{-\epsilon}\Big] \\ & + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_{i-1}^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_i^{[r]}}{t_i^{[r+1]}}\right) + \mathrm{Li}_2\left(1 - \frac{t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{r+1}}\right) \\ & + \mathrm{Li}_2\left(1 - \frac{t_{i+r+1}^{n-r-2}}{t_i^{[r+1]}}\right) - \mathrm{Li}_2\left(1 - \frac{t_i^{[r]} t_{i+r+1}^{[n-r-2]}}{t_{i-1}^{[r+1]} t_i^{[r+1]}}\right) + \frac{1}{2}\log^2\left(\frac{t_{i-1}^{[r+1]}}{t_i^{[r+1]}}\right) \; ; \qquad (1.9.6) \end{aligned}$$

where Li_2 is Euler's dilogarithm

$$\mathrm{Li}_2(z) := -\int_0^z dt \frac{\log(1-t)}{t} \; . \qquad (1.9.7)$$

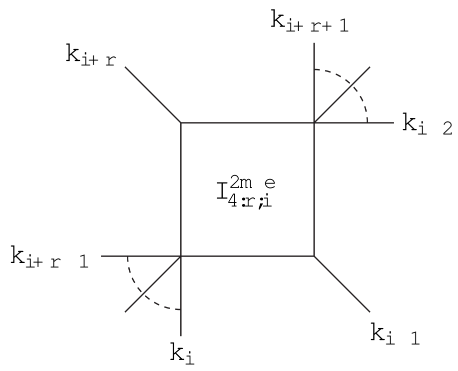


Figure 1.8: The 2-mass easy box function.

The one-loop MHV amplitudes were constructed in [38] from tree diagrams using cuts. A given cut results in singularities in the relevant momentum channels and by considering all possible cuts one can construct the full set of possible singularities. From this and unitarity one can deduce the amplitude as given in (1.9.1). More explicitly, consider a cut one-loop MHV diagram where the cut separates the external momenta k_{m_1} & k_{m_1-1}, and k_{m_2} & k_{m_2+1} (i.e. the set of external momenta $k_{m_1}; k_{m_1+1}; \ldots; k_{m_2}$ lie to the left of the cut, and the set $k_{m_2+1}; k_{m_2+2}; \ldots; k_{m_1-1}$ lie to the right, with momenta labelled clockwise and outgoing). This separates the diagram into two MHV tree diagrams connected only by two momenta l_1 and l_2 flowing across the cut, with

$$l_1 = l_2 + P_L \; ; \qquad (1.9.8)$$

where $P_L = \sum_{i=m_1}^{m_2} k_i$ is the sum of the external momenta on the left of the cut. The momenta l_1, l_2 are taken to be null. It is important to note that the resulting integrals are not equal to the corresponding Feynman integrals where l_1 and l_2 would be left off shell; however, the discontinuities in the channel under consideration are identical and this gives enough information to determine the full amplitude uniquely.

However, we will now sketch how to derive the MHV amplitudes using the method of MHV diagrams. This is quite similar, but not identical to the approach of BDDK using cut-constructibiliy, a brief review of which can be found in Appendix D.

1.9.3 MHV vertices at one-loop

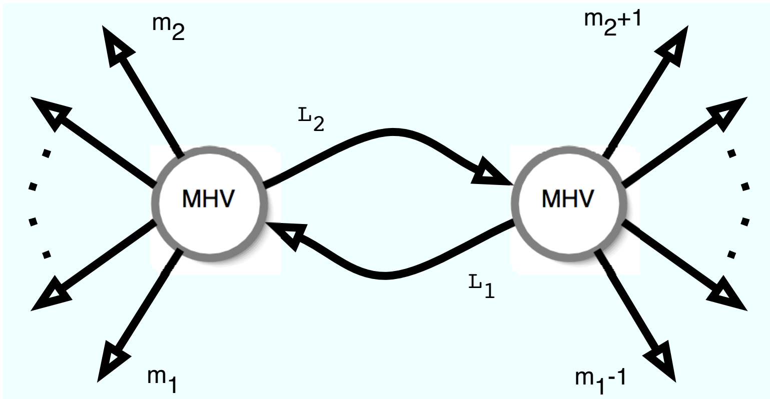


Figure 1.9: A one-loop MHV diagram computed using MHV amplitudes as interaction vertices. This diagram has the momentum structure of the cut referred to at the end of §1.9.2.

1. To each MHV vertex we associate the appropriate form of the MHV amplitude for that vertex, recalling that internal lines must be taken off-shell using the prescription described in §1.7.1. To each internal line we associate a scalar propagator and integrate over the appropriate loop momentum. The generic expression for the diagram of Figure 1.9 then reads:

$$\begin{aligned} A &= \int \frac{d^4L_1}{(2\pi)^4}\frac{d^4L_2}{(2\pi)^4}\frac{1}{L_1^2+i\varepsilon}\frac{1}{L_2^2+i\varepsilon}A_L A_R \\ &= \int \frac{d^4L_1}{L_1^2+i\varepsilon}\frac{d^4L_2}{L_2^2+i\varepsilon}\frac{iN_L\,\delta^{(4)}(L_2-L_1+P_L)}{D_L}\frac{iN_R\,\delta^{(4)}(L_1-L_2+P_R)}{D_R} \\ &= \delta^{(4)}(P_L+P_R)\int \frac{d^4L_1}{L_1^2+i\varepsilon}\frac{d^4L_2}{L_2^2+i\varepsilon}\,\delta^{(4)}(L_2-L_1+P_L)\frac{iN_L}{D_L}\frac{iN_R}{D_R}\,. \quad (1.9.9) \end{aligned}$$

Here L and R denote the left and right vertices respectively and we have $P_L := k_{m_1} + k_{m_1+1} + \ldots + k_{m_2}$ and $P_R := k_{m_2+1} + k_{m_2+2} + \ldots + k_{m_1-1}$. N and D denote the functions of spinor variables describing the numerator and denominator of each MHV vertex respectively and we have included a factor of $i(2\pi)^4$ with each vertex in keeping with Nair's supersymmetric description [31, 37, 175].

2. In [37] an approach using Nair super-vertices was used. Here we will just consider the usual MHV vertices for ease of transition to the later chapters where we will discuss MHV amplitudes in theories with less supersymmetry. In this case there are two possibilities to consider. The first is where both external negative-helicity gluons lie on one MHV vertex and the second is where they lie on different vertices (see e.g. Figure 2.4). After some manipulation (employing the Schouten identity stated in Appendix A), they can be shown to give the same contribution. Extracting an overall factor of

$$A_n^{\text{tree}} := i(2\pi)^4\delta^{(4)}(P_L+P_R)\frac{\langle i\,j\rangle^4}{\prod_{k=1}^{n}\langle k\;k+1\rangle}\,; \tag{1.9.10}$$

where i and j are the external negative-helicity gluons and regulating by promoting the integrals to $4-2\epsilon$ dimensions, (1.9.9) becomes

$$\mathcal{A}=\frac{i}{(2\pi)^4}A_n^{\text{tree}}\int d\mathcal{M}\,\hat{R}\,; \tag{1.9.11}$$

where $d\mathcal{M}$ is the measure (1.8.12) derived previously and

$$\hat{R}:=\frac{\langle m_1-1\;m_1\rangle\langle l_2\,l_1\rangle}{\langle m_1-1\;l_1\rangle\langle l_1\,m_1\rangle}\frac{\langle m_2\;m_2+1\rangle\langle l_1\,l_2\rangle}{\langle m_2\,l_2\rangle\langle l_2\;m_2+1\rangle}\,. \tag{1.9.12}$$

3. Following equations (2.11)–(2.16) of [37] we may finally write $\hat{R}$ as a signed sum (i.e. two terms come with plus signs and two with minus signs – see Eq. (2.13) of [37]) of terms of the form [28]

$$R(i;j):=\frac{\langle i\,l_2\rangle\langle j\,l_1\rangle}{\langle i\,l_1\rangle\langle j\,l_2\rangle}\,. \tag{1.9.13}$$

Once expressed in terms of momenta by multiplying top and bottom by appropriate anti-holomorphic spinor invariants, cancellations arise between different terms of the signed sum and we can schematically write $\hat{R}=\sum R\to\sum R_\epsilon$ with [37, 79]

$$R_\epsilon=\frac{1}{4}\frac{P_{L;z}^2(ij)-2(iP_{L;z})(jP_{L;z})}{(i\,l_1)(j\,l_2)}\,. \tag{1.9.14}$$

[28] Be careful to note that in the following expression i and j refer to the different possibilities m_1, m_2, m_2+1 and m_1-1, and not to the negative-helicity particles of the overall amplitude which now only arise in the factor of A_n^{tree}.

The notation (ab) here is shorthand for $(a \cdot b)$. (1.9.11) then becomes

$$\mathcal{A} = \frac{i}{(2\pi)^4} A_n^{\mathrm{tree}} \sum \int d\mathcal{M}\, \mathcal{R}_e \, . \qquad (1.9.15)$$

It is worth mentioning that the procedure of expressing $\sum \mathcal{R} \to \sum \mathcal{R}_e$ is a clever way of cancelling the triangle and bubble contributions in $\mathcal{R}$ to leave only box functions [37, 79] and is equivalent to the usual method of Passarino-Veltman reduction of [212]. (1.9.15) is then the basic integral that we have to work with and we will consider the specific term $\mathcal{R}_e(m_1, m_2)$ for definiteness.

4. Recall that the measure $d\mathcal{M}$ involves a dispersive part and an integral over Lorentz-invariant phase space (dLIPS). We wish to begin by performing the integral over this phase space. For this we go to the centre of mass frame for $P_{L;z}$ – $P_{L;z} = P_0(1, \vec{0})$ – and parametrize $l_1 = \frac{1}{2}P_0(1, \vec{v})$ and $l_2 = \frac{1}{2}P_0(-1, \vec{v})$ with $\vec{v} := (\sin\theta_1 \cos\theta_2, \sin\theta_1 \sin\theta_2, \cos\theta_1)$. In $4-2\epsilon$ dimensions, the LIPS measure (1.8.13) can be written in terms of the angles θ_1 and θ_2 as[29]

$$d^{4-2\epsilon}\mathrm{LIPS} = \frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)} \left(\frac{P_0^2}{4}\right)^{-\epsilon} d\theta_1 d\theta_2 (\sin\theta_1)^{1-2\epsilon} (\sin\theta_2)^{-2\epsilon} \, , \qquad (1.9.16)$$

and the denominator of (1.9.14) as

$$(m_1 l_1)(m_2 l_2) = -\frac{P_0^2}{4} m_{10}(1-\cos\theta_1)(A + B\sin\theta_1\cos\theta_2 + C\cos\theta_1) \, , \qquad (1.9.17)$$

where $m_1 := m_{10}(1,0,0,1)$ and $m_2 := (A, B, 0, C)$ with $A^2 = B^2 + C^2$. The numerator of (1.9.14) does not involve l_1 or l_2 and we leave it as $N(P_{L;z})$ for now. We thus have

$$\Omega_1 \int dW\, \Omega_2 \int \frac{d\theta_1 d\theta_2 (\sin\theta_1)^{1-2\epsilon} (\sin\theta_2)^{-2\epsilon}}{(1-\cos\theta_1)(A + B\sin\theta_1\cos\theta_2 + C\cos\theta_1)} \, , \qquad (1.9.18)$$

where

$$\Omega_1 := \frac{i}{(2\pi)^4} \frac{\pi^{1/2-\epsilon}}{4^{1-\epsilon}\,\Gamma(1/2-\epsilon)} A_n^{\mathrm{tree}} \, , \qquad (1.9.19)$$

$$\Omega_2 := -\frac{N(P_{L;z})}{m_{10} P_0^2} (P_0^2)^{-\epsilon} \, , \qquad (1.9.20)$$

$$dW := (2\pi i)\,\delta^{(+)}(P_{L;z}^2) \frac{dP_{L;z}^2}{P_{L;z}^2 - P_L^2 - i\varepsilon} \, . \qquad (1.9.21)$$

The integral over θ_1 and θ_2 has been performed in [213] and we borrow the result in a form from [214]. Converting A, B, C, m_{10} and P_0 back into Lorentz-invariants

[29] See Appendix C for details.

we obtain:

$$4 \quad {}_1\frac{1}{\ } \int dW\,(P^2_{L;z})\;\; {}_2F_1\;\; 1; \quad ;1 \quad ; a_z P^2_{L;z} \quad : \tag{1.9.22}$$

In Equations (1.9.16), (1.9.19) and (1.9.22) above, is the gamma function and ${}_2F_1$ the Gauss hypergeometric function. They can be defined by

$$(z) \;\coloneqq\; \int_0^1 dt\, t^{z\ 1} e^{\ t}\;;\; <[z] > 0\,; \tag{1.9.23}$$

$${}_2F_1(a;b;c;z) \;\coloneqq\; \frac{(c)}{(b)\ (c\quad b)} \int_0^1 dt\, t^{b\ 1}(1\quad t)^{\ b+c\ 1}(1\quad tz)^{\ a} \tag{1.9.24}$$

where the second definition holds when $<[c] > <[b] > 0$ and $|\arg(1\quad z)| <$. a_z is defined to be $a_z \coloneqq (ij)/N(P_{L;z})$ and so is equal to $(m_1 m_2)/N(P_{L;z})$ in this case.

5. Finally, we would like to evaluate this dispersive integral (1.9.22). In [37], this was done by combining different terms coming from different R_e to give a convergent integral. In fact the different R_e that one must combine come not from different i and j in R_e (ij) as obtained from $\hat{R} = \sum R \to \sum R_e$, but from R_e with the same i and j coming from different terms in the overall summation over all the cuts of the one-loop integral mentioned at the end of §1.8.2 and not so far alluded to in this section. This summation is just a summation over all cyclic partitions of the external particles between the two MHV vertices, but at the level of the integrals we have arrived at in (1.9.22) the summation over R_e (ij) with the same values of i and j from different orderings of the external particles serves to re-construct the 2me box functions from their different cuts.

   The integrals are explicitly done by expanding the hypergeometric functions above in an expansion in in terms of polylogarithms (generalisations of Li_2) and then combining different cuts of the same box function to give a convergent answer. A key ingredient in all this is the knowledge that the final result will be independent of . has already been eliminated from the dispersive integration measure by converting the integral over z and z^0 into an integral over $P_{L;z}$, so one may expect that even before we evaluate this dispersive integral we should be able to pick a particular value for to simplify the calculation. However, in [37] a stronger gauge invariance was proposed; namely that one may choose separately for each box function. This was checked numerically in [37] and independently (also numerically) in [209] and further evidence was provided in [79].[30] It means that one can write $N(P_{L;z}) = N(P_L)$ if one chooses $= m_1$ or $= m_2$ in all four R_e $(m_1 m_2)$ which contribute to that particular box function.

[30] See also Appendix F for an analytic proof of the same statement for triangle functions.

The final result (up to finite order in ϵ) given in Equation (5.16) of [37] is that the contribution of a particular box function (say a generic box function such as that in Figure 1.8, which would come from combining the four terms with $m_1 = k_{i+r}$ and $m_2 = k_{i-1}$) is

$$\begin{aligned} F^{2m\,e}_{n:r;i} = & -\frac{1}{\epsilon^2}\Big[(-t^{[r+1]}_{i-1})^{-\epsilon} + (-t^{[r+1]}_{i})^{-\epsilon} - (-t^{[r]}_{i})^{-\epsilon} - (-t^{[n-r-2]}_{i+r+1})^{-\epsilon}\Big] \\ & + \mathrm{Li}_2\left(1 - a\, t^{[r]}_{i}\right) + \mathrm{Li}_2\left(1 - a\, t^{[n-r-2]}_{i+r+1}\right) \\ & - \mathrm{Li}_2\left(1 - a\, t^{[r+1]}_{i-1}\right) - \mathrm{Li}_2\left(1 - a\, t^{[r+1]}_{i}\right) ; \end{aligned} \tag{1.9.25}$$

where

$$a = \frac{t^{[r]}_{i} + t^{[n-r-2]}_{i+r+1} - t^{[r+1]}_{i-1} - t^{[r+1]}_{i}}{t^{[r]}_{i}\, t^{[n-r-2]}_{i+r+1} - t^{[r+1]}_{i-1}\, t^{[r+1]}_{i}} : \tag{1.9.26}$$

Equation (1.9.25) is in fact equal to (1.9.6) but is an alternative form which was discovered in [215] and independently derived in [37] and involves one less dilogarithm and one less logarithm than (1.9.6). After summing over all partitions of the external particles between the two MHV vertices we recover (1.9.1).

The calculation outlined above is essentially what we will follow in Chapters 2 and 3 for the $\mathcal{N}=1$ and $\mathcal{N}=0$ MHV amplitudes. For full details of the amplitudes in $\mathcal{N}=4$ see [37] and for a short discussion on the overall ϵ-normalisation of the result obtained there compared with the one obtained originally in [38] see Appendix C.

CHAPTER 2

MHV AMPLITUDES IN $\mathcal{N} = 1$ SUPER-YANG-MILLS

In Chapter 1 we described some of the hidden simplicity of perturbative gauge theory – in particular in the context of maximally supersymmetric Yang-Mills – and saw how it may be applied to simplifying the calculation of perturbative quantities such as scattering amplitudes. The many techniques available to illuminate the perturbative structure included colour stripping, the use of a helicity scheme and supersymmetric decompositions. A perturbative duality with a twistor string theory highlighted the unexpected compactness of the MHV amplitudes at tree-level and provided motivation for a new perturbative expansion of gauge theory – the CSW rules.

The CSW rules have been shown to be valid even at loop level – despite the failure of the duality with twistor string theory – and the MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills were derived using these rules in [37] and shown to be identical to the original derivation of [38] using 2-particle cuts. As a bonus, the CSW rules also gave rise to a representation of the 2-mass easy box functions that is simpler to that originally used in [38]. However, at the time it was far from certain that these remarkable techniques would be applicable to other gauge theories. One might not have been surprised if such results only held for a theory with an extremely high amount of symmetry such as $\mathcal{N} = 4$ SYM.

In [40, 41] a first step towards establishing the general validity of the MHV-vertex formalism was taken and it was shown independently by Bedford, Brandhuber, Spence & Travaglini and Quigley & Rosali that the CSW rules correctly calculate the MHV amplitudes in theories with less supersymmetry such as $\mathcal{N} = 1$ and $\mathcal{N} = 2$ super-Yang-Mills. In particular the MHV amplitudes for scattering of external gluons with an $\mathcal{N} = 1$ chiral multiplet running in the loop was calculated and it was found that the results exactly agree with those originally obtained by BDDK in [42]. This chapter follows [40] and shows how the $\mathcal{N} = 1$ MHV amplitudes may be obtained from MHV vertices.

2.1 The $\mathcal{N} = 1$ MHV amplitudes at one-loop

The expression for the MHV amplitudes at one-loop in $\mathcal{N} = 1$ SYM was obtained for the first time by BDDK in [42] using the cut-constructibility method. We will shortly give their explicit result and then re-write it by introducing appropriate functions. This turns out to be useful when we compare the BDDK result to that which we will derive by using MHV diagrams.

In order to obtain the one-loop MHV amplitudes in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM it is sufficient to compute the contribution $A_n^{\mathcal{N}=1,\mathrm{chiral}}$ to the one-loop MHV amplitudes coming from a single $\mathcal{N} = 1$ chiral multiplet. This was calculated in [42], and the result turns out to be proportional to the Parke-Taylor MHV tree amplitude [170]

$$A_n^{\mathrm{tree}} := \frac{\langle i\, j\rangle^4}{\prod_{k=1}^{n} \langle k\; k+1\rangle} ; \tag{2.1.1}$$

as is also the case with the one-loop MHV amplitudes in $\mathcal{N} = 4$ SYM. However, in contradistinction with that case, the remaining part of the $\mathcal{N} = 1$ amplitudes depends non-trivially on the position of the negative-helicity gluons i and j. The result obtained in [42] is:

$$\begin{aligned}
A_n^{\mathcal{N}=1,\mathrm{chiral}} &= A_n^{\mathrm{tree}} \sum_{m=i+1}^{j-1} \sum_{s=j+1}^{i-1} b^{i,j}_{m,s}\, B\big(t_{m+1}^{[s-m]}, t_m^{[s-m]}, t_{m+1}^{[s-m-1]}, t_{s+1}^{[m-s-1]}\big) \\
&+ \sum_{m=i+1}^{j-1} \sum_{a\in D_m} c^{i,j}_{m,a} \frac{\log\big(t_{m+1}^{[a-m]}/t_m^{[a-m+1]}\big)}{t_{m+1}^{[a-m]} - t_m^{[a-m+1]}} \\
&+ \sum_{m=j+1}^{i-1} \sum_{a\in C_m} c^{i,j}_{m,a} \frac{\log\big(t_{a+1}^{[m-a]}/t_{a+1}^{[m-a-1]}\big)}{t_{a+1}^{[m-a]} - t_{a+1}^{[m-a-1]}} \\
&+ \frac{c^{i,j}_{i+1,i-1}}{t_i^{[2]}} K_0\big(t_i^{[2]}\big) + \frac{c^{i,j}_{i-1,i}}{t_{i-1}^{[2]}} K_0\big(t_{i-1}^{[2]}\big) \\
&+ \frac{c^{i,j}_{j+1,j-1}}{t_j^{[2]}} K_0\big(t_j^{[2]}\big) + \frac{c^{i,j}_{j-1,j}}{t_{j-1}^{[2]}} K_0\big(t_{j-1}^{[2]}\big) \quad ;
\end{aligned} \tag{2.1.2}$$

where $t_i^{[k]} := (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$ for $k \geq 0$, and $t_i^{[k]} = t_i^{[n-k]}$ for $k < 0$. The sums in the second and third line of (2.1.2) cover the ranges C_m and D_m defined by

$$C_m = \begin{cases} \{i, i+1, \ldots, j-2\}, & m = j+1 ; \\ \{i, i+1, \ldots, j-1\}, & j+2 \leq m \leq i-2 ; \\ \{i+1, i+2, \ldots, j-1\}, & m = i-1 ; \end{cases} \tag{2.1.3}$$

and

$$D_m = \begin{cases} \{j, j+1, \ldots, i-2\}, & m = i+1, \\ \{j, j+1, \ldots, i-1\}, & i+2 \le m \le j-2, \\ \{j+1, j+2, \ldots, i-1\}, & m = j-1. \end{cases} \tag{2.1.4}$$

The coefficients $b^{i,j}_{m,s}$ and $c^{i,j}_{m,a}$ are

$$b^{i,j}_{m,s} := -2\,\frac{\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_m\not{k}_s)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_s\not{k}_m)}{[(k_i+k_j)^2]^2\,[(k_m+k_s)^2]^2}\,; \tag{2.1.5}$$

$$c^{i,j}_{m,a} := \left[\frac{\mathrm{tr}_+(\not{k}_m\not{k}_{a+1}\not{k}_j\not{k}_i)}{(k_{a+1}+k_m)^2} - \frac{\mathrm{tr}_+(\not{k}_m\not{k}_a\not{k}_j\not{k}_i)}{(k_a+k_m)^2}\right]\frac{\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_m\not{q}_{m,a}) - \mathrm{tr}_+(\not{k}_i\not{k}_j\not{q}_{m,a}\not{k}_m)}{[(k_i+k_j)^2]^2}\,; \tag{2.1.6}$$

where $q_{r,s} := \sum_{l=r}^{s} k_l$. Notice that both coefficients $b^{i,j}_{m,s}$ and $c^{i,j}_{m,a}$ are symmetric under the exchange of i and j. In the case of b this is evident; for c it is also manifest as c is expressed as the product of two antisymmetric quantities. The function B in the first line of (2.1.2) is the "finite" part of the easy two-mass (2me) scalar box function $F(s,t,P^2,Q^2)$, with

$$F(s,t,P^2,Q^2) := -\frac{1}{\epsilon^2}\Big[(-s)^{-\epsilon} + (-t)^{-\epsilon} - (-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}\Big] + B(s,t,P^2,Q^2)\,. \tag{2.1.7}$$

As in [37] we have introduced the following convenient kinematical invariants:

$$s := (P+p)^2\,, \qquad t := (P+q)^2\,, \tag{2.1.8}$$

where p and q are null momenta and P and Q are in general massive. We also have momentum conservation in the form $p+q+P+Q=0$.[1] In [37] the following new expression for B was found:

$$B(s,t,P^2,Q^2) = \mathrm{Li}_2(1-aP^2) + \mathrm{Li}_2(1-aQ^2) - \mathrm{Li}_2(1-as) - \mathrm{Li}_2(1-at)\,, \tag{2.1.9}$$

where

$$a = \frac{P^2+Q^2-s-t}{P^2Q^2-st}\,. \tag{2.1.10}$$

The expression (2.1.9) contains one less dilogarithm and one less logarithm than the

[1] The kinematical invariant $s=(P+p)^2$ should not be confused with the label s which is also used to label an external leg (as in Figure 2.1 for example). The correct meaning will be clear from the context.

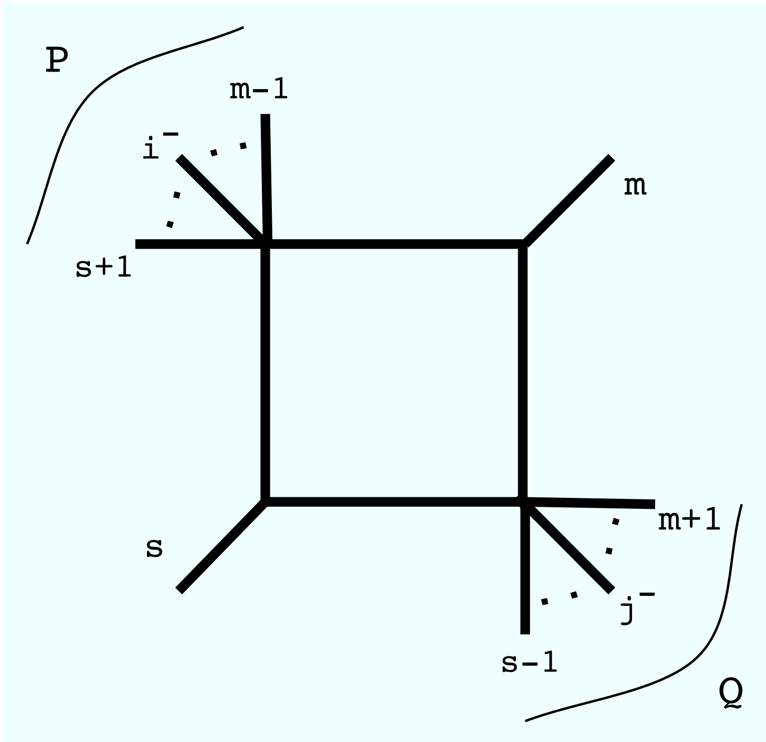


Figure 2.1: The box function F of (2.1.7), whose finite part B, Eq. (2.1.9), appears in the $\mathcal{N}=1$ amplitude (2.1.2). The two external gluons with negative helicity are labelled by i and j. The legs labelled by s and m correspond to the null momenta p and q respectively in the notation of (2.1.9). Moreover, the quantities $t_{m+1}^{[s-m]}$, $t_{m}^{[s-m]}$, $t_{m+1}^{[s-m-1]}$, $t_{s+1}^{[m-s-1]}$ appearing in the box function B in (2.1.19) correspond to the kinematical invariants $t := (Q+p)^2$, $s := (P+p)^2$, Q^2, P^2 in the notation of (2.1.9), with $p+q+P+Q=0$.

traditional form used by BDDK,

$$B(s,t,P^2,Q^2) = \mathrm{Li}_2\left(1-\frac{P^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{P^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{t}\right) - \mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right) + \frac{1}{2}\log^2\left(\frac{s}{t}\right) . \tag{2.1.11}$$

The agreement of (2.1.9) with (2.1.11) was discussed and proved in Section 5 of [37].[2] In Figure 2.1 we give a pictorial representation of the box function F defined in (2.1.7) (with the leg labels identified by $s \to p$, $m \to q$).

[2] More precisely, this agreement holds only in certain kinematical regimes e.g. in the Euclidean region where all kinematical invariants are negative. More care is needed when analytically continuing the amplitude to the physical region. The usual prescription of replacing a kinematical invariant s by $s+i\varepsilon$ and continuing s from negative to positive values gives the correct result only for our form of the box function (2.1.9), whereas (2.1.11) has to be amended by correction terms [216].

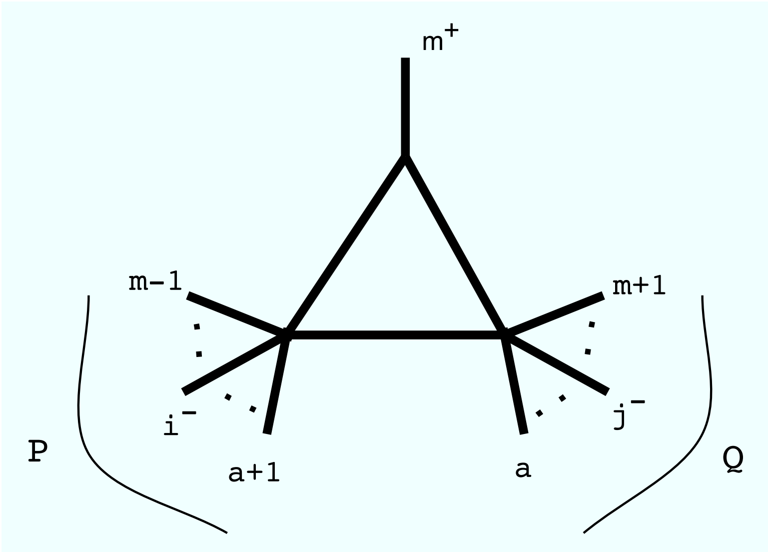


Figure 2.2: A triangle function, corresponding to the first term $T(p_m; q_{a+1,m-1}; q_{m+1,a})$ in the second line of (2.1.19). p, Q and P correspond to p_m, $q_{m+1,a}$ and $q_{a+1,m-1}$ in the notation of Eq. (2.1.19), where $j \in Q$, $i \in P$. In particular, $Q^2 \to t_{m+1}^{[a-m]}$ and $P^2 \to t_m^{[a-m+1]}$.

Finally, infrared divergences are contained in the bubble functions $K_0(t)$, defined by

$$K_0(t) := \frac{(-t)^{-\epsilon}}{\epsilon(1-2\epsilon)} : \tag{2.1.12}$$

We notice that in order to re-express (2.1.2) in a simpler form, it is useful to introduce the triangle function [73]

$$T(p;P;Q) := \frac{\log(Q^2/P^2)}{Q^2 - P^2} ; \tag{2.1.13}$$

with $p + P + Q = 0$. A diagrammatic representation of this function is given in Figure 2.2 (with $m^+ \to p$). We also find it useful to introduce an ϵ-dependent triangle function,[3]

$$T_\epsilon(p;P;Q) := \frac{1}{\epsilon} \frac{(-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}}{Q^2 - P^2} : \tag{2.1.14}$$

As long as P^2 and Q^2 are non-vanishing, one has

$$\lim_{\epsilon \to 0} T_\epsilon(p;P;Q) = T(p;P;Q) ; \qquad P^2 \neq 0 ; Q^2 \neq 0 : \tag{2.1.15}$$

[3] The function $T_\epsilon(p;P;Q)$ defined in (2.1.14) arises naturally in the twistor-inspired approach which will be developed in §2.2.

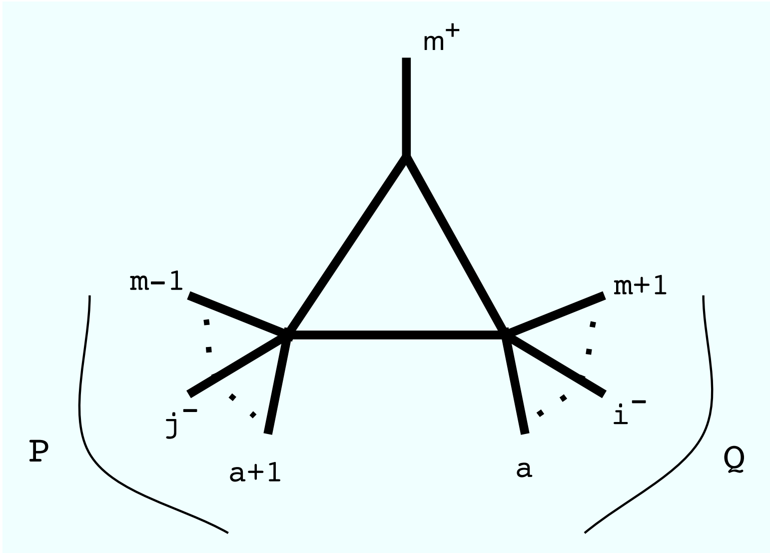


Figure 2.3: This triangle function corresponds to the second term in the second line of (2.1.19) – where i and j are swapped. As in Figure 2.2, p, Q and P correspond to p_m, $q_{m+1,a}$ and $q_{a+1,m-1}$ in the notation of Eq. (2.1.19), where now $i \in Q$, $j \in P$. In particular, $Q^2 \to t_{a+1}^{[m-a]}$ and $P^2 \to t_{a+1}^{[m-a-1]}$.

If either of the invariants vanishes, one has a different limit. For example, if $Q^2 = 0$ one has

$$T(p;P;Q)|_{Q^2=0} \to -\frac{1}{\epsilon}\frac{(-P^2)^{-\epsilon}}{P^2}\,, \qquad \epsilon \to 0\,. \tag{2.1.16}$$

We will call these cases "degenerate triangles".

The usefulness of the previous remark stems from the fact that precisely the quantity $(1/\epsilon)\,(-P^2)^{-\epsilon}/P^2$ appears in the last line of (2.1.2) – the bubble contributions. Therefore, these can be equivalently obtained as degenerate triangles i.e. triangles where one of the massive legs becomes massless.

Specifically, we notice that the four degenerate triangles (bubbles) in the last line of (2.1.2) can be precisely obtained by including the "missing" index assignments in D_m and C_m:

$$(m = i+1; a = i-1); \quad (m = j-1; a = j) \quad \text{for } D_m\,; \tag{2.1.17}$$

which correspond to two degenerate triangles, and

$$(m = j+1; a = j-1); \quad (m = i-1; a = i) \quad \text{for } C_m\,; \tag{2.1.18}$$

corresponding to two more degenerate triangles.

In conclusion, the previous remarks allow us to rewrite (2.1.2) in a more compact form as follows:

$$
\begin{aligned}
A_n^{\mathcal{N}=1,\mathrm{chiral}} = A_n^{\mathrm{tree}} \Bigg[& \sum_{m=i+1}^{j-1} \sum_{s=j+1}^{i-1} b_{m,s}^{i,j}\, B\big(t_{m+1}^{[s-m]}, t_m^{[s-m]}, t_{m+1}^{[s-m-1]}, t_{s+1}^{[m-s-1]}\big) \\
&+ \frac{1}{1-2\epsilon} \sum_{m=i+1}^{j-1} \sum_{a=j}^{i-1} c_{m,a}^{i,j}\, T(p_m, q_{a+1,m-1}, q_{m+1,a}) + (i \leftrightarrow j) \Bigg] .
\end{aligned}
\tag{2.1.19}
$$

In this expression it is understood that we only keep terms that survive in the limit $\epsilon \to 0$. This means that the factor $1/(1-2\epsilon)$ can be replaced by 1 whenever the term in the sum is finite, i.e. whenever the triangle is non-degenerate. However, in the case of degenerate triangles, which contain infrared-divergent terms, we have to expand this factor to linear order in ϵ. In the notation of (2.1.19), $q_{m+1,a}^2 = t_{m+1}^{[a-m]}$ and $q_{a+1,m-1}^2 = t_m^{[a-m+1]}$; in Figure 2.2, these invariants correspond to Q^2 and P^2 respectively, where $j \in Q$, $i \in P$. In the sum with $i \leftrightarrow j$, one would have $q_{m+1,a}^2 = t_{a+1}^{[m-a]}$, $q_{a+1,m-1}^2 = t_{a+1}^{[m-a-1]}$, corresponding respectively to Q^2 and P^2 in Figure 2.3, with $i \in Q$, $j \in P$. It is the expression (2.1.19) for the $\mathcal{N}=1$ chiral multiplet amplitude which we will derive using MHV diagrams.

2.2 MHV one-loop amplitudes in $\mathcal{N}=1$ SYM from MHV vertices

In §1.8 we reviewed how MHV vertices can be sewn together into one-loop diagrams, and how a particular decomposition of the loop momentum measure leads to a representation of the amplitudes strikingly similar to traditional dispersion formulæ. This method was tested successfully in [37] for the case of MHV one-loop amplitudes in $\mathcal{N}=4$ SYM as reviewed in §1.9. In the following we will apply the same philosophy to amplitudes in $\mathcal{N}=1$ SYM, in particular to the infinite sequence of MHV one-loop amplitudes, which were obtained using the cut-constructibility approach [42], and whose twistor space picture has been analysed in [73].

Similarly to the $\mathcal{N}=4$ case, the one-loop amplitude has an overall factor proportional to the MHV tree-level amplitude, but, as opposed to the $\mathcal{N}=4$ case, the remaining one-loop factor depends non-trivially on the positions i and j of the two external negative-helicity gluons. This is due to the fact that a different set of fields is allowed to propagate in the loop.

The MHV diagrams contributing to MHV one-loop amplitudes consist of two MHV vertices connected by two off-shell scalar propagators. If both negative-helicity gluons are on one MHV vertex, only gluons of a particular helicity can propagate in the loop. This is independent of the number of supersymmetries. On the other hand, for diagrams with one negative-helicity gluon on one MHV vertex and the other negative-helicity gluon on the other MHV vertex, all components of the supersymmetric multiplet propagate in the loop. In the case of $\mathcal{N}=4$ SYM this corresponds to helicities $h=-1,-1/2,0,1/2,1$ with multiplicities $1,4,6,4,1$, respectively; for the $\mathcal{N}=1$ vector multiplet the multiplicities are $1,1,0,1,1$. Hence, we can obtain the $\mathcal{N}=1$ (vector) amplitude by simply taking the $\mathcal{N}=4$ amplitude and subtracting three times the contribution of an $\mathcal{N}=1$ chiral multiplet, which has multiplicities $0,1,2,1,0$.[4]

This supersymmetric decomposition of general one-loop amplitudes is useful as it splits the calculation into pieces of increasing difficulty, and allows one to reduce a one-loop diagram with gluons circulating in the loop to a combination of an $\mathcal{N}=4$ vector amplitude, an $\mathcal{N}=1$ chiral amplitude and finally a non-supersymmetric amplitude with a scalar field running in the loop as in Equation (1.3.3).

In our case, the supersymmetric decomposition takes the form

$$A_n^{\mathcal{N}=1,\mathrm{vector}} = A_n^{\mathcal{N}=4} - 3A_n^{\mathcal{N}=1,\mathrm{chiral}}\,, \tag{2.2.1}$$

where n denotes the number of external lines. Since the $\mathcal{N}=4$ contribution is known, one only needs to determine $A_n^{\mathcal{N}=1,\mathrm{chiral}}$ using MHV diagrams. To be more precise, we are solely addressing the computation of the planar part of the amplitudes. However, this is sufficient since at one-loop level the non-planar partial amplitudes are obtained as appropriate sums of permutations of the planar partial amplitudes [38], as discussed in §1.1.

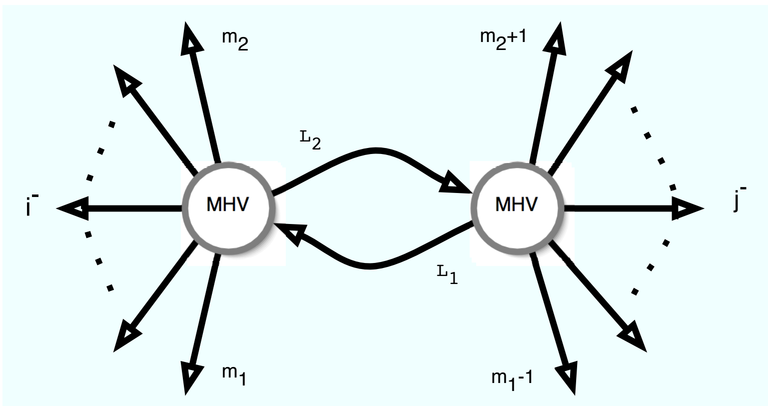


Figure 2.4: A one-loop MHV diagram, computed in (2.2.4) using MHV amplitudes as interaction vertices, with the CSW off-shell prescription. The two external gluons with negative helicity are labelled by i and j.

[4]We can also obtain the $\mathcal{N}=2$ amplitude in a completely similar way.

2.2.1 The procedure

Our task therefore consists of:

1. Evaluating the class of diagrams where we allow all the helicity states of a chiral multiplet,
$$h \in \{-1/2, 0, 0, 1/2\}\,; \tag{2.2.2}$$
to run in the loop. We depict the prototype of such diagrams in Figure 2.4.

2. Summing over all diagrams such that each of the two MHV vertices always has one external gluon of negative helicity. Assigning i^- to the left and j^- to the right, the summation range of m_1 and m_2 is determined to be:
$$j+1 \le m_1 \le i\,; \qquad i \le m_2 \le j-1\,. \tag{2.2.3}$$

Hence we get

$$\begin{aligned} A_n^{\mathcal{N}=1,\text{chiral}} = \sum_{m_1,m_2,h} \int & d\mathcal{M}\; A(-\ell_1, m_1, \ldots, i^-, \ldots, m_2, \ell_2) \\ & \times A(-\ell_2, m_2+1, \ldots, j^-, \ldots, m_1-1, \ell_1)\,; \end{aligned} \tag{2.2.4}$$

where the summation ranges of h, m_1 and m_2 are given in (2.2.2), (2.2.3). Notice that, in order to compute the loop amplitude (2.2.4), we make use of the integration measure $d\mathcal{M}$ given in (1.8.12).

After some spinor algebra and after performing the sum over the helicities h, the integrand of (2.2.4) becomes

$$iA_n^{\text{tree}} \cdot \frac{\langle m_2\, (m_2+1)\rangle \langle (m_1-1)\, m_1\rangle \langle i\, \ell_1\rangle \langle j\, \ell_1\rangle \langle i\, \ell_2\rangle \langle j\, \ell_2\rangle}{\langle i\, j\rangle^2 \langle m_1\, \ell_1\rangle \langle (m_1-1)\, \ell_1\rangle \langle m_2\, \ell_2\rangle \langle (m_2+1)\, \ell_2\rangle}\,. \tag{2.2.5}$$

The focus of the remainder of this section will be to evaluate the integral in (2.2.4) explicitly. Since iA_n^{tree} factors out completely, we will now drop it and only reinstate it at the very end of the calculation.

The integrand (without this factor) can be rewritten in terms of dot products of momentum vectors,

$$I = \frac{\mathcal{N}}{(i \cdot j)^2\, (m_1 \cdot \ell_1)\, ((m_1-1) \cdot \ell_1)\, (m_2 \cdot \ell_2)\, ((m_2+1) \cdot \ell_2)}\,; \tag{2.2.6}$$

with

$$\mathcal{N} = \mathrm{tr}_+ (\not\ell_1 \not k_{m_1-1} \not k_{m_1} \not\ell_1 \not k_j \not k_i)\, \mathrm{tr}_+ (\not\ell_2 \not k_{m_2} \not k_{m_2+1} \not\ell_2 \not k_j \not k_i)\,. \tag{2.2.7}$$

$\mathcal{N}$ is a product of Dirac traces, where the tr_+ symbol indicates that the projector $(1+\gamma^5)/2$ has been inserted.

Next, notice that each of these Dirac traces involving six momenta can be expressed in terms of simpler Dirac traces involving only four momenta. For the first factor of (2.2.7) we find

$$\mathrm{tr}_+(\not{l}_1\not{k}_{m_1-1}\not{k}_{m_1}\not{l}_1\not{k}_j\not{k}_i) = 2(m_1\, l_1)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1-1}\not{l}_1) - 2((m_1-1)\, l_1)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{l}_1)\,; \quad (2.2.8)$$

where

$$\mathrm{tr}_+(\not{k}_a\not{k}_b\not{k}_c\not{k}_d) = 2\left[(a\,b)(c\,d) - (a\,c)(b\,d) + (a\,d)(b\,c) - 2i\epsilon(a,b,c,d)\right]. \quad (2.2.9)$$

The second factor in (2.2.7) takes a similar form. Consequently, the integrand becomes a sum of four terms, one of which is

$$\frac{\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{l}_1)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_2}\not{l}_2)}{(i\,j)^2\,(m_1\,l_1)\,(m_2\,l_2)}\,. \quad (2.2.10)$$

The other three terms are obtained by replacing m_1 with m_1-1 and/or m_2 with m_2+1 in (2.2.10) and come with alternating signs. Note that the original expression (2.2.5) is symmetric in i, and j, although when we make use of the decomposition (2.2.10) this symmetry is no longer manifest. We will symmetrize over i and j at the end of the calculation in order to make this exchange symmetry manifest in the final expression.

In the next step we have to perform the phase space integration, which is equivalent to the calculation of a unitarity cut with momentum $P_{L;z} = \sum_{l=m_1}^{m_2} k_l - z\eta$ flowing through the cut. Note that, as explained in §1.7.1, the momentum is shifted by a term proportional to the reference momentum η. The term $(l_1\, m_1)(l_2\, m_2)$ in the denominator of (2.2.10) corresponds to two propagators, hence the denominator by itself corresponds to a cut box diagram. However, the numerator of (2.2.10) depends non-trivially on the loop momentum, so that in fact (2.2.10) corresponds to a tensor box diagram, not simply a scalar box diagram. Using the Passarino-Veltman method [212], we can reduce the expression (2.2.10), integrated with the LIPS measure, to a sum of cuts of scalar box diagrams, scalar and vector triangle diagrams, and scalar bubble diagrams. This procedure is somewhat technical and details are collected in Appendix E. Luckily, the final result takes a less intimidating form than the intermediate expressions. We will now present the result of these calculations after the LIPS integration.

2.2.2 Discontinuities

We first observe that loop integrations are performed in $4-2\epsilon$ dimensions. It turns out that singular $1/\epsilon$ terms appearing at intermediate steps of the phase space integration cancel out completely. Notice that this does not mean that the final result will be free of infrared divergences. In fact the dispersion integral can and does give rise to $1/\epsilon$ divergent terms but there cannot be any $1/\epsilon^2$ terms, as expected for the contribution of a chiral multiplet [42]. The $1/\epsilon$ divergences in the scattering amplitude correspond to the bubble contributions in (2.1.2), or degenerate triangles contributions in (2.1.19), as explained in §2.1. In Appendix E we show that the finite terms of the phase space integral combine into the following simple expression:

$$\hat{C} = C(m_1-1;m_2) - C(m_1;m_2) + C(m_1;m_2+1) - C(m_1-1;m_2+1)\,; \qquad (2.2.11)$$

with[5]

$$C(m_1;m_2) = \frac{2}{1-2\epsilon}\frac{(P^2_{L;z})^{-\epsilon}}{(i-j)(m_1-m_2)}\left[\frac{T(m_1;m_2;P_{L;z})}{(m_1 P_{L;z})}+\frac{T(m_2;m_1;P_{L;z})}{(m_2 P_{L;z})}\right]$$
$$-\frac{2\,T(m_1;m_2;m_2)}{(i-j)(m_1-m_2)^2}(P^2_{L;z})^{-\epsilon}\log\left(1-a_z P^2_{L;z}\right)\,; \qquad (2.2.12)$$

where

$$T(m_1;m_2;P) := \mathrm{tr}_+(k_i k_j k_{m_1} P)\,\mathrm{tr}_+(k_i k_j k_{m_2} k_{m_1})\,;$$
$$a_z := \frac{m_1-m_2}{N(P_{L;z})}\,; \qquad (2.2.13)$$

and

$$N(P) := (m_1-m_2)P^2 - 2(m_1 P)(m_2 P)\,. \qquad (2.2.14)$$

A closer inspection of (2.2.12) reveals that the first line of that expression corresponds to two cuts of scalar triangle integrals, up to an ϵ-dependent factor and the explicit z-dependence of the two numerators. The second line is a term familiar from [37], corresponding to the $P^2_{L;z}$-cut of the finite part B of a scalar box function, defined in (2.1.9) (see also (2.1.7)). The full result for the one-loop MHV amplitudes is obtained by summing over all possible MHV diagrams, as specified in (2.2.4) and (2.2.2), (2.2.3).

2.2.3 The full amplitude

We begin our analysis by focusing on the box function contributions in (2.2.12), and notice the following important facts:

1. By taking into account the four terms in (2.2.11) and summing over Feynman diagrams, we see each fixed finite box function B appears in exactly four phase

[5] In (2.2.12) we omit an overall, finite numerical factor that depends on ϵ. This factor, which can be read off from (E.2.12), is irrelevant for our discussion.

space integrals, one for each of its possible cuts, in complete similarity with [37]. It was shown in Section 5 of that paper that the corresponding dispersion integration over z will then yield the finite B part of the scalar box functions F. It was also noted in [37] that one can make a particular gauge choice for such that the z-dependence in N disappears. This happens when is chosen to be equal to one of the massless external legs of the box function. The question of gauge invariance is further discussed in Appendix F.

2. The coefficient multiplying the finite box function is precisely equal to $b^{i;j}_{m_1;m_2}$ defined in (2.1.5).

3. Finally, the functions B generated by summing over all MHV Feynman diagrams with the range dictated by (2.2.3) are precisely those included in the double sum for the finite box functions in the first line of (2.1.2) (or (2.1.19)) upon identifying m_1 and m_2 with s and m. To be precise, (2.2.3) includes the case where the indices s and/or m (in the notation of (2.1.2) and (2.1.19)) are equal to either i or j; but for any of these choices, it is easy to check that the corresponding coefficient $b^{i;j}_{m;s}$ vanishes.

This settles the agreement between the result of our computation with MHV vertices and (2.1.19) for the part corresponding to the box functions. Next we have to collect the cuts contributing to particular triangles, and show that the z-integration reproduces the expected triangle functions from (2.1.19), each with the correct coefficient.

To this end, we notice that for each fixed triangle function $T(p;P;Q)$, exactly four phase space integrals appear, two for each of the two possible cuts of the function. Moreover, a gauge invariance similar to that of the box functions also exists for triangle cuts (see Appendix F), so that we can choose in a way that the T numerators in (2.2.12) become independent of z. A particularly convenient choice is $= k_i$, since it can be kept fixed for all possible cuts. Choosing this gauge, we see that a sum, T, of terms proportional to cut-triangles is generated from (2.2.11) (up to a common normalisation):

$$T := T_A + T_B + T_C + T_D \; ; \tag{2.2.15}$$

where

$$T_A := \frac{S(i;j;m_1;m_2)}{(m_1 - m_2)} \quad \frac{S(i;j;m_1 - 1;m_2)}{((m_1 - 1) - m_2)} \quad S(i;j;m_2;P_L) \;\; {}_A \; ; \tag{2.2.16}$$

$$T_B := \frac{S(i;j;m_2;m_1)}{(m_1 - m_2)} \quad \frac{S(i;j;m_2 + 1;m_1)}{((m_2 + 1) - m_1)} \quad S(i;j;m_1;P_L) \;\; {}_B \; ;$$

$$T_C := \frac{S(i;j;m_2 + 1;m_1 - 1)}{((m_2 + 1) - (m_1 - 1))} \quad \frac{S(i;j;m_2;m_1 - 1)}{(m_2 - (m_1 - 1))} \quad S(i;j;m_1 - 1;P_L) \;\; {}_C \; ;$$

$$T_D := \left[\frac{S(i;j;m_1-1;m_2+1)}{((m_1-1)-(m_2+1))} - \frac{S(i;j;m_1;m_2+1)}{(m_1-(m_2+1))}\right] S(i;j;m_2+1;P_L)\,\Delta_D\;.$$

Here we have defined

$$S(a;b;c;d) = \mathrm{tr}_+(\not k_a \not k_b \not k_c \not k_d)\;; \tag{2.2.17}$$

and Δ_I, $I=A;\ldots;D$, are the following cut-triangles, all in the $P_{L;z}$-cut:

$$\begin{aligned}
\Delta_A &:= \frac{1}{(m_2-P_{L;z})} = Q^2\text{-cut of } T\left(m_2;P_{L;z}-m_2;-P_{L;z}\right); \qquad (2.2.18)\\
\Delta_B &:= \frac{1}{(m_1-P_{L;z})} = P^2\text{-cut of } T\left(m_1;-P_{L;z};P_{L;z}-m_1\right);\\
\Delta_C &:= \frac{1}{((m_1-1)-P_{L;z})} = Q^2\text{-cut of } T\left(m_1-1;-P_{L;z}-(m_1-1);P_{L;z}\right);\\
\Delta_D &:= \frac{1}{((m_2+1)-P_{L;z})} = P^2\text{-cut of } T\left(m_2+1;P_{L;z};-P_{L;z}-(m_2+1)\right):
\end{aligned}$$

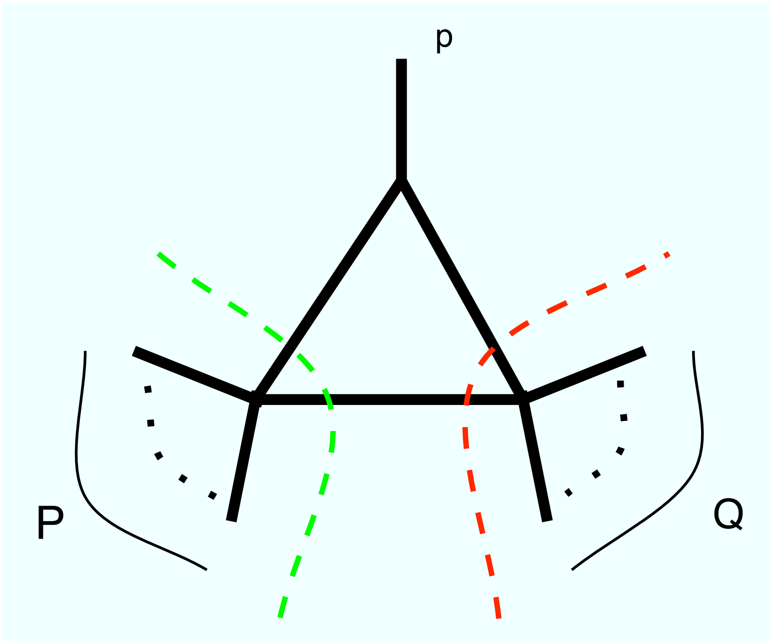


Figure 2.5: A triangle function with massive legs labelled by P and Q, and massless leg p. This function is reconstructed by summing two dispersion integrals, corresponding to the P_z^2– and Q_z^2-cut.

Next, we notice that the prefactors multiplying Δ_B, Δ_C become the same, up to a minus sign, upon shifting $m_1-1 \to m_1$ in the second prefactor; and so do the prefactors of Δ_A, Δ_D upon shifting $m_2 \to m_2+1$. Doing this, Δ_B and the shifted Δ_C become the two cuts of the same triangle function $T(m_1;-P_{L;z};P_{L;z}-m_1)$, and similarly, Δ_A and Δ_D give the two cuts of the function $T(m_2;P_{L;z}-m_2;-P_{L;z})$. Furthermore, in

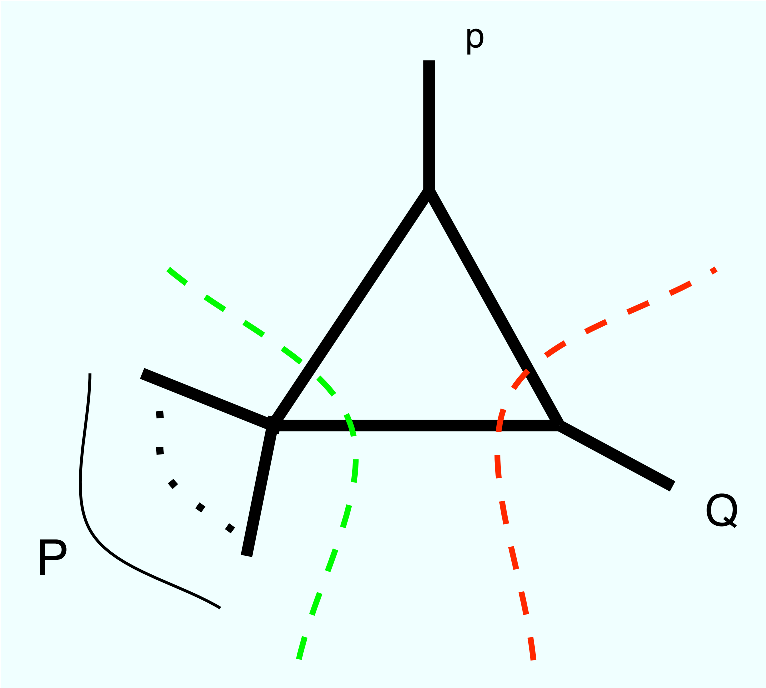


Figure 2.6: A degenerate triangle function. Here the leg labelled by P is still massive, but that labelled by Q becomes massless. This function is also reconstructed by summing over two dispersion integrals, corresponding to the P_z^2– and Q_z^2-cut.

Appendix F we will show that summing the two dispersion integrals of the two different cuts of a triangle indeed generates the triangle function { in fact this procedure gives a novel way of obtaining the triangle functions.[6] Specifically, the result derived in Appendix F is

$$\int \frac{dz}{z} \quad \frac{(P_z^2)}{(P_z p)} + \frac{(Q_z^2)}{(Q_z p)} \quad = 2 \quad \csc(\quad) T(p;P;Q) ; \qquad (2.2.19)$$

where the -dependent triangle function $T(p;P;Q)$ (with $p+P+Q=0$) was introduced in (2.1.14) and gives, as $\;\to 0$, the triangle function (2.1.13) (as well as the bubbles when either P^2 or Q^2 vanish). The result (2.2.19) holds for a generic choice of the reference vector , see (F.1.6)–(F.1.11). We give a pictorial representation of the non-degenerate and degenerate triangle functions in Figures 2.5 and 2.6, respectively.

[6] A remark is in order here. In our procedure the momentum appearing in each of the possible cuts is always shifted by an amount proportional to z ; the triangle is then reproduced by performing the appropriate dispersion integrals. Because of the above mentioned shift, we produce a non-vanishing cut (with shifted momentum) even when the cut includes only one external (massless) leg, say K, as the momentum owing in the cut is effectively $K_z = K \quad z$, so that $K_z^2 \neq 0$.

At this point, it should be noticed that for a gauge choice different from $\eta = k_i$ adopted so far, the numerators T in (2.2.12) do acquire an η-dependence. This gauge dependence should not be present in the final result for the scattering amplitude. Indeed, it is easy to check that, thanks to (F.1.6), the coefficient of the η-dependent terms actually vanishes.

Using (2.2.15)–(2.2.19) and collecting terms as specified above, we see that the generic term produced by this procedure takes the form

$$\left[\frac{S(i,j,a,p_m)}{(k_a \cdot p_m)} - \frac{S(i,j,a+1,p_m)}{(k_{a+1} \cdot p_m)}\right] S(i,j,p_m,Q)\, T(p_m,P,Q)\,, \tag{2.2.20}$$

with $P = q_{a+1,m-1}$ and $Q = q_{m+1,a}$.

Finally, we implement the symmetrization of the indices i, j, as explained earlier, and convert (2.2.20) into

$$c^{i,j}_{m,a}\, T(p_m,P,Q)\,, \tag{2.2.21}$$

where the coefficient $c^{i,j}_{m,a}$ is[7]

$$c^{i,j}_{m,a} \coloneqq \frac{1}{2}\left[\frac{S(i,j,a+1,p_m)}{(k_{a+1} \cdot p_m)} - \frac{S(i,j,a,p_m)}{(k_a \cdot p_m)}\right] \frac{S(i,j,p_m,q_{m,a}) - S(i,j,q_{m,a},p_m)}{[(k_i + k_j)^2]^2}\,; \tag{2.2.22}$$

which coincides with the definition of $c^{i,j}_{m,a}$ given in (2.1.6). Lastly, it is easy to see that in summing over the range given by (2.2.3), we produce exactly all the triangle functions appearing in the second line of (2.1.19). It is also important to notice that the bubbles, which appear in the last line of (2.1.2), are actually obtained as particular cases of triangle functions where one of the massive legs becomes massless, as observed at the end of §2.1.

In conclusion, we have seen that all the terms in (2.1.19), i.e. finite box contributions and triangle contributions – which include the bubbles as special (degenerate) cases – are precisely reproduced in our diagrammatic approach.

[7] In writing (2.2.22), we make also use of the fact that $S(i,j,q_{m-1,a},p_m) = S(i,j,q_{m,a},p_m)$.

CHAPTER 3

NON-SUPERSYMMETRIC MHV AMPLITUDES

Having seen that the CSW rules can be applied at loop level in supersymmetric gauge theories, the obvious question is whether the same also holds in non-supersymmetric gauge theories. To this end the one-loop MHV amplitudes in pure Yang-Mills with a scalar running in the loop were computed in [43]. This is the last contribution to the MHV amplitudes for gluon scattering in QCD in the supersymmetric decomposition of Eq. (1.3.3) and has only been computed previously in certain special cases in [42, 44].

In this chapter we follow [43] and apply the CSW rules to this scalar amplitude in the general case of n-gluon MHV scattering where the two negative-helicity gluons sit at arbitrary positions. We find that the results agree perfectly with those already obtained in [42, 44] and we go on to present the general result for the cut-constructible part of the one-loop MHV amplitudes in pure Yang-Mills. It turns out that the CSW rules only compute this cut-constructible part and the rational terms (which do not contain cuts) are not found. This is discussed in [43] and §3.2.1 below. They can and have, however, been recently computed using an on-shell unitarity bootstrap [45] which thus completely determines the one-loop MHV n-gluon amplitudes in QCD.

3.1 The scalar amplitude

In complete similarity with the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ cases – see Chapters 1 & 2 and e.g. [37, 40] – we can immediately write down the expression for the scalar amplitude in terms of MHV vertices as

$$A_n^{\text{scalar}} = \sum_{m_1, m_2} \int d\mathcal{M}\; A(-l_1, m_1, \ldots, i^-, \ldots, m_2, l_2)\, A(-l_2, m_2+1, \ldots, j^-, \ldots, m_1-1, l_1)\,, \tag{3.1.1}$$

where the ranges of summation of m_1 and m_2 are

$$j+1 \le m_1 \le i\,, \qquad i \le m_2 \le j-1\,. \tag{3.1.2}$$

A typical MHV diagram contributing to A_n^{scalar}, for fixed m_1 and m_2, is depicted in Figure 3.1. The off-shell vertices A in (3.1.1) correspond to having complex scalars

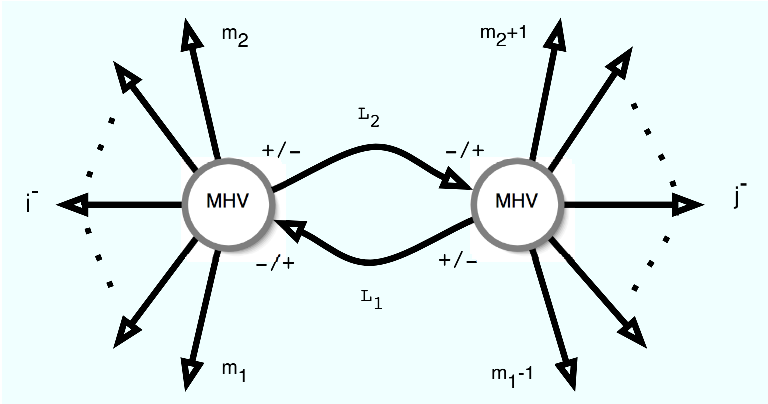


Figure 3.1: A one-loop MHV diagram with a complex scalar running in the loop, computed in Eq. (3.1.1). We have indicated the possible helicity assignments for the scalar particle.

running in the loop. It follows that there are two possible helicity assignments[1] for the scalar particles in the loop which have to be summed over. These two possibilities are denoted by $\pm$ in (3.1.1) and in the internal lines in Figure 3.1. It turns out that each of them gives rise to the same integrand for (3.1.1):

$$iA_n^{\mathrm{tree}} \frac{\langle m_2\, m_2+1\rangle\langle m_1-1\, m_1\rangle\langle i l_1\rangle^2\langle j l_1\rangle^2\langle i l_2\rangle^2\langle j l_2\rangle^2}{\langle i j\rangle^4\langle m_1\, l_1\rangle\langle m_1-1\, l_1\rangle\langle m_2\, l_2\rangle\langle m_2+1\, l_2\rangle\langle l_1\, l_2\rangle^2} : \qquad (3.1.3)$$

A crucial ingredient in (3.1.1) is (as before in Chapters 1 & 2) the integration measure $d\mathcal{M}$. This measure was constructed in [37, 79] using the decomposition $L := l + z\eta$ for a non-null four-vector L in terms of a null vector l and a real parameter z as reviewed in §1.8.2. We refer the reader to §1.8.2 and [37, 79] for the construction of this measure, and here we merely quote the result:

$$d\mathcal{M} = 2\pi i\, \delta(P^2_{L;z}) \frac{dP^2_{L;z}}{P^2_{L;z} - P^2_L - i\varepsilon}\, d^{4-2\epsilon}\mathrm{LIPS}(l_2, -l_1; P_{L;z}) : \qquad (3.1.4)$$

In order to calculate (3.1.1), we will first integrate the expression (3.1.3) over the Lorentz-invariant phase space (appropriately regularised to $4-2\epsilon$ dimensions), and then perform the dispersion integral.

For the sake of clarity, we will separate the analysis into two parts. Firstly, we will

[1] For scalar fields, the "helicity" simply distinguishes particles from antiparticles (see, for example, [154]).

present the (simpler) calculation of the amplitude in the case where the two negative-helicity gluons are adjacent. This particular amplitude has already been computed by Bern, Dixon, Dunbar and Kosower in [42] using the cut-constructibility approach; the result we will derive here will be in precise agreement with the result in that approach. Then, in §3.3 we will move on to address the general case, deriving new results.

3.2 The scattering amplitude in the adjacent case

The adjacent case corresponds to choosing $i = m_1$, $j = m_1 - 1$ in Figure 3.1. Therefore we now have a single sum over MHV diagrams, corresponding to the possible choices of m_2. We will also set $i = 2$, $j = 1$ for the sake of definiteness, and $m_2 = m$.

After conversion into traces, the integrand of (3.1.1) takes on the form:

$$\frac{\mathrm{tr}_+(\not k_1 \not k_2 \not P_{L;z} \not l_2)\,\mathrm{tr}_+(\not k_1 \not k_2 \not l_2 \not P_{L;z})}{2^5 (k_1 \cdot k_2)^3 (l_1 \cdot l_2)^2}\left\{\frac{\mathrm{tr}_+(\not k_1 \not k_2 \not k_{m+1} \not l_2)}{(l_2 \cdot k_{m+1})} - \frac{\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not l_2)}{(l_2 \cdot k_m)}\right\};\qquad (3.2.1)$$

where we note that $(l_1 \cdot l_2) = -P^2_{L;z}/2$ by momentum conservation.

The next step consists of performing the Passarino-Veltman reduction [212] of the Lorentz-invariant phase space integral of (3.2.1). This requires the calculation of the three-index tensor integral

$$I(m; P_{L;z}) = \int dLIPS(l_2; -l_1; P_{L;z})\,\frac{l_2\, l_2\, l_2}{(l_2 \cdot k_m)}:\qquad (3.2.2)$$

This calculation is performed in Appendix G. The result of this procedure gives the following term at $O(\epsilon^0)$, which we will later integrate with the dispersive measure:

$$\tilde{A}^{\mathrm{scalar}}_n = -\frac{}{3}(-P^2_{L;z})\,\frac{[\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not P_{L;z})]^2}{2^5 (k_1 \cdot k_2)^3}\left\{\frac{\mathrm{tr}_+(\not k_1 \not k_2 \not P_{L;z} \not k_m)}{(k_m \cdot P_{L;z})^3} + \frac{2(k_1 \cdot k_2)}{(k_m \cdot P_{L;z})^2}\right\}$$
$$-(m \leftrightarrow m+1);\qquad (3.2.3)$$

and we have dropped a factor of $4\,\hat{c}\,A^{\mathrm{tree}}$ on the right hand side of (3.2.3), where $\hat{c}$ is defined in (G.1.11). We can reinstate this factor at the end of the calculation. We also notice that (3.2.3) is a finite expression, i.e. it is free of infrared poles.

3.2.1 Rational terms

An important remark is in order here. On general grounds, the result of a phase space integral in, say, the P^2-channel, is of the form

$$I(\epsilon) = (-P^2)^{-\epsilon}\, f(\epsilon);\qquad (3.2.4)$$

where

$$f(\epsilon) = \frac{f_{-1}}{\epsilon} + f_0 + f_1\,\epsilon + \cdots\,; \tag{3.2.5}$$

and f_i are rational coefficients. In the case at hand, infrared poles generated by the phase space integrals cancel completely, so that we can in practice replace (3.2.5) by $f(\epsilon) \to f_0 + f_1\,\epsilon + \cdots$. The amplitude A is then obtained by performing a dispersion integral, which converts (3.2.4) into an expression of the form

$$A(\epsilon) = \frac{(-P^2)^{-\epsilon}}{\epsilon}\, g(\epsilon) = \frac{g_0}{\epsilon} - g_0 \log(-P^2) + g_1 + O(\epsilon)\,; \tag{3.2.6}$$

where $g(\epsilon) = g_0 + g_1\,\epsilon + \cdots$, and the coefficients g_i are rational functions, i.e. they are free of cuts. Importantly, errors can be generated in the evaluation of phase space integrals if one contracts $(4-2\epsilon)$-dimensional vectors with ordinary four-vectors. This does not affect the evaluation of the coefficient $g_0 := g(\epsilon = 0)$, and hence the part of the amplitude containing cuts is reliably computed; but the coefficients g_i for $i \geq 1$, in particular g_1, are in general affected. This implies that rational contributions to the scattering amplitude cannot be detected [42] in this construction. A notable exception to this is provided by the phase space integrals which appear in supersymmetric theories. These are "four-dimensional cut-constructible" [42], in the sense that the rational parts are unambiguously linked to the discontinuities across cuts, and can therefore be uniquely determined.[2] This occurs, for example, in the calculation of the $\mathcal{N} = 4$ MHV amplitudes at one-loop performed in [37] and reviewed in §1.9 and the $\mathcal{N} = 1$ MHV amplitudes at one-loop in Chapter 2. In the present case, however, the relevant phase space integrals violate the cut-constructibility criteria given in [42][3], since we encounter tensor triangles with up to three loop momenta in the numerator. Hence, we will be able to compute the part of the amplitude containing cuts, but not the rational terms. In practice this means that we will compute all phase space integrals up to $O(\epsilon^0)$ and discard $O(\epsilon)$ contributions, which would generate rational terms that cannot be determined correctly.

3.2.2 Dispersion integrals for the adjacent case

After this digression, we now move on to the dispersive integration. In the center of mass frame, where $P_{L;z} := P_{L;z}(1, \vec{0})$, all the dependence on $P_{L;z}$ in (3.2.3) cancels out, as there are equal powers of $P_{L;z}$ in the numerator as in the denominator of any term. As a consequence, the dependence on the arbitrary reference vector η disappears (see [41] for the application of this argument to the $\mathcal{N} = 1$ case). We are thus left with

[2] For more details about cut-constructibility, see the detailed analysis in Sections 3–5 of [42] and Appendix D of this thesis for a brief review.

[3] An example of an integral violating the power-counting criterion of [42] is provided by (G.1.3).

dispersion integrals of the form

$$I(P_L^2) := \int \frac{ds'}{s' - P_L^2}\,\rho(s') = \frac{1}{\epsilon}\left[-\pi\csc(\pi\epsilon)\right](-P_L^2)^{-\epsilon} : \qquad (3.2.7)$$

Taking this into account, the dispersion integral of (3.2.3) then gives

$$\tilde{A}_n^{\text{scalar}} = -\pi\csc(\pi\epsilon)\frac{\epsilon}{3}\frac{(-P_L^2)^{-\epsilon}}{\ldots}\Bigg[\frac{[\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m P\!\!\!/_L)]^2}{2^5 (k_1\cdot k_2)^3}\Bigg. $$
$$\Bigg.\frac{\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 P\!\!\!/_L k\!\!\!/_m)}{(m\cdot P_L)^3} + \frac{2(k_1\cdot k_2)}{(m\cdot P_L)^2}\Bigg] - (m \leftrightarrow m+1) : \qquad (3.2.8)$$

The momentum flow can be conveniently represented as in Figure 3.2, where we define

$$P := q_{2,m-1}\,; \qquad Q := q_{m+1,1} = -q_{2,m}\,; \qquad (3.2.9)$$

and $q_{p_1,p_2} := \sum_{l=p_1}^{p_2} k_l$. We also have $P_L := q_{2,m} = -Q$.

Now we wish to combine the terms written explicitly in (3.2.8) with those that arise under $m \leftrightarrow m+1$. Since (3.2.8) is summed over m, we simply shift $m+1 \to m$ in these latter terms. Let us now focus our attention on the second term in (3.2.3) (similar manipulations will be applied to the first term). Writing the $m \leftrightarrow m+1$ term explicitly, we obtain a contribution proportional to

$$(-P_L^2)^{-\epsilon}\left(\frac{[\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m P\!\!\!/_L)]^2}{(m\cdot P_L)^2} - \frac{[\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_{m+1} P\!\!\!/_L)]^2}{((m+1)\cdot P_L)^2}\right) : \qquad (3.2.10)$$

By shifting $m+1 \to m$ in the second term of (3.2.10), we change its P_L so that $P_L \to q_{2,m-1} = P$ (whereas, in the non-shifted term, $P_L = -Q$). The expression (3.2.10) then reads

$$\frac{[\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m Q\!\!\!/)]^2}{(m\cdot Q)^2}\left[(-Q^2)^{-\epsilon} - (-P^2)^{-\epsilon}\right] ; \qquad (3.2.11)$$

where we used $\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m Q\!\!\!/) = -\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m P\!\!\!/)$ and $Q\cdot m = -P\cdot m$. Notice also that $m\cdot Q = (1/2)(Q^2 - P^2)$.

Next we re-instate the antisymmetry of the amplitudes under the exchange of the indices $1 \leftrightarrow 2$ (which is manifest from equation (3.1.3)). Doing this we get

$$\left(\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m Q\!\!\!/)\right)^2 \to \frac{1}{2}\Big[\left(\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m Q\!\!\!/)\right)^2 - \left(\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 Q\!\!\!/ k\!\!\!/_m)\right)^2\Big] \qquad (3.2.12)$$
$$= -2(k_1\cdot k_2)(m\cdot Q)\Big[\mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 k\!\!\!/_m Q\!\!\!/) - \mathrm{tr}_+(k\!\!\!/_1 k\!\!\!/_2 Q\!\!\!/ k\!\!\!/_m)\Big] :$$

Following similar steps for the first term in (3.2.8), we arrive at the following expression

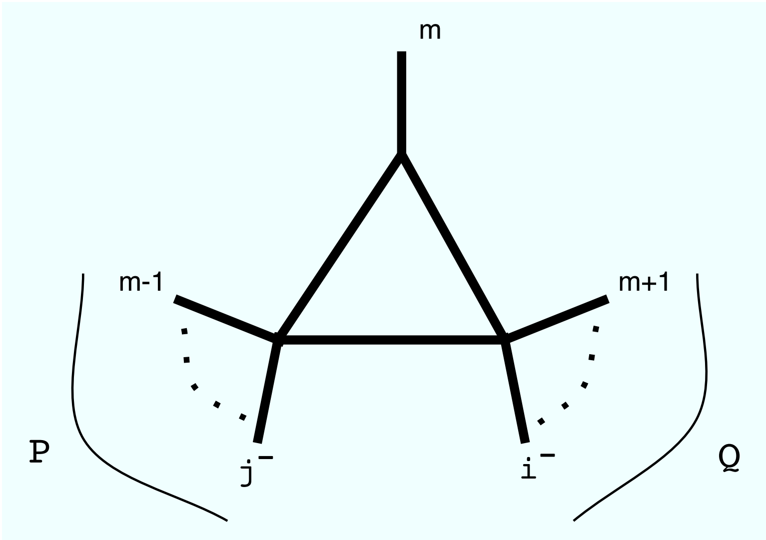


Figure 3.2: A triangle function contributing to the amplitude in the case of adjacent negative-helicity gluons. Here we have defined $P := q_{j,m-1}$, $Q := q_{m+1,i} = -q_{j,m}$ (in the text we set $i = 1$, $j = 2$ for definiteness).

for the amplitude before taking the $\epsilon \to 0$ limit:

$$A^{\epsilon} = A_{1;\epsilon} + A_{2;\epsilon}\,, \tag{3.2.13}$$

where

$$\begin{aligned}
A_{1;\epsilon} &= \frac{A^{\rm tree}}{t_1^{[2]}}\,\frac{\epsilon}{6}\Big[\mathrm{tr}_+(\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1}) - \mathrm{tr}_+(\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m)\Big]\, T(m; q_{2,m-1}, q_{2,m})\,;\\
A_{2;\epsilon} &= \frac{A^{\rm tree}}{(t_1^{[2]})^3}\,\frac{\epsilon}{3}\Big[\mathrm{tr}_+(\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1})^2\,\mathrm{tr}_+(\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m)\\
&\quad - \mathrm{tr}_+(\not{k}_1\not{k}_2\not{k}_m\not{q}_{m,1})\,\mathrm{tr}_+(\not{k}_1\not{k}_2\not{q}_{m,1}\not{k}_m)^2\Big]\, T^{(3)}(m; q_{2,m-1}, q_{2,m})\,;
\end{aligned} \tag{3.2.14}$$

and $t_1^{[2]}$ follows from the definition of equation (1.9.5). In order to write (3.2.14) in a compact from, we have introduced ϵ-dependent triangle functions [40] as in the previous chapter (c.f. Eq. (2.1.14))

$$T^{(r)}(p; P, Q) := \frac{1}{\epsilon}\,\frac{(-P^2)^{-\epsilon} - (-Q^2)^{-\epsilon}}{(Q^2 - P^2)^r}\,, \tag{3.2.15}$$

where $p + P + Q = 0$, and r is a positive integer.[4]

[4] For $r = 1$ we will omit the superscript (1) in $T^{(1)}$.

We can now take the $\epsilon \to 0$ limit. As long as P^2 and Q^2 are non-vanishing, one has

$$\lim_{\epsilon \to 0} T^{(r)}(p;P;Q) = T^{(r)}(p;P;Q)\,; \qquad P^2 \neq 0\,; Q^2 \neq 0\,; \tag{3.2.16}$$

where the ϵ-independent triangle functions are defined by

$$T^{(r)}(p;P;Q) := \frac{\log(Q^2/P^2)}{(Q^2 - P^2)^r}\,. \tag{3.2.17}$$

If either of the invariants vanishes, the limit of the ϵ-dependent triangle gives rise to an infrared-divergent term (which we call a "degenerate" triangle – this is one with two massless legs). For example, if $Q^2 = 0$, one has

$$T(p;P;Q)\big|_{Q^2=0} \to \frac{1}{\epsilon}\frac{(-P^2)^{-\epsilon}}{P^2}\,; \qquad \epsilon \to 0\,. \tag{3.2.18}$$

The two possible configurations which give rise to infrared-divergent contributions correspond to the following two possibilities:

a. $q_{2,m-1} = k_2$ (hence $q^2_{2,m-1} = 0$). In this case we also have $q^2_{2,m} = t_2^{[2]}$.

b. $q_{2,m} = k_1$ (hence $q^2_{2,m} = 0$). Therefore $q^2_{2,m-1} = t_n^{[2]}$.

We notice that infrared poles will appear only in terms corresponding to the triangle function T. Indeed, whenever one of the kinematical invariants contained in $T^{(3)}$ vanishes, the combination of traces multiplying this function in (3.2.14) vanishes as well.

In conclusion we arrive at the following result, where we have explicitly separated-out the infrared-divergent terms:[5]

$$A_n^{\text{scalar}} = A_{\text{poles}} + A_1 + A_2\,; \tag{3.2.19}$$

where

$$\begin{aligned}
A_{\text{poles}} &= \frac{1}{6}A^{\text{tree}}\frac{1}{\epsilon^2}\Big[(-t_2^{[2]})^{-\epsilon} + (-t_n^{[2]})^{-\epsilon}\Big]\,; \qquad (3.2.20)\\
A_1 &= \frac{1}{6}A^{\text{tree}}\frac{1}{t_1^{[2]}}\sum_{m=4}^{n-1}\Big[\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m,1}) - \mathrm{tr}_+(\not k_1 \not k_2 \not q_{m,1} \not k_m)\Big] T(m;q_{2,m-1};q_{2,m})\,;\\
A_2 &= \frac{1}{3}A^{\text{tree}}\frac{1}{(t_1^{[2]})^3}\sum_{m=4}^{n-1}\Big[\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m,1})^2\,\mathrm{tr}_+(\not k_1 \not k_2 \not q_{m,1} \not k_m)\\
&\qquad - \mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m,1})\,\mathrm{tr}_+(\not k_1 \not k_2 \not q_{m,1} \not k_m)^2\Big] T^{(3)}(m;q_{2,m-1};q_{2,m})\,.
\end{aligned}$$

---

[5] A factor of $4\,\hat{}$ will be understood on the right hand sides of Eqs. (3.2.19), (3.2.21) and (3.2.23), where $\hat{}$ is defined in (G.1.11).

More compactly, we can recognise that A_{poles} and A_1 reconstruct the contribution of an $\mathcal{N}=1$ chiral multiplet, and rewrite (3.2.19) as

$$A_n^{\text{scalar}} = \frac{1}{3}A_{12}^{\mathcal{N}=1;\text{chiral}} + \frac{1}{3}A_{12}^{\text{tree}}\frac{1}{(t_1^{[2]})^3}\sum_{m=4}^{n-1} B_{12}^m\, T^{(3)}(m;q_{2,m-1};q_{2,m})\,; \tag{3.2.21}$$

where

$$\begin{aligned} B_{12}^m = {} & \left[\operatorname{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m;1})\right]^2 \operatorname{tr}_+(\not k_1 \not k_2 \not q_{m;1} \not k_m) \\ & - \left[\operatorname{tr}_+(\not k_1 \not k_2 \not q_{m;1} \not k_m)\right]^2 \operatorname{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m;1})\,: \end{aligned} \tag{3.2.22}$$

and

$$\begin{aligned} A_{12}^{\mathcal{N}=1;\text{chiral}} = {} & \frac{1}{2}A_{12}^{\text{tree}}\frac{1}{t_1^{[2]}}\sum_{m=3}^{n}\Big\{ \operatorname{tr}_+(\not k_1 \not k_2 \not k_m \not q_{m;1}) - \operatorname{tr}_+(\not k_1 \not k_2 \not q_{m;1} \not k_m)\Big\} \\ & \qquad T(m;q_{2,m-1};q_{2,m})\,: \end{aligned} \tag{3.2.23}$$

This is our result for the cut-constructible part of the n-gluon MHV scattering amplitude with adjacent negative-helicity gluons in positions 1 and 2. This expression was first derived by Bern, Dixon, Dunbar and Kosower in [42], and our result agrees precisely with this. A remark is in order here. In [42], the final result is expressed in terms of a function

$$L_2(x) := \frac{\log x - (x - 1/x)/2}{(1-x)^3}\,; \tag{3.2.24}$$

which contains a rational part $-(x-1/x)/2(1-x)^3$ which removes a spurious third-order pole from the amplitude. With our approach however we did not expect to detect rational terms in the scattering amplitude, and indeed we do not find such terms.[6] Furthermore, we do not find the other rational terms which are known to be present in the one-loop scattering amplitude [44, 45].

3.3 The scattering amplitude in the general case

The situation where the negative-helicity gluons are not adjacent is technically more challenging. Our starting point will be (3.1.3), to which we will apply the Schouten identity (see Appendix A for a collection of spinor identities used). Eq. (3.1.3) can then

[6] In our notation L_2 corresponds to $T^{(3)}$, which, however, lacks a rational term.

be written as a sum of four terms:[7]

$$\hat{\mathscr{C}} = \mathscr{C}(m_1;m_2+1) - \mathscr{C}(m_1;m_2) - \mathscr{C}(m_1-1;m_2+1) + \mathscr{C}(m_1-1;m_2) \tag{3.3.1}$$

where

$$\mathscr{C}(a;b) := \frac{\langle i l_1\rangle\langle j l_1\rangle^2\langle i l_2\rangle^2\langle j l_2\rangle}{\langle ij\rangle^4\langle l_1 l_2\rangle^2}\,\frac{\langle ia\rangle\langle jb\rangle}{\langle l_1 a\rangle\langle l_2 b\rangle}\,. \tag{3.3.2}$$

The calculation of the phase space integral of this expression is discussed in Appendix G. The result is

$$\begin{aligned}
&\int d^{4-2\epsilon}\,\mathrm{LIPS}(l_2;-l_1;P_{L;z})\,\mathscr{C}(a;b)\\
&= \frac{1}{3}\frac{\mathrm{tr}_+(\not i\not j\not b\not a)}{(a\cdot b)}\left[\mathrm{tr}_+(\not i\not j\not P_{L;z}\not a)^2\left(\frac{\mathrm{tr}_+(\not i\not j\not a\not P_{L;z})}{(P_{L;z}\cdot a)^3}+\frac{2(i\cdot j)}{(P_{L;z}\cdot a)^2}\right)-(a\leftrightarrow b)\right]\\
&\quad+\frac{1}{2}\frac{\mathrm{tr}_+(\not i\not j\not b\not a)\,\mathrm{tr}_+(\not i\not j\not a\not b)}{(a\cdot b)^2}\left[\frac{\mathrm{tr}_+(\not i\not j\not P_{L;z}\not a)^2}{(P_{L;z}\cdot a)^2}+(a\leftrightarrow b)\right]\\
&\quad-\frac{\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not b\not a)}{(a\cdot b)^3}\left[\frac{\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not P_{L;z}\not a)}{(P_{L;z}\cdot a)}+(a\leftrightarrow b)\right]\\
&\quad+\frac{\mathrm{tr}_+(\not i\not j\not a\not b)^2\,\mathrm{tr}_+(\not i\not j\not b\not a)^2}{(a\cdot b)^4}\log\left(1-\frac{(a\cdot b)}{N}P_{L;z}^2\right)\,;
\end{aligned} \tag{3.3.3}$$

where $N \equiv N(P) := (a\cdot b)P^2 - 2(P\cdot a)(P\cdot b)$, and we have suppressed a factor of $4\pi^{\epsilon}\hat{\Gamma}(-P_L^2)^{-\epsilon}[2(i\cdot j)^4]^{-1}$ on the right hand side of (3.3.3), where $\hat{\Gamma}$ is defined in (G.1.11). We notice that (3.3.3) is symmetric under the simultaneous exchange of i with j and a with b. This symmetry is manifest in the coefficient multiplying the logarithm – the last term in (3.3.3); for the remaining terms, nontrivial gamma matrix identities are required. For instance, consider the terms in the second line of (3.3.3). These terms are present in the adjacent gluon case (3.2.3), and it is therefore natural to expect that the trace structure of this term is separately invariant when $i \leftrightarrow j$ and $a \leftrightarrow b$. Indeed this is the case thanks to the identity

$$\begin{aligned}
32(i\cdot j)^3 &= \mathrm{tr}_+(\not i\not j\not P_{L;z}\not a)^2\left(\frac{\mathrm{tr}_+(\not i\not j\not a\not P_{L;z})}{(P_{L;z}\cdot a)^3}+\frac{2(i\cdot j)}{(P_{L;z}\cdot a)^2}\right)\\
&\quad+\mathrm{tr}_+(\not i\not j\not a\not P_{L;z})^2\left(\frac{\mathrm{tr}_+(\not i\not j\not P_{L;z}\not a)}{(P_{L;z}\cdot a)^3}+\frac{2(i\cdot j)}{(P_{L;z}\cdot a)^2}\right)\,.
\end{aligned} \tag{3.3.4}$$

[7] We drop the factor of iA_n^{tree} from now on and reinstate it at the end of the calculation.

Similar identities show that the third and fourth line of (3.3.3) are invariant under the simultaneous exchange $i \leftrightarrow j$ and $a \leftrightarrow b$.

The next step is to perform the dispersion integral of (3.3.3), i.e. the integral over the variable z which has been converted to an integral over $P_{L;z}$. The relevant terms are thus those involving $P_{L;z}$ in (3.3.3), and in an overall factor $(P_{L;z}^2)^{-\epsilon}$ arising from the dimensionally regulated measure.

The integral over the term involving the logarithm has been evaluated in [37], with the result

$$\int \frac{dP_{L;z}^2}{P_{L;z}^2 - P_L^2}(P_{L;z}^2)^{-\epsilon} \log\left(1 - \frac{(a\cdot b)}{N}P_{L;z}^2\right) = -\mathrm{Li}_2\left(1 - \frac{(a\cdot b)}{N(P)}P_L^2\right) + O(\epsilon)\,. \quad (3.3.5)$$

Notice that these terms were not present in the adjacent negative-helicity gluon case considered in §3.2.

Next we move on to the remaining terms in (3.3.3). Inspecting their z-dependence, we see that, in complete similarity with the adjacent case of §3.2, in each term there are the same powers of $P_{L;z}$ in the numerator as in the denominator. Hence, in the centre of mass frame in which $P_{L;z} := P_{L;z}(1,\vec{0})$, one finds that $P_{L;z}$ cancels completely. Note that this also immediately resolves the question of gauge invariance for these terms – this occurs only through the η dependence in $P_{L;z} = P_L - z\eta$. Furthermore, the box functions coming from (3.3.5) are separately gauge-invariant [37]. The conclusion is that our expression for the amplitude below, built from sums over MHV diagrams of the dispersion integral of (3.3.3), will be gauge-invariant. Moreover, apart from (3.3.5), the only other dispersion integral we will need is that computed in (3.2.7).

It follows from this discussion that the result of the dispersion integral of (3.3.3) is (suppressing a factor of $-4\hat{c}_\Gamma(-P_L^2)^{-\epsilon}\,\langle i\,j\rangle^8\,[2(i\cdot j)^4]^{-1}\,[-\pi\csc(\pi\epsilon)]$):

$$\int \frac{dz}{z}\int d^{4-2\epsilon}\,\mathrm{LIPS}(l_2,-l_1;P_{L;z})\,\mathscr{C}(a,b)$$

$$= -\frac{1}{\epsilon}(-P_L^2)^{-\epsilon}\Bigg\{\frac{1}{3}\frac{\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})}{(a\cdot b)}\left[-\mathrm{tr}_+(\not{i}\not{j}\not{P}_L\not{a})^2\frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{P}_L)}{(P_L\cdot a)^3} + \frac{2(i\cdot j)}{(P_L\cdot a)^2}\right] - (a\leftrightarrow b)$$

$$+\frac{1}{2}\frac{\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})}{(a\cdot b)^2}\left[\frac{\mathrm{tr}_+(\not{i}\not{j}\not{P}_L\not{a})^2}{(P_L\cdot a)^2} + (a\leftrightarrow b)\right]$$

$$-\frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})}{(a\cdot b)^3}\left[\frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})\mathrm{tr}_+(\not{i}\not{j}\not{P}_L\not{a})}{(P_L\cdot a)} + (a\leftrightarrow b)\right]\Bigg\}$$

$$+\frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2}{(a\cdot b)^4}\,\mathrm{Li}_2\left(1-\frac{(a\cdot b)}{N(P_L)}P_L^2\right)\,. \quad (3.3.6)$$

Now, due to the four terms in (3.3.1), the sum over MHV diagrams will include a signed sum over four expressions like (3.3.6). Let us begin by considering the last line of (3.3.6). This is a term familiar from [37] and [40] and corresponds to one of the four dilogarithms in the novel expression found in [37] for the finite part B of a scalar box function,

$$\begin{aligned} B(s,t,P^2,Q^2) = {} & \mathrm{Li}_2\left(1-\frac{(a\cdot b)}{N(P)}P^2\right) + \mathrm{Li}_2\left(1-\frac{(a\cdot b)}{N(P)}Q^2\right) \\ & - \mathrm{Li}_2\left(1-\frac{(a\cdot b)}{N(P)}s\right) - \mathrm{Li}_2\left(1-\frac{(a\cdot b)}{N(P)}t\right) , \end{aligned} \tag{3.3.7}$$

with $s := (P+a)^2$, $t := (P+b)^2$, and $P+Q+a+b=0$. By taking into account the four terms in (3.3.1) and summing over MHV diagrams as specified in (3.1.1) and (3.1.2), one sees that each of the four terms in any finite box function B appears exactly once, in complete similarity with [37] and [40]. The final contribution of this term will then be[8]

$$\sum_{m_1=j+1}^{i-1}\sum_{m_2=i+1}^{j-1} \frac{1}{2}\left(b^{ij}_{m_1m_2}\right)^2 B\left(q^2_{m_1,m_2-1}, q^2_{m_1+1,m_2}, q^2_{m_1+1,m_2-1}, q^2_{m_2+1,m_1-1}\right) , \tag{3.3.8}$$

where $t_i^{[k]} := (p_i + p_{i+1} + \cdots + p_{i+k-1})^2$ for $k \geq 0$, and $t_i^{[k]} = t_i^{[n-k]}$ for $k < 0$. In writing (3.3.8), we have taken into account that the dilogarithm in (3.3.6) is multiplied by a coefficient proportional to the square of $b^{ij}_{m_1m_2}$, where

$$b^{ij}_{m_1m_2} := -2\,\frac{\mathrm{tr}_+(k_ik_jk_{m_1}k_{m_2})\,\mathrm{tr}_+(k_ik_jk_{m_2}k_{m_1})}{[(k_i+k_j)^2]^2\,[(k_{m_1}+k_{m_2})^2]^2} . \tag{3.3.9}$$

We notice that $b^{ij}_{m_1m_2}$ is the coefficient of the box functions in the one-loop $\mathcal{N}=1$ MHV amplitude, originally calculated by Bern, Dixon, Dunbar and Kosower in [42], and derived in [40, 41] using the MHV diagram approach for loops proposed in [37]. Furthermore, we observe that $b^{ij}_{m_1m_2}$ is holomorphic in the spinor variables, and as such has simple localisation properties in twistor space. Indeed, from (3.3.9) it follows that

$$b^{ij}_{m_1m_2} = -2\,\frac{\langle m_1 i\rangle\langle m_2 i\rangle\langle j m_1\rangle\langle j m_2\rangle}{\langle ij\rangle^2\langle m_1 m_2\rangle^2} . \tag{3.3.10}$$

Summing over the four terms for the remainder of (3.3.6) can be done in complete similarity with §2.2 (and Section 4 of [40]).[9] We will skip the details of this derivation and now present our result.

[8] We multiply our final results by a factor of 2, which takes into account the two possible helicity assignments for the scalars in the loop.

[9] In §3.2 we have illustrated in detail how this sum is performed for the simpler case of adjacent negative-helicity gluons.

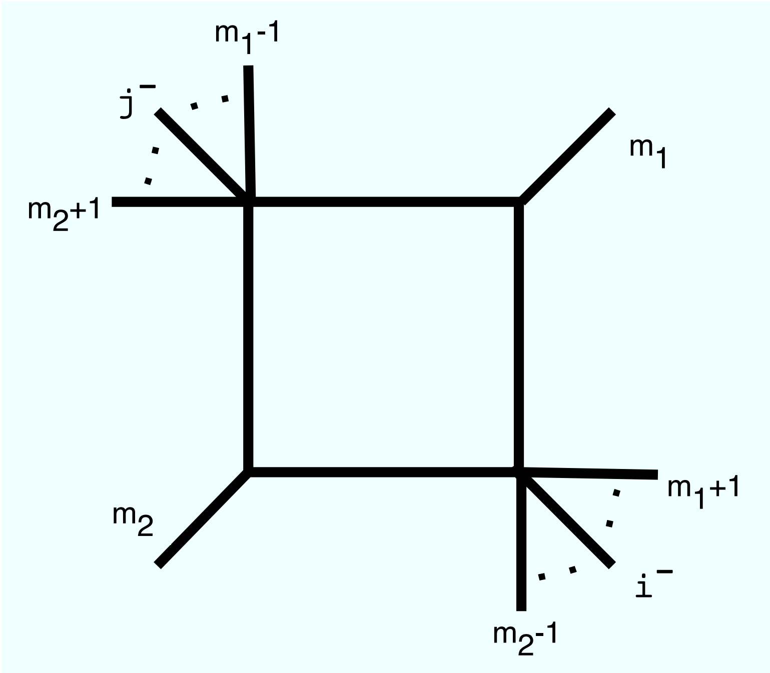


Figure 3.3: A box function contributing to the amplitude in the general case. The negative-helicity gluons, i and j, cannot be in adjacent positions, as the figure shows.

In order to do this, we find it convenient to define the following expressions:

$$A^{ij}_{m_1 m_2} := \frac{(i j m_2+1\, m_1)}{((m_2+1)-m_1)} - \frac{(i j m_2 m_1)}{(m_2 - m_1)} \tag{3.3.11}$$

$$= -2[ij]\langle m_1 i\rangle\langle m_1 j\rangle \frac{\langle m_2\, m_2+1\rangle}{\langle m_2+1\, m_1\rangle\langle m_1 m_2\rangle};$$

$$S^{ij}_{m_1 m_2} := \frac{(i j m_1\, m_2+1)(i j\, m_2+1\, m_1)}{((m_2+1)-m_1)^2} - \frac{(i j m_1 m_2)(i j m_2 m_1)}{(m_2-m_1)^2}; \tag{3.3.12}$$

$$I^{ij}_{m_1 m_2} := \frac{(i j m_1\, m_2+1)^2 (i j\, m_2+1\, m_1)}{((m_2+1)-m_1)^3} - \frac{(i j m_1 m_2)^2 (i j m_2 m_1)}{(m_2-m_1)^3}; \tag{3.3.13}$$

where for notational simplicity we set $(a_1 a_2 a_3 a_4) := \mathrm{tr}_+(\not{a}_1\not{a}_2\not{a}_3\not{a}_4)$ in the above. We also note the symmetry properties

$$A^{ji}_{m_1 m_2} = -A^{ij}_{m_1 m_2}; \qquad S^{ji}_{m_1 m_2} = S^{ij}_{m_1 m_2}: \tag{3.3.14}$$

The momentum flow is best described using the triangle diagram in Figure 3.4, where we use the following definitions:

$$P := q_{m_2+1,m_1-1} = -q_{m_1,m_2}\,; \qquad (3.3.15)$$
$$Q := q_{m_1+1,m_2}\,.$$

The triangle in Figure 3.5 also appears in the calculation, and can be converted into a triangle as in Figure 3.4 – but with i and j swapped – if one shifts $m_1-1\to m_1$, and then swaps $m_1 \leftrightarrow m_2$.

Next we introduce the coefficients

$$\mathcal{A}^{ij}_{m_1m_2} := 2^{-8}\langle i\,j\rangle^4 A^{ij}_{m_1m_2}\Big[\langle i|m_1Q|j\rangle^2\langle i|Qm_1|j\rangle - \langle i|m_1Q|j\rangle\langle i|Qm_1|j\rangle^2\Big]\,; \qquad (3.3.16)$$

$$\tilde{\mathcal{A}}^{ij}_{m_1m_2} := 2^{-8}\langle i\,j\rangle^4 A^{ij}_{m_1m_2}\Big[\langle i|m_1Q|j\rangle^2 - \langle i|Qm_1|j\rangle^2\Big]\,; \qquad (3.3.17)$$

$$\mathcal{S}^{ij}_{m_1m_2} = 2^{-8}\langle i\,j\rangle^4 S^{ij}_{m_1m_2}\Big[\langle i|m_1Q|j\rangle^2 + \langle i|Qm_1|j\rangle^2\Big]\,; \qquad (3.3.18)$$

$$\mathcal{I}^{ij}_{m_1m_2} := 2^{-8}\langle i\,j\rangle^4\Big[I^{ij}_{m_1m_2}\langle i|Qm_1|j\rangle + I^{ji}_{m_1m_2}\langle i|m_1Q|j\rangle\Big]\,. \qquad (3.3.19)$$

We will also make use of the $\;$-dependent triangle functions introduced in (3.2.15), whose $\;\to 0$ limits have been considered in (3.2.16)–(3.2.18). This is in order to write a compact expression which also incorporates the infrared-divergent terms.[10]

We can now present our result for the one-loop MHV amplitude:

$$\frac{A_{\rm scalar}}{A_{\rm tree}} = \sum_{m_1=j+1}^{i-1}\sum_{m_2=i+1}^{j-1}\frac{1}{2}\,b^{ij}_{m_1m_2}\,{}^{2}B(q^2_{m_1,m_2-1},q^2_{m_1+1,m_2},q^2_{m_1+1,m_2-1},q^2_{m_2+1,m_1-1})$$
$$-\frac{8}{3}\sum_{m_1=j+1}^{i-1}\sum_{m_2=i}^{j-1}\Big[\mathcal{A}^{ij}_{m_1m_2}T^{(3)}(m_1;P;Q) - (i\;j)\tilde{\mathcal{A}}^{ij}_{m_1m_2}T^{(2)}(m_1;P;Q)\Big]$$
$$+2\sum_{m_1=j+1}^{i-1}\sum_{m_2=i}^{j-1}\Big[\mathcal{S}^{ij}_{m_1m_2}T^{(2)}(m_1;P;Q) + \mathcal{I}^{ij}_{m_1m_2}T(m_1;P;Q)\Big] + (i\leftrightarrow j)\,; \qquad (3.3.20)$$

where on the right hand side of (3.3.20) a factor of $\;4\,\hat{\ }$ is understood and $\hat{\ }$ is defined

[10] The infrared-divergent terms will be described below and used to check that our result has the correct infrared pole structure.

in (G.1.11). We can also introduce the coefficient

$$c^{ij}_{m_1 m_2} \equiv \frac{1}{2}\left[\frac{(ij m_2+1\, m_1)}{(m_2+1)\, m_1} - \frac{(ij m_2 m_1)}{(m_2\, m_1)}\right]\frac{(ij m_1 Q) - (ij Q m_1)}{[(i+j)^2]^2}\;, \qquad (3.3.21)$$

which already appears as the coefficient multiplying the triangle function T in the $N = 1$ amplitude, (see e.g. Eq. (2.1.19)), and rewrite (3.3.20) as

$$\begin{aligned} F \;=\; & \sum_{m_1=j+1}^{i-1}\sum_{m_2=i+1}^{j-1} \frac{1}{2}\, b^{ij}_{m_1 m_2}\, \Delta^2 B(q^2_{m_1,m_2-1}, q^2_{m_1+1,m_2}, q^2_{m_1+1,m_2-1}, q^2_{m_2+1,m_1-1}) \\ & - \frac{1}{2}\sum_{m_1=j+1}^{i-1}\sum_{m_2=i}^{j-1} \frac{1}{3}\, c^{ij}_{m_1 m_2} \left[\frac{(ij m_1 Q)(ij Q m_1)}{2(i\cdot j)^2}\, T^{(3)}(m_1, P, Q) + T(m_1, P, Q)\right] \\ & + 2\sum_{m_1=j+1}^{i-1}\sum_{m_2=i}^{j-1}\left[S^{ij}_{m_1 m_2}\, T^{(2)}(m_1, P, Q) + I^{ij}_{m_1 m_2}\, T(m_1, P, Q)\right] + (i \leftrightarrow j)\;, \end{aligned}$$

$$(3.3.22)$$

where $F = A_{\mathrm{scalar}}/A_{\mathrm{tree}}$.

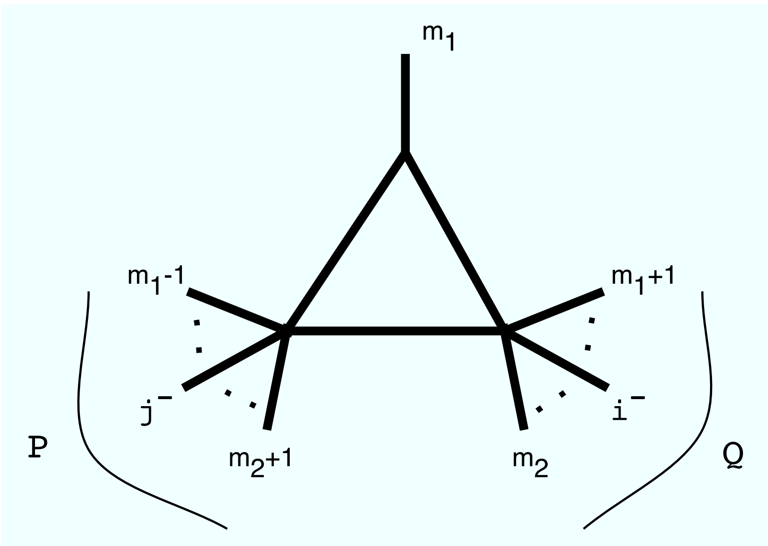


Figure 3.4: One type of triangle function contributing to the amplitude in the general case, where $i \in Q$, and $j \in P$.

Several remarks are in order.

1. As usual, the variables $q^2_{m_1,m_2-1}$, $q^2_{m_1+1,m_2}$ correspond to the s- and t-channel of the finite part of the "easy two-mass" box function with massless legs m_1 and m_2, and massive legs $q^2_{m_1+1,m_2-1}$, $q^2_{m_2+1,m_1-1}$ (Figure 3.3).

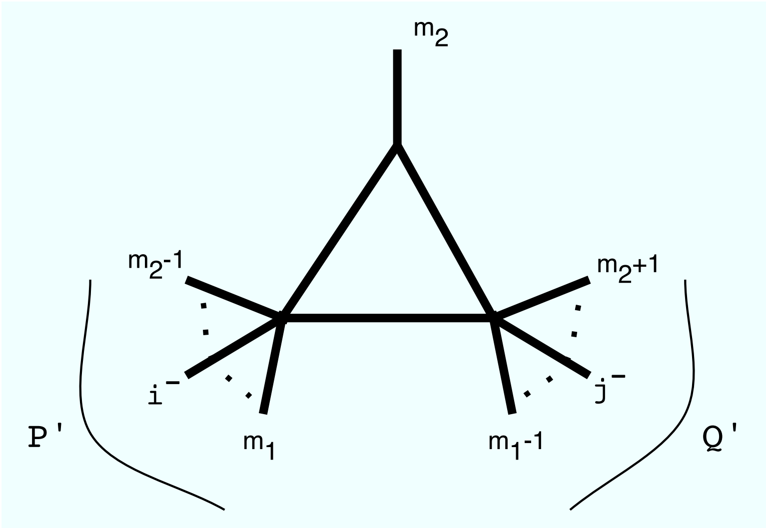


Figure 3.5: Another type of triangle function contributing to the amplitude in the general case. By first shifting $m_1 - 1 \to m_1$, and then swapping $m_1 \leftrightarrow m_2$, we convert this into a triangle function as in Figure 3.4 – but with i and j swapped. These are the triangle functions responsible for the $i \leftrightarrow j$ swapped terms in (3.3.20) – or (3.3.22).

2. Compared to the ranges of m_1 and m_2 indicated in (3.1.2), we have omitted $m_1 = i$ in the summation of the triangles as for this value the coefficients A, S, I defined in (3.3.16)–(3.3.19) vanish. Notice also that we have $i \in Q$ and $j \in P$.

3. In the case of adjacent negative-helicity gluons, the only surviving terms are those containing the coefficient $c^{ij}_{m_1 m_2}$, on the second line of (3.3.20) or (3.3.22). We will return to this point in §3.4.

4. We comment that, in contrast to the adjacent case (see (3.2.21)), in the general case the $\mathcal{N} = 1$ chiral amplitude does not separate out naturally in the final result. One can quickly see this from the coefficient of the box function B in (3.3.20) for example.

Next we wish to explicitly separate out the infrared divergences from (3.3.20). We can immediately anticipate that there will be four infrared-divergent terms, corresponding to the four possible degenerate triangles. Two of these degenerate triangles occur when either P^2 or Q^2 happen to vanish. The other two originate from the $i \leftrightarrow j$ swapped terms.

Let us first consider the terms arising from the summation with $i \leftrightarrow j$ unswapped. When $Q^2 = 0$, it follows that $m_1 = i - 1$ and $m_2 = i$ (see Figure 3.4). When $P^2 = 0$, it follows that $m_1 = j+1$ and $m_2 = j-1$ (see Figure 3.4). Hence

$$T^{(r)}(p;P;Q) \to (-)^r \frac{1}{\epsilon^2}\frac{(-t_{i-1}^{[2]})^{-\epsilon}}{(t_{i-1}^{[2]})^r}\,; \qquad Q^2 \to 0\,; \tag{3.3.23}$$

$$T^{(r)}(p;P;Q) \to \frac{1}{\epsilon^2}\frac{(-t_{j}^{[2]})^{-\epsilon}}{(t_{j}^{[2]})^r}\,; \qquad P^2 \to 0\,:$$

The infrared-divergent terms coming from $Q^2 = 0$ are then easily extracted, and are

$$\frac{1}{\epsilon^2}(-t_{i-1}^{[2]})^{-\epsilon}\left[4(i-j)\frac{(i|j|i-1|i+1)}{(i+1)\cdot(i-1)}\right. \tag{3.3.24}$$
$$\left.-\frac{8}{3}(i-j)^2 - 2\frac{(i|j|i+1|i-1)}{(i+1)\cdot(i-1)}(i-j) + \frac{(i|j|i+1|i-1)(i|j|i-1|i+1)}{(i+1)\cdot(i-1)^2}\right];$$

and from $P^2 = 0$

$$\frac{1}{\epsilon^2}(-t_{j}^{[2]})^{-\epsilon}\left[4(i-j)\frac{(i|j|j-1|j+1)}{(j+1)\cdot(j-1)}\right. \tag{3.3.25}$$
$$\left.-\frac{8}{3}(i-j)^2 - 2\frac{(i|j|j+1|j-1)}{(j+1)\cdot(j-1)}(i-j) + \frac{(i|j|j+1|j-1)(i|j|j-1|j+1)}{(j+1)\cdot(j-1)^2}\right]:$$

Likewise, from the "swapped" degenerate triangles we obtain the following infrared-divergent terms:

$$\frac{1}{\epsilon^2}(-t_{j-1}^{[2]})^{-\epsilon}\left[4(i-j)\frac{(i|j|j+1|j-1)}{(j+1)\cdot(j-1)}\right. \tag{3.3.26}$$
$$\left.-\frac{8}{3}(i-j)^2 - 2\frac{(i|j|j-1|j+1)}{(j+1)\cdot(j-1)}(i-j) + \frac{(i|j|j-1|j+1)(i|j|j+1|j-1)}{(j+1)\cdot(j-1)^2}\right];$$

and

$$\frac{1}{\epsilon^2}(-t_{i}^{[2]})^{-\epsilon}\left[4(i-j)\frac{(i|j|i+1|i-1)}{(i+1)\cdot(i-1)}\right. \tag{3.3.27}$$
$$\left.-\frac{8}{3}(i-j)^2 - 2\frac{(i|j|i-1|i+1)}{(i+1)\cdot(i-1)}(i-j) + \frac{(i|j|i-1|i+1)(i|j|i+1|i-1)}{(i+1)\cdot(i-1)^2}\right]:$$

3.3.1 Comments on twistor space interpretation

We would like to make some brief comments on the interpretation in twistor space of our result (3.3.22).

1. As noticed earlier, the coefficient $b^{ij}_{m_1 m_2}$ appears already in the $\mathcal{N} = 1$ chiral multiplet contribution to a one-loop MHV amplitude, where it multiplies the box function. It was noticed in Section 4 of [73] that $b^{ij}_{m_1 m_2}$ is a holomorphic func-

tion and hence it does not affect the twistor space localisation of the finite box function.

2. The coefficient $c^{ij}_{m_1 m_2}$ also appears in the $\mathcal{N} = 1$ amplitude as the coefficient of the triangles (see e.g. Eq. (2.19) of [40]). Its twistor space interpretation was considered in Section 4 of [73], where it was found that $c^{ij}_{m_1 m_2}$ has support on two lines in twistor space. Furthermore, it was also found that the corresponding term in the amplitude has a derivative of a delta function support on coplanar configurations.

3. The combination $c^{ij}_{m_1 m_2}\langle i|j m_1|Q\rangle\langle i|j Q|m_1\rangle/\langle i\, j\rangle^2$ already appears in the case of adjacent negative-helicity gluons. The localisation properties of the corresponding term in the amplitude were considered in Section 5.3 of [73] and found to have, similarly to the previous case, derivative of a delta function support on coplanar configurations.

4. On general grounds, we can argue that the remaining terms in the amplitude have a twistor space interpretation which is similar to that of the terms already considered. The gluons whose momenta sum to P are contained on a line; likewise, the gluons whose momenta sum to Q localise on another line.

We observe that the rational parts of the amplitude are not generated from the MHV diagram construction presented here. Such rational terms were not present for the $\mathcal{N} = 1$ and $\mathcal{N} = 4$ amplitudes derived in [37, 40, 41]. However, for the amplitude studied here, rational terms are required to ensure the correct factorisation properties [42]. These terms have recently been computed using an on-shell unitarity bootstrap [45] which makes use of the cut-constructible part (3.3.20) (or (3.3.22)) as input.

3.4 Checks of the general result

In this section we present three consistency checks that we have performed for the result (3.3.20) (or (3.3.22)) for the one-loop scalar contribution to the MHV scattering amplitude. These checks are:

1. For adjacent negative-helicity gluons, the general expression (3.3.20) should reproduce the previously calculated form (3.2.21).

2. In the case of five gluons in the configuration $(1^- 2^+ 3^- 4^+ 5^+)$, the result (3.3.20) should reproduce the known amplitude given in [44].

3. The result (3.3.20) should have the correct infrared-pole structure.

We next discuss these requirements in turn.

3.4.1 Adjacent case

The amplitude where the two negative-helicity external gluons are adjacent is given in Section 7 of [42] and was explicitly rederived in x3.2 of this thesis by combining MHV vertices, see Eq. (3.2.21). It is easy to show that our general result (3.3.22) reproduces (3.2.21) correctly as a special case.

To start with, recall that our result (3.3.22) is expressed in terms of box functions and triangle functions, see Figure 3.3 and Figures 3.4, 3.5 respectively. In the adjacent case, the box functions are not present. Indeed, in the sum (3.3.8) the negative-helicity gluons can never be in adjacent positions (see Figure 3.3).

Next, we focus on the triangles of Figure 3.4. In terms of these triangles, requiring i and j to be adjacent eliminates the sum over m_2, as we must have $m_2 = i$ and $m_2 + 1 = j$. Moreover, in this case $Q = q_{m_1+1;i}$, $P = q_{j;m_1\ 1}$ and one has

$$A_{ij}^{m_1 m_2} = \quad 4 (i \quad j) \, ;$$

$$S_{ij}^{m_1 m_2} = 0 \, ; \qquad I_{ij}^{m_1 m_2} = 0 \, ; \tag{3.4.1}$$

(for $m_2 = i$, and $m_2 + 1 = j$). Similar simplications occur for the swapped triangle. Hence the only surviving terms are those in the second line of (3.3.20) (or (3.3.22)), and it is then easy to see that they generate the same amplitude (3.2.8) already calculated in x3.2.

3.4.2 Five-gluon amplitude

The other special case is the non-adjacent ve-gluon amplitude $(1 \quad 2^+ \, 3 \quad 4^+ \, 5^+)$, given in Equation (9) of [44]. This amplitude may be written as[11] $c \, A_{\rm tree}$ times[12]

$$\frac{1}{6} \qquad \frac{1}{6}\log(\quad s_{34}) + \frac{\mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})^2 \, \mathrm{tr}_+(\not{1}\not{3}\not{5}\not{2})^2}{2^7 (2 \quad 5)^4 (1 \quad 3)^4} \, B(s_{51}; s_{12}; 0; s_{34})$$

$$\frac{1}{3} \, \frac{\mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5}) \, \mathrm{tr}_+(\not{1}\not{3}\not{5}\not{2})}{2^4 (2 \quad 5)(1 \quad 3)^4} \qquad \mathrm{tr}_+(\not{1}\not{3}\not{5}\not{2})^2 \, \frac{\log(s_{12}=s_{34})}{(s_{12} \quad s_{34})^3}$$

$$+ \, \mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})^2 \, \frac{\log(s_{34}=s_{51})}{(s_{34} \quad s_{51})^3}$$

---

[11] $c = r = (4 \quad)^2$ is given in terms of Eq. (C.3.1).

[12] The derivation in [44] used string-based methods which a ect the coe cient of the pole term . In Eq. (3.4.2) we have written the pole coe cient which matches the adjacent case.

$$
\begin{aligned}
&+ \quad \frac{1}{3}\,\frac{1}{2^3(1\quad 3)^3}\quad \mathrm{tr}_+(\not{1}\not{3}\not{4}\not{2})\,\mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})^2\,\frac{\log(s_{34}=s_{51})}{(s_{34}\quad s_{51})^3} \\
&\qquad \frac{\mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})^2\ \mathrm{tr}_+(\not{1}\not{3}\not{5}\not{2})^2}{2^6(2\quad 5)^2(1\quad 3)^4}\quad \frac{\log(s_{12}=s_{34})}{(s_{12}\quad s_{34})^2}\quad \frac{\log(s_{34}=s_{51})}{(s_{34}\quad s_{51})^2} \\
&+ \quad \frac{\mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})^2\ \mathrm{tr}_+(\not{1}\not{3}\not{5}\not{2})^2}{2^6(2\quad 5)^3(1\quad 3)^4}\quad \frac{\log(s_{12}=s_{34})}{(s_{12}\quad s_{34})} + \frac{\log(s_{34}=s_{51})}{(s_{34}\quad s_{51})} \\
&\qquad \frac{1}{3}\,\frac{1}{2^2(1\quad 3)}\quad \mathrm{tr}_+(\not{1}\not{3}\not{2}\not{5})\,\frac{\log(s_{34}=s_{51})}{(s_{34}\quad s_{51})} \\
&+ \quad (1;4) \leftrightarrow (3;5) \quad ;
\end{aligned}
\tag{3.4.2}
$$

where the interchange on the last line applies to all terms above it in this equation, including the first two terms. The box function B is defined in (3.3.7). In deriving (3.4.2) from [44], we have used the dilogarithm identity

$$
\mathrm{Li}_2(1-r)+\mathrm{Li}_2(1-s)+\log(r)\log(s) = \mathrm{Li}_2\frac{1-r}{s}+\mathrm{Li}_2\frac{1-s}{r}-\mathrm{Li}_2\frac{1-s}{r}\frac{1-r}{s} \quad :
$$

We have checked explicitly that our expression for the n-gluon non-adjacent amplitude (3.3.20), when specialised to the case with five gluons in the configuration ($1^-\ 2^+\ 3^-\ 4^+\ 5^+$), yields precisely the result (3.4.2) above. For the terms involving dilogarithms, this is easily done. For the remaining terms, which contain logarithms, a more involved calculation is necessary using various spinor identities from Appendix A. A straightforward method of doing this calculation begins with the explicit sum over MHV diagrams in this case, isolating the coefficients of each logarithmic function such as e.g. $\log(s_{12})$, and then checking that these coefficients match those in (3.4.2). The remaining $1/\epsilon$ term arises from the following discussion.

3.4.3 Infrared-pole structure

The infrared-divergent terms (poles in $1/\epsilon$) can easily be extracted from (3.3.24){(3.3.27) by simply replacing $(-t_r^{[2]})^{-\epsilon} \to 1$ ($r = i-1; i; j-1; j$). Consider first the terms in (3.3.25) and (3.3.26). After a little algebra, and using

$$
(i\,j\,j+1\,j-1) + (i\,j\,j-1\,i-1) = 4(i\ j)(j-1)\quad(j+1)\,; \tag{3.4.3}
$$

one finds that these two contributions add up to

$$
\frac{64}{3}\,(i\quad j)^4 : \tag{3.4.4}
$$

Similarly, the pole contribution arising from (3.3.24) and (3.3.27) gives an additional contribution of $(64/3\ )\,(i\ \ j)^4$. Reinstating a factor of $2\ \ 2^8 (i\ \ j)^4\ \ A_{tree}$, we see that the pole part of (3.3.20) is simply given by

$$A_{scalar}\big|_{pole} = \frac{A_{tree}}{3}\,. \tag{3.4.5}$$

Hence our result (3.3.20) has the expected infrared-singular behaviour.

3.5 The MHV amplitudes in QCD

We conclude by mentioning that the full one-loop n-gluon MHV amplitudes (with arbitrary positions for the negative-helicity gluons) in QCD can now be constructed. These are given by:

$$A^{MHV}_{QCD} = A^{MHV}_{N=4}\ \ 4A^{MHV}_{N=1,chiral} + A^{MHV}_{scalar}\,; \tag{3.5.1}$$

where in contradistinction with (1.3.3) we have written the scalar contribution in terms of a complex scalar rather than a real scalar. The individual pieces (to finite order in) can be found as follows:

- $A^{MHV}_{N=4}$ was first computed in [38] and can be found there as Equation (4.1). Alternatively it is given as Equation (1.9.1) in Chapter 1 of this thesis. Note that an alternative form to Eq. (1.9.6) for the 2me box functions is given by Eq. (1.9.25).

- $A^{MHV}_{N=1,chiral}$ was first computed in [42] and can be found there as Equation (5.12) or more compactly as Equation (2.1.19) in Chapter 2 of this thesis.

- In contrast to the $N = 4$ and $N = 1$ cases, A^{MHV}_{scalar} is an amplitude in a non-supersymmetric theory and as such its cuts are not uniquely determined by its cut-constructible part ($A^{MHV}_{s\text{-}cut}$). $A^{MHV}_{s\text{-}cut}$ was first computed in [43] and can be found there as Equation (4.20) or Equation (4.22). Alternatively it can be found earlier in this chapter as Equation (3.3.20) or Equation (3.3.22).

- Building on the results of [43], the rational part of A^{MHV}_{scalar} ($A^{MHV}_{s\text{-}rational}$) was computed in [45]. In doing this it was found that it is useful to 'complete' the cut-constructible parts obtained in [43] by introducing certain preliminary rational terms in order to remove spurious singularities. The cut-completion of $A^{MHV}_{s\text{-}cut}$ is given by Equation (A1) of Appendix A of [45] and the full amplitude is then obtained by adding the remaining rational terms as given in Equation (5.30) of that paper. Explicitly, the full scalar amplitude is given by Equation (5.1) (for negative-helicity gluons 1 and m), where $\hat{C}$ is given by (A1) and $\hat{R}$ by Equation (5.30) of [45].

A^{MHV}_{QCD} can then be found using the decomposition (3.5.1) and

$$A^{MHV}_{N=4} = \text{Eq. (4.1) of [38]}$$
$$A^{MHV}_{N=1,\text{chiral}} = \text{Eq. (5.12) of [42]}$$
$$A^{MHV}_{\text{scalar}} = \text{Eq. (5.1) of [45]}:$$

CHAPTER 4

RECURSION RELATIONS IN GRAVITY

The proposal of a twistor string dual to perturbative Yang-Mills in [31] led not-only to the advances described in Chapters 1–3 of the so-called MHV rules for perturbation theory, but to many others as well. The support of many quantities such as scattering amplitudes, their integral functions and the coefficients of these functions in twistor space has led to many insights [31, 43, 47, 53, 72, 73, 75, 76, 91, 179, 180, 181, 182, 183, 184] as has the use of signature $+ + - -$ (or equivalently the restriction of momenta to be complex rather than real). In particular, this second technique of using complex momenta has proved very powerful, leading to the idea of generalised unitarity [47, 84] and then to the tree-level on-shell recursion relations [48, 49] which will be central to this chapter.

Recursion relations have been known for some time in field theory since Berends and Giele proposed them in terms of off-shell currents [171]. However, the gluon recursion relations introduced by Britto, Cachazo and Feng in [48] (stemming from observations in [46]) and then proved in [49] are in some ways much more powerful. They apply directly to on-shell scattering amplitudes and are particularly apt when the amplitudes are written in the spinor helicity formalism, which as we have seen in the preceding chapters is a formalism which tends to favour simple and compact expressions.

The proof of the on-shell recursion relation for gluons presented in [49] is very simple, only relying (essentially) on the ability to express an amplitude as a function of a complex variable z and then the asymptotic behaviour of this function as $z \to \infty$. As such, it is natural to ask whether such a recursive structure might persist in other field theories and even in gravity.[1] This question was answered independently in [50] and [51] in the affirmative, where the authors of [50] (including the present author) used it to present a new compact formula for n-graviton MHV amplitudes at tree-level in general relativity (GR). Such compact formulae are particularly interesting as gravity is very-much more complicated than Yang-Mills – the 3-point vertex of GR for example contains 171 terms in total, while the 4-point vertex has 2850 altogether [165].

In this chapter we will follow [50] and describe the recursion relation in Einstein gravity at tree-level. We will not summarise the proof of the relation in Yang-Mills as

[1] Here we mean gravity as a field theory (rather than as a string theory).

it is almost identical to that in gravity. Any differences between the two are pointed out in what follows.

4.1 The recursion relation

In this section we closely follow the proof of the recursion relation in Yang-Mills [49], which we will extend to the case of gravity amplitudes. As we shall see, the main new ingredient is that gravity amplitudes depend on more kinematical invariants than the corresponding Yang-Mills amplitudes, namely those which are sums of non-cyclically adjacent momenta; hence, more multi-particle channels should be considered.

To derive a recursion relation for scattering amplitudes, we start by introducing a one-parameter family of scattering amplitudes, $M(z)$ [49], where we choose z in such a way that $M(0)$ is the amplitude we wish to compute. We work in complexified Minkowski space and regard $M(z)$ as a complex function of z and the momenta. One can then consider the contour integral [103]

$$C_\infty := \frac{1}{2\pi i}\oint dz\,\frac{M(z)}{z}\,, \tag{4.1.1}$$

where the integration is taken around the circle at infinity in the complex z plane. Assuming that $M(z)$ has only simple poles at $z = z_i$, the integration gives

$$C_\infty = M(0) + \sum_i \frac{[\mathrm{Res}\,M(z)]_{z=z_i}}{z_i}\,. \tag{4.1.2}$$

In the important case of Yang-Mills amplitudes, $M(z) \to 0$ as $z \to \infty$, and hence $C_\infty = 0$ [49].

Notice that up to this point the definition of the family of amplitudes $M(z)$ has not been given – we have not even specified the theory whose scattering amplitudes we are computing.

There are some obvious requirements for $M(z)$. The main point is to define $M(z)$ in such a way that poles in z correspond to multi-particle poles in the scattering amplitude $M(0)$. If this occurs then the corresponding residues can be computed from factorisation properties of scattering amplitudes (see, for example, [3, 154]). In order to accomplish this, $M(z)$ was defined in [48, 49] by shifting the momenta of two of the external particles in the original scattering amplitude. For this procedure to make sense, we have to make sure that even with these shifts overall momentum conservation is preserved and that all particle momenta remain on-shell. We are thus led to define $M(z)$ as the scattering amplitude $M(p_1, \ldots, p_k(z), \ldots, p_l(z), \ldots, p_n)$, where the momenta of particles k and l are shifted to

$$p_k(z) := p_k + z\eta\,, \qquad p_l(z) := p_l - z\eta\,. \tag{4.1.3}$$

Momentum conservation is then maintained. As in [48], we can solve $p_k^2(z) = p_l^2(z) = 0$ by choosing $\eta = \lambda_l\tilde{\lambda}_k$ (or $\eta = \lambda_k\tilde{\lambda}_l$), which makes sense in complexified Minkowski space. Equivalently,

$$\lambda_k(z) := \lambda_k + z\lambda_l\,; \qquad \tilde{\lambda}_l(z) := \tilde{\lambda}_l - z\tilde{\lambda}_k\,; \tag{4.1.4}$$

with λ_l and $\tilde{\lambda}_k$ unshifted.

More general families of scattering amplitudes can also be defined, as pointed out in [103]. For instance, one can single out three particles k, l, m, and define

$$p_k(z) := p_k + z\,\eta_k\,; \qquad p_l(z) := p_l + z\,\eta_l\,; \qquad p_m(z) := p_m + z\,\eta_m\,; \tag{4.1.5}$$

where η_k, η_l and η_m are null and $\eta_k + \eta_l + \eta_m = 0$. Imposing $p_k^2(z) = p_l^2(z) = p_m^2(z) = 0$, one finds the solution

$$\eta_k = \alpha\lambda_k\tilde{\lambda}_l - \beta\lambda_k\tilde{\lambda}_m\,; \qquad \eta_l = -\alpha\lambda_k\tilde{\lambda}_l\,; \qquad \eta_m = \beta\lambda_k\tilde{\lambda}_m\,; \tag{4.1.6}$$

for arbitrary α and β. This has been used in [103]. In the following we will limit ourselves to shifting only two momenta as in [48] and [49].

At tree level, scattering amplitudes in field theory can only have simple poles in multi-particle channels; for $M(z)$, these generate poles in z (unless the channel contains both particles k and l, or none). Indeed, if $P(z)$ is a sum of momenta including $p_l(z)$ but not $p_k(z)$, then $P^2(z) = P^2 - 2z(P\cdot\eta)$ vanishes at $z_P = P^2/2(P\cdot\eta)$ [49]. In Yang–Mills theory, one considers colour-ordered partial amplitudes which have a fixed cyclic ordering of the external legs. This implies that a generic Yang-Mills partial amplitude can only depend on kinematical invariants made of sums of cyclically adjacent momenta. Hence, tree-level Yang-Mills amplitudes can only have poles in kinematical channels made of cyclically adjacent sums of momenta.

For gravity amplitudes this is not the case as there is no such notion of ordering for the external legs. Therefore, the multi-particle poles which produce poles in z are those obtained by forming all possible combinations of momenta which include $p_k(z)$ but not $p_l(z)$. This is the only modification to the BCFW recursion relation we need to make in order to derive a gravity recursion relation.

For any such multi-particle channel $P^2(z)$, we have

$$M(z) \to \sum_h M_L^h(z_P)\,\frac{1}{P^2(z)}\,M_R^{-h}(z_P)\,; \tag{4.1.7}$$

as $P^2(z) \to 0$ (or, equivalently, $z \to z_P$). The sum is over the possible helicity assignments on the two sides of the propagator which connects the two lower-point tree-level amplitudes $\mathcal{M}_L^h$ and $\mathcal{M}_R^{-h}$. It follows that

$$[\mathrm{Res}\,\mathcal{M}(z)]_{z=z_P} = \sum_h \mathcal{M}_L^h(z_P)\frac{z_P}{P^2}\mathcal{M}_R^{-h}(z_P)\,, \tag{4.1.8}$$

so that finally

$$\mathcal{M}(0) = C_\infty + \sum_{P,h} \frac{\mathcal{M}_L^h(z_P)\mathcal{M}_R^{-h}(z_P)}{P^2}\,. \tag{4.1.9}$$

The sum is over all possible decompositions of momenta such that $p_k \in P$ but $p_1 \notin P$.

If $C_\infty = 0$ there is no boundary term in the recursion relation and

$$\mathcal{M}(0) = \sum_{P,h} \frac{\mathcal{M}_L^h(z_P)\mathcal{M}_R^{-h}(z_P)}{P^2}\,. \tag{4.1.10}$$

In [49] it was shown that for Yang-Mills amplitudes the boundary terms C_∞^{YM} always vanish. Two different proofs were presented, the first based on the use of CSW diagrams [33] and the second on Feynman diagrams. An MHV-vertex formulation of gravity only recently appeared [77], so at the time the authors of [50] could only rely on Feynman diagrams. This is also the case for other field theories we might be interested in (such as ϕ^4, for example).

As we have remarked, $C_\infty = 0$ if $\mathcal{M}(z) \to 0$ as $z \to \infty$. $\mathcal{M}(z)$ is a scattering amplitude with shifted, z-dependent external null momenta. One can then try to estimate the behaviour of $\mathcal{M}(z)$ for large z by using power counting (different theories will of course give different results). In ϕ^4 the Feynman vertices are momentum independent and $C_\infty = 0$ (see §4.3); in quantum gravity, however, vertices are quadratic in momenta, and one cannot determine a priori whether or not a boundary term is present.

From the previous discussion, it follows that the behaviour of $\mathcal{M}(z)$ as $z \to \infty$ is related to the high-energy behaviour of the scattering amplitude (and hence to the renormalisability of the theory). The ultraviolet behaviour of quantum gravity, however, is full of surprises (for a summary, see for example Section 2.2 of [217] and also more recent results of [55, 56, 57, 58, 59, 60]). We may therefore expect a more benign behaviour of $\mathcal{M}(z)$ as $z \to \infty$. Specifically, in the next section we will focus on the n-graviton MHV scattering amplitudes which have been computed by Berends, Giele and Kuijf (BGK) in [218]. Performing the shifts (4.1.3) explicitly in the BGK formula, one finds the surprising result[2]

$$\lim_{z\to\infty} \mathcal{M}_{MHV}(z) = 0\,. \tag{4.1.11}$$

[2] We have checked that $\mathcal{M}(z) \sim O(1/z^2)$ as $z \to \infty$, analytically for $n \le 7$ legs and numerically for $n \le 11$ legs.

In more general amplitudes one can (at least in principle) use the (field theory limit of the) KLT relations [219], which connect tree-level gravity amplitudes to tree-level amplitudes in Yang-Mills, to estimate the large-z behaviour of the scattering amplitude.[3] As an example, we have considered the next-to-MHV gravity amplitude $M(1^-;2^-;3^-;4^+;5^+;6^+)$, and performed the shifts as in (4.1.4), with $k = 1$ and $l = 2$. Similarly to the MHV case, we find that

$$\lim_{z\to\infty} M(1^-;2^-;3^-;4^+;5^+;6^+)(z) = 0 \, . \tag{4.1.12}$$

In [51] it was shown that $M(z)$ vanishes as $z \to \infty$ for all amplitudes up to eight gravitons and also for all n-point MHV and NMHV amplitudes. Further to this, recent work [52] provides a proof of this statement for all tree-level n-graviton amplitudes thus establishing the validity of the recursion relation in gravity unambiguously.

In the next section we will apply the recursion relation (4.1.10) to the case of MHV amplitudes in gravity and show that it does generate correct expressions for the amplitudes. As a bonus, we will derive a new closed-form expression for the n-particle scattering amplitude.

4.2 Application to MHV gravity amplitudes

In the following we will compute the MHV scattering amplitude $M(1^-;2^-;3^+;\ldots;n^+)$ for n gravitons. We will choose the two negative-helicity gravitons 1^- and 2^- as reference legs. This is a particularly convenient choice as it reduces the number of terms arising in the recursion relation to a minimum. The shifts for the momenta of particles 1 and 2 are

$$p_1 \to p_1 + z\lambda_2\tilde\lambda_1 \, ; \qquad p_2 \to p_2 - z\lambda_2\tilde\lambda_1 \, . \tag{4.2.1}$$

In terms of spinors, the shifts are realised as

$$\lambda_1 \to \hat\lambda_1 := \lambda_1 + z\lambda_2 \, ; \qquad \tilde\lambda_2 \to \hat{\tilde\lambda}_2 := \tilde\lambda_2 - z\tilde\lambda_1 \, ; \tag{4.2.2}$$

with λ_2 and $\tilde\lambda_1$ unmodified.

Let us consider the possible recursion diagrams that can arise. There are only two possibilities, corresponding to the two possible internal helicity assignments, $(+\,-)$ and $(-\,+)$:

1. The amplitude on the left is googly $(+\,+\,-)$ whereas on the right there is an MHV gravity amplitude with $n-1$ legs (see Figure 4.1).

[3] See Appendix H for explicit examples of KLT relations for four, five and six legs.

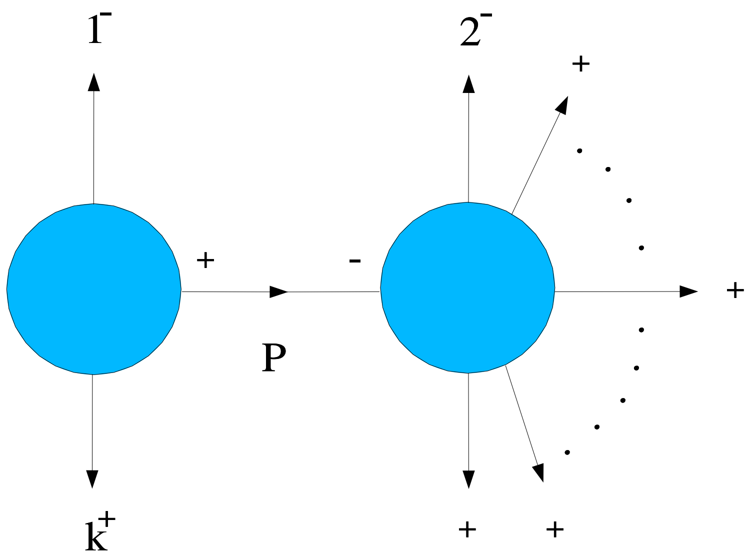


Figure 4.1: One of the terms contributing to the recursion relation for the MHV amplitude $M(1^-;2^-;3^+;\ldots;n^+)$. The gravity scattering amplitude on the right is symmetric under the exchange of gravitons of the same helicity. In the recursion relation, we sum over all possible values of k, i.e. $k = 3;\ldots;n$. This amounts to summing over cyclical permutations of $(3;\ldots;n)$.

2. The amplitude on the right is googly and the amplitude on the left is MHV (see Figure 4.2).

We recall that a gravity amplitude is symmetric under the interchange of identical helicity gravitons; this implies that we have to sum $n-2$ diagrams for each of the configurations in Figures 4.1 and 4.2. Each diagram is then completely specified by choosing k, with $k = 3;\ldots;n$.

However, it is easy to see that diagrams of type 2 actually give a vanishing contribution. Indeed, they are proportional to

$$[k\hat{P}] = \frac{[k|\hat{P}|\hat{2}\rangle}{\langle\hat{P}\,\hat{2}\rangle} = \frac{[k|P|2\rangle}{\langle\hat{P}\,\hat{2}\rangle} = 0\;; \qquad (4.2.3)$$

where the last equality follows from $P = p_k + p_2$. Hence we will have to compute diagrams of type 1 only. We will do this in the following.

4.2.1 Four-, five- and six-graviton scattering

To show explicitly how our recursion relation generates amplitudes we will now derive the 4-, 5- and 6-point MHV scattering amplitudes.

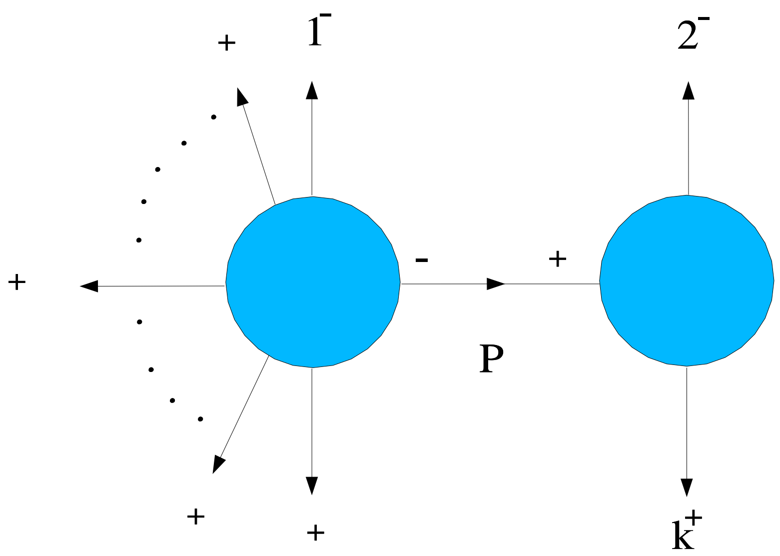


Figure 4.2: This class of diagrams also contributes to the recursion relation for the MHV amplitude $M(1^-;2^-;3^+;\ldots;n^+)$; however, each of these diagrams vanishes if the shifts (4.2.2) are performed.

We start with the four point case. There are two diagrams to sum, one of which is represented in Figure 4.3; the other is obtained by swapping the labels 4 and 3. For the diagram in Figure 4.3, we have

$$M^{(4)} = M_L \frac{1}{P^2} M_R ; \tag{4.2.4}$$

where the superscript denotes the label on the positive-helicity leg in the trivalent googly MHV vertex,

$$M_L = \left(\frac{[\hat{P}\,4]^3}{[4\,1][1\,\hat{P}]} \right)^2 ; \tag{4.2.5}$$

$$M_R = \left(\frac{\langle \hat{P}\,2\rangle^3}{\langle 2\,3\rangle\langle 3\,\hat{P}\rangle} \right)^2 ;$$

and $P^2 = (p_1 + p_4)^2$. Using

$$\langle i\,\hat{P}\rangle = \frac{\langle i|P|1]}{[\hat{P}\,1]} ; \tag{4.2.6}$$

we find, after a little algebra,

$$M^{(4)} = \frac{\langle 12\rangle^6 [14]}{\langle 14\rangle\langle 23\rangle^2\langle 34\rangle^2} : \tag{4.2.7}$$

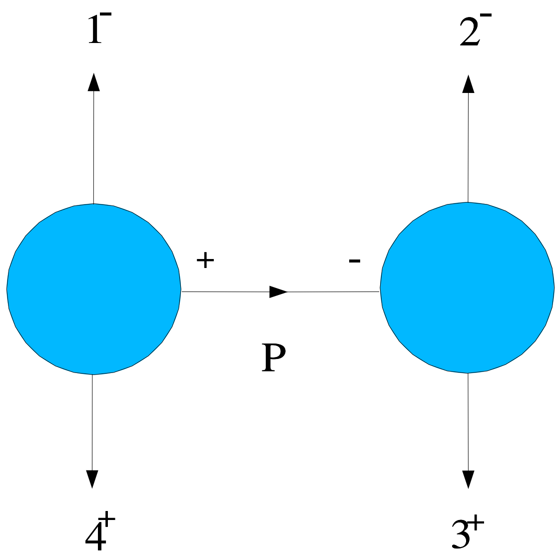


Figure 4.3: One of the two diagrams contributing to the recursion relation for the MHV amplitude $M(1^-,2^-,3^+,4^+)$. The other is obtained from this by cyclically permuting the labels $(3,4)$ — i.e. swapping 3 with 4.

The full amplitude is $M(1^-,2^-,3^+,4^+) = M^{(3)} + M^{(4)}$. Thus, we conclude that the four point MHV amplitude generated by our recursion relation is given by

$$M(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^6 [14]}{\langle 14\rangle\langle 23\rangle^2\langle 34\rangle^2} + 3 \leftrightarrow 4\,. \tag{4.2.8}$$

It is easy to check that this agrees with the conventional formula for this amplitude

$$M(1^-,2^-,3^+,4^+) = \frac{\langle 12\rangle^8 [12]}{N(4)\langle 34\rangle}\,, \tag{4.2.9}$$

where

$$N(n) := \prod_{1\le i<j\le n} \langle ij\rangle\,, \tag{4.2.10}$$

or, equivalently, with the expression from the appropriate KLT relation, Eq. (H.0.2).

For the five-graviton scattering case our recursion relation yields a sum of three diagrams. A calculation similar to that illustrated previously for the four-point case leads to the result

$$M(1^-,2^-,3^+,4^+,5^+) = \frac{\langle 12\rangle^6 [15][34]}{\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle} + P^C(3,4,5)\,, \tag{4.2.11}$$

where $P^C(3,4,5)$ means that we have to sum over cyclic permutations of the labels

3,4,5. The conventional formula for the five graviton MHV scattering amplitude is

$$M(1^-,2^-,3^+,4^+,5^+) = \frac{\langle 12\rangle^8}{N(5)}\left([12][34]\langle 13\rangle\langle 24\rangle - [13][24]\langle 12\rangle\langle 34\rangle\right) . \tag{4.2.12}$$

Using standard spinor identities and momentum conservation, it is straightforward to check that our expression (4.2.11) agrees with this (alternatively, one can use the KLT relation (H.0.3)).

For the six graviton scattering amplitude, our recursion relation yields a sum of four terms,

$$\begin{aligned} M(1^-,2^-,3^+,4^+,5^+,6^+) &= \frac{\langle 12\rangle^6[16]}{\langle 16\rangle}\frac{1}{\langle 26\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle} \\ &\quad \left(\frac{[34]}{\langle 23\rangle\langle 24\rangle}\frac{\langle 2|3+4|5]}{\langle 56\rangle} + \frac{[45]}{\langle 24\rangle\langle 25\rangle}\frac{\langle 2|4+5|3]}{\langle 36\rangle} + \frac{[53]}{\langle 23\rangle\langle 25\rangle}\frac{\langle 2|5+3|4]}{\langle 46\rangle}\right) \\ &\quad + P^c(3,4,5,6) . \end{aligned} \tag{4.2.13}$$

The known formula for this amplitude is

$$M^{\text{6-point}}_{MHV} = \langle 12\rangle^8 \frac{[12][45]\langle 3|4+5|6\rangle}{\langle 15\rangle\langle 16\rangle\langle 12\rangle\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 36\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle} + P(2,3,4) \quad ; \tag{4.2.14}$$

where $P(2,3,4)$ indicates permutations of the labels 2,3,4. We have checked numerically that the formula (4.2.13) agrees with this expression.

4.2.2 General formula for MHV scattering

Recursion relations of the form given in [48], or the graviton recursion relation given here, naturally produce general formulæ for scattering amplitudes. For a suitable choice of reference spinors, these new formulæ can often be simpler than previously known examples. For the choice of reference spinors 1,2, which we have made above, the graviton recursion relation is particularly simple as it produces only one term at each step. This immediately suggests that one can use it to generate an explicit expression for the n-point amplitude. This turns out to be the case, and experience with the use of our recursion relation leads us to propose the following new general formula for the n-graviton MHV scattering amplitude. This is (labels 1,2 carry negative helicity, the remainder carry positive helicity)

$$\begin{aligned} M(1,2,i_1,\ldots,i_{n-2}) &= \frac{\langle 12\rangle^6[1\, i_{n-2}]}{\langle 1\, i_{n-2}\rangle} G(i_1,i_2,i_3) \prod_{s=3}^{n-3} \frac{\langle 2|i_1+\cdots+i_{s-1}|i_s]}{\langle i_s i_{s+1}\rangle\langle 2 i_{s+1}\rangle} \\ &\quad + P(i_1,\ldots,i_{n-2}) ; \end{aligned} \tag{4.2.15}$$

where
$$G(i_1;i_2;i_3) = \frac{1}{2}\frac{[i_1 i_2]}{\langle 2 i_1\rangle\langle 2 i_2\rangle\langle i_1 i_2\rangle\langle i_2 i_3\rangle\langle i_1 i_3\rangle} \, . \tag{4.2.16}$$

(For $n = 5$ the product term is dropped from (4.2.15)). It is straightforward to check that this amplitude satisfies the recursion relation with the choice of reference legs 1^- and 2^-.

The known general MHV amplitude for two negative-helicity gravitons, 1 and 2, and the remaining $n-2$ with positive helicity is given by [218]

$$M(1^-;2^-;3^+;\ldots;n^+) = \langle 12\rangle^8 \left[\frac{[12][n-2\; n-1]}{\langle 1\; n-1\rangle} \frac{1}{N(n)} \prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1} \langle ij\rangle F + \mathcal{P}(2;\ldots;n-2) \right] ; \tag{4.2.17}$$

where
$$F = \begin{cases} \prod_{l=3}^{n-3} [l|(p_{l+1}+p_{l+2}+\cdots+p_{n-1})|n\rangle & n \geq 6 \\ 1 & n = 5 \end{cases} \tag{4.2.18}$$

We have checked numerically, up to $n = 11$, that our formula (4.2.15) gives the same results as (4.2.17).

It is interesting to note that the very existence of this recursion relation in gravity – described here and in [50, 51] – has something to say about the divergences of quantum gravity. A central feature of the recursion relation is that it requires $M(\infty) = 0$, and the behaviour of $M(z)$ as $z \to \infty$ is related to the high-energy behaviour (and hence the renormalisability) of the theory. It is not a priori clear that gravity has this behaviour, though the analyses of [50, 51] and more recently the complete analysis of [52] show that indeed $M(\infty) = 0$ for any tree-level amplitude in gravity. This means that at tree-level, gravity has divergences in the UV that are perhaps better than one might expect. This supports recent arguments that gravity may not be as divergent as previously thought and more specifically that 4-dimensional $\mathcal{N} = 8$ supergravity may be finite [55, 56, 57, 58, 59, 60].

4.3 Applications to other field theories

One of the striking features of the BCFW proof of the BCF recursion relations is that the specification of the theory with which one is dealing is almost unnecessary. Indeed in [49] the only step where specifying the theory did matter was in the estimate of the behaviour of the scattering amplitudes $M(z)$ as $z \to \infty$, which was important to assess the possible existence of boundary terms in the recursion relation. This leads us to conjecture that recursion relations could be a more generic feature of massless (or spontaneously broken) field theories in four dimensions.[4] After all, the BCF recursion

[4] This was also suggested in [103].

relations – as well as the recursion relation for gravity amplitudes discussed in this chapter and in [51] – just reconstruct a tree-level amplitude (which is a rational function) from its poles.

Let us focus on massless $(\phi\phi^\dagger)^2$ theory in four dimensions. We use the spinor helicity formalism, meaning that each momentum will be written as $p_{a\dot{a}} = \lambda_a\tilde{\lambda}_{\dot{a}}$. A scalar propagator $1/P^2$ connects states of opposite "helicity", which here just means that the propagator is $\langle\phi(x)\,\phi^\dagger(0)\rangle$, with $\langle\phi(x)\,\phi(0)\rangle = \langle\phi^\dagger(x)\,\phi^\dagger(0)\rangle = 0$. Now consider a Feynman diagram contributing to an n-particle scattering amplitude, and let us shift the momenta of particles k and l as in (4.1.3). As for the Yang-Mills case discussed in [49], there is a unique path of propagators going from particle k to particle l. Each of these propagators contributes $1/z$ at large z, whereas vertices are independent of z. We thus expect Feynman diagrams contributing to the amplitude to vanish in the large-z limit.

An exception to the above reasoning is represented by those Feynman diagrams where the shifted legs belong to the same vertex; these diagrams are z-independent, and hence not suppressed as $z \to \infty$. In order to deal with this problematic situation, and ensure that the full amplitude $M(z)$ computed from Feynman diagrams vanishes as $z \to \infty$ we propose two alternatives.

Firstly, if one considers $(\phi\phi^\dagger)^2$ theory without any group structure, one can remove the problem by performing multiple shifts. This possibility has already been used in the context of the rational part of one-loop amplitudes in pure Yang-Mills [103]. In our case, it is sufficient to shift at least four external momenta.

Alternatively, we can consider $(\phi\phi^\dagger)^2$ theory with global symmetry group $U(N)$ and ϕ in the adjoint. In this case we can group the amplitude into colour-ordered partial amplitudes, as in the Yang-Mills case. Then, for any colour-ordered amplitude one can always find a choice of shifts such that the shifted legs do not belong to the same Feynman vertex. The procedure can be repeated for any colour ordering, and the complete amplitude is obtained by summing over non-cyclic permutations of the external legs.

In this way, the appearance of a boundary term C_∞ can be avoided, and one can thus derive a recursion relation for scattering amplitudes akin to (4.1.10). A similar analysis can be carried out in other theories, possibly in the presence of spontaneous symmetry breaking etc. We expect this to play an important rôle in future studies.

4.4 CSW as BCFW

Finally, we would like to point out the connection between the CSW rules at tree-level [33] and the BCFW recursion relation introduced in [48, 49] and discussed for gravity in this chapter. This was hinted at in [49] where it was noted that the existence of BCFW recursion (which can construct any gluon amplitude solely from a knowledge of its singularities) provides an indirect proof of the CSW rules since the CSW rules provide results which are Lorentz-invariant, gauge-invariant and have the correct singularities. It was also briefly touched on in [50] where some formal observations were made regarding the relation between the way that the shifts of Eq. (4.1.3) are performed – so as to keep the corresponding momenta on-shell in the BCFW recursion relation – and the way that the internal legs in the CSW rules are shifted (Eq. (1.7.1)).

However, Risager showed that the CSW rules are in fact a special case of the BCFW recursion relation when specific shifts of momenta are made [34]. The most natural shifts to make when using the recursion relations are those which minimise the number of terms appearing and thus the work that one has to do. In [34], however, a different set of shifts was employed which affects every propagator that may appear in a CSW diagram. The propagators are defined by the momenta that flow through them and thus by a set of consecutive external particles. In the case of CSW diagrams, the vertices are MHV vertices and thus this set of consecutive particles (and its compliment on the other vertex to-which the propagator is attached) must contain at least one gluon of negative helicity each. Exactly this set of propagators is affected if every external negative-helicity gluon is shifted, provided that the sum of any subset of the shifts does not vanish. In addition, the shifts must all involve the anti-holomorphic spinors so that all 3-point googly amplitudes drop out.

Using these shifts (see Eq. (5.1) of [34] for an explicit example of the shifts for an NMHV amplitude), Risager used induction to prove the CSW rules directly thus highlighting their connection with the BCFW recursion relation. In [77] these ideas were then used to construct an MHV-vertex formalism for gravity, thus accentuating the remarkable similarities between gauge theory and gravity despite the latter's more complicated structure.

CHAPTER 5

CONCLUSIONS AND OUTLOOK

In the previous chapters we have studied gluon scattering amplitudes in perturbative gauge theory and have seen how they can be stripped of colour and written in terms of spinor variables to illuminate their basic structure in a unified context. Their twistor-space localisation then allows for an understanding of the unexpected simplicity of many n-point processes. The tree-level MHV amplitudes were seen to lie on simple straight lines in twistor space and it was shown how they could be calculated from a topological string theory as an integral over the moduli space of holomorphically embedded, degree 1, genus 0 curves. This in turn motivated a new perturbative expansion of Yang-Mills gauge theory where tree-level MHV amplitudes are taken off-shell and joined with scalar propagators to create tree-level amplitudes with successively greater numbers of negative-helicity particles. The MHV vertices effectively combine many Feynman diagrams into one and thus provide a great simplification which aids calculation and highlights the underlying geometrical structure.

We saw how these techniques could be applied at loop-level to calculate the MHV amplitudes in $\mathcal{N} = 4$ super-Yang-Mills, which is a slightly surprising result as the duality with the twistor string theory constructed in [31] (and also that in [112]) fails at loop-level. These string theories contain conformal supergravity states which do not decouple at one-loop and this suggests that the application of the CSW rules to loops might fail or simply calculate amplitudes in some theory of Yang-Mills coupled to conformal supergravity. Indeed, a recent calculation of various loop amplitudes in Berkovits' twistor string theory appears to give amplitudes in such a theory [114].

One might also expect that such a surprising result would only apply to maximally supersymmetric Yang-Mills. However in Chapter 2 we saw that MHV vertices can be used to calculate amplitudes at loop-level in theories with less supersymmetry such as $\mathcal{N} = 1$ super-Yang-Mills. There we calculated the one-loop MHV amplitudes and found complete agreement with the known results in [42]. The calculation itself is more involved than the corresponding one in $\mathcal{N} = 4$ presented in [37] and reviewed in Chapter 1 because the reduction in supersymmetry leads to fewer cancellations. Happily though, this does not spoil the technique of using MHV amplitudes as effective vertices.

In Chapter 3 we applied the loop-level CSW rules to pure Yang-Mills with a scalar running in the loop. Pure Yang-Mills is a non-supersymmetric theory and as such the calculation is even more involved than before. This still does not invalidate the process, although it was found that the use of MHV vertices only calculates the cut-constructible part of the amplitude. The rational parts, which are intrinsically linked to the cuts for supersymmetric theories, were thus missed. Nonetheless, the results obtained match perfectly with the known (special) cases [42, 44] and provide the cut-constructible part of the MHV amplitude in pure Yang-Mills with arbitrary positions for the negative-helicity gluons for the first time. The rational part of the amplitude has since been calculated in [45] building on the results described in Chapter 3.

In Chapter 4 we turned our attention to gravity and another interesting development stemming from twistor string theory, namely that of on-shell recursion relations. Recursion relations have been used before in the construction of amplitudes [171], but it wasn't until recently that they were used to recursively turn on-shell amplitudes into amplitudes with a larger number of external legs. They were introduced in [48] at tree-level and have since been used in a bootstrap approach to loop amplitudes which was in fact one of the techniques applied in [45].

We saw that on-shell recursion relations can also be applied at tree-level in gravity and it is a beauty of the proof of these relations in gauge theory [49] which means that they can be proved in gravity without too much (!) extra work. The main additional ingredient is a proof of the behaviour of tree-level n-graviton amplitudes as a function of a complex variable z as $z \to \infty$. In Chapter 4 we argued the case for many amplitudes of interest, but a recent proof that $\lim_{z \to \infty} M_n(z) = 0$ establishes that the recursion relation in gravity can construct any tree-level n-graviton amplitude [52].

We showed how this recursion relation could be used to construct MHV amplitudes with successively more external gravitons and as a by-product constructed a new compact form for the n-graviton MHV amplitudes which provides an interesting alternative to the previously-known form in [218]. We finished by commenting on the relation between the tree-level CSW rules and on-shell recursion relations both in field theory and in gravity and also made some observations on the existence of recursion relations in other theories such as scalar ϕ^4 theory.

Unsurprisingly, this is not the end of the story. In the introduction we already mentioned some of the directions that have been explored following from and related to the material presented here. This includes the construction of twistor string theories describing $\mathcal{N} = 4$ Yang-Mills as well as ones describing other field theories such as a recent description of Einstein supergravity [39], the use of on-shell recursion relations at loop level in both gauge theory and gravity [111, 220] and improvements to the unitarity method [47]. It may be particularly interesting to note that in [39], one of the theories for which a twistor description is found is $\mathcal{N} = 4$ Yang-Mills coupled to $\mathcal{N} = 4$ Einstein supergravity. It appears that there exists a decoupling limit for this theory which gives pure Yang-Mills and thus opens the door to the possibility of understanding loops in Yang-Mills from twistor strings.

From the point of view of the MHV diagram formulation of gauge theory there has also been some considerable progress. Their use at tree-level is already well-established and a Lagrangian formulation now exists [35, 80, 203]. In this scenario, a non-local change of variables is made to the light-cone Yang-Mills Lagrangian which yields a kinetic term describing a scalar propagator connecting positive and negative helicities and interaction terms consisting of the infinite sequence of MHV amplitudes.

Quantisation of this Lagrangian, however, is still an open problem. One of the main points here is the fact that – as demonstrated in Chapter 3 – the use of MHV diagrams alone is not enough to generate a complete amplitude at the quantum level in non-supersymmetric theories and rational terms are missed. As such, one might ask how one could compute the one-loop all-plus (and $- + \ldots +$) amplitude in pure Yang-Mills from MHV diagrams. At tree-level this vanishes, but at one-loop it is a purely rational function – see e.g. Equation (3.4) of [84]. Construction of a one-loop amplitude from MHV diagrams will always give q negative-helicity gluons that satisfies $q \geq 2$, and thus the all-plus amplitude (and also the $- + \ldots +$ amplitude) cannot be constructed from MHV vertices alone. In [73] it was conjectured that perhaps the all-plus amplitude could be elevated to the status of a vertex to generate these missing amplitudes, but at the time an appropriate off-shell continuation for this amplitude could not be found.

Recently, however, more progress has been made in this direction [81, 82, 83]. It appears that the all-plus amplitude is intimately connected with the regularisation procedure needed to evaluate loop diagrams as was initially hinted-at by the fact that the parity conjugate of this amplitude, the all-minus amplitude, arises from an $\epsilon \times 1/\epsilon$ cancellation in dimensional regularisation [81]. Inspired by this, Brandhuber, Spence, Travaglini and Zoubos showed in [82] that a certain one-loop two-point Lorentz-violating counterterm is the generating function for the infinite sequence of one-loop all-plus amplitudes in pure Yang-Mills although there must be another contribution in this story to correctly explain the origin of the $- + \ldots +$ amplitude. In their approach it was found that a certain four-dimensional regularisation scheme (rather than dimensional regularisation) [221, 222, 223] was most useful. It may be interesting and insightful to see if a light-cone approach and such a regularisation scheme is also helpful for computing the cut-constructible terms of amplitudes using MHV diagrams.

Despite these advances, the MHV diagram technique is still practically-speaking limited to tree-level amplitudes and the cut-constructible part of one-loop MHV amplitudes. This is largely because of the intrinsic complexity of loop calculations, though there are other complications. For example, the topologies involved in calculating the cut-constructible part of amplitudes with more than 2 negative-helicity gluons can include (in the case of the NMHV amplitude say) triangle diagrams where each vertex is an MHV vertex. In such diagrams one has 3 different internal particles to take off-shell and it is not clear whether the measure can be found in terms of LIPS integrals and dispersion integrals such as that described in [37, 79] which has been so instrumental in the application of the CSW rules at loop-level so far. Such issues are common to one-loop amplitudes which have $q > 2$ negative-helicity gluons and higher loops as well. It would be desirable from both a theoretical and a phenomenological perspective to understand how the MHV rules can be used to calculate such quantities and would also help to give the MHV rules a more solid footing.

Another interesting avenue of exploration is the suggestion that (planar) higher-loop amplitudes in $\mathcal{N} = 4$ Yang-Mills may be expressed (essentially) as an exponential of certain one-loop amplitudes [224, 225, 226, 227, 228, 229]. Such expressions are termed cross-order relations and may be remarkably powerful if more generally applicable than has been found to date. They could allow the summation of amplitudes in $\mathcal{N} = 4$ Yang-Mills to all orders in perturbation theory and so to non-perturbative information which may be connected to perturbative string theory via the AdS/CFT correspondence.[1] It would be interesting to see how the known cross-order relations arise from MHV diagrams. It is possible that the different terms in the cross-order relations may arise naturally from MHV diagrams which might then provide a framework for proving their validity more generally.

The situation for gravity is in some ways even more exciting, with the possibility that there may exist UV-finite field theories of gravity. Such proposals have recently been made for $\mathcal{N} = 8$ supergravity [53, 54, 55, 56, 57, 58, 59, 60] and it would be interesting to make contact between this and the twistor approach. One such point of contact may be the recent proposal of a twistor string theory describing $\mathcal{N} = 8$ supergravity [39]. Another possibility is that of loop amplitudes from MHV vertices in ($\mathcal{N} = 8$ super-) gravity. These have not yet been understood and their explication would provide a new prescriptive method for the calculation of loop amplitudes in gravity which could shed light on their UV properties.[2]

A further possibility that has not been explored so far (in either gauge theory or gravity) is a more direct connection between recursion relations and loop amplitudes than those already mentioned. Risager [34] showed that the CSW rules at tree-level are really just a specific case of the on-shell recursion relations proposed by Britto, Cachazo and Feng [48]. As we have seen throughout this thesis, the CSW rules can naturally be extended to loop amplitudes which begs the question of whether the same can be done for other cases of the on-shell recursion relation in either Yang-Mills or in gravity.

[1] A very recent paper [230] by Alday and Maldacena appears to have taken a step in this direction. They show how to calculate gluon scattering amplitudes at strong coupling from a classical string configuration via the AdS/CFT correspondence. As a result the full finite form of the four-gluon scattering amplitude in $\mathcal{N} = 4$ super-Yang-Mills is presented. See also [231] which addresses the n-point case.

[2] Shortly after the completion of this work [232] appeared which deals with precisely this point.

In this thesis we have seen some of the improvements that can be made to perturbative techniques in field theory and gravity and the power that they can have. These also hint at new underlying structures whose elucidation could prove extremely interesting if not revolutionary in our understanding. Could such structures presage the existence of new symmetries and will they end up replacing Feynman diagrams entirely in the future? Whatever the outcome, these are exciting timely developments that are sure to aid the discovery of new physics at colliders such as the LHC and deepen our understanding of nature.

APPENDIX A

SPINOR AND DIRAC-TRACE IDENTITIES

In this appendix we present some useful identities pertaining to the spinor helicity formalism and also to help in dealing with Dirac traces.

A.1 Spinor identities

We take the metric to be the usual field-theory one $\eta^{\mu\nu} = (1;-1;-1;-1)$ and the epsilon tensors with which we raise and lower indices to be

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \tag{A.1.1}$$

with $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta} \Rightarrow \epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}$ and

$$\begin{aligned} \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \\ \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \\ \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{A.1.2}$$

We also have $\sigma^{\mu}_{\alpha\dot\alpha} = (1;\vec\sigma)$ (with $\vec\sigma = (\sigma^1;\sigma^2;\sigma^3)$), giving

$$\begin{aligned} P_{\alpha\dot\alpha} &= P_{\mu}\sigma^{\mu}_{\alpha\dot\alpha} \\ &= \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & P_0 - P_3 \end{pmatrix} \\ &= \begin{pmatrix} P^0 - P^3 & -P^1 + iP^2 \\ -P^1 - iP^2 & P^0 + P^3 \end{pmatrix}; \end{aligned} \tag{A.1.3}$$

and $\bar\sigma^{\mu\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$, $\sigma^\mu_{\alpha\dot\alpha} = (1;\vec\sigma)$, giving

$$
\begin{aligned}
P_{\alpha\dot\alpha} &= P_\mu \sigma^\mu_{\alpha\dot\alpha} \\
&= \begin{pmatrix} P_0 - P_3 & -P_1 + iP_2 \\ -P_1 - iP_2 & P_0 + P_3 \end{pmatrix} \\
&= \begin{pmatrix} P^0 + P^3 & P^1 - iP^2 \\ P^1 + iP^2 & P^0 - P^3 \end{pmatrix}.
\end{aligned}
\tag{A.1.4}
$$

Some useful identities involving σ and $\bar\sigma$ are:

$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma_\mu^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta} \tag{A.1.5}$$

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}, \tag{A.1.6}$$

which means that we can interpret $g_{\mu\nu}$ as acting as $2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$ and $g^{\mu\nu}$ as acting as $\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}/2$ in spinor space, giving $g^\mu_\mu = \delta_\alpha^\alpha\delta_{\dot\alpha}^{\dot\alpha} = 4$ as it should.

We are concerned with massless particles for which we can write

$$
\begin{aligned}
p_{\alpha\dot\alpha} &= \lambda_\alpha\tilde\lambda_{\dot\alpha} \\
&= \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \tilde\lambda_{\dot 1} & \tilde\lambda_{\dot 2} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1\tilde\lambda_{\dot 1} & \lambda_1\tilde\lambda_{\dot 2} \\ \lambda_2\tilde\lambda_{\dot 1} & \lambda_2\tilde\lambda_{\dot 2} \end{pmatrix},
\end{aligned}
\tag{A.1.7}
$$

which implies (by raising indices) that

$$
\begin{aligned}
p^{\dot\alpha\alpha} &= \tilde\lambda^{\dot\alpha}\lambda^\alpha \\
&= \begin{pmatrix} \tilde\lambda^{\dot 1} \\ \tilde\lambda^{\dot 2} \end{pmatrix} \begin{pmatrix} \lambda^1 & \lambda^2 \end{pmatrix} \\
&= \begin{pmatrix} \tilde\lambda^{\dot 1}\lambda^1 & \tilde\lambda^{\dot 1}\lambda^2 \\ \tilde\lambda^{\dot 2}\lambda^1 & \tilde\lambda^{\dot 2}\lambda^2 \end{pmatrix} \\
&= \begin{pmatrix} \tilde\lambda_{\dot 2}\lambda_2 & -\tilde\lambda_{\dot 2}\lambda_1 \\ -\tilde\lambda_{\dot 1}\lambda_2 & \tilde\lambda_{\dot 1}\lambda_1 \end{pmatrix},
\end{aligned}
\tag{A.1.8}
$$

which follows from having $\lambda = (\epsilon\lambda)^T = \lambda^T\epsilon^T$ and $\tilde\lambda^{\dot\alpha} = (\tilde\lambda_{\dot\beta}\epsilon^{\dot\beta\dot\alpha})^T = \epsilon^{\dot\alpha\dot\beta}\tilde\lambda^T_{\dot\beta}$

For scalar products we take

$$\begin{aligned} \langle\ \ \rangle &= \\ &= {}^{1}\ \ {}^{2} \begin{pmatrix} {}^{1} \\ {}^{2} \end{pmatrix} \\ &= {}^{T}\ \ ; \end{aligned} \qquad \text{(A.1.9)}$$

and

$$\begin{aligned} [\tilde{}\ \tilde{}\,] &= \tilde{}\,\tilde{} \\ &= \tilde{}_{1}\ \ \tilde{}_{2} \begin{pmatrix} \tilde{}^{1} \\ \tilde{}^{2} \end{pmatrix} \\ &= \tilde{}\,\tilde{}^{T} \end{aligned} \qquad \text{(A.1.10)}$$

Note that and $\tilde{}$ are most naturally associated with column vectors, while $\tilde{}$ and are most naturally associated with row vectors.

For spinor manipulations, the Schouten identity is very useful:[1]

$$\begin{aligned} \langle ij\rangle\langle kl\rangle &= \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle ; \\ [ij][kl] &= [ik][jl] + [il][kj] : \end{aligned} \qquad \text{(A.1.11)}$$

For other introductions to the spinor helicity formalism see e.g. [153, 154].

A.2 The holomorphic delta function

Consider the $x - y$ plane in real coordinates and let $(x;y) = (x^1;x^2)$. Now change to complex coordinates by letting

$$z = x^1 + ix^2 \qquad \text{(A.2.1)}$$

$$\bar{z} = x^1 - ix^2 : \qquad \text{(A.2.2)}$$

Also define derivatives

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right) = \frac{1}{2}(\partial_1 - i\partial_2) \qquad \text{(A.2.3)}$$

$$\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right) = \frac{1}{2}(\partial_1 + i\partial_2) ; \qquad \text{(A.2.4)}$$

[1] Recall that we have the shorthand notation $\langle\ _i\ \ _j\rangle = \langle ij\rangle$ etc.

which have the properties that

$$\partial_z z = 1\,;\quad \partial_z \bar z = 0\,;\quad \partial_{\bar z} \bar z = 1\,;\quad \partial_{\bar z} z = 0\,.$$

We take the area element in the $x-y$ plane to be $d^2x = dx^1dx^2 = |dx^1 \wedge dx^2|$, where $||$ just indicates that one picks a plus sign to define the orientation. For the area element in the $z-\bar z$ plane we take $d^2z = i|dz \wedge d\bar z|$ so that we have $d^2z = 2d^2x$.

We normalise delta functions in the $x-y$ plane as

$$\int d^2x\, \delta^{(2)}(x-a) = 1\,; \tag{A.2.5}$$

where $\delta^{(2)}(x-a) := \delta(x^1-a^1)\,\delta(x^2-a^2)$, and after transforming this to the $z-\bar z$ plane (with $b = a^1 + ia^2$ and $\bar b = a^1 - ia^2$) we have

$$\int d^2z\, \delta^{(2)}(z-b) = 1\,; \tag{A.2.6}$$

where

$$\begin{aligned} \delta^{(2)}(z-b) &:= \delta(z-b)\,\delta(\bar z-\bar b) \\ &= \frac{1}{2}\delta^{(2)}(x-a)\,. \end{aligned} \tag{A.2.7}$$

We now define a holomorphic delta function as

$$\bar\delta(z-b) = \delta^{(2)}(z-b)d\bar z\,; \tag{A.2.8}$$

which gives us

$$\begin{aligned} \int dz\, \bar\delta(z-b) &= \int dz \wedge d\bar z\, \delta^{(2)}(z-b) \\ &= -i\int d^2z\, \delta^{(2)}(z-b) \\ &= -i\,. \end{aligned} \tag{A.2.9}$$

As can be seen the holomorphic delta function is a $\bar\partial$-closed $(0,1)$-form and is defined for a general holomorphic function f by $\bar\delta(f) = \delta^{(2)}(f)d\bar f$.

A representation of this holomorphic delta function which will be particularly useful for us is the following [33]. Consider a momentum-vector described by λ and $\tilde\lambda$ with $\tilde\lambda = \bar\lambda$ in order to ensure that $p_{\alpha\dot\alpha} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$ is real. Go to coordinates where $\lambda = (1,z)$ and choose an arbitrary spinor $\mu = (1,b)$ with b a complex number. The tilded spinors are then

$$\tilde\lambda = \begin{pmatrix}1\\ z\end{pmatrix} \qquad \tilde\eta = \begin{pmatrix}1\\ b\end{pmatrix};$$

and we have $\langle\tilde\lambda\,\tilde\eta\rangle = z - b$ and $\langle\tilde\lambda\, d\tilde\lambda\rangle = dz$. So, a more covariant statement of (A.2.9) is

$$\begin{aligned}\int \langle\tilde\lambda\, d\tilde\lambda\rangle\, \delta(\langle\tilde\lambda\,\tilde\eta\rangle) F(\tilde\lambda) &= \int dz\, \delta(z-b) F(z)\\ &= iF(b)\\ &= iF(\tilde\eta)\,. \end{aligned} \qquad \text{(A.2.10)}$$

A.3 Dirac traces

Some basic formulae for converting between spinor invariants and Dirac traces are

$$\langle ij\rangle[ji] = \mathrm{tr}_+(\not{k}_i\not{k}_j)\,; \qquad \text{(A.3.1)}$$

$$\langle ij\rangle[jl]\langle lm\rangle[mi] = \mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_l\not{k}_m)\,; \qquad \text{(A.3.2)}$$

$$\langle ij\rangle[jl]\langle lm\rangle[mn]\langle np\rangle[pi] = \mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_l\not{k}_m\not{k}_n\not{k}_p)\,; \qquad \text{(A.3.3)}$$

for momenta $k_i, k_j, k_l, k_m, k_n, k_p$ and where the + sign indicates the insertion of $(1+\gamma_5)/2$:

$$\mathrm{tr}_+(\not{k}_i\not{k}_j) := \frac{1}{2}\mathrm{tr}_+((1+\gamma_5)\not{k}_i\not{k}_j)\,. \qquad \text{(A.3.4)}$$

We also note that

$$\mathrm{tr}_+(\not{k}_i\not{k}_j) = 2(k_i\cdot k_j) \qquad \text{(A.3.5)}$$

$$\begin{aligned}\mathrm{tr}_+(\not{k}_a\not{k}_b\not{k}_c\not{k}_d) &= 2(k_a\cdot k_b)(k_c\cdot k_d) - 2(k_a\cdot k_c)(k_b\cdot k_d)\\ &\quad + 2(k_a\cdot k_d)(k_b\cdot k_c) - 2i\varepsilon(k_a,k_b,k_c,k_d)\,. \end{aligned} \qquad \text{(A.3.6)}$$

The following identities are additionally useful:

$$\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_l\not{k}_m) = \mathrm{tr}_+(\not{k}_m\not{k}_l\not{k}_j\not{k}_i) = \mathrm{tr}_+(\not{k}_l\not{k}_m\not{k}_i\not{k}_j)\,; \qquad \text{(A.3.7)}$$

$$\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_l\not{k}_m) = 4(k_i\cdot k_j)(k_l\cdot k_m) - \mathrm{tr}_+(\not{k}_j\not{k}_i\not{k}_l\not{k}_m)\,; \qquad \text{(A.3.8)}$$

$$\mathrm{tr}_+(\not{i}\not{j}\gamma^\mu\not{P})\,\mathrm{tr}_+(\not{i}\not{j}\gamma_\mu\not{m}) = 0\,; \qquad \text{(A.3.9)}$$

$$\mathrm{tr}_+(\not{i}\not{j}\gamma^\mu\not{P})\,\mathrm{tr}_+(\not{i}\not{j}\not{m}\gamma_\mu) = 4(i\cdot j)\,\mathrm{tr}_-(\not{i}\not{j}\not{m}\not{P}) \qquad \text{(A.3.10)}$$

for similarly generic momenta and where we use the shorthand $\mathrm{tr}_+(\not{k}_i\not{k}_j) = \mathrm{tr}_+(\not{i}\not{j})$ etc. If $k_i; k_j; k_{m_1}$ and k_{m_2} are massless, while P_L is not necessarily so, then we have the remarkable identity:

$$
\begin{aligned}
& 2(k_{m_1}\cdot k_{m_2})\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{P}_L)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_2}\not{P}_L) \\
& \qquad + P_L^2\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{k}_{m_2})\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_2}\not{k}_{m_1}) \\
& - 2(k_{m_1}\cdot P_L)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{k}_{m_2})\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_2}\not{P}_L) \\
& - 2(k_{m_2}\cdot P_L)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_1}\not{P}_L)\,\mathrm{tr}_+(\not{k}_i\not{k}_j\not{k}_{m_2}\not{k}_{m_1}) \;=\; 0\,. \qquad \text{(A.3.11)}
\end{aligned}
$$

We also have, for null momenta $i; j; k; a; b$;

$$
\frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})\,\mathrm{tr}_+(\not{j}\not{a}\not{k}\not{b})}{(j\cdot a)} = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})\,\mathrm{tr}_+(\not{i}\not{a}\not{k}\not{b})}{(i\cdot a)}\,. \qquad \text{(A.3.12)}
$$

APPENDIX B

FEYNMAN RULES IN THE SPINOR HELICITY FORMALISM

In this appendix we present the Feynman rules for massless SU(N_c) Yang-Mills theory in Feynman gauge written in the spinor helicity formalism for comparison with those laid out at the start of Chapter 1. As mentioned in a footnote in §1.1 we will use the normalisation $\mathrm{tr}(T^a T^b) = {}^{ab}$ for the Lie-algebra in order to reduce the proliferation of factors of 2.

B.1 Wavefunctions

External Scalar:

$$ = 1 \tag{B.1.1}$$

External outgoing fermion i, helicity plus:

$${}^{+}_{i} = \tilde{}_{i\dot{}} = [i| \tag{B.1.2}$$

External outgoing fermion i, helicity minus:

$${}_{i} = {}_{i} = \langle i| \tag{B.1.3}$$

External outgoing anti-fermion j, helicity plus:

$${}^{+}_{j} = \tilde{}^{\dot{}}_{j} = |j] \tag{B.1.4}$$

External outgoing anti-fermion j, helicity minus:

$${}_{j} = {}_{j} = |j\rangle \tag{B.1.5}$$

External outgoing vector $p = \tilde{}$, helicity plus:

$${}^{+}_{\dot{}} = \frac{\sqrt{2}\,\tilde{}_{\dot{}}}{\langle\ \rangle} = \frac{\sqrt{2}\,|\ \rangle[\tilde{}\,|}{\langle\ \rangle} \tag{B.1.6}$$

External outgoing vector $p = \;\tilde{}$, helicity minus:

$$_{-} = \sqrt{2}\,\frac{\tilde{}}{[\tilde{}\;\tilde{}\,]} = \sqrt{2}\,\frac{j\;i[\tilde{}\,j}{[\tilde{}\;\tilde{}\,]}\,; \qquad \text{(B.1.7)}$$

where $q = \;\tilde{}$ is an arbitrary reference spinor that can be chosen independently for each external particle. All the above wavefunctions are understood to be multiplied by a factor of exp(ix $\_$ $\tilde{}\,^{-}$), where $p\,\_ = \;\tilde{}\,\_$ is the momentum of the particle.

B.2 Propagators

Scalars with kinetic term $(@\;)^2=2$:

$$\frac{i}{p^2} \qquad \text{(B.2.1)}$$

Fermions with $p = \;\tilde{}$ and kinetic term $i\;@\;$:

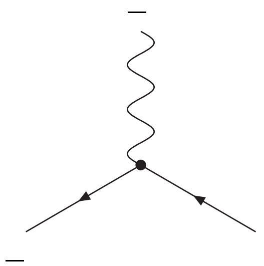
$$\frac{i}{p} = \frac{ip\,\_}{2p^2} = \frac{ij\;i[\tilde{}\,j}{2p^2} \qquad \text{(B.2.2)}$$

Vectors with kinetic term $(@A\;)^2=4$:

$$\frac{2i\;\_\_}{p^2} \qquad \text{(B.2.3)}$$

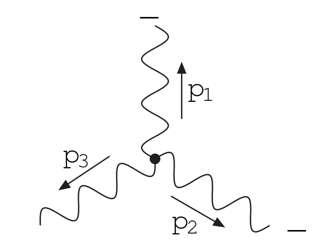
B.3 Vertices

Fermion Vertex:



$$= ig\sqrt{2}\qquad {}_{-}^{\;-}$$

3-Boson Vertex:



$$= \frac{g}{2\sqrt{2}}[\;{}^{-}{}^{-}\;(p_1\;\;p_2)^{-}$$
$$+\;{}^{-}{}_{-}\;(p_2\;\;p_3)^{-}$$
$$+\;\text{—}\;(p_3\;\;p_1)^{-}\;]$$

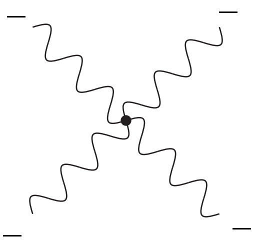
4-Boson Vertex:  $= \frac{ig^2}{8}[2 \quad \ldots \quad].$

Figure B.1: The Vertices of the colour-stripped scheme in terms of spinors.

For more details on how these arise see for example [153].

B.4 Examples

4-Point MHV gluon scattering

Let us consider how we get the $A(1_g; 2_g; 3_g^+; 4_g^+)$ gluon amplitude. The diagrams contributing to this amplitude are shown in Figure B.2

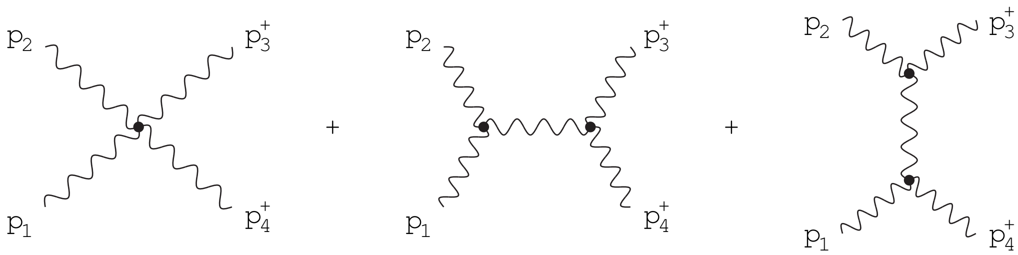


Figure B.2: The diagrams contributing to the 4-gluon MHV tree-amplitude. All external momenta are taken to be outgoing.

In order to calculate the amplitude we need to specify external wavefunctions as prescribed by the Feynman rules and for gluons this includes a choice of reference momentum. In order to minimise the number of terms we need to consider we will make the choices $q_1 = q_2 = p_4$ and $q_3 = q_4 = p_1$. This means that the wavefunctions are

$$\epsilon_1^- = \sqrt{2}\frac{\lambda_1\tilde\lambda_4}{[41]} \qquad \epsilon_2^- = \sqrt{2}\frac{\lambda_2\tilde\lambda_4}{[42]}$$

$$\epsilon_3^+ = \sqrt{2}\frac{\lambda_1\tilde\lambda_3}{\langle 13\rangle} \qquad \epsilon_4^+ = \sqrt{2}\frac{\lambda_1\tilde\lambda_4}{\langle 14\rangle}\;;$$

while momentum conservation for the second (going from left to right) two diagrams reads $P_{12} = -(p_1 + p_2) = p_3 + p_4$ and $P_{14} = -(p_1 + p_4) = p_2 + p_3$ where P_{12} and P_{14} are the momenta of the propagators of the respective diagrams. By writing down the Feynman rules for the different diagrams it can quickly be seen that the contributions of the 1st and the 3rd diagrams both vanish. The 2nd diagram gives:

$$\begin{aligned}
A_4 &= \left(\frac{g}{\sqrt{2}\sqrt{2}}\right)^2 \sqrt{2}^4 \frac{\lambda_1\tilde\lambda_4\,\lambda_2\tilde\lambda_4\,\lambda_1\tilde\lambda_3\,\lambda_1\tilde\lambda_4}{[41][42]\langle 14\rangle\langle 13\rangle}\Big[\,\cdot\,(p_1 - p_2)\cdot \\
&\quad + \cdot\,(p_2 - P_{12})\cdot + \cdot\,(P_{12} - p_1)\cdot\Big]\frac{2i}{P_{12}^2} \\
&\quad \Big[\,\cdot\,(p_3 - p_4)\cdot + \cdot\,(p_4 + P_{12})\cdot + \cdot\,(-P_{12} - p_3)\cdot\Big] \\
&= \frac{4ig^2}{\langle 12\rangle[21]}\frac{(\lambda_2\lambda_1)(\tilde\lambda_4\tilde\lambda_2)(\lambda_3\lambda_1)(\tilde\lambda_4\tilde\lambda_3)(\tilde\lambda_3\tilde\lambda_4)(\lambda_1\lambda_2)}{[41][42]\langle 14\rangle\langle 13\rangle} \\
&= 4ig^2\frac{\langle 12\rangle[34]^2}{\langle 14\rangle[12][41]}\;: \qquad\qquad (B.4.1)
\end{aligned}$$

This is our answer, though it is in a rather unfamiliar form! We can convert it into something more familiar by multiplying both top and bottom by $\langle 23\rangle\langle 34\rangle$. We then use momentum conservation in the numerator in the form $\langle 23\rangle[34] = -\langle 21\rangle[14]$ and recognise that $s_{34} := 2(p_3 \cdot p_4) = \langle 34\rangle[43] = s_{12} = \langle 12\rangle[21]$ to give

$$\begin{aligned}
A_4 &= 4ig^2\frac{\langle 12\rangle(\langle 23\rangle[34])(\langle 34\rangle[43])}{[12]\langle 23\rangle\langle 34\rangle\langle 41\rangle[41]} \\
&= 4ig^2\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}\;; \qquad\qquad (B.4.2)
\end{aligned}$$

which is the usual form for the Parke–Taylor amplitude at 4-point.

MHV $q\bar q \to gg$

Again we take all momenta to be outgoing. Momentum conservation is the same as for diagrams 2 and 3 of the previous example and for the gluon wavefunctions we take $q_3 = p_4$ and $q_4 = p_1$. This gives polarisation vectors

$$\epsilon_3^- = \sqrt{2}\frac{\lambda_3\tilde\lambda_4}{[43]} \qquad \epsilon_4^+ = \sqrt{2}\frac{\lambda_1\tilde\lambda_4}{\langle 14\rangle}\;:$$

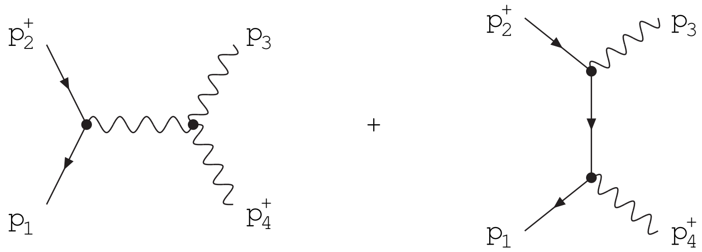


Figure B.3: The diagrams for $\tilde{A}(1_q; 2_q^+; 3_g; 4_g^+)$.

The second diagram can be seen to vanish while the first gives:

$$\begin{aligned}
\tilde{A}_4 &= (\tilde{\ }_2 \quad {}^1)(ig\sqrt{2} \quad)\frac{2i}{P_{12}^2}\frac{g}{\sqrt{2}\sqrt{2}}\quad (p_3 \quad p_4) \\
&+ \quad (p_4 + P_{12}) + \quad (P_{12} \quad p_3) \quad (\sqrt{2})^2 \frac{{}^3\tilde{\ }{}^4 \quad {}^1\tilde{\ }{}^4}{[43]\langle14\rangle} \\
&= 4g^2 \frac{(\tilde{\ }_2 \quad \tilde{\ }^4)(\ _1 \quad {}^3)({}^1 \ _3)(\tilde{\ }^4\tilde{\ }_3)}{\langle12\rangle[21][43]\langle14\rangle} \\
&= 4g^2 \frac{\langle13\rangle^2[42]}{\langle12\rangle[21]\langle41\rangle} \\
&= 4g^2 \frac{\langle13\rangle^3}{\langle12\rangle\langle34\rangle\langle41\rangle} \\
&= i\frac{\langle23\rangle}{\langle13\rangle} A(1_g; 2_g^+; 3_g; 4_g^+)\,;
\end{aligned} \tag{B.4.3}$$

thus verifying the relations between amplitudes that we derived from supersymmetric Ward identities in §1.4.

MHV $qq \to qq$

As a final example let us consider the amplitude $\hat{A}(1_q; 2_q^+; 3_q; 4_q^+)$. This time both of the diagrams are non-zero. The first one gives

$$\begin{aligned}
\hat{A}_4^1 &= (\tilde{\ }_2 \quad {}^1)(ig\sqrt{2} \quad)\frac{2i}{P_{12}^2}(ig\sqrt{2} \quad)(\tilde{\ }_4 \quad {}^3) \\
&= 4ig^2 \frac{(\tilde{\ }^2\tilde{\ }_4)(\ _3 \ {}^1)}{\langle12\rangle[21]} \\
&= 4ig^2 \frac{[24]\langle31\rangle}{\langle12\rangle[21]} \\
&= 4ig^2 \frac{\langle13\rangle^2}{\langle12\rangle\langle34\rangle}\,:
\end{aligned} \tag{B.4.4}$$

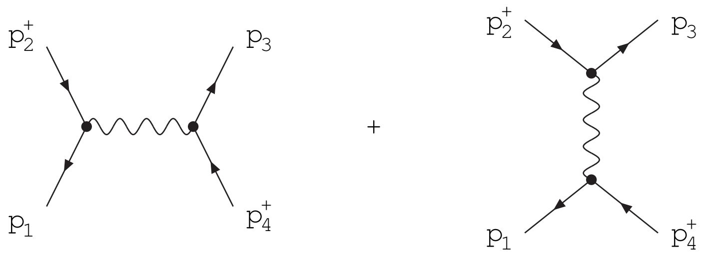


Figure B.4: The diagrams for $\hat{A}(1_q, 2_q^+, 3_q, 4_q^+)$.

A similar calculation – or equivalently the realisation that diagrams two is simply the same as diagram one with $2 \leftrightarrow 4$ – gives

$$\hat{A}_4^2 = 4ig^2 \frac{\langle 13\rangle^2}{\langle 23\rangle\langle 41\rangle} \qquad \text{(B.4.5)}$$

for the second diagram and thus the total is

$$\begin{aligned} \hat{A}_4 &= \hat{A}_4^1 + \hat{A}_4^2 \\ &= 4ig^2 \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\left(\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 41\rangle\right) \\ &= A(1_g, 2_g^+, 3_g, 4_g^+) \quad \frac{\langle 12\rangle\langle 34\rangle + \langle 23\rangle\langle 41\rangle}{\langle 13\rangle^2} \quad . \end{aligned} \qquad \text{(B.4.6)}$$

APPENDIX C

D-DIMENSIONAL LORENTZ-INVARIANT PHASE SPACE

In this appendix we expound on the D-dimensional measure for Lorentz-invariant two-body phase space, ultimately focussing on the case $D = 4 - 2\epsilon$.

C.1 D-spheres

One thing that we will need to consider is the volume of a D-dimensional unit sphere $V(S^D)$. We mean this in the sense of a D-sphere regarded as a manifold. Thus the volume we are talking about is the volume of that manifold rather than the volume enclosed by it when it is regarded as being embedded in one-dimension higher. Thus $V(S^1) = 2\pi$ – the circumference of a circle – and $V(S^2) = 4\pi$, the surface area of a sphere such as the Earth.

In fact we can parametrize a round D-sphere in terms of D angles θ_i. In this case the volume element of an S^D is given by

$$dV(S^D) = d\theta_1 \ldots d\theta_D (\sin\theta_1)^{D-1}(\sin\theta_2)^{D-2} \ldots (\sin\theta_{D-1})^1 ; \qquad \text{(C.1.1)}$$

with the result

$$\begin{aligned} V(S^D) &= \int_{\theta_i = 0}^{\theta_j = \pi, j \neq D;\ \theta_D = 2\pi} dV(S^D) \\ &= \frac{2\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}\right)} : \qquad \text{(C.1.2)} \end{aligned}$$

C.2 dLIPS

Recall from Chapter 1, Equation (1.8.13) that

$$d^D LIPS(l_2 ; -l_1^+ ; P) = d^D l_1 d^D l_2\, \delta^{(+)}(l_1^2)\, \delta^{(-)}(l_2^2)\, \delta^{(D)}(P + l_2 - l_1) ; \qquad \text{(C.2.1)}$$

where $\theta^{(+)}(l^2) := \theta(l_0)\,\delta(l^2)$, θ is the unit step function[1] and l_0 the 0-component (energy) of l. If we also remember that

$$\int dx\, g(x)\,\delta(f(x)-a) = \frac{g(x)}{\left|\frac{df}{dx}\right|}\Bigg|_{x=x_0;\, f(x_0)=a} \tag{C.2.2}$$

then we can integrate over the 0-components of l_1 and l_2 to get

$$d^D LIPS = \frac{d^{D-1}\vec{l}_1}{2|l_{10}|}\frac{d^{D-1}\vec{l}_2}{2|l_{20}|}\,\delta^{(D)}(P+l_2-l_1)\Bigg|_{\substack{l_{10}=|\vec{l}_1| \\ l_{20}=-|\vec{l}_2|}} ; \tag{C.2.3}$$

where $\vec{l}$ represents the spatial components of the D-vector l. Furthermore, going to the center of mass frame for the vector P, $P=(P_0;\vec{0})$ we can use $D-1$ of the remaining delta functions to localise the integral:

$$\begin{aligned} d^D LIPS &= \frac{d^{D-1}\vec{l}_1}{2|\vec{l}_1|}\frac{d^{D-1}\vec{l}_2}{2|\vec{l}_1|}\,\delta^{(D-1)}(\vec{l}_2-\vec{l}_1)\,\delta(P_0-2|\vec{l}_1|) \\ &= \frac{1}{2}\frac{d^{D-1}\vec{l}_1}{4|\vec{l}_1|^2}\,\delta\!\left(|\vec{l}_1|-\frac{P_0}{2}\right) . \end{aligned} \tag{C.2.4}$$

Now, for $d^n\vec{l}$ we can write

$$d^n\vec{l} = d|\vec{l}|\,|\vec{l}|^{n-1}\,dV(S^{n-1}) ; \tag{C.2.5}$$

so we have

$$\begin{aligned} d^{D-1}|\vec{l}_1| &= d|\vec{l}_1|\,|\vec{l}_1|^{D-2}\,d\theta_1 d\theta_2 (\sin\theta_1)^{D-3}(\sin\theta_2)^{D-4} \\ &\quad d\theta_3 \ldots d\theta_{D-2}(\sin\theta_3)^{D-5}\ldots(\sin\theta_{D-3}) \\ &= d|\vec{l}_1|\,|\vec{l}_1|^{D-2}\,d\theta_1 d\theta_2 (\sin\theta_1)^{D-3}(\sin\theta_2)^{D-4}\,dV(S^{D-4}) . \end{aligned} \tag{C.2.6}$$

For our case of a 2-particle phase space in $4-2\epsilon$ dimensions, 2 angles θ_1 and θ_2 are sufficient and none of the momenta will depend on any of the other angles. We can thus integrate over them to get

$$\begin{aligned} d^{D-1}|\vec{l}_1| &= d|\vec{l}_1|\,|\vec{l}_1|^{D-2}\,d\theta_1 d\theta_2 (\sin\theta_1)^{D-3}(\sin\theta_2)^{D-4}\,V(S^{D-4}) \\ &= \frac{2\pi^{\frac{D-3}{2}}}{\Gamma\left(\frac{D-3}{2}\right)}\,d|\vec{l}_1|\,|\vec{l}_1|^{D-2}\,d\theta_1 d\theta_2 (\sin\theta_1)^{D-3}(\sin\theta_2)^{D-4} . \end{aligned} \tag{C.2.7}$$

[1] Not to be confused with the angles θ_i of (C.1.1) and (C.1.2).

With $D = 4-2\epsilon$ this leads us to

$$\begin{aligned} d^{4-2\epsilon}\text{LIPS} &= \frac{1}{2}\frac{d^{3-2\epsilon}l_1}{4|l_1|^2}\delta\left(|l_1|-\frac{P_0}{2}\right) \\ &= \frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{P_0^2}{4}\right)^{-\epsilon} d\theta_1 d\theta_2 (\sin\theta_1)^{1-2\epsilon}(\sin\theta_2)^{-2\epsilon} \\ &= \frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{P^2}{4}\right)^{-\epsilon} d\theta_1 d\theta_2 (\sin\theta_1)^{1-2\epsilon}(\sin\theta_2)^{-2\epsilon}\,; \end{aligned} \tag{C.2.8}$$

and

$$\int d^{4-2\epsilon}\text{LIPS} = \frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{P^2}{4}\right)^{-\epsilon}\int_{\theta_1=0}^{\pi}\int_{\theta_2=0}^{\pi} d\theta_1 d\theta_2 (\sin\theta_1)^{1-2\epsilon}(\sin\theta_2)^{-2\epsilon}\,. \tag{C.2.9}$$

C.3 Overall amplitude normalisation

In the original papers of [38, 42], the one-loop amplitudes derived are normalised with a factor of $c_\Gamma = r_\Gamma/(4\pi)^{2-\epsilon}$ where

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{C.3.1}$$

In [37, 40, 43] and this thesis, however, the normalisation most naturally arises as

$$\frac{\pi\epsilon}{\sin\pi\epsilon}\frac{1}{\Gamma\left(\frac{1}{2}-\epsilon\right)}\,; \tag{C.3.2}$$

where the gamma function comes from the LIPS measure described above and the factor of $\pi\epsilon\csc\pi\epsilon$ comes from performing the dispersion integral (see e.g. Section 5 of [37]). We are mostly interested in the results of these amplitude calculations up to order ϵ^0, and as (C.3.2) $= 1/\sqrt{\pi} + O(\epsilon)$ we have usually dropped it as an uninteresting overall factor. Nonetheless, the all-orders in ϵ results can be useful and we will here show how the two are related.

To start with there is the product identity for gamma functions:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}\,; \tag{C.3.3}$$

which can be combined with the well-known recurrence relation $z\Gamma(z) = \Gamma(z+1)$ to give

$$\frac{\pi\epsilon}{\sin\pi\epsilon} = \Gamma(1+\epsilon)\Gamma(1-\epsilon)\,. \tag{C.3.4}$$

There is also the Legendre duplication formula:

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)\,; \tag{C.3.5}$$

which implies that

$$(1/2\quad) = \frac{(1\quad 2\quad)\sqrt{\quad 2^2}}{(1\quad\quad)} \; : \tag{C.3.6}$$

This therefore leads us to

$$\begin{aligned} \frac{}{\sin}\frac{1}{\frac{1}{2}} &= \frac{1}{4\sqrt{\quad}}\frac{(1+\quad)^2(1\quad\quad)}{(1\quad 2\quad)} \\ &= \frac{\sqrt{\quad}}{4\sqrt{\quad}} \; ; \end{aligned} \tag{C.3.7}$$

and we can see that the two are the same up to a simple factor.

APPENDIX D

UNITARITY

Unitarity is a well-known and useful tool in quantum field theory [210, 233, 234, 235, 236, 237].[1] The unitarity of the S-matrix, $S^\dagger S = 1$, is the basic starting point and leads to the possibility of being able to reconstruct scattering amplitudes from the knowledge of their properties as functions of complex momenta. In certain cases this can lead to a purely algebraic construction of amplitudes.

It can be checked that each Feynman diagram contributing to an S-matrix element S is purely real unless some denominator vanishes, in which case the $i\varepsilon$ prescription for treating poles becomes relevant. We thus get an imaginary part for S only when virtual particles in a Feynman diagram go on-shell.

Consider now $S(s)$ as an analytic function of a complex variable s. s is the square of the centre of mass energy, and while this is physically real we will consider it to be complex for now. If s_0 is the minimum (square of the) energy for production of the lightest multiparticle state (i.e. the minimum energy for the creation of an intermediate multiparticle state such as when a loop is formed in a Feynman diagram), then for real s lying below s_0, the intermediate state cannot go on-shell. $S(s)$ is thus real and we have

$$S(s) = \overline{S}(s)\,. \tag{D.0.1}$$

However, as we are regarding $S(s)$ as an analytic function of s, we can analytically continue this equation to anywhere in the complex plane. If we explicitly split $S(s)$ into its real and imaginary parts, $S(s) = \Re[S(s)] + i\Im[S(s)]$, then at a point $s > s_0$ that is ε away from the real line (D.0.1) implies that

$$\begin{aligned} \Re[S(s+i\varepsilon)] &= \Re[S(s-i\varepsilon)]\,; \\ \Im[S(s+i\varepsilon)] &= -\Im[S(s-i\varepsilon)]\,. \end{aligned} \tag{D.0.2}$$

There is thus a branch cut along the positive real axis starting at s_0 and the discontinuity D of $S(s)$ across the cut is

$$D[S(s)] = 2i\Im[S(s+i\varepsilon)]\,. \tag{D.0.3}$$

[1] Note that some of this appendix is based on Section 7.3 of [2].

It turns out that this discontinuity – which only arises because we have intermediate multiparticle states and thus loop contributions to Feynman diagrams – can be related to simpler amplitudes which may be known already or more easily computed. This is the content of the optical theorem which we review below.

D.1 The optical theorem

The S-matrix is a unitary operator which evolves the initial states k_a so that one may compute their overlap with the final states p_i in a scattering process:

$$ {}_{out}\langle p_i | k_a \rangle_{in} = \langle p_i | S | k_a \rangle . \tag{D.1.1} $$

It is conventional to split S into the part that describes unimpeded propagation of the initial particles and a part T due to interactions, $S = 1 + iT$. The matrix element (D.1.1) taken with the interacting part of S is what then gives a scattering amplitude. More concretely, we can write

$$ \langle p_i | T | k_a \rangle = (2\pi)^4 \delta^{(4)}\left(\sum p_i + k_a\right) S(k_a \to p_i) ; \tag{D.1.2} $$

where we have taken all particles to be outgoing.

Unitarity of S, $S^\dagger S = 1$ implies

$$ -i(T - T^\dagger) = T^\dagger T ; \tag{D.1.3} $$

and we may extract some useful information by taking the matrix element of this between some particle states p_i and k_a. The LHS of (D.1.3) gives

$$ \begin{aligned} -i(\langle p_i | T | k_a \rangle - \langle p_i | T^\dagger | k_a \rangle) &= -i\left(\langle p_i | T | k_a \rangle - \overline{\langle k_a | T | p_i \rangle}\right) \\ &= -i(2\pi)^4 \delta^{(4)}\left(\sum p_i + k_a\right)\left(S(k_a \to p_i) - \overline{S}(p_i \to k_a)\right) \\ &= -i(2\pi)^4 \delta^{(4)}\left(\sum p_i + k_a\right) D\left[S(p_i; k_a)\right] . \end{aligned} \tag{D.1.4} $$

On the RHS of (D.1.3) we can insert the identity operator as a sum over a complete set of intermediate states to obtain

$$
\begin{aligned}
\langle p_i|T^\dagger T|k_a\rangle &= \sum_n \left(\prod_{j=1}^{n} \int \frac{d^4 l_j}{(2\pi)^4}\, \theta(l_j^0)\, \delta(l_j^2 - m_j^2) \right) \langle p_i|T^\dagger|\{l_j\}\rangle\langle\{l_j\}|T|k_a\rangle \\
&= (2\pi)^4 \sum_n \left(\prod_{j=1}^{n} \int \frac{d^4 l_j}{(2\pi)^4}\, \theta(l_j^0)\, \delta(l_j^2 - m_j^2) \right) \overline{S}(p_i \to \{l_j\}) S(k_a \to \{l_j\}) \\
&\quad \delta^{(4)}\Big(p_i + \sum l_j\Big)\, \delta^{(4)}\Big(k_a - \sum l_j\Big) \\
&= (2\pi)^4\, \delta^{(4)}\Big(\sum (p_i + k_a)\Big) \sum_n \int dLIPS(n)\, \overline{S}(p_i \to \{l_j\}) S(k_a \to \{l_j\})\,;
\end{aligned}
\tag{D.1.5}
$$

where dLIPS(n) is the n-body Lorentz-invariant phase space measure. Putting the LHS and RHS of (D.1.3) back together again we find[2]

$$
i\,\mathrm{Disc}\,[S(p_i;k_a)] = \sum_n \int dLIPS(n)\, \overline{S}(p_i \to \{l_j\}) S(k_a \to \{l_j\})\,. \tag{D.1.6}
$$

Equation (D.1.6) says that the discontinuity of a scattering amplitude may be obtained as a sum of integrals over the phase spaces of intermediate multiparticle states of the amplitudes for scattering of the initial and final states into these intermediate states. In particular, for a one-loop process, the amplitudes arising on the RHS of (D.1.6) are tree-level amplitudes and the phase space is a 2-particle one.

D.2 Cutting rules

Cutkosky showed that using some cutting rules, one may compute the physical discontinuity of any Feynman diagram and prove the optical theorem to all orders in perturbation theory [210]. The rules are as follows [2]:

1. Cut through a diagram in all possible ways such that the cut propagators may be put on-shell.
2. For each cut (massive) propagator replace $1/(p^2 - m^2 + i\varepsilon)$ with a delta function $-2\pi i\,\delta(p^2 - m^2)$. This explicitly provides the delta functions which generate the dLIPS measure in (D.1.5). The off-shell vertices that are separated by the cut are thus put on-shell. For massless momenta the replacement is simply $1/(p^2 + i\varepsilon) \to -2\pi i\,\delta(p^2)$.

[2] In fact the optical theorem is usually stated in terms of the forward scattering amplitude, in which case we have $k_a = p_i$. The theorem is more general than this though and can be applied to generic asymptotic states.

3. Sum the contributions of all possible cuts.

For example, for a Feynman diagram in massless ϕ^3 theory such as Figure D.1, the

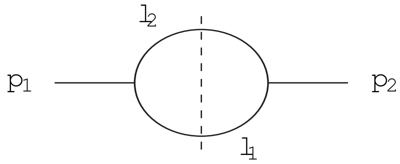


Figure D.1: The cut of a bubble diagram in massless ϕ^3 theory.

Feynman rules would give

$$A \propto \delta^{(4)}(p_1+p_2)\int \frac{d^4 l_1}{(2\pi)^4}\frac{d^4 l_2}{(2\pi)^4}\frac{1}{l_1^2}\frac{1}{l_2^2}\,\delta^{(4)}(p_1+l_1-l_2)\,. \tag{D.2.1}$$

Cutkosky's rules on the other hand would give

$$\begin{aligned} D[A] &\propto \delta^{(4)}(p_1+p_2)\int \frac{d^4 l_1}{(2\pi)^4}\frac{d^4 l_2}{(2\pi)^4}\,\delta(l_1^2)\,\delta(l_2^2)\,\delta^{(4)}(p_1+l_1-l_2) \\ &\propto \lambda^2\,\delta^{(4)}(p_1+p_2)\int dLIPS(l_2;-l_1;p_1)\,, \end{aligned} \tag{D.2.2}$$

which allows one to calculate the discontinuity of the diagram concerned.

D.2.1 BDDK's unitarity cuts

In [38, 42] Cutkosky's rules were applied at the level of amplitudes to derive one-loop MHV amplitudes in supersymmetric and non-supersymmetric gauge theories. In this case the factors on either side of the cut are not vertices (e.g. the λ factors of (D.2.2)), but full amplitudes. In fact for the one-loop MHV amplitudes these factors are tree-level MHV amplitudes.

Consider for concreteness the n-point one-loop MHV amplitudes for gluon scattering in $N = 4$ super-Yang-Mills as reviewed in §1.9. We would like to see how these can be obtained from 2-particle cuts as in [38].

By analogy with the Cutkosky rules, the procedure is to consider 'cuts' in every possible kinematical channel and then add the contributions without overcounting. We are then left with LIPS integrals as above (but this time with non-trivial kinematic factors in the integrand) which can in-principle be evaluated to reveal the discontinuities of the amplitude. However, BDDK recover the amplitude by using an algebraic

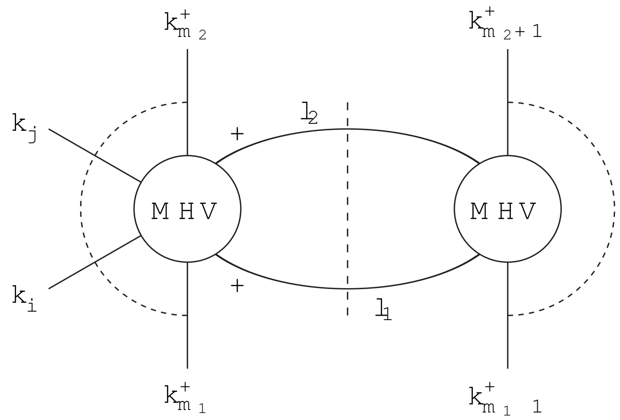


Figure D.2: The cut of a one-loop MHV amplitude in the $t_{m_1}^{[m_2-m_1+1]}$ channel.

procedure which means that these dispersion integrals do not actually need to be done. This involves replacing the delta functions associated with the cuts with propagators (a procedure that is known as 'reconstruction of the Feynman integral') which then produces Feynman integrals rather than LIPS integrals. These integrals contain cuts in the channel being considered (as well as cuts in other channels too) and by considering all channels and avoiding over-counting the amplitude can be re-constructed.

When we cut the amplitudes, we must assign helicities to the particles that were in the loop. Since we use conventions in which all particles are outgoing, the helicities of these internal particles are reversed. For the one-loop MHV amplitudes there are two distinct cases. Case (a) is where the negative-helicity external particles i and j are on the same side of the cut, and case (b) is where they are on opposite sides of the cut. Case (a), is a priori the simpler of the two as the two internal particles must have the same helicities and thus amplitude relations of equations (1.4.9) and (1.4.10) mean that only gluons can circulate in the loop. This is the situation regardless of the amount of supersymmetry present. Case (b) involves the entire multiplet circulating in the loop and for maximally supersymmetric Yang-Mills it turns out that this case is the same as case (a) after applying identities such as the Schouten identity (A.1.11). For the case being considered of $\mathcal{N} = 4$ Yang-Mills it is thus enough for us to treat case (a) only.

Consider now a cut in the channel where P_L, the momentum on the left of the cut, is given by $P_L^2 = (k_{m_1} + k_{m_1+1} + \dots + k_{m_2-1} + k_{m_2})^2 = t_{m_1}^{[m_2-m_1+1]}$ and where $k_i, k_j \in P_L$.

This situation is shown in Figure D.2 and the rules that we have outlined above give

$$
\begin{aligned}
D[A(t_{m_1}^{[m_2-m_1+1]})] &= \int \frac{d^4 l_1}{(2\pi)^4}\frac{d^4 l_2}{(2\pi)^4} A^{MHV}_{tree}(-l_1^+, m_1^+, \ldots, i^-, \ldots, j^-, \ldots, m_2^+, l_2^+) \\
&\quad \delta^{(+)}(l_1^2)\,\delta^{(+)}(l_2^2)\, A^{MHV}_{tree}(-l_2, m_2+1^+, \ldots, m_1-1^+, l_1) \\
&\quad \frac{i}{(2\pi)^4} A^{MHV}_{tree}(i^-, j^-) \int dLIPS(l_2, -l_1; P_L)\, \hat{R} \qquad \text{(D.2.3)} \\
&\to \frac{i}{(2\pi)^4} A^{MHV}_{tree}(i^-, j^-) \int d^4 l_1 d^4 l_2 \frac{1}{l_1^2}\frac{1}{l_2^2}\hat{R}\,; \qquad \text{(D.2.4)}
\end{aligned}
$$

where

$$
\hat{R} \coloneqq \frac{\langle m_1-1\, m_1\rangle\langle l_2\, l_1\rangle}{\langle m_1-1\, l_1\rangle\langle l_1\, m_1\rangle}\frac{\langle m_2\, m_2+1\rangle\langle l_1\, l_2\rangle}{\langle m_2\, l_2\rangle\langle l_2\, m_2+1\rangle} \qquad \text{(D.2.5)}
$$

as in (1.9.12) and the MHV amplitudes for negative-helicity gluons l, s are defined as in (1.9.10):

$$
A^{MHV}_{tree}(\lambda_l, \lambda_s) \coloneqq i(2\pi)^4\,\delta^{(4)}\left(\sum_i k_i\right)\frac{\langle l s\rangle^4}{\prod_{r=1}^{n}\langle r\, r+1\rangle}\,. \qquad \text{(D.2.6)}
$$

Note that Equation (D.2.4) is a Feynman integral rather than a LIPS integral.

Now recall from §1.9 and [38, 42] that the basis of integral functions at one-loop is known and the Feynman integrals can be done to give explicit expressions (see e.g. Appendix I of [42]). The Feynman integrals generated in (D.2.4) (and for other channels) can then be compared with the Feynman integrals for the known integral functions and the amplitude recreated. Since the integral functions are already known one can reconstruct the amplitude in a purely algebraic manner. As a strong check of the final expression, the results can be compared with the known behaviour (on general grounds) for the collinear ($p_a, p_b \to p_a \parallel p_b$) and soft ($p_a \to 0$) limits of such an amplitude.

For supersymmetric theories any terms which do not contain cuts are uniquely linked to the cut-containing terms and thus the entire amplitude is reconstructed. In particular, the $N = 4$ amplitudes discussed above can be completely constructed in this way leading to (1.9.1). In non-supersymmetric theories more information is needed to get the rational (cut-free) terms and thus only the cut-constructible part may be obtained this way.

D.3 Dispersion relations

Imagine now that we stop at (D.2.3) and proceed to do the LIPS integral rather than uplift to Feynman integrals. If we can actually do this integral we can calculate the discontinuity of the amplitude directly. However, we would really like to know the whole amplitude rather than just the imaginary part of it and the natural question is whether it is possible to arrive at this from what we have so far. For a function with a branch cut, it is in fact possible to reconstruct the real part from the imaginary part and the relations which allow one to do this are known as dispersion relations (or sometimes Kramers-Kronig relations).

By considering a function $\mathscr{A}(z)$ which is analytic in the complex plane with a branch cut along the positive real axis starting at x_0, it is possible to show using complex analysis that

$$\Re[\mathscr{A}(x)] = \frac{1}{\pi} P \int_{x_0}^{\infty} \frac{dx'}{x' - x} \Im[\mathscr{A}(x')] + \frac{1}{2\pi i} I_\infty \quad ; \qquad \text{(D.3.1)}$$

where $x \in \mathbb{R}$ in the range[3] $(x_0; \infty)$ and P denotes the Cauchy principal value prescription (i.e. the value of the integral without consideration of the pole at $x' = x$).[4] I_∞ is the contribution from the contour at infinity which represents the ambiguity due to possible rational terms (i.e. terms which are cut-free functions of the kinematic invariants).

I_∞ vanishes in any supersymmetric gauge theory, and while these do contain rational terms they are fixed uniquely by the supersymmetry once one knows the cut-containing terms [38, 42]. Such theories are said to be cut-constructible (in 4 dimensions). Non-supersymmetric theories are not cut-constructible in 4 dimensions, but are in $4-2\epsilon$ dimensions with $\epsilon \neq 0$ [86, 87, 213]. While this is a powerful statement, it does mean that one has to consider the prospect of using amplitudes with particles continued to $4-2\epsilon$ dimensions which are not simple.

In a sense, the one-loop CSW rules make BDDK's approach prescriptive for the MHV amplitudes. The imaginary part of the amplitude is constructed as a phase space integral and then the dispersion integral over $P^2_{L;z}$ in (1.8.12) performs (D.3.1) with I_∞ absent. For supersymmetric theories this is sufficient to construct the full amplitude, while in non-supersymmetric theories we must find other methods to calculate the rational part.

[3] For purely massless theories, $x_0 = 0$.

[4] See e.g. [238] for a fuller explanation of these ideas.

APPENDIX E

INTEGRALS FOR THE $\mathcal{N}=1$ AMPLITUDE

In this appendix we give details of the integrals needed to compute the discontinuities of the $\mathcal{N}=1$ amplitude discussed in Chapter 2.

E.1 Passarino–Veltman reduction

In §2.2 we saw that a typical term in the $\mathcal{N}=1$ amplitude is the dispersion integral of the following phase space integral:

$$C(m_1,m_2) \equiv \int dLIPS(l_2,-l_1;P_{L;z})\, \frac{\mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not l_1)\,\mathrm{tr}_+(\not k_i \not k_j \not k_{m_2} \not l_2)}{\langle i\,j\rangle^2 (m_1-l_1)^2 (m_2-l_2)^2}\,. \tag{E.1.1}$$

The full amplitude is then obtained by adding the dispersion integrals of three more terms similar to (E.1.1) but with m_1 replaced by m_1-1 and/or m_2 replaced by m_2+1. The goal of this appendix is to perform the Passarino–Veltman reduction [212] of (E.1.1), which will lead us to re-express $C(m_1,m_2)$ in terms of cut-boxes, cut-triangles and cut-bubbles.

The explicit forms for the Dirac traces involve Lorentz contractions over the various momenta, so in a short-hand notation we can write these as

$$T(i,j,m_1)\cdot l_1 \equiv \mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not l_1)\,. \tag{E.1.2}$$

$C(m_1,m_2)$ can then be recast as

$$C(m_1,m_2) = \frac{T(i,j,m_1)^\mu\, T(i,j,m_2)^\nu}{\langle i\,j\rangle^2}\, I_{\mu\nu}(m_1,m_2;P_{L;z})\,, \tag{E.1.3}$$

where[1]

$$I_{\mu\nu}(m_1,m_2;P_L) = \int dLIPS(l_2,-l_1;P_L)\, \frac{l_1^\mu\, l_2^\nu}{(m_1-l_1)^2 (m_2-l_2)^2}\,. \tag{E.1.4}$$

$I_{\mu\nu}(m_1,m_2;P_L)$ contains three independent momenta m_1, m_2 and P_L. On general

[1] For the rest of this appendix we drop the subscript z in $P_{L;z}$ for the sake of brevity.

grounds we can therefore decompose it as

$$
\begin{aligned}
I \quad = \quad & I_0 + m_1 m_1 I_1 + m_2 m_2 I_2 + P_L P_L I_3 + m_1 m_2 I_4 \\
& + m_2 m_1 I_5 + m_1 P_L I_6 + P_L m_1 I_7 + m_2 P_L I_8 + P_L m_2 I_9 \; ; \qquad \text{(E.1.5)}
\end{aligned}
$$

for some coefficients I_i, $i = 0, \ldots, 9$. One can then contract with different combinations of the independent momenta in order to solve for the I_i. For instance, two of the integrals that we will end up having to do are I and $m_1 m_1 I$. Using momentum conservation $l_2 - l_1 + P_L = 0$ and the identity $a \cdot b = (a+b)^2/2 = -(a-b)^2/2$ for a, b massless momenta, we can convert these integrals into ones which have the general form

$$
I^{(a,b)} = \int \frac{dLIPS(l_2, -l_1; P_L)}{(l_1 - m_1)^a (l_2 - m_2)^b} \; ; \qquad \text{(E.1.6)}
$$

possibly with a kinematical-invariant coefficient, and with a and b ranging over the values $1, 0, -1$. The results of these integrals are collected in §E.2. As an example, we find that

$$
m_1 m_1 I \quad = \int dLIPS(l_2, -l_1; P_L) \frac{(l_1 - m_1)}{(l_2 - m_2)} - (m_1 \cdot P_L) \int \frac{dLIPS(l_2, -l_1; P_L)}{(l_2 - P_L)} \; . \qquad \text{(E.1.7)}
$$

Considering the values (a, b), the case $(1, 1)$ is a cut scalar box, $(1, 0)$ and $(0, 1)$ are cut scalar triangles, $(1, -1)$ and $(-1, 1)$ are cut vector triangles, whilst $(0, 0)$ is a cut scalar bubble.

Because of the structure of $T(i, j, m_1)$ and $T(i, j, m_2)$, terms with coefficients such as $T(i, j, m_1) \cdot T(i, j, m_2)\, m_1 \cdot m_2$ are zero, and thus some of the I_i do not contribute to the final answer. The only contributing terms are found to be I_3, I_5, I_7 and I_8, and we find that

$$
\begin{aligned}
C(m_1, m_2) \quad = \quad & \frac{\mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not P_L)\,\mathrm{tr}_+(\not k_i \not k_j \not k_{m_2} \not P_L)}{\langle i\, j\rangle^2} I_3 \\
& + \frac{\mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not k_{m_2})\,\mathrm{tr}_+(\not k_i \not k_j \not k_{m_2} \not k_{m_1})}{\langle i\, j\rangle^2} I_5 \\
& + \frac{\mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not P_L)\,\mathrm{tr}_+(\not k_i \not k_j \not k_{m_2} \not k_{m_1})}{\langle i\, j\rangle^2} I_7 \\
& + \frac{\mathrm{tr}_+(\not k_i \not k_j \not k_{m_1} \not k_{m_2})\,\mathrm{tr}_+(\not k_i \not k_j \not k_{m_2} \not P_L)}{\langle i\, j\rangle^2} I_8 \; . \qquad \text{(E.1.8)}
\end{aligned}
$$

The inversion of (E.1.5) in order to find the coefficients is tedious and somewhat lengthy, so we just present the results for the relevant I_i in (E.1.8) above:

$$
\begin{aligned}
I_3 &= \frac{1}{N^2}\Big\{ 2(m_1 - m_2)P_L^2\,\tilde{I}^{(0,0)} - N(m_1 - \not{P}_L)\tilde{I}^{(1,0)} + N(m_2 - \not{P}_L)\tilde{I}^{(0,1)} \\
&\quad + 2(m_2 - \not{P}_L)^2\tilde{I}^{(-1,1)} + 2(m_1 - \not{P}_L)^2\tilde{I}^{(1,-1)} \Big\} ; \qquad \text{(E.1.9)}
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \frac{1}{(m_1 - m_2)^2 N^2}\Big\{ 4(m_1 - \not{P}_L)^2(m_2 - \not{P}_L)^2 \\
&\quad - 6(m_1 - \not{P}_L)(m_2 - \not{P}_L)(m_1 - m_2)P_L^2 + 3(m_1 - m_2)^2\left(P_L^2\right)^2 \Big]\tilde{I}^{(0,0)} \\
&\quad + \Big[2(m_1 - \not{P}_L)^2(m_2 - \not{P}_L) - \frac{3}{2}(m_1 - m_2)P_L^2\Big] N(m_1 - \not{P}_L)\tilde{I}^{(1,0)} \\
&\quad - \Big[2(m_1 - \not{P}_L)^2(m_2 - \not{P}_L) - \frac{3}{2}(m_1 - m_2)P_L^2\Big] N(m_2 - \not{P}_L)\tilde{I}^{(0,1)} + \frac{N^3}{4}\tilde{I}^{(1,1)} \\
&\quad + 2\Big[(m_1 - m_2)P_L^2 - (m_1 - \not{P}_L)(m_2 - \not{P}_L)\Big](m_2 - \not{P}_L)^2\tilde{I}^{(-1,1)} \\
&\quad + 2\Big[(m_1 - m_2)P_L^2 - (m_1 - \not{P}_L)(m_2 - \not{P}_L)\Big](m_1 - \not{P}_L)^2\tilde{I}^{(1,-1)} \Big\} ; \qquad \text{(E.1.10)}
\end{aligned}
$$

$$
\begin{aligned}
I_7 &= \frac{1}{(m_1 - \not{P}_L)(m_1 - m_2)N^2}\Big\{ 2(m_1 - \not{P}_L)^2(m_2 - \not{P}_L)^2 \\
&\quad - 3(m_1 - \not{P}_L)(m_2 - \not{P}_L)(m_1 - m_2)P_L^2 \\
&\quad \tilde{I}^{(0,0)} + \frac{1}{2}(m_1 - m_2)P_L^2 N(m_1 - \not{P}_L)\tilde{I}^{(1,0)} - (m_1 - \not{P}_L)N(m_2 - \not{P}_L)^2\tilde{I}^{(0,1)} \\
&\quad - 2(m_1 - \not{P}_L)(m_2 - \not{P}_L)^3\tilde{I}^{(-1,1)} - (m_1 - m_2)P_L^2(m_1 - \not{P}_L)^2\tilde{I}^{(1,-1)} \Big\} ; \qquad \text{(E.1.11)}
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= \frac{1}{(m_2 - \not{P}_L)(m_1 - m_2)N^2}\Big\{ 2(m_1 - \not{P}_L)^2(m_2 - \not{P}_L)^2 \\
&\quad - 3(m_1 - \not{P}_L)(m_2 - \not{P}_L)(m_1 - m_2)P_L^2 \\
&\quad \tilde{I}^{(0,0)} + (m_2 - \not{P}_L)N(m_1 - \not{P}_L)^2\tilde{I}^{(1,0)} - \frac{1}{2}(m_1 - m_2)P_L^2 N(m_2 - \not{P}_L)\tilde{I}^{(0,1)} \\
&\quad - (m_1 - m_2)P_L^2(m_2 - \not{P}_L)^2\tilde{I}^{(-1,1)} - 2(m_1 - \not{P}_L)^3(m_2 - \not{P}_L)\tilde{I}^{(1,-1)} \Big\} ; \qquad \text{(E.1.12)}
\end{aligned}
$$

where $N = (m_1 - m_2)P_L^2 - 2(m_1 - \not{P}_L)(m_2 - \not{P}_L)$. The explicit expressions for the relevant $\tilde{I}^{(a,b)}$ are summarised in §E.2.

Combining (E.1.8) and (E.1.9)–(E.1.12) with the identity (A.3.11) and the explicit expressions for the integrals $\tilde{I}^{(a,b)}$ in §E.2, we arrive at the final result (2.2.12).

E.2 Box & triangle discontinuities from phase space integrals

The integrals that arise in the Passarino-Veltman reduction in §E.1 have the general form:

$$I^{(a,b)} = \int \frac{d^{4-2\epsilon}\mathrm{LIPS}(l_2,-l_1;P_{L;z})}{(l_1-m_1)^a(l_2-m_2)^b}\,, \qquad \text{(E.2.1)}$$

where we have introduced dimensional regularisation in dimension $D = 4-2\epsilon$ [239] in order to deal with infrared divergences.

There are six cases to deal with: $I^{(0,0)}$, $I^{(1,0)}$, $I^{(0,1)}$, $I^{(1,1)}$, $I^{(-1,1)}$, $I^{(1,-1)}$, though due to symmetry we can transform $I^{(1,0)}$ into $I^{(0,1)}$, and $I^{(-1,1)}$ into $I^{(1,-1)}$, so we only need consider four cases overall.

Generically we will evaluate these integrals in convenient special frames following Appendix B of [37], with a convenient choice for m_1 and m_2. For instance, in the case of $I^{(1,1)}$ it is convenient to transform to the centre of mass frame of the vector l_1-l_2, so that

$$l_1 = \frac{1}{2}P_{L;z}\left(1;\vec{v}\right)\,, \qquad l_2 = \frac{1}{2}P_{L;z}\left(-1;\vec{v}\right)\,, \qquad \text{(E.2.2)}$$

and write

$$\vec{v} = (\sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \cos\theta_1)\,. \qquad \text{(E.2.3)}$$

Using a further spatial rotation we write

$$m_1 = (m_1, 0, 0, m_1)\,, \qquad m_2 = (A, B, 0, C)\,, \qquad \text{(E.2.4)}$$

with the mass-shell condition $A^2 = B^2 + C^2$.

After integrating over all angular coordinates except θ_1 and θ_2, the two-body phase space measure in $4-2\epsilon$ dimensions becomes (see Appendix C)

$$d^{4-2\epsilon}\mathrm{LIPS}(l_2,-l_1;P_{L;z}) = \frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{P_{L;z}}{2}\right)^{-2\epsilon} d\theta_1\, d\theta_2\,(\sin\theta_1)^{1-2\epsilon}(\sin\theta_2)^{-2\epsilon}\,. \qquad \text{(E.2.5)}$$

As a result of this and of our parametrizations of l_1, l_2, m_1 and m_2, the integrals take the form

$$I^{(a,b)} = \mathcal{N}^{(a,b)}\frac{\pi^{\frac{1}{2}-\epsilon}}{4\,\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{P_{L;z}}{2}\right)^{-2\epsilon} J^{(a,b)}\,, \qquad \text{(E.2.6)}$$

where

$$
\begin{aligned}
{}^{(0;0)} &= 1\,; \\
{}^{(1;0)} &= \frac{2}{P_{L,z}m_1}\,; \\
{}^{(0;1)} &= \frac{2}{P_{L,z}m_2}\,; \\
{}^{(1;1)} &= \frac{4}{P_{L,z}^2 m_1}\,; \\
{}^{(\ 1;1)} &= m_1\,; \\
{}^{(1;\ 1)} &= m_2\,;
\end{aligned}
\tag{E.2.7}
$$

and $J^{(a;b)}$ is the angular integral

$$
J^{(a;b)} \coloneqq \int_0 d_1 \int_0 d_2 \frac{(\sin\ _1)^{1\ \ 2}(\sin\ _2)^{\ \ 2}}{(1\quad \cos\ _1)^a(A + C\cos\ _1 + B\sin\ _1\cos\ _2)^b}\,: \tag{E.2.8}
$$

The integrals (E.2.8) have been evaluated in [213] for the values of a and b specified above, and we borrow the results in a form from [214]:

$$
\begin{aligned}
J^{(0;0)} &= \frac{2}{1\quad 2}\,; \\
J^{(1;0)} &= -\,; \\
J^{(1;1)} &= -\frac{1}{A}\,{}_2F_1\ \ 1;1;1\quad ;\frac{A\quad C}{2A}\quad ; \\
J^{(\ 1;1)} &= \frac{2\ (1\qquad)}{(1\quad 2\ )}\,{}_2F_1\quad 1;1;1\quad ;\frac{A\quad C}{2A}\quad :
\end{aligned}
\tag{E.2.9}
$$

Here, A and C will differ depending on which case we are considering and our particular parametrization for it, but in all cases the combinations that arise can be re-expressed in terms of Lorentz-invariant quantities using suitable identities. In the case of $J^{(1;1)}$ for example, one uses the easily verified identities

$$
N(P_{L,z}) = \quad P_{L,z}^2(A + C)m_1\,;\quad m_1\quad m_2 = m_1(A\quad C)\,; \tag{E.2.10}
$$

where $N(P_{L,z})$ was defined in (2.2.14).

Eventually, after re-expressing A and C in this way, and upon application of some standard hypergeometric identities we nd the following:

$$
\begin{aligned}
{}^{1}\tilde{I}^{(0;0)} &= \frac{2}{1\quad 2}\,; \qquad\qquad (E.2.11)\\
{}^{1}\tilde{I}^{(1;0)} &= \frac{1}{\ }\,\frac{2}{m_1 \quad P_{L,z}}\,;\\
{}^{1}\tilde{I}^{(0;1)} &= \frac{1}{\ }\,\frac{2}{m_2 \quad P_{L,z}}\,;\\
{}^{1}\tilde{I}^{(1;1)} &= \frac{8}{N(P_{L,z})}\left(\frac{1}{\ } + \log\left(1 \quad \frac{(m_1 \quad m_2)P_{L,z}^2}{N(P_{L,z})}\right) + O(\ )\right);\\
{}^{1}\tilde{I}^{(\ 1;1)} &= \frac{}{(m_1 \quad P_{L,z})^2}\quad\frac{N(P_{L,z})}{}\\
&+ \frac{2}{1\quad 2}\,(m_1 \quad P_{L,z})(m_2 \quad P_{L,z}) \quad (m_1 \quad m_2)P_{L,z}^2\,;\\
{}^{1}\tilde{I}^{(1;\ 1)} &= \frac{}{(m_2 \quad P_{L,z})^2}\quad\frac{N(P_{L,z})}{}\\
&+ \frac{2}{1\quad 2}\,(m_1 \quad P_{L,z})(m_2 \quad P_{L,z}) \quad (m_1 \quad m_2)P_{L,z}^2\,;
\end{aligned}
$$

where is the ubiquitous factor

$$
\coloneqq \frac{\frac{1}{2}}{4\quad \frac{1}{2}}\;\frac{P_{L,z}}{2}^{\;2}\,: \qquad (E.2.12)
$$

APPENDIX F

GAUGE–INVARIANT TRIANGLE RECONSTRUCTION

In this appendix we find a new representation of the triangle function

$$T(p;P;Q) = \frac{\log(Q^2/P^2)}{Q^2 - P^2} ; \qquad \text{(F.0.1)}$$

as the dispersion integral of a sum of two cut-triangles.[1]

A comment on gauge (in)dependence is in order here. Recall from §1.7.1, Equation (1.7.1), that in the approach of [37] to loop diagrams one introduces an arbitrary null vector η in order to perform loop integrations. The corresponding gauge dependence should disappear in the expression for scattering amplitudes. In what follows we will work in an arbitrary gauge, and show analytically that gauge-dependent terms disappear in the final result for the triangle function. Perhaps unsurprisingly, this gauge invariance will also hold for the finite-ϵ version of $T(p;P;Q)$, which we define in (2.1.14).

F.1 Gauge-invariant dispersion integrals

To begin with, recall from (2.2.18) that the basic quantity we have to compute reads

$$\mathscr{R} := \int \frac{dz}{z} \left[\frac{(P_z^2)}{(P_z p)} + \frac{(Q_z^2)}{(Q_z p)} \right] ; \qquad \text{(F.1.1)}$$

where $P + Q + p = 0$. We will work in an arbitrary gauge, where

$$P_z := P - z\eta ; \quad Q_z := Q + z\eta : \qquad \text{(F.1.2)}$$

A short calculation shows that

$$P_z p = P p \left[1 - b_P (P^2 - P_z^2) \right] ; \qquad \text{(F.1.3)}$$

$$Q_z p = Q p \left[1 - b_Q (Q^2 - Q_z^2) \right] ; \qquad \text{(F.1.4)}$$

[1] For a review of dispersion relations see [237] and Appendix D.

where

$$b_P := \frac{\eta\cdot p}{2(\eta\cdot P)(p\cdot P)}\,; \qquad b_Q := \frac{\eta\cdot p}{2(\eta\cdot Q)(p\cdot Q)}\,. \tag{F.1.5}$$

It is also useful to notice the relation

$$\frac{1}{b_Q} = \frac{1}{b_P} + Q^2 - P^2\,; \tag{F.1.6}$$

as well as $(P\cdot p) = -(Q\cdot p) = (1/2)(Q^2 - P^2)$, which trivially follows from momentum conservation. We can then rewrite (F.1.1) as

$$\mathscr{R} = \mathscr{I}_1 - \mathscr{I}_2\,; \tag{F.1.7}$$

where

$$\begin{aligned} \mathscr{I}_1 &:= \frac{1}{(P\cdot p)}\int ds'\,\rho(s')\,\frac{1}{(s'-P^2)\left[1-b_P(P^2-s')\right]} \\ &= -\frac{\pi\csc(\pi\epsilon)}{(P\cdot p)}\,(-P^2)^{\epsilon}\left(\frac{b_P}{b_P P^2-1}\right)^{\epsilon}\,; \end{aligned} \tag{F.1.8}$$

$$\begin{aligned} \mathscr{I}_2 &:= \frac{1}{(P\cdot p)}\int ds'\,\rho(s')\,\frac{1}{(s'-Q^2)\left[1-b_Q(Q^2-s')\right]} \\ &= -\frac{\pi\csc(\pi\epsilon)}{(P\cdot p)}\,(-Q^2)^{\epsilon}\left(\frac{b_Q}{b_Q Q^2-1}\right)^{\epsilon}\,. \end{aligned} \tag{F.1.9}$$

But (F.1.6) implies

$$\frac{b_P}{b_P P^2-1} = \frac{b_Q}{b_Q Q^2-1}\,; \tag{F.1.10}$$

so that we can finally recast (F.1.1) as:

$$\mathscr{R} = -2\pi\epsilon\csc(\pi\epsilon)\,\frac{1}{\epsilon}\,\frac{(-P^2)^{\epsilon}-(-Q^2)^{\epsilon}}{Q^2-P^2} = -2\pi\epsilon\csc(\pi\epsilon)\,T_{\epsilon}(p;P,Q)\,; \tag{F.1.11}$$

where the ϵ-dependent triangle function is[2]

$$T_{\epsilon}(p;P,Q) := \frac{1}{\epsilon}\,\frac{(-P^2)^{\epsilon}-(-Q^2)^{\epsilon}}{Q^2-P^2}\,. \tag{F.1.12}$$

This is the result we were after. Notice that all the gauge dependence, i.e. any dependence on the arbitrary null vector η, has completely cancelled out in (F.1.11).

We now discuss the $\epsilon\to 0$ limit of the final expression (F.1.11). As already discussed in §2.1 (see (2.1.15) and (2.1.16)), in studying the $\epsilon\to 0$ limit of $\mathscr{R}$ (and hence of $T_{\epsilon}(p;P,Q)$) we need to distinguish the case where P^2 and Q^2 are both nonvanishing

[2] The ϵ-dependent triangle function already appeared in (2.1.14).

from the case where one of the two, say Q^2, vanishes. In the former case, we get precisely the triangle function $T(p;P;Q)$ defined in (F.0.1):

$$\lim_{\to 0} \mathscr{R} = 2T(p;P;Q); \qquad P^2 \neq 0; Q^2 \neq 0: \tag{F.1.13}$$

In the latter case, where $Q^2 = 0$, we have instead

$$\lim_{\to 0} \mathscr{R} = -\frac{2(\ P^2)}{P^2}; \qquad P^2 \neq 0; Q^2 = 0; \tag{F.1.14}$$

which corresponds to a degenerate triangle.

The final issue is that of the gauge invariance of the contributions to the amplitude from the box functions B (this is also relevant to the issue of gauge invariance in the $N = 4$ calculation of [37], and in that paper a general argument for gauge invariance was also given – further evidence can be found in [79]). We expect that an explicit analytic proof of the gauge invariance of the box function contribution to the amplitude could be constructed using identities such as those in Appendix B of [37]. In the meantime, numerical tests have shown that gauge invariance is present [209]. Indeed, it would be surprising if this were not the case given that the correct, gauge-invariant, amplitudes are derived with the choices of gauge we have made here and in [37]. We have also carried out the MHV diagram analysis of this paper using the alternative gauge choice $= k_{m_2}$; one obtains (2.1.19).

APPENDIX G

INTEGRALS FOR THE NON–SUPERSYMMETRIC AMPLITUDE

In this appendix we give details of the integrals needed to compute the discontinuities of the non-supersymmetric amplitude discussed in Chapter 3.

G.1 Passarino–Veltman reduction

In §3.3 we saw that a typical term in the cut-constructible part of the Yang–Mills amplitude is the dispersion integral of the following phase space integral:

$$\mathscr{C}(m) := \int dLIPS(l_2; -l_1; P_{L;z}) \frac{\mathrm{tr}_+(\not k_1 \not k_2 \not P_{L;z} \not l_2)\,\mathrm{tr}_+(\not k_1 \not k_2 \not l_2 \not P_{L;z})\,\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \not l_2)}{(l_2 \cdot m)(k_1 \cdot k_2)^3 (P_{L;z}^2)^2} \tag{G.1.1}$$

The goal of this appendix is to perform the Passarino–Veltman reduction [212] of (G.1.1). To this end, we rewrite $\mathscr{C}(m)$ as

$$\mathscr{C}(m) = \frac{\mathrm{tr}_+(\not k_1 \not k_2 \not P_{L;z} \gamma_\mu)\,\mathrm{tr}_+(\not k_1 \not k_2 \gamma_\nu \not P_{L;z})\,\mathrm{tr}_+(\not k_1 \not k_2 \not k_m \gamma_\rho)}{(k_1 \cdot k_2)^3 (P_{L;z}^2)^2}\, I^{\mu\nu\rho}(m; P_{L;z})\,; \tag{G.1.2}$$

where[1]

$$I^{\mu\nu\rho}(m; P_L) = \int dLIPS(l_2; -l_1; P_L) \frac{l_2^\mu l_2^\nu l_2^\rho}{(l_2 \cdot m)}\,: \tag{G.1.3}$$

On general grounds, $I^{\mu\nu\rho}(m; P_L)$ can be decomposed as

$$\begin{aligned} I^{\mu\nu\rho} = {} & m^\mu m^\nu m^\rho \mathscr{J}_1 + (m^\mu m^\nu P_L^\rho + m^\mu P_L^\nu m^\rho + P_L^\mu m^\nu m^\rho)\mathscr{J}_2 \\ & + (m^\mu P_L^\nu P_L^\rho + P_L^\mu m^\nu P_L^\rho + P_L^\mu P_L^\nu m^\rho)\mathscr{J}_3 + P_L^\mu P_L^\nu P_L^\rho \mathscr{J}_4 \\ & + (\eta^{\mu\nu} m^\rho + \eta^{\mu\rho} m^\nu + \eta^{\nu\rho} m^\mu)\mathscr{J}_5 + (\eta^{\mu\nu} P_L^\rho + \eta^{\mu\rho} P_L^\nu + \eta^{\nu\rho} P_L^\mu)\mathscr{J}_6 \end{aligned} \tag{G.1.4}$$

[1] For the rest of this appendix we will generally drop the subscript z in $P_{L;z}$ for the sake of brevity.

for some coefficients $\mathscr{J}_i$; $i = 0;\ldots;6$. One can then contract with different combinations of the independent momenta in order to solve for the $\mathscr{J}_i$. Introducing the quantities

$$
\begin{aligned}
A &\coloneqq m\; m\; m\; I \quad ; \\
B &\coloneqq m\; m\; P_L\; I \quad ; \\
C &\coloneqq m\; P_L\; P_L\; I \quad ; \\
D &\coloneqq P_L\; P_L\; P_L\; I \quad ; \\
E &\coloneqq \quad m\; I \quad = 0\,; \\
F &\coloneqq \quad P_L\; I \quad = 0\,;
\end{aligned}
\qquad \text{(G.1.5)}
$$

the result for the Passarino-Veltman reduction of $\{\mathscr{J}_1;\ldots;\mathscr{J}_6\}$ in the basis $\{A;\ldots;D\}$ is:

$$
\begin{aligned}
\mathscr{J}_2 &= \quad 5(P_L^2)^2 = 2(m \quad P_L)^5 \;\; ; \;\; 6P_L^2 = (m \quad P_L)^4 \,; 3 = (m \quad P_L)^3 \,; 0 \quad ; \\
\mathscr{J}_3 &= \quad 2P_L^2 = (m \quad P_L)^4 \,; 3 = (m \quad P_L)^3 \,; 0 \,; 0 \quad ; \\
\mathscr{J}_4 &= \quad 1 = (m \quad P_L)^3 \,; 0 \,; 0 \,; 0 \\
\mathscr{J}_5 &= \quad (P_L^2)^2 = 2(m \quad P_L)^4 \;\; ; 3P_L^2 = 2(m \quad P_L)^3 \;\; ; \;\; 1 = (m \quad P_L)^2 \,; 0 \quad ; \\
\mathscr{J}_6 &= \quad P_L^2 = 2(m \quad P_L)^3) \,; \;\; 1 = (m \quad P_L)^2 \,; 0 \,; 0 \quad :
\end{aligned}
\qquad \text{(G.1.6)}
$$

We omit the decomposition for $\mathscr{J}_1$ as the corresponding term in (G.1.4) drops out of all future expressions due to $k_m^2 = 0$.

Finally, using the methods of [40] and the results of §G.3, the integrals in (G.1.5) are found to be, keeping only terms to $O(\;^0)$,

$$A = (m \quad P_L)^2 \frac{4}{3} \quad \hat{} \;; \qquad \text{(G.1.7)}$$

$$B = P_L^2 (m \quad P_L) \quad \hat{} \;; \qquad \text{(G.1.8)}$$

$$C = (P_L^2)^2 \quad \hat{} \;; \qquad \text{(G.1.9)}$$

$$D = \frac{(P_L^2)^3}{8(m \quad P_L)} \frac{4}{\quad} \hat{} \;; \qquad \text{(G.1.10)}$$

where

$$\hat{} \coloneqq \frac{\frac{1}{2}}{4^1 \qquad \frac{1}{2}} \;: \qquad \text{(G.1.11)}$$

G.2 Evaluating the integral of $\mathscr{C}(a;b)$

The basic expression which arises in the MHV diagram construction in this paper is

$$\mathscr{C}(a;b) = \frac{\langle i l_1\rangle\langle j l_1\rangle^2\langle i l_2\rangle^2\langle j l_2\rangle}{\langle i j\rangle^4\langle l_1 l_2\rangle^2}\frac{\langle i a\rangle\langle j b\rangle}{\langle l_1 a\rangle\langle l_2 b\rangle}: \tag{G.2.1}$$

We wish to integrate this expression over the Lorentz-invariant phase space. We begin by simplifying it, using multiple applications of the Schouten identity. First note that using this identity twice, one deduces that

$$\begin{aligned}\frac{\langle i l_2\rangle\langle j l_1\rangle}{\langle l_1 a\rangle\langle l_2 b\rangle}\langle a b\rangle^2 &= \langle i a\rangle\langle b j\rangle + \langle i a\rangle\langle a j\rangle\frac{\langle l_1 b\rangle}{\langle a l_1\rangle} + \langle b j\rangle\langle i b\rangle\frac{\langle l_2 a\rangle}{\langle b l_2\rangle} \\ &+ \langle a j\rangle\langle i b\rangle - \langle a j\rangle\langle i b\rangle\frac{\langle l_1 l_2\rangle\langle a b\rangle}{\langle a l_1\rangle\langle b l_2\rangle}:\end{aligned} \tag{G.2.2}$$

Now use this identity in $\mathscr{C}(a;b)$. This generates five terms, which we will label (in correspondence with the ordering arising from the order of terms in (G.2.2) above) as $T_i; i = 1;:::;4$, and U. The T_i have dependence on the loop momenta such that we may use the phase space integrals of §G.3 to calculate them. The term U is more complicated; however, one may again use the identity (G.2.2), generating another five terms, which we will label $T_5;:::;T_8$, and V. Again, the expressions in $T_i; i = 5;:::;8$ may be calculated using the integrals of §G.3. Finally, the term V may be simplified, here using the identity (G.2.2) with i and j interchanged. This generates a further five terms, which we label $T_9;:::;T_{13}$. The explicit forms of these terms follow:

$$T_1 = \frac{\mathrm{tr}_+(\not i\not j\not b\not a)^2\,\mathrm{tr}_+(\not i\not j\not l_1\not l_2)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)}{2^8(i\cdot j)^4(a\cdot b)^2(l_1\cdot l_2)^2}; \tag{G.2.3}$$

$$T_2 = \frac{\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not b\not a)\,\mathrm{tr}_+(\not i\not j\not l_1\not l_2)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)\,\mathrm{tr}_+(\not i\not b\not l_1\not a)}{2^{10}(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)^2(i\cdot b)(a\cdot l_1)}; \tag{G.2.4}$$

$$T_3 = \frac{\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not b\not a)\,\mathrm{tr}_+(\not i\not j\not l_1\not l_2)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)\,\mathrm{tr}_+(\not j\not a\not l_2\not b)}{2^{10}(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)^2(j\cdot a)(b\cdot l_2)}; \tag{G.2.5}$$

$$T_4 = \frac{\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not b\not a)\,\mathrm{tr}_+(\not i\not j\not l_1\not l_2)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)}{2^8(i\cdot j)^4(a\cdot b)^2(l_1\cdot l_2)^2}; \tag{G.2.6}$$

and

$$T_5 = \frac{\mathrm{tr}_+(\not i\not j\not b\not a)^2\,\mathrm{tr}_+(\not i\not j\not a\not b)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)}{2^8(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)}; \tag{G.2.7}$$

$$T_6 = \frac{\mathrm{tr}_+(\not i\not j\not a\not b)^2\,\mathrm{tr}_+(\not i\not j\not b\not a)\,\mathrm{tr}_+(\not i\not j\not l_2\not l_1)\,\mathrm{tr}_+(\not i\not b\not l_1\not a)}{2^{10}(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)(i\cdot b)(a\cdot l_1)}; \tag{G.2.8}$$

$$T_7 = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{l}_1)\,\mathrm{tr}_+(\not{i}\not{a}\not{l}_2\not{b})}{2^{10}(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)(i\cdot a)(b\cdot l_2)}\,; \tag{G.2.9}$$

$$T_8 = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{l}_1)}{2^{8}(i\cdot j)^4(a\cdot b)^3(l_1\cdot l_2)}\,; \tag{G.2.10}$$

and

$$T_9 = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^3\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})}{2^{8}(i\cdot j)^4(a\cdot b)^4}\,; \tag{G.2.11}$$

$$T_{10} = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2\,\mathrm{tr}_+(\not{j}\not{b}\not{l}_1\not{a})}{2^{10}(i\cdot j)^4(a\cdot b)^4(j\cdot b)(a\cdot l_1)}\,; \tag{G.2.12}$$

$$T_{11} = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2\,\mathrm{tr}_+(\not{i}\not{a}\not{l}_2\not{b})}{2^{10}(i\cdot j)^4(a\cdot b)^4(i\cdot a)(b\cdot l_2)}\,; \tag{G.2.13}$$

$$T_{12} = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2}{2^{8}(i\cdot j)^4(a\cdot b)^4}\,; \tag{G.2.14}$$

$$T_{13} = \frac{\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})^2\,\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})^2\,\mathrm{tr}_+(\not{b}\not{l}_2\not{l}_1\not{a})}{2^{10}(i\cdot j)^4(a\cdot b)^4(a\cdot l_1)(b\cdot l_2)}\,; \tag{G.2.15}$$

The expression $\mathscr{C}(a;b)$ is then the sum of the terms $T_i, i = 1,\ldots,13$.

Before performing the phase space integrals, it proves convenient to collect the resulting expressions in pairs as T_1+T_2, T_3+T_4, T_5+T_6, T_7+T_8, T_9+T_{11} and $T_{10}+T_{12}$. This leads us to the following decomposition:

$$\begin{aligned}\mathscr{C}(a;b) &= \frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{l}_2)\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{l}_1)\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{a})\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{l}_2)}{2^{8}(i\cdot j)^4(l_1\cdot l_2)^2(l_1\cdot a)(l_2\cdot b)} \\ &= \frac{1}{2^{8}(i\cdot j)^4}(H_1+\cdots+H_4)\,;\end{aligned} \tag{G.2.16}$$

where

$$\begin{aligned}
H_1 &:= \frac{\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{l}_2)\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{l}_1)}{(l_1\cdot l_2)^2(a\cdot b)}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{a})}{(l_1\cdot a)}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{b})}{(l_2\cdot b)}\,;\\
H_2 &:= \frac{\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b})\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})\,\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{l}_1)}{(l_1\cdot l_2)(a\cdot b)^2}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{a})}{(l_1\cdot a)}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{b})}{(l_2\cdot b)}\,;\\
H_3 &:= \frac{(\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b}))^2\,\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a})}{(a\cdot b)^3}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_1\not{a})}{(l_1\cdot a)}\,\frac{\mathrm{tr}_+(\not{i}\not{j}\not{l}_2\not{b})}{(l_2\cdot b)}\,;\\
H_4 &:= \frac{(\mathrm{tr}_+(\not{i}\not{j}\not{a}\not{b}))^2(\mathrm{tr}_+(\not{i}\not{j}\not{b}\not{a}))^2\,\mathrm{tr}_+(\not{l}_1\not{a}\not{b}\not{l}_2)}{4(a\cdot b)^2(l_1\cdot a)(l_2\cdot b)}\,.
\end{aligned} \tag{G.2.17}$$

Finally, we perform the phase space integrals of the above expressions, using the formul in §G.3 below. One quickly finds that the divergent (as $\to 0$) part of the total expression is zero. The finite part, after further spinor manipulations, becomes the expression we have given in (3.3.3).

G.3 Phase space integrals

The basic method which we use for evaluating Lorentz-invariant phase space integrals has been outlined in [37, 40] and also discussed in §1.9 and Appendix E. Here we will just quote the results which we need. In the following we will use a shorthand notation where $\int \equiv \int d^{4-2} \mathrm{LIPS}(l_2; l_1; P_{L;z})$, and a common factor of $4\hat{\ }(P_{L;z})^{2}$ is understood to multiply all expressions, where $\hat{\ }$ is the ubiquitous factor of (G.1.11). We also define $= (a \cdot P)$, $= (b \cdot P)$, $N(P) = (a \cdot b)P^2 - 2(a \cdot P)(b \cdot P)$ and drop the $L;z$ subscripts from $P_{L;z}$ for clarity.

Firstly we quote the results from Appendix B of [40] up to terms of $O(^{0})$:

$$\int 1 = 1 ; \qquad \int \frac{1}{(a \cdot l_1)} = \frac{1}{} ; \qquad \int \frac{1}{(b \cdot l_2)} = \frac{1}{} ; \tag{G.3.1}$$

$$\int \frac{1}{(a \cdot l_1)(b \cdot l_2)} = \frac{4}{N(P)} \left(\frac{1}{} + L \right) ;$$

where

$$L = \log\left(1 - \frac{(a \cdot b)}{N} P^2 \right) :$$

From this, we can recursively derive the following integrals (up to $O(^{0})$):

$$\int l_1 = \frac{1}{2} P ; \qquad \int l_2 = \frac{1}{2} P ; \tag{G.3.2}$$

$$\int l_1 l_1 = \int l_2 l_2 = \frac{1}{3} \left(P P - \frac{1}{4} P^2 \right) ;$$

$$\int \frac{l_1}{(a \cdot l_1)} = \frac{P^2}{2^{2}} a + \frac{1}{} \left(P - \frac{P^2}{2} a \right) ;$$

$$\int \frac{l_2}{(b \cdot l_2)} = \frac{P^2}{2^{2}} b + \frac{1}{} \left(P - \frac{P^2}{2} b \right) ;$$

and

$$\int \frac{l_1 l_1}{(a \cdot l_1)} = \frac{P^4}{4^{3}} a a + \frac{1}{2} P P + \frac{P^2}{2^{2}} P^{(} a^{)} - \frac{3P^4}{4^{3}} a a - \frac{P^2}{4} ;$$

$$\int \frac{l_2 l_2}{(b \cdot l_2)} = \frac{P^4}{4^{3}} b b - \frac{1}{2} P P - \frac{P^2}{2^{2}} P^{(} b^{)} + \frac{3P^4}{4^{3}} b b + \frac{P^2}{4} ; \tag{G.3.3}$$

$$\int \frac{l_2}{(a \cdot l_1)(b \cdot l_2)} = \frac{1}{N} \left(2P - \frac{P^2}{} a + \frac{P^2}{} b \right) + \frac{2L}{N} \left(P - \frac{}{(a \cdot b)} a + \frac{}{(a \cdot b)} b \right) :$$

Finally, there are integrals involving cubic powers of loop momenta in the numerator. The first is

$$\int \frac{l_1 l_1 l_1}{(a \quad l_1)} = \frac{P^4}{4 \quad ^3} P^{(} \; a \; a^{)} \quad + \quad \frac{P^2}{4 \quad ^2} P^{(} \; P \quad a^{)} + \frac{1}{3} P \quad P \quad P$$
$$\qquad \frac{P^4}{8 \quad ^2} {}^{(} \quad a^{)} \qquad \frac{P^2}{4} {}^{(} \quad P^{)}; \tag{G.3.4}$$

where we have suppressed terms cubic in a as they prove not to contribute when this integral is contracted into the products of Dirac traces which appear in the expressions in xG.2. The second cubic integral required is

$$\int \frac{l_2 l_2 l_2}{(b \quad l_2)} = \frac{P^4}{4 \quad ^3} P^{(} \; b \; b^{)} \quad + \quad \frac{P^2}{4 \quad ^2} P^{(} \; P \quad b^{)} + \frac{1}{3} P \quad P \quad P$$
$$\qquad \frac{P^4}{8 \quad ^2} {}^{(} \quad b^{)} \qquad \frac{P^2}{4} {}^{(} \quad P^{)}; \tag{G.3.5}$$

again suppressing terms cubic in b which will not contribute.

APPENDIX H

KLT RELATIONS

For completeness, in this appendix we write the eld theory limit of the KLT relations [219] for the cases of four, ve and six points:

$$M(1;2;3) = iA(1;2;3)A(1;2;3); \tag{H.0.1}$$

$$M(1;2;3;4) = is_{12}A(1;2;3;4)A(1;2;4;3); \tag{H.0.2}$$

$$\begin{aligned} M(1;2;3;4;5) &= is_{12}s_{34}A(1;2;3;4;5)A(2;1;4;3;5) \\ &+ is_{13}s_{24}A(1;3;2;4;5)A(3;1;4;2;5); \end{aligned} \tag{H.0.3}$$

$$\begin{aligned} M(1;2;3;4;5;6) &= is_{12}s_{45}A(1;2;3;4;5;6)\, s_{35}A(2;1;5;3;4;6) \\ &+ (s_{34}+s_{35})A(2;1;5;4;3;6) \\ &+ P(2;3;4): \end{aligned} \tag{H.0.4}$$

In these formul, $M(i)$ ($A(i)$) denotes a tree-level gravity (Yang-Mills colour-ordered) amplitude, $s_{ij} \coloneqq (p_i + p_j)^2$, and $P(2;3;4)$ stands for permutations of $(2;3;4)$. The relation for a generic number of particles can be found in Appendix A of [240].

References

[1] W. N. Cottingham and D. A. Greenwood, An introduction to the standard model of particle physics. Cambridge, UK: Univ. Pr., 2007.

[2] M. E. Peskin and D. V. Schroeder, An introduction to quantum field theory. Reading, USA: Addison-Wesley, 1995.

[3] S. Weinberg, The quantum theory of fields. Vol. 1: Foundations. Cambridge, UK: Univ. Pr., 1995.

[4] S. Weinberg, The quantum theory of fields. Vol. 2: Modern applications. Cambridge, UK: Univ. Pr., 1996.

[5] S. Weinberg, The quantum theory of fields. Vol. 3: Supersymmetry. Cambridge, UK: Univ. Pr., 2000.

[6] Particle Data Group Collaboration, W. M. Yao et al., Review of particle physics, J. Phys. G 33 (2006) 1–1232.

[7] L. Susskind, Dynamics of spontaneous symmetry breaking in the Weinberg-Salam theory, Phys. Rev. D 20 (1979) 2619–2625.

[8] E. Farhi and L. Susskind, A technicolored G.U.T., Phys. Rev. D 20 (1979) 3404–3411.

[9] J. Wess and J. Bagger, Supersymmetry and supergravity. Princeton, USA: Univ. Pr., 1992.

[10] M. F. Sohnius, Introducing supersymmetry, Phys. Rept. 128 (1985) 39–204.

[11] J. M. Figueroa-O'Farrill, BUSSTEPP lectures on supersymmetry, hep-th/0109172.

[12] L. Alvarez-Gaume and S. F. Hassan, Introduction to S-duality in N = 2 supersymmetric gauge theories: A pedagogical review of the work of Seiberg and Witten, Fortsch. Phys. 45 (1997) 159–236, [hep-th/9701069].

[13] J. Rosiek, Complete set of Feynman rules for the minimal supersymmetric extension of the standard model, Phys. Rev. D 41 (1990) 3464.

[14] J. Rosiek, Complete set of Feynman rules for the MSSM – erratum, hep-ph/9511250.

[15] L. Bombelli, J.-H. Lee, D. Meyer, and R. Sorkin, Space-time as a causal set, Phys. Rev. Lett. 59 (1987) 521.

[16] J. Henson, The causal set approach to quantum gravity, gr-qc/0601121.

[17] A. Ashtekar, New variables for classical and quantum gravity, Phys. Rev. Lett. 57 (1986) 2244–2247.

[18] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A status report, Class. Quant. Grav. 21 (2004) R53, [gr-qc/0404018].

[19] M. B. Green, J. H. Schwarz, and E. Witten, Superstring theory. Vol. 1: Introduction. Cambridge, UK: Univ. Pr., 1987.

[20] M. B. Green, J. H. Schwarz, and E. Witten, Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology. Cambridge, UK: Univ. Pr., 1987.

[21] J. Polchinski, String theory. Vol. 1: An introduction to the bosonic string. Cambridge, UK: Univ. Pr., 1998.

[22] J. Polchinski, String theory. Vol. 2: Superstring theory and beyond. Cambridge, UK: Univ. Pr., 1998.

[23] E. Witten, String theory dynamics in various dimensions, Nucl. Phys. B 443 (1995) 85–126, [hep-th/9503124].

[24] J. Polchinski, Dirichlet-branes and Ramond-Ramond charges, Phys. Rev. Lett. 75 (1995) 4724–4727, [hep-th/9510017].

[25] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72 (1974) 461.

[26] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231–252, [hep-th/9711200].

[27] G. 't Hooft, Dimensional reduction in quantum gravity, In: Salamfest (1993) 284–296, [gr-qc/9310026].

[28] R. Bousso, The holographic principle, Rev. Mod. Phys. 74 (2002) 825–874, [hep-th/0203101].

[29] D. J. Gross and F. Wilczek, Ultraviolet behavior of non-Abelian gauge theories, Phys. Rev. Lett. 30 (1973) 1343–1346.

[30] H. D. Politzer, Reliable perturbative results for strong interactions?, Phys. Rev. Lett. 30 (1973) 1346–1349.

[31] E. Witten, Perturbative gauge theory as a string theory in twistor space, Commun. Math. Phys. 252 (2004) 189–258, [hep-th/0312171].

[32] R. Penrose, Twistor algebra, J. Math. Phys. 8 (1967) 345.

[33] F. Cachazo, P. Svrcek, and E. Witten, MHV vertices and tree amplitudes in gauge theory, JHEP 09 (2004) 006, [hep-th/0403047].

[34] K. Risager, A direct proof of the CSW rules, JHEP 12 (2005) 003, [hep-th/0508206].

[35] P. Mansfield, The Lagrangian origin of MHV rules, JHEP 03 (2006) 037, [hep-th/0511264].

[36] N. Berkovits and E. Witten, Conformal supergravity in twistor-string theory, JHEP 08 (2004) 009, [hep-th/0406051].

[37] A. Brandhuber, B. J. Spence, and G. Travaglini, One-loop gauge theory amplitudes in N = 4 super Yang-Mills from MHV vertices, Nucl. Phys. B 706 (2005) 150–180, [hep-th/0407214].

[38] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, One-loop n-point gauge theory amplitudes, unitarity and collinear limits, Nucl. Phys. B 425 (1994) 217–260, [hep-ph/9403226].

[39] M. Abou-Zeid, C. M. Hull, and L. J. Mason, Einstein supergravity and new twistor string theories, hep-th/0606272.

[40] J. Bedford, A. Brandhuber, B. J. Spence, and G. Travaglini, A twistor approach to one-loop amplitudes in N = 1 supersymmetric Yang-Mills theory, Nucl. Phys. B 706 (2005) 100–126, [hep-th/0410280].

[41] C. Quigley and M. Rozali, One-loop MHV amplitudes in supersymmetric gauge theories, JHEP 01 (2005) 053, [hep-th/0410278].

[42] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Fusing gauge theory tree amplitudes into loop amplitudes, Nucl. Phys. B 435 (1995) 59–101, [hep-ph/9409265].

[43] J. Bedford, A. Brandhuber, B. J. Spence, and G. Travaglini, Non-supersymmetric loop amplitudes and MHV vertices, Nucl. Phys. B 712 (2005) 59–85, [hep-th/0412108].

[44] Z. Bern, L. J. Dixon, and D. A. Kosower, One-loop corrections to five gluon amplitudes, Phys. Rev. Lett. 70 (1993) 2677–2680, [hep-ph/9302280].

[45] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, All one-loop maximally helicity violating gluonic amplitudes in QCD, Phys. Rev. D 75 (2007) 016006, [hep-ph/0607014].

[46] R. Roiban, M. Spradlin, and A. Volovich, Dissolving $N = 4$ loop amplitudes into QCD tree amplitudes, Phys. Rev. Lett. 94 (2005) 102002, [hep-th/0412265].

[47] R. Britto, F. Cachazo, and B. Feng, Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills, Nucl. Phys. B 725 (2005) 275–305, [hep-th/0412103].

[48] R. Britto, F. Cachazo, and B. Feng, New recursion relations for tree amplitudes of gluons, Nucl. Phys. B 715 (2005) 499–522, [hep-th/0412308].

[49] R. Britto, F. Cachazo, B. Feng, and E. Witten, Direct proof of tree-level recursion relation in Yang-Mills theory, Phys. Rev. Lett. 94 (2005) 181602, [hep-th/0501052].

[50] J. Bedford, A. Brandhuber, B. J. Spence, and G. Travaglini, A recursion relation for gravity amplitudes, Nucl. Phys. B 721 (2005) 98–110, [hep-th/0502146].

[51] F. Cachazo and P. Svrcek, Tree level recursion relations in general relativity, hep-th/0502160.

[52] P. Benincasa, C. Boucher-Veronneau, and F. Cachazo, Taming tree amplitudes in general relativity, hep-th/0702032.

[53] Z. Bern, N. E. J. Bjerrum-Bohr, and D. C. Dunbar, Inherited twistor-space structure of gravity loop amplitudes, JHEP 05 (2005) 056, [hep-th/0501137].

[54] N. E. J. Bjerrum-Bohr, D. C. Dunbar, and H. Ita, Six-point one-loop $N = 8$ supergravity NMHV amplitudes and their IR behaviour, Phys. Lett. B 621 (2005) 183–194, [hep-th/0503102].

[55] N. E. J. Bjerrum-Bohr, D. C. Dunbar, and H. Ita, Perturbative gravity and twistor space, Nucl. Phys. Proc. Suppl. 160 (2006) 215–219, [hep-th/0606268].

[56] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins, and K. Risager, The no-triangle hypothesis for $N = 8$ supergravity, JHEP 12 (2006) 072, [hep-th/0610043].

[57] Z. Bern, L. J. Dixon, and R. Roiban, Is $N = 8$ supergravity ultraviolet finite?, Phys. Lett. B 644 (2007) 265–271, [hep-th/0611086].

[58] Z. Bern et al., Three-loop superfiniteness of N = 8 supergravity, Phys. Rev. Lett. 98 (2007) 161303, [hep-th/0702112].

[59] M. B. Green, J. G. Russo, and P. Vanhove, Non-renormalisation conditions in type II string theory and maximal supergravity, JHEP 02 (2007) 099, [hep-th/0610299].

[60] M. B. Green, J. G. Russo, and P. Vanhove, Ultraviolet properties of maximal supergravity, Phys. Rev. Lett. 98 (2007) 131602, [hep-th/0611273].

[61] C.-J. Zhu, The googly amplitudes in gauge theory, JHEP 04 (2004) 032, [hep-th/0403115].

[62] G. Georgiou and V. V. Khoze, Tree amplitudes in gauge theory as scalar MHV diagrams, JHEP 05 (2004) 070, [hep-th/0404072].

[63] J.-B. Wu and C.-J. Zhu, MHV vertices and scattering amplitudes in gauge theory, JHEP 07 (2004) 032, [hep-th/0406085].

[64] J.-B. Wu and C.-J. Zhu, MHV vertices and fermionic scattering amplitudes in gauge theory with quarks and gluinos, JHEP 09 (2004) 063, [hep-th/0406146].

[65] D. A. Kosower, Next-to-maximal helicity violating amplitudes in gauge theory, Phys. Rev. D 71 (2005) 045007, [hep-th/0406175].

[66] G. Georgiou, E. W. N. Glover, and V. V. Khoze, Non-MHV tree amplitudes in gauge theory, JHEP 07 (2004) 048, [hep-th/0407027].

[67] Y. Abe, V. P. Nair, and M.-I. Park, Multigluon amplitudes, N = 4 constraints and the WZW model, Phys. Rev. D 71 (2005) 025002, [hep-th/0408191].

[68] L. J. Dixon, E. W. N. Glover, and V. V. Khoze, MHV rules for Higgs plus multi-gluon amplitudes, JHEP 12 (2004) 015, [hep-th/0411092].

[69] Z. Bern, D. Forde, D. A. Kosower, and P. Mastrolia, Twistor-inspired construction of electroweak vector boson currents, Phys. Rev. D 72 (2005) 025006, [hep-ph/0412167].

[70] T. G. Birthwright, E. W. N. Glover, V. V. Khoze, and P. Marquard, Multi-gluon collinear limits from MHV diagrams, JHEP 05 (2005) 013, [hep-ph/0503063].

[71] T. G. Birthwright, E. W. N. Glover, V. V. Khoze, and P. Marquard, Collinear limits in QCD from MHV rules, JHEP 07 (2005) 068, [hep-ph/0505219].

[72] F. Cachazo, P. Svrcek, and E. Witten, Gauge theory amplitudes in twistor space and holomorphic anomaly, JHEP 10 (2004) 077, [hep-th/0409245].

[73] F. Cachazo, P. Svrcek, and E. Witten, Twistor space structure of one-loop amplitudes in gauge theory, JHEP 10 (2004) 074, [hep-th/0406177].

[74] F. Cachazo, Holomorphic anomaly of unitarity cuts and one-loop gauge theory amplitudes, hep-th/0410077.

[75] R. Britto, F. Cachazo, and B. Feng, Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts, Phys. Rev. D 71 (2005) 025012, [hep-th/0410179].

[76] S. J. Bidder, N. E. J. Bjerrum-Bohr, L. J. Dixon, and D. C. Dunbar, N = 1 supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts, Phys. Lett. B 606 (2005) 189–201, [hep-th/0410296].

[77] N. E. J. Bjerrum-Bohr, D. C. Dunbar, H. Ita, W. B. Perkins, and K. Risager, MHV-vertices for gravity amplitudes, JHEP 01 (2006) 009, [hep-th/0509016].

[78] S. Giombi, R. Ricci, D. Robles-Llana, and D. Trancanelli, A note on twistor gravity amplitudes, JHEP 07 (2004) 059, [hep-th/0405086].

[79] A. Brandhuber, B. Spence, and G. Travaglini, From trees to loops and back, JHEP 01 (2006) 142, [hep-th/0510253].

[80] J. H. Ettle and T. R. Morris, Structure of the MHV-rules Lagrangian, JHEP 08 (2006) 003, [hep-th/0605121].

[81] A. Brandhuber, B. Spence, and G. Travaglini, Amplitudes in pure Yang-Mills and MHV diagrams, JHEP 02 (2007) 088, [hep-th/0612007].

[82] A. Brandhuber, B. Spence, G. Travaglini, and K. Zoubos, One-loop MHV rules and pure Yang-Mills, JHEP 07 (2007) 002, [arXiv:0704.0245 [hep-th]].

[83] J. H. Ettle, C.-H. Fu, J. P. Fudger, P. R. W. Mansfield, and T. R. Morris, S-matrix equivalence theorem evasion and dimensional regularisation with the canonical MHV Lagrangian, hep-th/0703286.

[84] A. Brandhuber, S. McNamara, B. J. Spence, and G. Travaglini, Loop amplitudes in pure Yang-Mills from generalised unitarity, JHEP 10 (2005) 011, [hep-th/0506068].

[85] E. I. Buchbinder and F. Cachazo, Two-loop amplitudes of gluons and octa-cuts in N = 4 super Yang-Mills, JHEP 11 (2005) 036, [hep-th/0506126].

[86] Z. Bern and A. G. Morgan, Massive loop amplitudes from unitarity, Nucl. Phys. B 467 (1996) 479–509, [hep-ph/9511336].

[87] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, One-loop self-dual and N = 4 super-Yang-Mills, Phys. Lett. B 394 (1997) 105–115, [hep-th/9611127].

[88] Z. Bern, L. J. Dixon, and D. A. Kosower, Progress in one-loop QCD computations, Ann. Rev. Nucl. Part. Sci. 46 (1996) 109–148, [hep-ph/9602280].

[89] Z. Bern, L. J. Dixon, and D. A. Kosower, A two-loop four-gluon helicity amplitude in QCD, JHEP 01 (2000) 027, [hep-ph/0001001].

[90] Z. Bern, A. De Freitas, and L. J. Dixon, Two-loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory, JHEP 03 (2002) 018, [hep-ph/0201161].

[91] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, One-loop gluon scattering amplitudes in theories with N < 4 supersymmetries, Phys. Lett. B 612 (2005) 75–88, [hep-th/0502028].

[92] R. Britto, E. Buchbinder, F. Cachazo, and B. Feng, One-loop amplitudes of gluons in SQCD, Phys. Rev. D 72 (2005) 065012, [hep-ph/0503132].

[93] R. Britto, B. Feng, and P. Mastrolia, The cut-constructible part of QCD amplitudes, Phys. Rev. D 73 (2006) 105004, [hep-ph/0602178].

[94] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, D-dimensional unitarity cut method, Phys. Lett. B 645 (2007) 213–216, [hep-ph/0609191].

[95] R. Britto and B. Feng, Unitarity cuts with massive propagators and algebraic expressions for coefficients, Phys. Rev. D 75 (2007) 105006, [hep-ph/0612089].

[96] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt, and P. Mastrolia, Unitarity cuts and reduction to master integrals in D dimensions for one-loop amplitudes, JHEP 03 (2007) 111, [hep-ph/0612277].

[97] R. Britto, B. Feng, R. Roiban, M. Spradlin, and A. Volovich, All split helicity tree-level gluon amplitudes, Phys. Rev. D 71 (2005) 105017, [hep-th/0503198].

[98] M.-x. Luo and C.-k. Wen, Recursion relations for tree amplitudes in super gauge theories, JHEP 03 (2005) 004, [hep-th/0501121].

[99] M.-x. Luo and C.-k. Wen, Compact formulas for all tree amplitudes of six partons, Phys. Rev. D 71 (2005) 091501, [hep-th/0502009].

[100] A. P. Hodges, Twistor diagram recursion for all gauge-theoretic tree amplitudes, hep-th/0503060.

[101] A. P. Hodges, Twistor diagrams for all tree amplitudes in gauge theory: A helicity-independent formalism, hep-th/0512336.

[102] A. P. Hodges, Scattering amplitudes for eight gauge fields, hep-th/0603101.

[103] Z. Bern, L. J. Dixon, and D. A. Kosower, On-shell recurrence relations for one-loop QCD amplitudes, Phys. Rev. D 71 (2005) 105013, [hep-th/0501240].

[104] Z. Bern, L. J. Dixon, and D. A. Kosower, The last of the finite loop amplitudes in QCD, Phys. Rev. D 72 (2005) 125003, [hep-ph/0505055].

[105] Z. Bern, L. J. Dixon, and D. A. Kosower, Bootstrapping multi-parton loop amplitudes in QCD, Phys. Rev. D 73 (2006) 065013, [hep-ph/0507005].

[106] Z. Bern, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and H. Ita, Recursive calculation of one-loop QCD integral coefficients, JHEP 11 (2005) 027, [hep-ph/0507019].

[107] D. Forde and D. A. Kosower, All-multiplicity one-loop corrections to MHV amplitudes in QCD, Phys. Rev. D 73 (2006) 061701, [hep-ph/0509358].

[108] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, Bootstrapping one-loop QCD amplitudes with general helicities, Phys. Rev. D 74 (2006) 036009, [hep-ph/0604195].

[109] C. F. Berger, V. Del Duca, and L. J. Dixon, Recursive construction of Higgs+multiparton loop amplitudes: The last of the ϕ-nite loop amplitudes, Phys. Rev. D 74 (2006) 094021, [hep-ph/0608180].

[110] S. D. Badger, E. W. N. Glover, and K. Risager, One-loop ϕ-MHV amplitudes using the unitarity bootstrap, JHEP 07 (2007) 066, [arXiv:0704.3914 [hep-ph]].

[111] A. Brandhuber, S. McNamara, B. Spence, and G. Travaglini, Recursion relations for one-loop gravity amplitudes, JHEP 03 (2007) 029, [hep-th/0701187].

[112] N. Berkovits, An alternative string theory in twistor space for N = 4 super-Yang-Mills, Phys. Rev. Lett. 93 (2004) 011601, [hep-th/0402045].

[113] N. Berkovits and L. Motl, Cubic twistorial string field theory, JHEP 04 (2004) 056, [hep-th/0403187].

[114] L. Dolan and P. Goddard, Tree and loop amplitudes in open twistor string theory, JHEP 06 (2007) 005, [hep-th/0703054].

[115] W. Siegel, Untwisting the twistor superstring, hep-th/0404255.

[116] A. Neitzke and C. Vafa, N = 2 strings and the twistorial Calabi-Yau, hep-th/0402128.

[117] M. Aganagic and C. Vafa, Mirror symmetry and supermanifolds, hep-th/0403192.

[118] I. Bars, Twistor superstring in 2t-physics, Phys. Rev. D 70 (2004) 104022, [hep-th/0407239].

[119] M. Kulaxizi and K. Zoubos, Marginal deformations of $N = 4$ SYM from open/closed twistor strings, Nucl. Phys. B 738 (2006) 317{349, [hep-th/0410122].

[120] P. Gao and J.-B. Wu, (Non)-supersymmetric marginal deformations from twistor string theory, hep-th/0611128.

[121] J. Park and S.-J. Rey, Supertwistor orbifolds: Gauge theory amplitudes and topological strings, JHEP 12 (2004) 017, [hep-th/0411123].

[122] S. Giombi, M. Kulaxizi, R. Ricci, D. Robles-Llana, D. Trancanelli, and K. Zoubos, Orbifolding the twistor string, Nucl. Phys. B 719 (2005) 234{252, [hep-th/0411171].

[123] C.-h. Ahn, $N = 1$ conformal supergravity and twistor string theory, J. High Energy Phys. 10 (2004) 064, [hep-th/0409195].

[124] C.-h. Ahn, $N = 2$ conformal supergravity from twistor-string theory, Int. J. Mod. Phys. A 21 (2006) 3733{3760, [hep-th/0412202].

[125] M. Abou-Zeid and C. M. Hull, A chiral perturbation expansion for gravity, JHEP 02 (2006) 057, [hep-th/0511189].

[126] V. P. Nair, A note on MHV amplitudes for gravitons, Phys. Rev. D 71 (2005) 121701, [hep-th/0501143].

[127] A. D. Popov and M. Wolf, Topological B-model on weighted projective spaces and self-dual models in four dimensions, JHEP 09 (2004) 007, [hep-th/0406224].

[128] D.-W. Chiou, O. J. Ganor, Y. P. Hong, B. S. Kim, and I. Mitra, Massless and massive three dimensional super Yang-Mills theory and mini-twistor string theory, Phys. Rev. D 71 (2005) 125016, [hep-th/0502076].

[129] A. D. Popov, C. Sämann, and M. Wolf, The topological B{model on a mini{supertwistor space and supersymmetric Bogomolny monopole equations, J. High Energy Phys. 0510 (2005) 058, [hep-th/0505161].

[130] C. Sämann, On the mini-superambitwistor space and $N = 8$ super Yang{Mills theory, hep-th/0508137.

[131] O. Lechtenfeld and C. Sämann, Matrix models and D-branes in twistor string theory, J. High Energy Phys. 0603 (2006) 002, [hep-th/0511130].

[132] A. D. Popov, Sigma models with N = 8 supersymmetries in 2+1 and 1+1 dimensions, Phys. Lett. B 647 (2007) 509–514, [hep-th/0702106].

[133] D. W. Chiou, O. J. Ganor, and B. S. Kim, A deformation of twistor space and a chiral mass term in N = 4 super Yang–Mills theory, J. High Energy Phys. 0603 (2006) 027, [hep-th/0512242].

[134] R. Roiban, M. Spradlin, and A. Volovich, A googly amplitude from the B-model in twistor space, JHEP 04 (2004) 012, [hep-th/0402016].

[135] R. Roiban and A. Volovich, All googly amplitudes from the B-model in twistor space, Phys. Rev. Lett. 93 (2004) 131602, [hep-th/0402121].

[136] R. Roiban, M. Spradlin, and A. Volovich, On the tree-level S-matrix of Yang-Mills theory, Phys. Rev. D 70 (2004) 026009, [hep-th/0403190].

[137] R. Boels, L. Mason, and D. Skinner, Supersymmetric gauge theories in twistor space, JHEP 02 (2007) 014, [hep-th/0604040].

[138] R. Boels, L. Mason, and D. Skinner, From twistor actions to MHV diagrams, Phys. Lett. B 648 (2007) 90–96, [hep-th/0702035].

[139] R. Boels, A quantization of twistor Yang-Mills theory through the background field method, hep-th/0703080.

[140] L. J. Mason, Twistor actions for non-self-dual fields: A derivation of twistor-string theory, JHEP 10 (2005) 009, [hep-th/0507269].

[141] L. J. Mason and D. Skinner, An ambitwistor Yang-Mills Lagrangian, Phys. Lett. B 636 (2006) 60–67, [hep-th/0510262].

[142] M. Wolf, Self-dual supergravity and twistor theory, arXiv:0705.1422 [hep-th].

[143] L. J. Mason and M. Wolf, A twistor action for N = 8 self-dual supergravity, arXiv:0706.1941 [hep-th].

[144] M. Wolf, On hidden symmetries of a super gauge theory and twistor string theory, JHEP 02 (2005) 018, [hep-th/0412163].

[145] A. D. Popov and M. Wolf, Hidden symmetries and integrable hierarchy of the N = 4 supersymmetric Yang-Mills equations, Commun. Math. Phys. 275 (2007) 685–708, [hep-th/0608225].

[146] O. Lechtenfeld and A. D. Popov, Supertwistors and cubic string field theory for open N = 2 strings, Phys. Lett. B 598 (2004) 113, [hep-th/0406179].

[147] C. Samann, The topological B-model on fattened complex manifolds and subsectors of N = 4 self-dual Yang-Mills theory., J. High Energy Phys. 0501 (2005) 042, [hep-th/0410292].

[148] P. A. Grassi and G. Policastro, Super Chern-Simons theory as superstring theory, hep-th/0412272.

[149] T. Tokunaga, String theories on at supermanifolds, hep-th/0509198.

[150] R. Ricci, Super Calabi-Yau's and special Lagrangians, JHEP 03 (2007) 048, [hep-th/0511284].

[151] C.-N. Yang and R. L. Mills, Conservation of isotopic spin and isotopic gauge invariance, Phys. Rev. 96 (1954) 191{195.

[152] R. Kleiss and H. Kuijf, Multi-gluon cross-sections and ve jet production at hadron colliders, Nucl. Phys. B 312 (1989) 616.

[153] M. L. Mangano and S. J. Parke, Multiparton amplitudes in gauge theories, Phys. Rept. 200 (1991) 301{367, [hep-th/0509223].

[154] L. J. Dixon, Calculating scattering amplitudes e ciently, hep-ph/9601359.

[155] M. L. Mangano, S. J. Parke, and Z. Xu, Duality and multi-gluon scattering, Nucl. Phys. B 298 (1988) 653.

[156] Z. Bern and D. A. Kosower, Color decomposition of one-loop amplitudes in gauge theories, Nucl. Phys. B 362 (1991) 389{448.

[157] J. E. Paton and H.-M. Chan, Generalized Veneziano model with isospin, Nucl. Phys. B 10 (1969) 516{520.

[158] Z. Koba and H. B. Nielsen, Manifestly crossing-invariant parametrization of n-meson amplitude, Nucl. Phys. B 12 (1969) 517{536.

[159] E. Witten, \Onassis Lectures on Strings and Fields, 5-9 July 2004, Foundation for Research and Technology - Hellas, Heraklion, http://www.forth.gr/onassis/lectures/2004-07-05/programme.html."

[160] M. Jacob and G. C. Wick, On the general theory of collisions for particles with spin, Ann. Phys. 7 (1959) 404{428.

[161] F. Cachazo and P. Svrcek, Lectures on twistor strings and perturbative Yang-Mills theory, PoS RTN 2005 (2005) 004, [hep-th/0504194].

[162] V. V. Khoze, Gauge theory amplitudes, scalar graphs and twistor space, hep-th/0408233.

[163] Z. Xu, D.-H. Zhang, and L. Chang, Helicity amplitudes for multiple bremsstrahlung in massless non-Abelian gauge theories, Nucl. Phys. B 291 (1987) 392.

[164] S. J. Parke and T. R. Taylor, Perturbative QCD utilizing extended supersymmetry, Phys. Lett. B 157 (1985) 81.

[165] B. S. DeWitt, Quantum theory of gravity III: Applications of the covariant theory, Phys. Rev. 162 (1967) 1239{1256.

[166] M. T. Grisaru, H. N. Pendleton, and P. van Nieuwenhuizen, Supergravity and the S matrix, Phys. Rev. D 15 (1977) 996.

[167] M. T. Grisaru and H. N. Pendleton, Some properties of scattering amplitudes in supersymmetric theories, Nucl. Phys. B 124 (1977) 81.

[168] S. J. Bidder, D. C. Dunbar, and W. B. Perkins, Supersymmetric Ward identities and NMHV amplitudes involving gluinos, JHEP 08 (2005) 055, [hep-th/0505249].

[169] M. L. Mangano and S. J. Parke, Quark-gluon amplitudes in the dual expansion, Nucl. Phys. B 299 (1988) 673.

[170] S. J. Parke and T. R. Taylor, An amplitude for n-gluon scattering, Phys. Rev. Lett. 56 (1986) 2459.

[171] F. A. Berends and W. T. Giele, Recursive calculations for processes with n gluons, Nucl. Phys. B 306 (1988) 759.

[172] R. Penrose and M. A. H. MacCallum, Twistor theory: An approach to the quantization of elds and space-time, Phys. Rept. 6 (1972) 241{316.

[173] R. Penrose, Twistor quantization and curved space-time, Int. J. Theor. Phys. 1 (1968) 61{99.

[174] R. Penrose, \Talk given at: Twistor String Theory, The Mathematical Institute, University of Oxford, 10–14 January 2005, http://www.maths.ox.ac.uk/ lmason/Tws/Penrose1.pdf."

[175] V. P. Nair, A current algebra for some gauge theory amplitudes, Phys. Lett. B 214 (1988) 215.

[176] R. Penrose and W. Rindler, Spinors and space-time. Vol. 1: Two-spinor calculus and relativistic elds. Cambridge, UK: Univ. Pr., 1984.

[177] R. Penrose and W. Rindler, Spinors and space-time. Vol. 2: Spinor and twistor methods in space-time geometry. Cambridge, UK: Univ. Pr., 1986.

[178] S. A. Huggett and K. P. Tod, An introduction to twistor theory. Cambridge, UK: Univ. Pr., 1985.

[179] I. Bena, Z. Bern, and D. A. Kosower, Twistor-space recursive formulation of gauge theory amplitudes, Phys. Rev. D 71 (2005) 045008, [hep-th/0406133].

[180] Z. Bern, L. J. Dixon, and D. A. Kosower, All next-to-maximally-helicity-violating one-loop gluon amplitudes in N = 4 super-Yang-Mills theory, Phys. Rev. D 72 (2005) 045014, [hep-th/0412210].

[181] Z. Bern, V. Del Duca, L. J. Dixon, and D. A. Kosower, All non-maximally-helicity-violating one-loop seven-gluon amplitudes in N = 4 super-Yang-Mills theory, Phys. Rev. D 71 (2005) 045006, [hep-th/0410224].

[182] R. Britto, F. Cachazo, and B. Feng, Coplanarity in twistor space of N = 4 next-to-MHV one-loop amplitude coefficients, Phys. Lett. B 611 (2005) 167–172, [hep-th/0411107].

[183] I. Bena, Z. Bern, D. A. Kosower, and R. Roiban, Loops in twistor space, Phys. Rev. D 71 (2005) 106010, [hep-th/0410054].

[184] S. J. Bidder, N. E. J. Bjerrum-Bohr, D. C. Dunbar, and W. B. Perkins, Twistor space structure of the box coefficients of N = 1 one-loop amplitudes, Phys. Lett. B 608 (2005) 151–163, [hep-th/0412023].

[185] E. Witten, Topological sigma models, Commun. Math. Phys. 118 (1988) 411.

[186] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B 340 (1990) 281–332.

[187] M. Vonk, A mini-course on topological strings, hep-th/0504147.

[188] M. Nakahara, Geometry, topology and physics. Bristol, UK: Hilger, 1990.

[189] P. Candelas, Lectures on complex manifolds, In: Trieste 1987, Proceedings, Superstrings '87 (1987) 1–88.

[190] G. T. Horowitz, What is a Calabi-Yau space?, In: Proc. of Workshop on Unified String Theories, Santa Barbara, CA, Jul 29 - Aug 16, 1985.

[191] V. Bouchard, Lectures on complex geometry, Calabi-Yau manifolds and toric geometry, In: Proceedings of the Modave Summer School in Mathematical Physics (2005) [hep-th/0702063].

[192] W. Lerche, C. Vafa, and N. P. Warner, Chiral rings in N = 2 superconformal theories, Nucl. Phys. B 324 (1989) 427.

[193] L. J. Dixon, "Some world-sheet properties of superstring compactifications, on orbifolds and otherwise: Lectures given at the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology, Trieste, Italy, Jun 29 – Aug 7, 1987."

[194] K. Hori et al., Mirror symmetry. Providence, USA: AMS, 2003.

[195] E. Witten, Chern-Simons gauge theory as a string theory, Prog. Math. 133 (1995) 637–678, [hep-th/9207094].

[196] E. Witten, Noncommutative geometry and string field theory, Nucl. Phys. B 268 (1986) 253.

[197] A. D. Popov and C. Samann, On supertwistors, the Penrose-Ward transform, and N = 4 super Yang-Mills theory, Adv. Theor. Math. Phys. 9 (2005) 931, [hep-th/0405123].

[198] W. Siegel, N = 2(4) string theory is self-dual N = 4 Yang-Mills theory, Phys. Rev. D 46 (1992) R3235, [hep-th/9205075].

[199] S. Gukov, L. Motl, and A. Neitzke, Equivalence of twistor prescriptions for super Yang-Mills, hep-th/0404085.

[200] E. S. Fradkin and A. A. Tseytlin, Conformal supergravity, Phys. Rept. 119 (1985) 233–362.

[201] D. L. Bennett, H. B. Nielsen, and R. P. Woodard, The initial value problem for maximally non-local actions, Phys. Rev. D 57 (1998) 1167–1170, [hep-th/9707088].

[202] R. Roiban, "Talk given at: From Twistors to Amplitudes, Queen Mary, University of London, 3–5 November 2005, http://www.strings.ph.qmul.ac.uk/ andreas/FTTA/RRoiban.pdf."

[203] A. Gorsky and A. Rosly, From Yang-Mills Lagrangian to MHV diagrams, JHEP 01 (2006) 101, [hep-th/0510111].

[204] H. Feng and Y.-t. Huang, MHV Lagrangian for N = 4 super Yang-Mills, hep-th/0611164.

[205] M. B. Green, J. H. Schwarz, and L. Brink, N = 4 Yang-Mills and N = 8 supergravity as limits of string theories, Nucl. Phys. B 198 (1982) 474–492.

[206] R. P. Feynman, Quantum theory of gravitation, Acta Phys. Polon. 24 (1963) 697–722.

[207] R. P. Feynman, Closed loop and tree diagrams, In: J R Klauder, Magic Without Magic, San Francisco 1972 (1972) 355–375.

[208] R. P. Feynman, Problems in quantizing the gravitational eld, and the massless Yang-Mills eld, In: J R Klauder, Magic Without Magic, San Francisco 1972 (1972) 377{408.

[209] L. J. Dixon, \Private communication (2004)."

[210] R. E. Cutkosky, Singularities and discontinuities of Feynman amplitudes, J. Math. Phys. 1 (1960) 429{433.

[211] J. C. Collins, D. E. Soper, and G. Sterman, Factorization of hard processes in QCD, Adv. Ser. Direct. High Energy Phys. 5 (1988) 1{91, [hep-ph/0409313].

[212] G. Passarino and M. J. G. Veltman, One-loop corrections for $e^+ e$ annihilation into $^+$ in the Weinberg model, Nucl. Phys. B 160 (1979) 151.

[213] W. L. van Neerven, Dimensional regularization of mass and infrared singularities in two loop on-shell vertex functions, Nucl. Phys. B 268 (1986) 453.

[214] W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, QCD corrections to heavy quark production in pp collisions, Phys. Rev. D 40 (1989) 54{82.

[215] G. Duplancic and B. Nizic, Dimensionally regulated one-loop box scalar integrals with massless internal lines, Eur. Phys. J. C 20 (2001) 357{370, [hep-ph/0006249].

[216] T. Binoth, J. P. Guillet, and G. Heinrich, Reduction formalism for dimensionally regulated one-loop n-point integrals, Nucl. Phys. B 572 (2000) 361{386, [hep-ph/9911342].

[217] Z. Bern, Perturbative quantum gravity and its relation to gauge theory, Living Rev. Rel. 5 (2002) 5, [gr-qc/0206071].

[218] F. A. Berends, W. T. Giele, and H. Kuijf, On relations between multi-gluon and multi-graviton scattering, Phys. Lett. B 211 (1988) 91.

[219] H. Kawai, D. C. Lewellen, and S. H. H. Tye, A relation between tree amplitudes of closed and open strings, Nucl. Phys. B 269 (1986) 1.

[220] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, On-shell unitarity bootstrap for QCD amplitudes, Nucl. Phys. Proc. Suppl. 160 (2006) 261{270, [hep-ph/0610089].

[221] C. B. Thorn, Notes on one-loop calculations in light-cone gauge, hep-th/0507213.

[222] D. Chakrabarti, J. Qiu, and C. B. Thorn, Scattering of glue by glue on the light-cone worldsheet, I: Helicity non-conserving amplitudes, Phys. Rev. D 72 (2005) 065022, [hep-th/0507280].

[223] D. Chakrabarti, J. Qiu, and C. B. Thorn, Scattering of glue by glue on the light-cone worldsheet, II: Helicity conserving amplitudes, Phys. Rev. D 74 (2006) 045018, [hep-th/0602026].

[224] C. Anastasiou, Z. Bern, L. J. Dixon, and D. A. Kosower, Planar amplitudes in maximally supersymmetric Yang-Mills theory, Phys. Rev. Lett. 91 (2003) 251602, [hep-th/0309040].

[225] Z. Bern, L. J. Dixon, and V. A. Smirnov, Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond, Phys. Rev. D 72 (2005) 085001, [hep-th/0505205].

[226] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory, Phys. Rev. D 75 (2007) 085010, [hep-th/0610248].

[227] F. Cachazo, M. Spradlin, and A. Volovich, Hidden beauty in multiloop amplitudes, JHEP 07 (2006) 007, [hep-th/0601031].

[228] F. Cachazo, M. Spradlin, and A. Volovich, Iterative structure within the five-particle two-loop amplitude, Phys. Rev. D 74 (2006) 045020, [hep-th/0602228].

[229] F. Cachazo, M. Spradlin, and A. Volovich, Four-loop cusp anomalous dimension from obstructions, Phys. Rev. D 75 (2007) 105011, [hep-th/0612309].

[230] L. F. Alday and J. M. Maldacena, Gluon scattering amplitudes at strong coupling, JHEP 06 (2007) 064, [arXiv:0705.0303 [hep-th]].

[231] S. Abel, S. Forste, and V. V. Khoze, Scattering amplitudes in strongly coupled N = 4 SYM from semiclassical strings in AdS, arXiv:0705.2113 [hep-th].

[232] A. Nasti and G. Travaglini, One-loop N = 8 supergravity amplitudes from MHV diagrams, arXiv:0706.0976 [hep-th].

[233] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring, Use of causality conditions in quantum theory, Phys. Rev. 95 (1954) 1612–1627.

[234] L. D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. 13 (1959) 181–192.

[235] S. Mandelstam, Determination of the pion-nucleon scattering amplitude from dispersion relations and unitarity. General theory, Phys. Rev. 112 (1958) 1344–1360.

[236] S. Mandelstam, Analytic properties of transition amplitudes in perturbation theory, Phys. Rev. 115 (1959) 1741–1751.

[237] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, The analytic S-matrix. Cambridge, UK: Univ. Pr., 1966.

[238] G. B. Arfken and H. J. Weber, Mathematical methods for physicists. Harcourt, USA: Academic Pr., 1966.

[239] G. 't Hooft and M. J. G. Veltman, Regularization and renormalization of gauge fields, Nucl. Phys. B 44 (1972) 189–213.

[240] Z. Bern, L. J. Dixon, M. Perelstein, and J. S. Rozowsky, Multi-leg one-loop gravity amplitudes from gauge theory, Nucl. Phys. B 546 (1999) 423–479, [hep-th/9811140].