COMMUNICATION SCIENCES

AND

ENGINEERING

XIII. STATISTICAL COMMUNICATION THEORY*

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M. J. Schaffer
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D. W. Steele
C. I. M. Uchoa

A. WORK COMPLETED

1. AN INVESTIGATION OF OPTIMUM QUANTIZATION

This study has been completed by J. D. Bruce. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Doctor of Science, Department of Electrical Engineering, M.I.T., May 1964.

2. SUPERPOSITION IN A CLASS OF NONLINEAR SYSTEMS

This study has been completed by A. V. Oppenheim. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Doctor of Science, Department of Electrical Engineering, M.I.T., May 1964.

A. G. Bose

3. STUDY OF AN ADAPTIVE FILTER FOR NONSTATIONARY NOISE

This study has been completed by M. Chessman. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M.I.T., May 1964.

4. FILTERS USING POLYNOMIAL SEGMENTS TO RECONSTRUCT SAMPLE SIGNALS

This study has been completed by M. J. Schaffer. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M.I.T., May 1964.

5. LINEAR SYSTEMS WITH ADAPTIVE GAIN

This study has been completed by P. M. Rego and C. I. M. Uchoa. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science,

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Department of Electrical Engineering and Naval Engineering, M. I. T., May 1964.

M. Schetzen

6. COMPUTER SIMULATION AND EVALUATION OF A TWO-STATE MODULATION SYSTEM

This study has been completed by J. E. Schindall. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M. I. T., May 1964.

7. THE EFFECT OF PRE- AND POST-EMPHASIS FILTERING ON AN OPTIMUM QUANTIZATION SYSTEM

This study has been completed by G. F. Crimi. In May 1964, he submitted the results to the Department of Electrical Engineering, M. I. T., as a thesis in partial fulfillment of the requirements for the degree of Master of Science.

J. D. Bruce

8. MEASUREMENTS OF POWER DENSITY SPECTRA BY USE OF LAGUERRE FUNCTIONS

This study has been completed by J. D. Gruber. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Bachelor of Science, Department of Electrical Engineering, M. I. T., May 1964.

9. NONLINEAR SYSTEM EVALUATION BY DIGITAL COMPUTER

This study has been completed by C. R. Mock. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Bachelor of Science, Department of Electrical Engineering, M. I. T., May 1964.

10. INVESTIGATION OF NONLINEAR DEVICES FOR USE IN LEVEL DETECTING SYSTEMS

This study has been completed by R. I. Ollins. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Bachelor of Science, Department of Electrical Engineering, M. I. T., May 1964.

M. Schetzen

B. CALCULATION OF POWER DENSITY SPECTRA FOR A CLASS OF RANDOMLY JITTERED WAVEFORMS

The power density spectrum has been calculated for an impulse train in which the impulse areas and intervals between successive impulses are random variables. Spectra corresponding to a variety of signals have been obtained as special cases of this result. With certain restrictions, a method for computing the spectra with the aid of a Smith chart is presented.

A special case of the general method is that of a two-state waveform in which the time intervals between successive transitions are given by a random variable. Three other cases related to specific pulse trains will be examined in detail.

1. The Waveform Considered

We shall consider the waveform g(t) shown in Fig. XIII-1. This signal consists of a sequence of impulses having areas given by the random variables $\{a_k\}$. The time intervals between adjacent impulses will be called interarrival times. The k^{th} interarrival

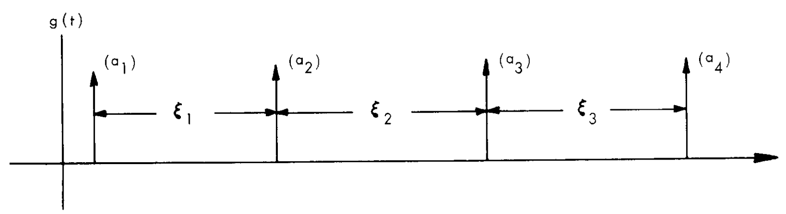

Fig. XIII-1. Impulse train with random interarrival times and areas.

time will be specified by the random variable ξ_k. We call the randomness in the interarrival time jitter.

We require that this waveform be stationary in the respect that the processes that generate the sets $\{a_k\}$ and $\{\xi_k\}$ are stationary. That is, we require $p_n[a_k, a_{k+n}]$ and $p_{\xi_k+\xi_{k+1}+\dots+\xi_{k+n}}(x)$ to be independent of k for any value of n, where $p_n[a_k,a_{k+n}]$ is the joint probability density for the variables a_k and a_{k+n}, and $P_{\xi_k+\xi_{k+1}+\dots+\xi_{k+n}}(x)$ is the probability density describing the waiting time between two pulses n intervals apart. We shall call $\sum_{i=1}^{n} \xi_{k+i}$ the waiting time until the n^{th} event and use a new random variable, η_n, to denote this waiting time. We also define

$$p_{\eta_n}(x) = p_{\xi_{k+1}+\xi_{k+2}+\dots+\xi_{k+n}}(x).$$

An exact expression for the power density spectrum of g(t) will be obtained. For certain cases, this result may be written in closed form. By suitably choosing $\{a_k\}$ and $\{\xi_k\}$, spectra for a variety of processes that are of physical interest may be obtained. A number of such cases will be discussed.

2. Derivation of the Spectrum of g(t)

The derivation is carried out in two parts. First, we calculate the autocorrelation, $R_g(\tau)$, of g(t). Then the desired spectrum is obtained by taking the Fourier transform of $R_g(\tau)$. Since the calculation of $R_g(\tau)$ is somewhat involved, only an outline of the method will be presented.

We first define an ergodic ensemble for g(t). Each member function of the ensemble is sampled at times t_1 and $t_1 + \tau$. We let s_{t_1} and $s_{t_1+\tau}$ be the random variables describing, respectively, the values taken on by each pair of samples. Then $R_g(\tau)$ is obtained as

$$R_g(\tau) = E\left[s_{t_1} s_{t_1+\tau}\right]. \tag{1}$$

This sampling procedure is not really defined for impulses because of their zero width and infinite height. If the impulses are represented as rectangles with width δ and finite height $\frac{1}{\delta}$, we can compute the autocorrelation function corresponding to this new pulse train. By letting $\delta \to 0$ the desired $R_g(\tau)$ is obtained.[1]

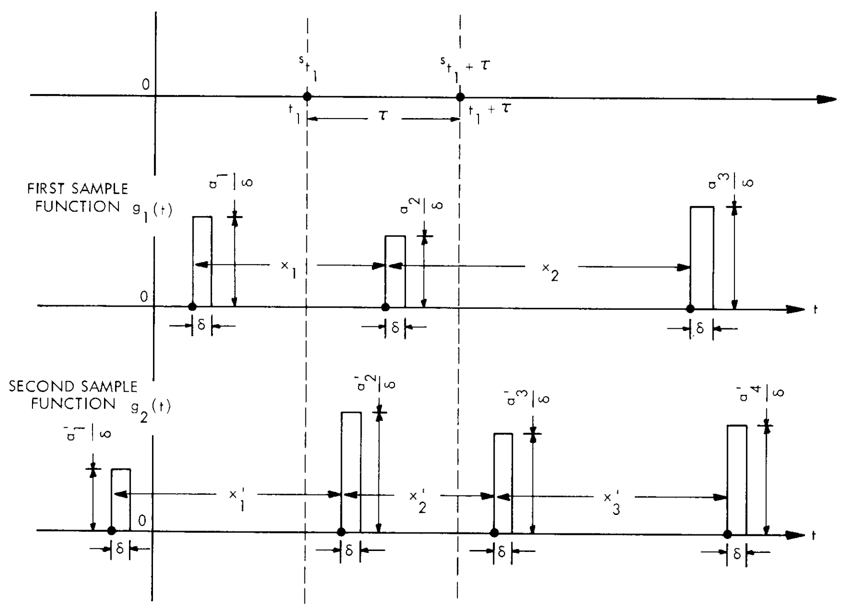

Fig. XIII-2. Some functions from the ensemble defined for calculating $R_g(\tau)$.

The member functions of the ensemble must be considered to have uniformly distributed starting times to ensure ergodicity. A few functions from the ensemble thus

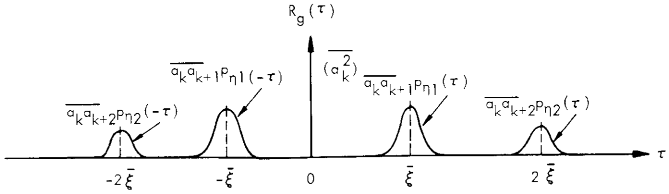

Fig. XIII-3. Typical form for $R_g(\tau)$.

defined are shown in Fig. XIII-2.

By computing the autocorrelation of the rectangular pulse waveform and letting $\delta \to 0$, we obtain

$$R_g(\tau) = \frac{1}{\bar{\xi}}\left[\overline{a_k^2} + \sum_{n=1}^{\infty} \overline{a_k a}_{k+n}\, p_{\eta_n}(\tau) + \sum_{n=1}^{\infty} \overline{a_k a}_{k+n}\, p_{\eta_n}(-\tau)\right], \tag{2}$$

where

$$\overline{a_k a}_{k+n} = E[a_k a_{k+n}].$$

$R_g(\tau)$ may typically appear as shown in Fig. XIII-3.

3. Spectrum of g(t)

By taking the Fourier transform of $R_g(\tau)$ term by term, we obtain $S_g(\omega)$, the spectrum of g(t). Observing that the Fourier transform of $p_{\eta_n}(\tau)$ is simply the conjugate of its characteristic function, we can write

$$S_g(\omega) = \frac{1}{\bar{\xi}}\left[\overline{a_k^2} + \sum_{n=1}^{\infty} \overline{a_k a}_{k+n}\left(M_{\eta_n}(\omega) + M^*_{\eta_n}(\omega)\right)\right]. \tag{3}$$

The asterisk denotes complex conjugation. $M_{\eta_n}(\omega)$ is the characteristic function corresponding to $p_{\eta_n}(\tau)$:

$$M_{\eta_n}(\omega) = E\left[e^{i\eta_n\omega}\right]$$

$$= \int_0^{\infty} p_{\eta_n}(\tau)\, e^{j\omega\tau}\, d\tau. \tag{4}$$

Let

$$\overline{a_k a}_{k+n} = \sigma_a^2\, {}_n\rho_n + \bar{a}^2, \tag{5}$$

where $\sigma_a^2 = \text{var}\,[a_k]$, $\bar{a} = E[a_k]$, and ρ_n = coefficient of correlation between a_k and a_{k+n}. Then Eq. 3 can be written

$$S_g(\omega) = \frac{\bar{a}^2}{\bar{\xi}}\left[1 + 2\,\text{Re}\sum_{n=1}^{\infty} M_{\eta_n}(\omega)\right] + \frac{\sigma_a^2}{\bar{\xi}}\left[1 + 2\,\text{Re}\sum \rho_n M_{\eta_n}(\omega)\right]. \tag{6}$$

The two series in (6) do not in general converge at all values of ω, since we obtained them be reversing the order of integration and summation. It can be shown, however, that whenever they do converge, the functions to which they converge are the desired functions.[2]

For the special cases considered in this report the result may either be written in closed form or expressed as recognizable Fourier series. In these instances, convergence does not present any difficulties.

CASE I: Uniform-Area Impulse Train with Statistically Independent Interarrival Times

We consider this particular signal first because its spectrum takes on a form that occurs in several of the cases considered here. Some of the general properties of this form will be developed so that they may be utilized later.

We obtain uniform-area impulses by requiring $a_k = 1$ for all k. This means that $\sigma_a^2 = 0$ and $\bar{a} = 1$. The requirement of independent interarrival times implies that we can write

$$\begin{aligned} M_{\eta_n}(\omega) &= E\left[e^{i\omega(\xi_1+\xi_2+\dots+\xi_n)}\right] \\ &= E\,[e^{i\omega\xi}]^n \\ &= [M_\xi(\omega)]^n. \end{aligned} \tag{7}$$

With these restrictions, Eq. 6, giving $S_g(\omega)$, becomes

$$S_g(\omega) = \frac{1}{\bar{\xi}}\left\{1 + 2\,\text{Re}\sum_{n=1}^{\infty}[M_\xi(\omega)]^n\right\}. \tag{8}$$

Since this series is a geometric series that always converges, except possibly when $|M_\xi(\omega)| = 1$, we can write

$$S_g(\omega) = \frac{1}{\bar{\xi}}\left[1 + 2\,\text{Re}\left(\frac{M_\xi(\omega)}{1 - M_\xi(\omega)}\right)\right] \qquad \text{and}$$

$$S_g(\omega) = \frac{1}{\xi} \text{Re} \left[\frac{1 + M_\xi(\omega)}{1 - M_\xi(\omega)} \right]. \tag{9}$$

Let

$$F(\lambda) = \frac{1 + M_\xi(\lambda)}{1 - M_\xi(\lambda)}, \tag{10}$$

where $\lambda = \omega + j\sigma$.

$F(\lambda)$ has some interesting properties that we shall now examine.

a. Some Characteristics of $F(\lambda)$

Observe that Eq. 10 is a one-to-one bilinear transformation that maps the region $|M_\xi(\lambda)| \leq 1$ onto the region Re $[F(\lambda)] \geq 0$. Note also that

$$\begin{aligned} |M(\lambda)| &\leq \int_0^\infty p_\xi(\tau)\, e^{-\sigma\tau}\, d\tau < 1 \qquad && \text{for } \sigma > 0 \\ |M(\lambda)| &\leq 1 && \text{for } \sigma = 0. \end{aligned} \tag{11}$$

By using these facts, it can be easily shown[3] that $F(\lambda)$ is a positive real function. That is, (i) $F(\lambda)$ is analytic for $\sigma > 0$; (ii) Re $[F(\lambda)] > 0$ for $\sigma > 0$; and (iii) Re $[F(\lambda)] = 0$ for $\sigma = 0$ except at poles that must be simple and have positive imaginary residues. It can be shown that Re $[F(\omega)]$ has impulses at these ω-axis poles.

Thus given any interarrival time distribution $p_\xi(x)$ for which $M_\xi(\omega)$ exists, the spectrum $S_g(\omega)$ exists except possibly at certain values of ω at which impulses appear.

b. A Method for Computing Re $[F(\omega)]$ from $M_\xi(\omega)$

The relationship between $F(\omega)$ and $M_\xi(\omega)$ is exactly analogous to a relationship that occurs in transmission-line theory. Let us consider the transmission line shown in

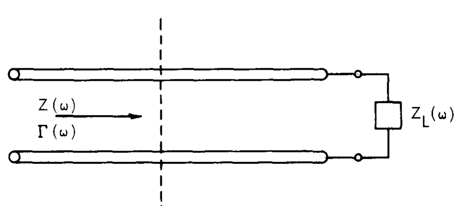

Fig. XIII-4. Transmission line for which the relationship between $\Gamma(\omega)$ and $Z(\omega)$ is analogous to that between $M_\xi(\omega)$ and $F(\omega)$.

Fig. XIII-4. At any particular point on this line the relationship between the impedance $Z(\omega)$ and the reflection coefficient at $\Gamma(\omega)$ is

$$Z(\omega) = \frac{1 + \Gamma(\omega)}{1 - \Gamma(\omega)}. \tag{12}$$

$\Gamma(\omega)$ has the property that $|\Gamma(\omega)| \leq 1$.

The Smith chart, a one-to-one mapping of Re $[Z(\omega)] \geq 0$ onto $|\Gamma(\omega)| \leq 1$, is shown in Fig. XIII-5. Both the real and imaginary parts $Z(\omega)$ corresponding to a given $\Gamma(\omega)$ can be read directly from the chart.

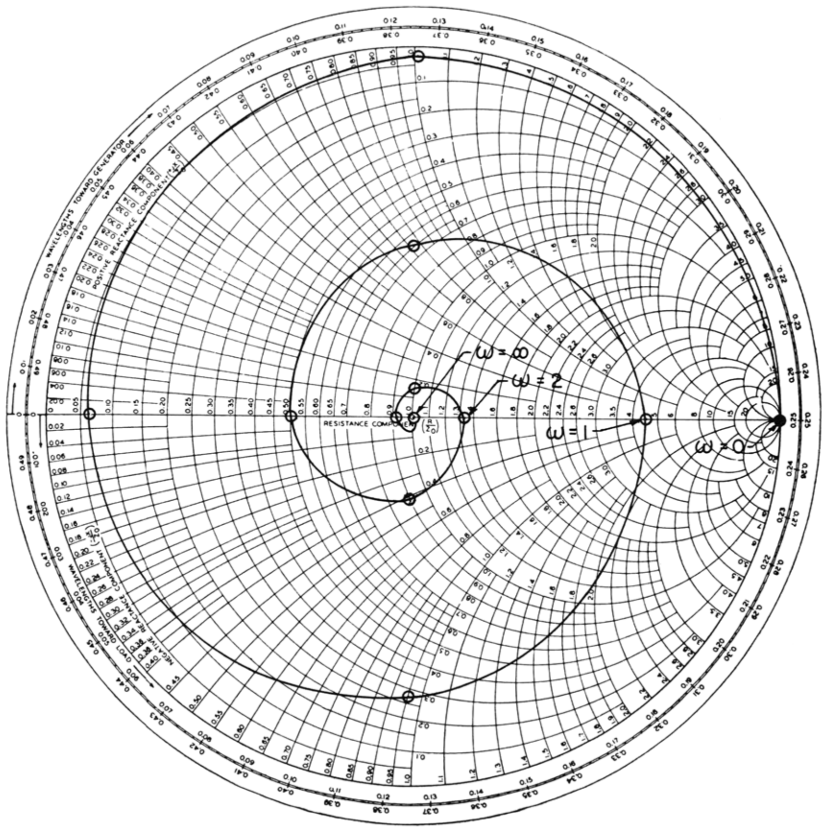

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Fig. XIII-5. Graph of the locus of $M_\xi(\omega)$ corresponding to a Gaussian jitter distribution. For this case, $\sigma_\xi = 1$ and $\overline{\xi} = 2\pi$.

If at a particular frequency, $\omega = \omega_1$, the complex number $M_\xi(\omega_1)$ is plotted on the chart, the value of Re $[F(\omega_1)]$ may be read directly from the chart. Consequently, $S_g(\omega)$ can be easily computed if the locus of $M_\xi(\omega)$ as ω varies from zero to infinity can be plotted. Thus one can determine by inspection the behavior of $S_g(\omega)$ for almost any well-behaved waiting time distribution.

When the jitter has a Gaussian distribution[4]

$$p_\xi(x) = \frac{1}{\sqrt{2\pi}\,\sigma_\xi} \exp\left[-\frac{(x-\bar{\xi})^2}{2\sigma_\xi^2}\right],$$

we have

$$M_\xi(\omega) = \exp\left(-\frac{1}{2}\sigma_\xi^2\omega^2 + j\bar{\xi}\omega\right). \tag{13}$$

The locus of this characteristic function as ω varies from zero to infinity is shown in Fig. XIII-5.

c. Asymptotic Behavior of Re $[F(\omega)]$

It can be shown that if $p_\xi(x)$ is bounded, then $M_\xi(\omega) \to 0$ as $\omega \to \infty$. This implies that $F(\omega) \to 1$ as $\omega \to \infty$. Thus, for large ω, the asymptotic form of our spectrum is

$$S_g(\omega) \cong \frac{1}{\bar{\xi}}. \tag{14}$$

d. Behavior of $F(\omega)$ for Small ω

$F(\lambda)$ may be expanded in a Laurent series about $\omega = 0$. The first two terms of the expansion are

$$F(\lambda) \cong \frac{2j}{\bar{\xi}}\frac{1}{\lambda} + \left(\frac{\sigma_\xi}{\bar{\xi}}\right)^2 + \text{terms in } \lambda \text{ of order} \geq 2. \tag{15}$$

This can be interpreted as

$$S_g(\omega) \cong \frac{1}{\bar{\xi}^2} u_o(\omega) + \frac{\sigma_\xi^2}{\bar{\xi}^3} + \text{terms in } \omega \text{ of order} \geq 2. \tag{16}$$

e. Spectrum of Jitter-Free Impulse Train

If we let the jitter go to zero, corresponding to a periodic train of impulses a distance $\bar{\xi}$ apart, we have

$$M_\xi(\omega) = e^{j\bar{\xi}\omega}$$

and (17)

$$F(\omega) = j \cot \frac{\bar{\xi}\omega}{2}.$$

In this case $F(\lambda)$ has poles and zeros that alternate along the ω-axis. Re $[F(\omega)]$ consists of unit-area impulses occurring at the poles of $F(\lambda)$. Thus

$$S_g(\omega) = \frac{1}{\bar{\xi}^2} \sum_{k=-\infty}^{\infty} u_o\left(\omega - \frac{k}{\bar{\xi}}\right), \tag{18}$$

which is indeed the spectrum corresponding to a periodic impulse train.

CASE II: Uniform-Area Impulses of Alternating Sign with Independent Interarrival Times

The only difference between this and the previous case is that we now have $a_k = (-1)^k$. This means that $\bar{a} = 0$, $\sigma_a^2 = 1$, and $\rho_k = (-1)^k$. Using these restrictions, as well as that of Eq. 7, we obtain

$$S_g(\omega) = \frac{1}{\bar{\xi}} \text{Re}\left[\frac{1 - M_\xi(\omega)}{1 + M_\xi(\omega)}\right] = \frac{1}{\bar{\xi}} \text{Re}\left[\frac{1}{F(\omega)}\right]. \tag{19}$$

The only difference between this result and that of Case I is that the sign of $M_\xi(\omega)$ is reversed.

In using the Smith chart for computing the spectrum we now plot $-M_\xi(\omega)$ rather than $+M_\xi(\omega)$. This situation is exactly analogous in the transmission-line case to the plotting of $\Gamma(\omega)$ to obtain the corresponding admittance $Y(\omega)$.

The asymptotic behavior for large ω is the same as for the previous case. If we expand $\frac{1}{F(\omega)}$ in a Taylor series about $\omega = 0$, we obtain

$$S_g(\omega) \cong \frac{1}{\bar{\xi}}\left[\frac{\sigma_\xi^2 \omega^2}{4} + \text{terms in } \omega \text{ of order} \geq 4\right]. \tag{20}$$

CASE III: Symmetric Rectangular Wave with Independent Interarrival Times between Zero Crossings[5]

We obtain the square wave, f(t), shown in Fig. XIII-6 by integrating the alternating-sign impulse train of Case II.

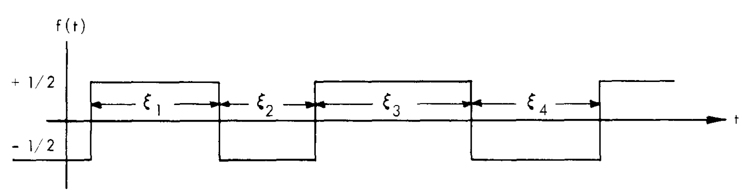

Fig. XIII-6. Symmetric rectangular wave considered in Case III.

The spectrum for this wave is obtained by dividing the result in (19) by ω^2 and is given by

$$S_f(\omega) = \frac{1}{\overline{\xi}\omega^2} \operatorname{Re}\left[\frac{1 - M_\xi(\omega)}{1 + M_\xi(\omega)}\right]. \tag{21}$$

The asymptotic form of $S_f(\omega)$ for large ω is

$$S_f(\omega) \cong \frac{1}{\overline{\xi}\omega^2}, \tag{22}$$

since $\operatorname{Re}\left[\frac{1 - M_\xi(\omega)}{1 + M_\xi(\omega)}\right] \to 1$ as $\omega \to \infty$.

Using (20), we see that the behavior of $S_f(\omega)$ for small ω is

$$S_f(\omega) \cong \frac{\sigma_\xi^2}{4\overline{\xi}} + \text{terms in } \omega \text{ of order} \geq 2 \tag{23}$$

CASE IV: Impulse Train with Independent Interarrival Times and Exponentially Correlated Areas

The signal for this case is obtained by choosing

$$\rho_n = e^{-\beta n}$$

and (24)

$$M_{\xi_n}(\omega) = [M_\xi(\omega)]^n.$$

Substituting these parameters in Eq. 6, we obtain

$$S_g(\omega) = \frac{\overline{a^2}}{\overline{\xi}}\left\{1 + 2 \operatorname{Re} \sum_{n=1}^{\infty} [M_\xi(\omega)]^n\right\} + \frac{\sigma_a^2}{\overline{\xi}}\left\{1 + 2 \operatorname{Re} \sum_{n=1}^{\infty} \left[e^{-\beta} M_\xi(\omega)\right]^n\right\}. \tag{25}$$

Observe that the expression consists of two geometric series, each of which can be simplified as before to yield

$$S_g(\omega) = \frac{\overline{a^2}}{\overline{\xi}} \operatorname{Re}\left[\frac{1 + M_\xi(\omega)}{1 - M_\xi(\omega)}\right] + \frac{\sigma_a^2}{\overline{\xi}} \operatorname{Re}\left[\frac{1 + e^{-\beta} M_\xi(\omega)}{1 - e^{-\beta} M_\xi(\omega)}\right]. \tag{26}$$

Each term in the expression can be evaluated in the same manner as before by using the

Smith chart. The contribution from the second term is obtained by plotting $M_{\xi}(\omega)$ attenuated by $e^{-\beta}$.

4. Concluding Remarks

The results presented here are general in the respect that no particular form for the interarrival time distribution has been assumed. Spectra corresponding to Cases I, II, and III have been obtained previously for Poisson-distributed impulses.[6]

When the characteristic function,

$$M_{\xi}(\omega) = \frac{1}{1 - j\bar{\xi}\omega},$$

corresponding to this particular distribution is substituted in our expressions, the previously calculated spectra are obtained.

5. Note Added in Proof

Subsequent to the writing of this report we have found several papers related to the topic presented here.

G. Biorci and P. Mazetti ("Correlation Function of Nonindependent Overlapping Pulse Trains," L'Elettrotecnica, Vol. 48, p. 469, June 1961; and "Study of Nonindependent Random Pulse Trains, with Application to Barkhausen Noise," Il Nuovo cimento, Vol. 25, p. 1322, September 1962) obtained an expression for the autocorrelation of a pulse train in which the interarrival times are independent and specified by the same probability distribution, and in which the pulse amplitudes are randomly scaled. The spectrum of this train was obtained by using a pulse shape and a spacing distribution that were experimentally observed for Barkhausen noise.

E. Banta ("A Note on the Correlation Function of Nonindependent, Overlapping Pulse Trains," IRE Trans. on Information Theory, Vol. IT-10, pp. 160-161, April 1964) showed that Biorci and Mazetti's expression for the autocorrelation could be presented in a much simpler form.

General expressions for the spectra of some related forms of pulse trains were obtained by D. Middleton (An Introduction to Statistical Communication Theory, McGraw-Hill Book Company, Inc., New York, 1960, pp. 228-238).

The present work yields a general expression for the spectrum that encompasses the cases considered by Biorci, Mazetti, and Banta. Also, methods for calculating and interpreting the spectra for a wide variety of cases are discussed.

D. E. Nelsen

Footnotes and References

1. One can obtain $R_g(\tau)$ by other methods without assuming any specific shape for the impulses. The present method is used because it is conceptually easier to present.

2. T. M. Apostol, Mathematical Analysis (Addison-Wesley Publishing Company, Inc., South Reading, Mass., 1957); Theorem 14-31, p. 451.

3. E. A. Guillemin, Mathematics of Circuit Analysis (John Wiley and Sons, Inc., New York, 1959), pp. 395-399.

4. Since this Gaussian density is nonzero for $x \leq 0$, it is not really a realizable waiting-time distribution. It may be used, however, to represent physical processes if $\bar{\xi} \gg \sigma_{\xi}$.

5. An equivalent form for the spectrum of this waveform is presented in J. L. Lawson and G. E. Uhlenbeck, Threshold Signals, Radiation Laboratory Series Vol. 24 (McGraw-Hill Book Company, Inc., New York, 1950), p. 45.

6. See, for instance, Y. W. Lee, Statistical Theory of Communication (John Wiley and Sons, Inc., New York, 1960), pp. 221-224; 240-248.