PHASE STABILITY

J. Le Duff
LAL, Orsay, France

Abstract
The principle of phase stability was established independently and almost simultaneously by V. Veksler [1] and E.M. McMillan [2]. The first electron synchrotron which was designed by McMillan and built at the University of California came to full energy (320 MeV) operation early 1949.

1. INTRODUCTION

The phase stability occurs in synchronous accelerators where the acceleration is made by using radio-frequency electric fields. If successive accelerating gaps (radio-frequency cavities) are arranged such that a given particle always sees the same RF phase and gets the same energy gain, that particle is called synchronous particle. In the case of a synchrotron a single cavity will be successively traversed by the particles turn after turn and for the particular case of ultra-relativistic electrons a fixed RF frequency can fit the revolution frequency.

Other particles, either deviated in phase or in energy from the synchronous one, will oscillate in phase and energy with respect to the synchronous particle (also called reference particle) and this is the mechanism of phase stability. However stability requires specific input conditions as it will be seen in the present lecture.

2. RADIO FREQUENCY ACCELERATION

2.1 Energy gain

In relativistic dynamics the total energy ($E = mc^2$), the rest energy ($E_0 = m_0 c^2$) and the momentum ($p = mv$) satisfy:

$$E^2 = E_0^2 + p^2c^2$$

which by differentiation gives:

$$dE = vdp$$

From Newton-Lorentz force:

$$\frac{dp}{dt} = v\frac{dp}{dz} = \frac{dE}{dz} = eE_z$$

and the energy gain due to the electric field component E_z is:

$$\Delta E = e\int E_z dz = eV$$

where V is the voltage across the accelerating gap.

Considering an oscillating accelerating electric field (TM mode in a cavity) one can write:

$$E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin\phi(t)$$

Neglecting the transit time through the gap one can also write:

$$\Delta E = e\hat{V}\sin\phi$$

$$\hat{V} = \int \hat{E}_z dz$$

and ϕ represents the RF phase as seen by the particle while going through the gap.

2.2 Principle of phase stability

Considering now periodic gap crossings and a reference particle for which the synchronism condition is satisfied for an entrance phase ϕ_s, the energy gain in each gap can be represented by the RF signal shown on Fig. 1.

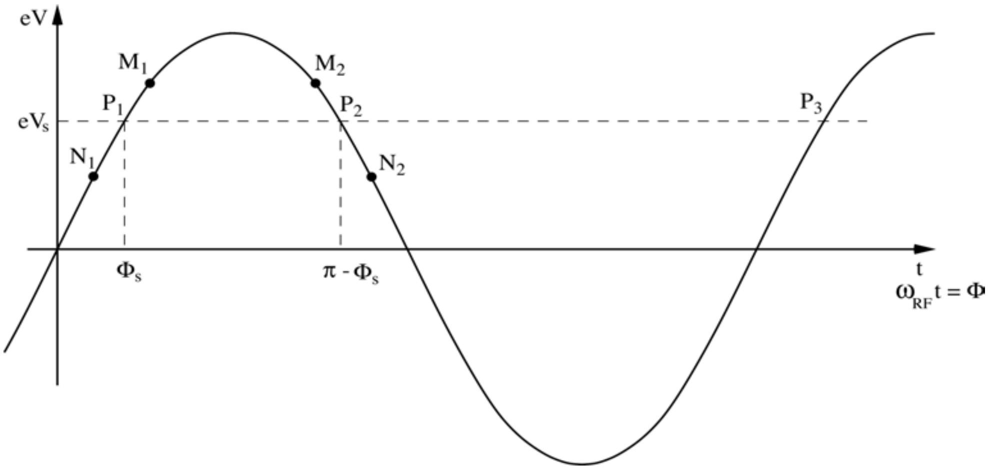

Fig. 1 Energy gain versus RF phase during gap crossing.

There are two possible synchronous phases (ϕ_s and $\pi - \phi_s$) per half accelerating periods, which repeat every period. Particles P_1, P_2 …P_n are called synchronous particles and at each gap they get the amount of energy that bring them to the next gap with an identical phase.

As for other particles which are deviated from the synchronous ones they will get a different story. For example particle M_1 which has arrived later in time in the first gap is getting more energy gain so it will run faster (assuming right now that an increase in energy, followed by an increase in velocity, reduces the time it takes to reach the next gap) and it will get closer to the synchronous particle at the next gap and so on. Particle N_1 arriving earlier in the first gap is getting less energy gain, hence will slow down compare to the synchronous one and get also closer to it at the next gap. It gets clear that the tendency for non synchronous particles is to oscillate in phase and energy around the synchronous one. That is true on the positive slope of the RF signal but the same type of approach made for particles M_2 and N_2 shows the reverse process as they tend, gap after gap, to go further away from P_2.

The capture phenomena which occurs around P_1 is called "phase stability" and is similar to a focusing effect where the focusing force is just the slope of the RF signal. Since it applies on the phase and energy variables it can be called "longitudinal focusing" by analogy with magnetic focusing in the transverse plane.

2.3 Consequence of phase stability

Noticing that a particle which is behind the reference one ($\Delta z < 0$) arrives later in the accelerating gap ($\Delta t > 0$); hence the positive RF slope which gives stability is translated into a negative slope when changing variable:

$$\frac{\partial V}{\partial t} > 0 \qquad \rightarrow \qquad \frac{\partial E_z}{\partial z} < 0$$

According to Maxell's equations:

$$\nabla \vec{E} = 0$$

leading to:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} > 0$$

showing that one or another (or both) of these two gradients has to be positive leading to a transverse defocusing.

3. THE ELECTRON SYNCHROTRON

3.1 Principle of operation

The synchrotron, as sketched on Fig.2, is a synchronous circular accelerator and the reference (synchronous) particle, at nominal energy, travels on a fixed closed orbit. In order to do so, while ramping in energy, the synchronous particle is such that its synchronous RF phase provides the exact amount of energy gain, at each turn, that fits the increase of the magnetic fields.

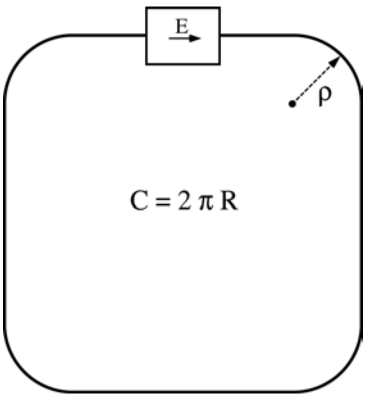

Fig. 2 The synchrotron

The cavity located in a straight section will give a synchronous energy gain:

$$\Delta E = e\hat{V}\sin\phi_s$$

In order to stay on the given circumference $C = 2\pi R$, where R is defined as the physical radius, the following relation needs to be satisfied:

$$eB\rho = p$$

where B is the bending field, ρ the bending radius and p the particle momentum.

Ramping in energy, while ρ remains constant, also means:

$$\frac{dp}{dt} = e\rho\dot{B}$$

where $\dot{B} = \dfrac{dB}{dt}$.

Since generally the revolution period, T_r, and the momentum gained per turn, $(\Delta p)_{turn}$, are rather small quantities one can approximate as follows:

$$(\Delta p)_{turn} \approx e\rho \dot{B} T_r = \frac{2\pi e \rho R \dot{B}}{v}$$

where v is the particle velocity.

Since:

$$\Delta E = v\Delta p$$

one gets the relationship between the required synchronous phase and the ramping rate of the magnetic field:

$$2\pi\rho R\dot{B} = \hat{V}\sin\phi_s$$

which can be fulfilled if $\hat{V}$ is sufficiently large.

Let's also mention that since, in general, the synchronism requires a constant phase ϕ_s at each turn, the RF frequency needs to be an integer multiple of the revolution frequency ($\omega_{RF} = h\omega_r$). In practice the RF frequency needs to be varied to follow the increase of the particle velocity. However in the case of electrons ,which are ultra-relativistic, their velocity will remain constant ($v = c$) during the energy ramping and consequently the RF frequency will be kept constant.

3.2 Dispersion effects due to the bending magnets

Since bending magnets, like spectrometers, will analyze particle energies, any particle which is slightly shifted in energy (or momentum) with respect to the reference particle will perform a different orbit. Fig. 3, though the sketch is very rough, is intended to show the change in circumference due to an energy deviation.

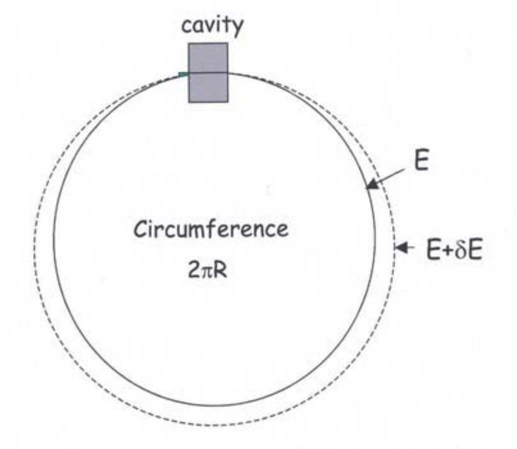

Fig. 3 Orbit versus energy

It is usual to introduce two parameters to describe the dispersion effects. The "momentum compaction factor" gives, to first order, the relative change in the circumference due to a relative momentum deviation:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

while the parameter "η" gives the corresponding relative change in the revolution frequency, which takes also account of the change in velocity:

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

3.2.1 Momentum compaction

Consider two particles, with different energies, going through a bending magnet as shown on Fig. 4. To a momentum deviation dp corresponds an orbit shifted by an amount $x(s_0)$ in the plane of curvature. The corresponding relative change in path length is:

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho}$$

Integrating over the whole circumference gives:

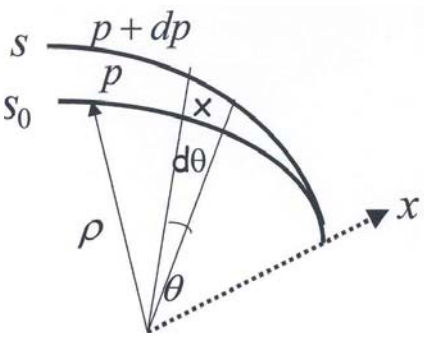

Fig. 4 Orbits in bending magnets

$$\int dl = 2\pi dR = \int \frac{x}{\rho} ds_0 = \frac{1}{\rho} \int_m x ds_0$$

leading to:

$$dR = < x >_m$$

where the subscript « m » tells that the integral is performed only in the bending magnets where ρ is finite. Introducing the dispersion function D_x , property of the lattice arrangement:

$$x = D_x \frac{dp}{p}$$

the momentum compaction simply becomes:

$$\alpha = \frac{< D_x >_m}{R}$$

3.2.2 Revolution frequency versus momentum

Since $f_r = \frac{\beta c}{2\pi R}$, with $\beta = \frac{v}{c}$, a relative change in the revolution frequency can be expressed as:

$$\frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

Expressing the momentum:

$$p = mv = \beta\gamma \frac{E_0}{c}$$

one gets for its relative change:

$$\frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1} \frac{d\beta}{\beta}$$

and finally:

$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right) \frac{dp}{p}$$

From the definition of the parameter η one gets:

$$\eta = \frac{1}{\gamma^2} - \alpha$$

showing that there is an energy, called “transition energy”, for which the synchrotron gets isochronous ($\eta = 0$):

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

3.3 Phase stability in an electron synchrotron

Since in an electron synchrotron γ is generally very large, one can easily assume that $\eta \approx -\alpha$. Since all operating electron synchrotrons have a positive momentum compaction they do have $\eta < 0$ and so they stand above transition energy.

In such conditions a small positive momentum deviation ($dp > 0$) will be followed by a small negative revolution frequency change ($df_r < 0$), or in other words by a longer revolution time. This is explained by the fact that the velocity of the particle is no longer depending on the energy ($v \approx c$) and then the revolution time only depends on the circumference, which indeed gets longer for a small increase in energy.

Back to Fig. 1, if P_1 still represents a synchronous particle for the synchrotron, then particle M_1 which arrives later in the cavity will have a higher energy gain, hence a longer revolution period, and will be delayed even more with respect to the reference particle; as for particle N_1, a similar story will happen and it will get more and more deviated from P_1 as they travel around the synchrotron. Clearly P_1 has become an unstable reference point.

It is straightforward to demonstrate that the reverse process will occur around P_2 , which turns out to be the stable reference now. In other words the phase stability in an electron synchrotron is obtained on the negative slope of the RF signal. The stable synchronous phase has become $\pi - \phi_s$.

4. LONGITUDINAL DYNAMICS

4.1 Energy and phase variables

The RF acceleration process clearly emphasizes the correlation between the RF phase experienced by a particle and its energy gain. Since by definition there is a well defined synchronous particle which always feel the same RF phase at each turn, and which has the nominal energy, it is then sufficient to follow other particles with respect to that reference particle. Consequently, in what follows, one will make use of reduced variables:

Revolution frequency:	$\Delta f_r = f_r - f_{rs}$
Particle RF phase:	$\Delta\phi = \phi - \phi_s$
Particle momentum :	$\Delta p = p - p_s$
Particle energy :	$\Delta E = E - E_s$
Azimuthal angle :	$\Delta\theta = \theta - \theta_s$

4.2 First energy-phase equation

The azimuthal variable θ (Fig. 5) is defined such that $ds_0 = Rd\theta$, where s_0 is the position along the nominal circumference $2\pi R$, no matter if the synchrotron is a pure circle or not. Introducing the angular revolution frequency $\omega_r = 2\pi f_r$, one can write:

$$\theta = \int \omega_r dt$$

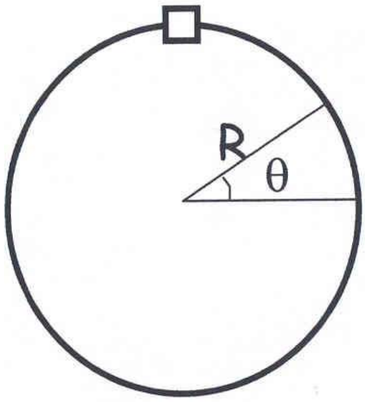

Fig. 5 Azimuthal variable

Assuming the RF frequency is an integer multiple of the revolution frequency, $\omega_{RF} = h\omega_{rs}$, h being the harmonic number, then the RF phase will rotate h times faster than the azimuthal angle. Distance between two particles can now be either expressed through one or another variable:

$$\Delta\phi = -h\Delta\theta$$

where the – sign just shows that a particle behind ($\Delta\theta < 0$) arrives later in the cavity ($\Delta\phi > 0$). For any particle with respect to the reference, one can write:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\dot{\phi}$$

Since in the electron case ($v \approx c$):

$$\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s = \frac{E_s}{\omega_{rs}}\left(\frac{d\omega_r}{dE}\right)_s \approx -\alpha$$

one gets a first order differential equation that relates the relative energy deviation to the time derivative of the phase deviation:

$$\frac{\Delta E}{E_s} = \frac{1}{\omega_{rs}\alpha h}\frac{d\phi}{dt} = \frac{R}{c\alpha h}\dot{\phi}$$

knowing that the synchronous phase is a constant.

4.3 Second energy-phase equation

The energy gained by a particle at each turn is $e\hat{V}\sin\phi$, and when compare to the reference's one it becomes:

$$(\Delta E)_{turn} = e\hat{V}(\sin\phi - \sin\phi_s)$$

The rate of relative energy gain can be approximated to first order:

$$\frac{d(\Delta E)}{dt} \approx (\Delta E)_{turn} f_{rs} = \frac{c}{2\pi R}e\hat{V}(\sin\phi - \sin\phi_s)$$

leading to the second first order differential equation that relates the time derivative of the relative energy deviation to the phase deviation:

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) = \frac{ec\hat{V}}{2\pi R E_s}\left(\sin\phi - \sin\phi_s\right)$$

where it is assumed that E_s is either constant or slow varying.

Note that in the case of an electron synchrotron the synchronous phase will be always finite, to compensate for the energy losses due to synchrotron radiation, even if E_s is kept constant ($\dot{B} = 0$).

4.4 Small amplitude oscillations

Considering small phase deviations from the reference particle one can expand the trigonometric function in the previous differential equation:

$$\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right) \approx \frac{ec\hat{V}\cos\phi_s}{2\pi R E_s}\Delta\phi$$

By differentiating the first energy-phase equation one gets:

$$\ddot{\phi} = \frac{c\alpha h}{R}\frac{d}{dt}\left(\frac{\Delta E}{E_s}\right)$$

keeping in mind that $\frac{d^2}{dt^2}\left(\Delta\phi\right) = \ddot{\phi}$ since ϕ_s is constant. Combining the above two equations leads to a second order equation for the phase motion:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

$$\Omega_s^2 = -\left(\frac{c}{R}\right)^2 \frac{\alpha h e\hat{V}\cos\phi_s}{2\pi E_s}$$

Provided Ω_s^2 is a real and positive number the equation will describe a simple harmonic oscillation. Since α is positive, a stable oscillation needs $\cos\phi_s < 0$, which corresponds to the negative slope of the RF signal ($\frac{\pi}{2} < \phi_s < \pi$), as already mentioned before. Note that $\sin\phi_s > 0$ is the choice previously made for having an acceleration.

Since the first derivative of the phase is proportional to the energy , the energy will oscillate around the nominal one with a $\frac{\pi}{2}$ phase difference as compared to the phase oscillation. In other words, when the phase deviation is at maximum the energy deviation is zero and vice versa.

4.5 Large amplitude oscillations

For large phase deviations the second order equation is non linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

Multiplying by $\dot{\phi}$ and integrating leads to an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi \sin\phi_s\right) = I$$

which, for small phase deviations, reduces to the quadratic form:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta\phi)^2}{2} = I$$

Since $\dot{\phi} \propto \frac{\Delta E}{E_s}$, the invariant represents a closed trajectory in the phase space ($\Delta\phi, \frac{\Delta E}{E_s}$). These trajectories are ellipses for the small amplitude case but they change shape as the amplitude gets larger and need to be numerically calculated. The corresponding curves, also known as Bohm and Foldy diagram [3], with normalized variables ($\frac{\dot{\phi}}{\Omega_s}, \phi$), are shown on Fig. 6 for the particular case $\phi_s = 150^0$.

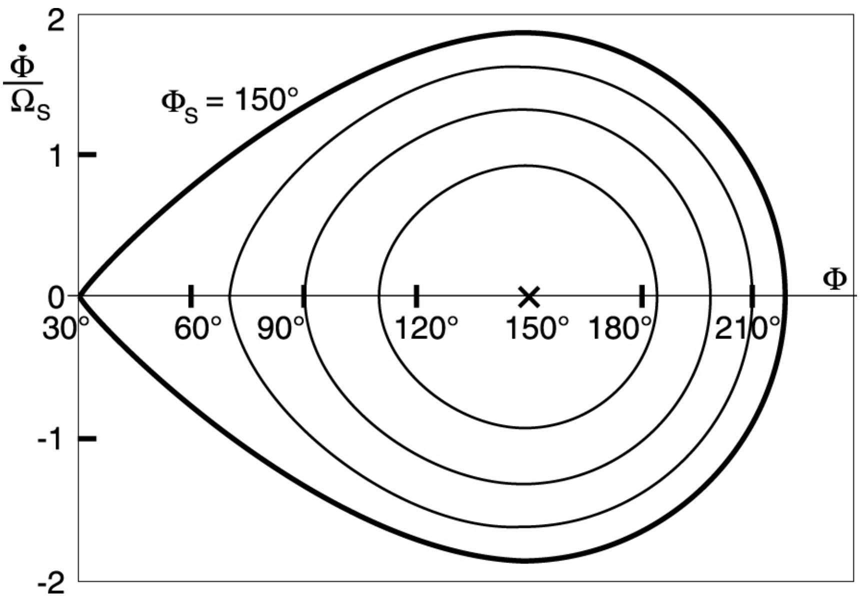

Fig. 6 Bohm & Foldy Diagram

In the second order equation of phase motion it can be seen that when the phase ϕ reaches the value $\pi - \phi_s$, the driving force goes to zero and beyond it changes sign and become a defocusing type force.

Hence $\pi - \phi_s$ represents an extreme amplitude for stable motion and the corresponding curve in the phase space is called the separatrix (the outer curve on Fig. 6). Inside the area described by the separatrix the trajectories are closed curves. Outside they slip in phase showing that particles get out of synchronism (Fig. 7) and in practice will be lost.

The equation of the separatrix is simply obtained by injecting one known point of the curve into the general equation to find the corresponding value of the invariant. This point is ($0, \pi - \phi_s$) leading to:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left\{\cos\left(\pi - \phi_s\right) + \left(\pi - \phi_s\right)\sin\phi_s\right\}$$

where the right hand side represent the maximum value of the invariant for stable oscillations. The second value ϕ_m, where the separatrix crosses the horizontal axis is obtained by solving the trigonometric equation:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos\left(\pi - \phi_s\right) + \left(\pi - \phi_s\right)\sin\phi_s$$

4.6 Energy acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches a maximum when $\ddot{\phi} = 0$, which corresponds to $\phi = \phi_s$. Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{\max}^2 = 2\Omega_s^2\left\{2 + \left(2\phi_s - \pi\right)\tan\phi_s\right\}$$

Using the first energy-phase equation gives the maximum acceptable energy deviation:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \pm\left\{-\frac{e\hat{V}}{\pi\alpha h E_s}G\left(\phi_s\right)\right\}^{\frac{1}{2}}$$

$$G\left(\phi_s\right) = \left[2\cos\phi_s + \left(2\phi_s - \pi\right)\sin\phi_s\right]$$

This “RF acceptance” strongly depends on ϕ_s and plays an important role for the electron capture at injection and for the beam lifetime.

Fig 7 shows, for an electron synchrotron, the drastic reduction of the stable area when ϕ_s approaches the value $\frac{\pi}{2}$. This fact can be understood from the slope of the RF signal which vanishes at $\frac{\pi}{2}$ and no restoring force is then available for stable oscillations.

Note that the vertical coordinate which is shown on Fig. 7 has the following meaning:

$$W = 2\pi\frac{\Delta E}{\omega_{rs}}$$

and is often used in the literature.

The stable area limited by the separatrix is often called “bucket” and is maximum for $\phi_s = \pi$, which corresponds to no acceleration. However, such a case does not happen in an electron synchrotron since particles radiate part of their energy at each turn, while going through the bending magnets, which is automatically compensated by the RF system. The synchronous particle in this case is the one which enters the cavity at an RF phase which provides the necessary energy gain to compensate for the loss per turn. The above treatment remains valid for any other particles.

The height of the bucket increases with increasing RF voltage, which allows the capture of larger energy spread from injected bunches.

The bucket also shows the phase extension of captured particles (bunch) and there will be h such bunches, equally spaced, circulating around the synchrotron. The distance between bunches equal the RF wavelength.

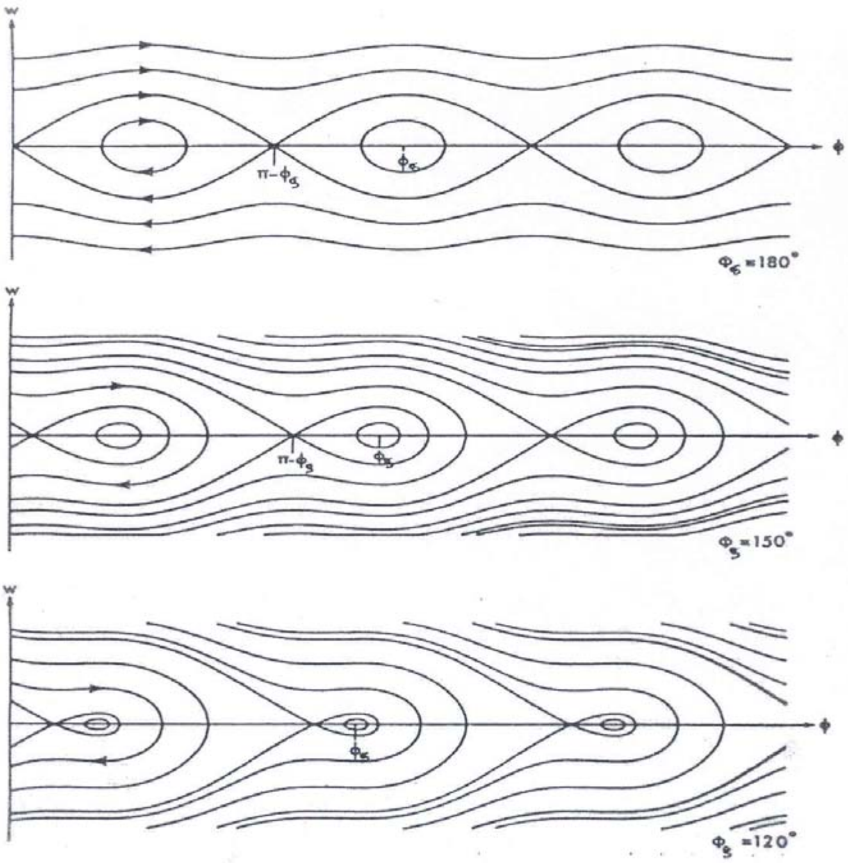

Fig 7 RF acceptance versus synchronous phase angle

5. FROM SYNCHROTRON TO LINAC

In an electron linac there are no bending magnets, hence there are no dispersion effects and $\alpha = 0$. Provided the accelerating cavities are periodically spaced to fulfill the synchronism condition, the longitudinal dynamics treatment remains valid and one ends up with a phase oscillation frequency $\Omega_s = 0$. In other words the distance in phase between particles is frozen, while the energy increases, which is simply due to a frozen velocity $v = c$.

In an ultra-relativistic electron linac it is then important to concentrate the injected bunches on the crest of the RF signal such that all particles will get the same energy.

However at small kinetic energies, where $v \neq c$, which is the case for heavier particles (protons, ions) or electrons generated at the gun voltage, the term $\frac{1}{\gamma^2}$ which appears in the definition of the η parameter, can not be neglected anymore. Then, phase and energy oscillations will exist with $\Omega_s \propto \gamma^{-\frac{3}{2}}$ (neglecting the contribution from $\beta = \frac{v}{c}$). Note that in a linac the distance between accelerating gaps will be mostly equal to an RF wavelength, which is more efficient, and then h=1.

6. ADIABATIC DAMPING

Though there are many physical processes that can damp the longitudinal oscillations, one of them appears to be directly generated by the acceleration process itself. It will happen in the electron synchrotron when ramping from injection energy to operating energy, but not in the ultra-relativistic linac.

As a matter of fact when E_s varies with time, one needs to be more careful when combining the two first order energy-phase equations into one second order equation:

$$\frac{d}{dt}\left(E_s\dot{\phi}\right) = -\Omega_s^2 E_s \Delta\phi$$

$$E_s\ddot{\phi} + \dot{E}_s\dot{\phi} + \Omega_s^2 E_s \Delta\phi = 0$$

which after dividing by E_s :

$$\ddot{\phi} + \frac{\dot{E}_s}{E_s}\dot{\phi} + \Omega_s^2 \Delta\phi = 0$$

shows the existence of a damping coefficient proportional to the energy ramping rate and from the formula giving the angular oscillation frequency one has:

$$\frac{\dot{E}_s}{E_s} = -2\frac{\dot{\Omega}_s}{\Omega_s}$$

This shows that the phase amplitude of the oscillation will be damped during acceleration, and the bunch length will be reduced provided all other parameters remain untouched.

Considering an adiabatic ramping, which means slow enough compared to the longitudinal oscillation period, it can be shown that the area in the phase space remains constant which finally tells that the bunch energy spread will grow during acceleration, but not the relative energy spread.

However this adiabatic damping of the longitudinal oscillations can hardly compete with the radiation damping that is generated by the synchrotron radiation in electron synchrotrons.

REFERENCES

[1] V. Veksler, *Compt. Rend. Acad. Sci. U.S.S.R.,* **43:**444 (1944); **44:**393 (1944); *J. Phys. (U.S.S.R.),* **9:**153 (1945).

[2] E. M. Mc Millan, *Phys. Rev.,* **68 :**143 (1945)

[3] D. Bohm and L. Foldy, *Phys. Rev.,* **70:**249 (1946) and **72:**649 (1947)

BIBLIOGRAPHY

M.Stanley Livingston, John P. Blewett, Particle Accelerators, Mc Graw-Hill, Book Company, INC. N.Y.,1962

H. Bruck, Accelerateurs Circulaires de Particules, Presses Universitaires de France, Paris 1966

E. Persico, E. Ferrari, S.E. Segre, Principles of Particle Accelerators, W.A. Benjamin, INC N.Y. 1968

M. Sands, The Physics of Electron Storage Rings, SLAC Report 121 (1970); also in Proceedings of the International School of Physics (Enrico Fermi), Varenna 1969, B. Touschek, Editor, Academic Press N.Y. 1971

B.W. Montague, RF Acceleration, Proceedings of the first course of the International School of Particle Accelerators, Erice, CERN 77-13, 19 July 1977

J. Le Duff, Longitudinal Beam Dynamics in Circular Accelerators, Proceedings of the CERN Accelerator School (Fifth General Accelerator Physics Course), University of Jyvaskyla, Finland, 7-18 September 1992, CERN 94-01, Vol. 1