TRANSVERSE MOTION

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Abstract
Transverse dynamics lies at the heart of modern synchrotron design. Considerable economies are to be had by intelligent choice of the arrangement of focusing magnets: the lattice. In this lecture we concentrate on the description of the magnetic focusing systems of a synchrotron. The effect of momentum spread on the beam's central orbit (dispersion) and the change in betatron oscillation frequency with momentum (chromaticity) are also analysed. We leave the effect of synchrotron radiation emission and the beam growth and damping to later lectures.

1. DESCRIPTION OF MOTION

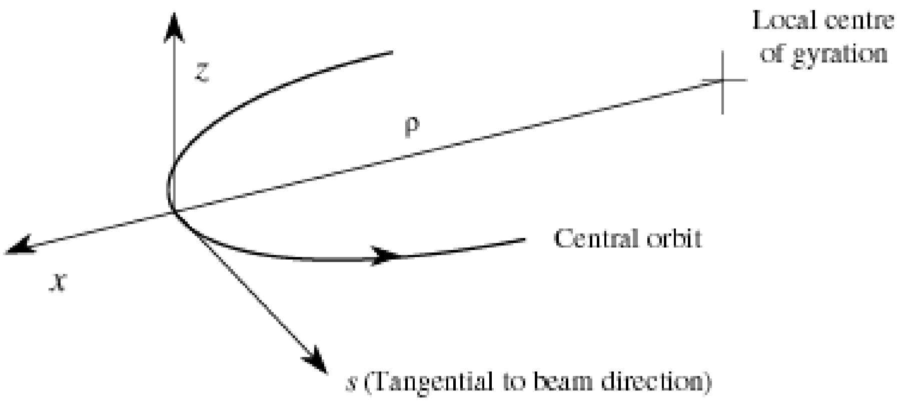

Fig. 1 Charged particle orbit in a magnetic field.

The bending fields of a synchrotron are usually vertically directed, causing the particle to follow a curved path in the horizontal plane (Fig. 1). The force acting on the particle is horizontal and is given by:

$$\mathbf{F} = e\ \mathbf{v} \times \mathbf{B}$$

where: **v** is the velocity of the charged particle in the direction tangential to its path and **B** is the magnetic guide field.

If the guide field is uniform, the ideal motion of the particle is simply a circle of radius of curvature, ρ. but we can also define a local radius of curvature, $\rho(s)$, to describe motion in a non-uniform field. We shall suppose that it is possible to find an orbit or curved path for the particle which closes on itself around the synchrotron which we call the equilibrium orbit. The machine is usually designed with this orbit at the centre of its vacuum chamber.

2. BENDING MAGNETS AND MAGNETIC RIGIDITY

Let us now examine how a particle is deflected in a simple dipole bending magnet. Suppose the particle has a relativistic momentum vector **p** and travels perpendicular to a field **B** which is into the plane of the diagram (Figure 2).

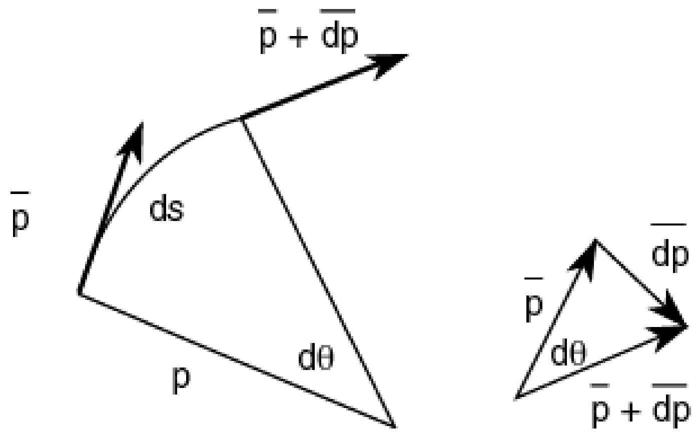

Fig. 2 Vector diagram showing differential changes in momentum for a particle trajectory.

After time, *dt*, it has followed a curved path of radius ρ whose length is *ds* and its new momentum is **p+dp**. Since we may equate the force and rate of change of momentum:

$$\frac{d\mathbf{p}}{dt} = |\mathbf{p}|\frac{d\theta}{dt} = \frac{|\mathbf{p}|}{\rho}\frac{ds}{dt} \ ,$$

On the other hand, if the field and plane of motion are normal, the magnitude of the force may be written:

$$e|\mathbf{v} \times \mathbf{B}| = e|\mathrm{B}|\frac{ds}{dt} \ .$$

Equating the right hand sides of the two expressions above, we find we can define the quantity known as magnetic rigidity:

$$(B\rho) = \frac{p}{e} \ .$$

Strictly we should use the units Newton-second for p and express e in Coulombs to give ($B\rho$) in Tesla.metres. However, in charged particle dynamics we often talk about the 'momentum' pc which has the dimensions of an energy and is expressed in units of GeV. A useful rule of thumb for magnetic rigidity is formula based on these units is:

$$B\rho \ [T.m] = 3.3356 pc \ [GeV]$$

Figure 3 shows the trajectory of a particle in a bending magnet or dipole of length, l. Usually the magnet is placed symmetrically about the arc of the particle's path. One may see from the geometry that:

$$\sin(\theta/2) = \frac{l}{2\rho} = \frac{lB}{2(B\rho)} \ ,$$

and if θ << π/2

$$\theta \approx \frac{lB}{(B\rho)}$$

The bending magnet aperture must be wide enough to contain the sagitta of the beam which is the distance between the apex of the arc and the chord:

$$l = \pm\rho(1-\cos(\theta/2)) \approx \pm\frac{\rho\theta^2}{8} \approx \frac{l\theta}{8}\ .$$

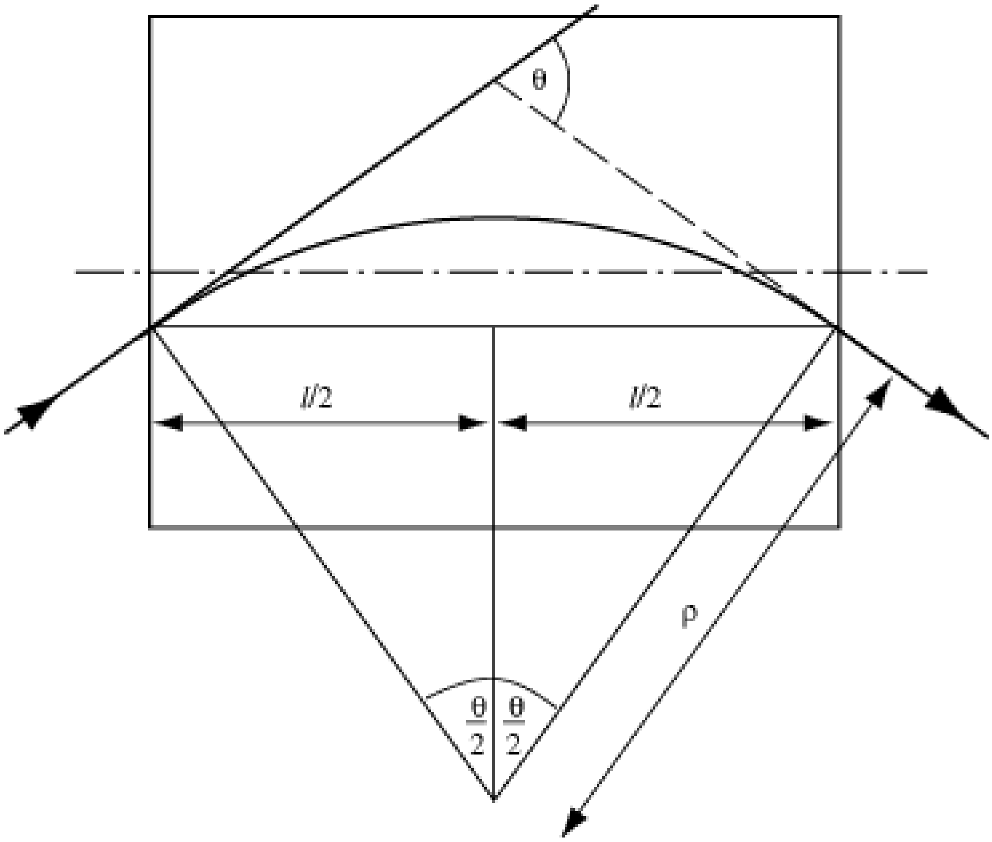

Fig. 3 Geometry of a particle trajectory in a bending magnet of length, *l*

The ends of bending magnets are often parallel but in some machines are designed to be normal to the beam. There is a focusing effect at the end which depends on the angle of these faces. We will come back to this later.

3. FOCUSING

3.1 Displacement and divergence

A beam of particles enters the machine as a bundle of trajectories spread about the ideal orbit. At any instant a particle may be displaced horizontally by x and vertically by z from the ideal position and may also have divergence angles horizontally and vertically:

$$x' = dx/ds \quad \text{and} \quad z' = dz/ds\ .$$

and we see how divergence is defined in the left half of Figure 6.

Such mis-steering would cause particles to leave the vacuum pipe were it not for the carefully shaped field which restores them back towards the beam centre so that they oscillate about the ideal orbit. The design of the restoring fields determines the transverse excursions of the beam and the size of the cross section of the magnets and is therefore of crucial importance to the cost of a project.

3.2 Weak Focussing

The first generation of synchrotrons upon the curved shape of the field at the open side of their "C" shaped magnets to provide vertical focusing forces to restore particles towards the mid-plane of the synchrotron. The curvature of the lines ensured that the force on circulating particles had a vertical component – deflecting downward above the mid-plane and upward below. (Figure 4)

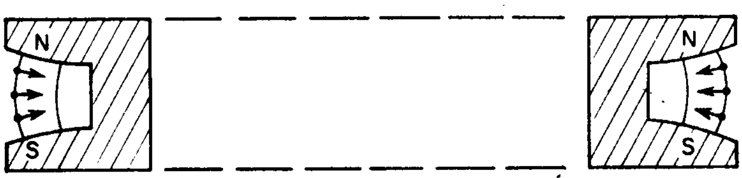

Fig. 4. Cross section of a constant gradient synchrotron ring illustrating focusing

The curvature was enhanced by tapering the pole gap towards the outside to produce a constant field gradient both across the aperture and around the machine. Horizontal focusing was provided by the effect of an imbalance between the force due to the guide field and the smaller radial acceleration of a particle in an outer and larger radius orbit. The acceleration towards the reference orbit, radius ρ, is just:

$$\frac{1}{\rho^2}x$$

Unfortunately the field gradient, though focusing in the vertical plane produces a defocusing effect in the horizontal direction and could easily swamp this weak horizontal focusing effect. Thus the focusing forces had to be kept rather weak, which led to large amplitudes of oscillation of particles about the centre line of the machine. The cross section of the vacuum chamber and the magnet gap had to be often 20 cm by 100 cm.

Nevertheless this kind of machine has the virtue that it is easy to understand the motion of particles as they oscillate about the axis in a uniform focussing environment.

3.3 The gutter analogy

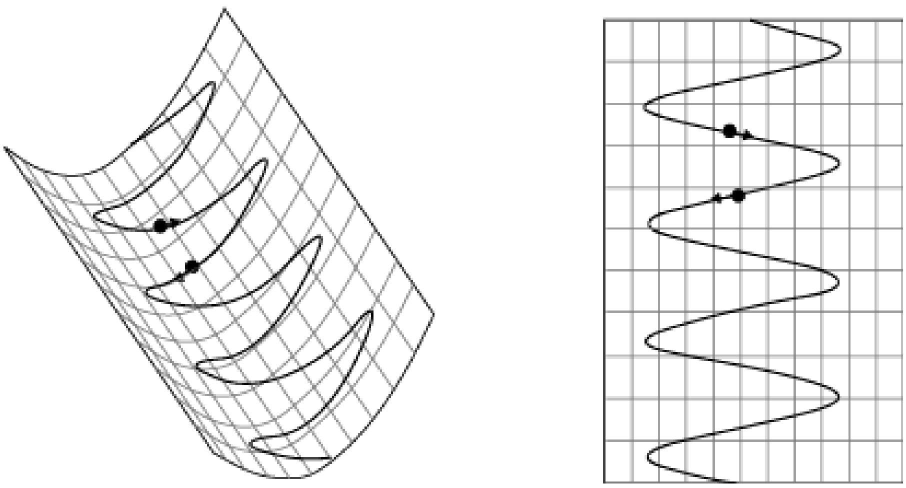

Fig. 5 Two views of a sphere rolling down a gutter as it is focused by the walls.

It is important to start with a tangible concept of focusing and so we consider an analogy to the focusing system of the weak focusing synchrotron. We suppose that a particle oscillates in this focusing system like a small sphere rolling down a slightly inclined gutter with constant speed. Figure 5 shows two views of this motion and from the right hand view we recognise the motion as a sine wave. Note too that the sphere makes four complete oscillations along the gutter. In the language of accelerators we shall learn to characterise this aspect of its motion by the wave number, $Q = 4$.

Now let us extend this analogy by bending the gutter into a circle rather like the brim of a hat. Suppose we provide the necessary instrumentation to measure the displacement of the sphere from the centre of the gutter each time it passes a given mark on the brim and we also have a means to measure its transverse velocity. With the aid of a computer, we might convert this information into the divergence angle shown in Fig. 6:

$$x' = \frac{dx}{ds} = \frac{v_{\perp}}{v_{\parallel}} \quad .$$

Suppose also that we make the brim of a hat out of a slightly different length of gutter than is shown so that Q is not an integer. We can plot a point (x, x') each time the sphere makes a turn of the brim in what we call a ‘phase space diagram’ of transverse motion. The sphere has a large transverse velocity as it crosses the axis of the gutter and has almost zero transverse velocity as it reaches its maximum displacement.

The locus of these ‘observations’ will be an ellipse (Fig. 6) and the phase of the motion (the angle subtended at the origin) will advance by Q revolutions each time the particle returns to the detector. Of course, only the fractional part of Q may be deduced from our observations since we are blind to what happens round the rest of the hat’s brim – a situation we shall find is common in the real life of accelerators.

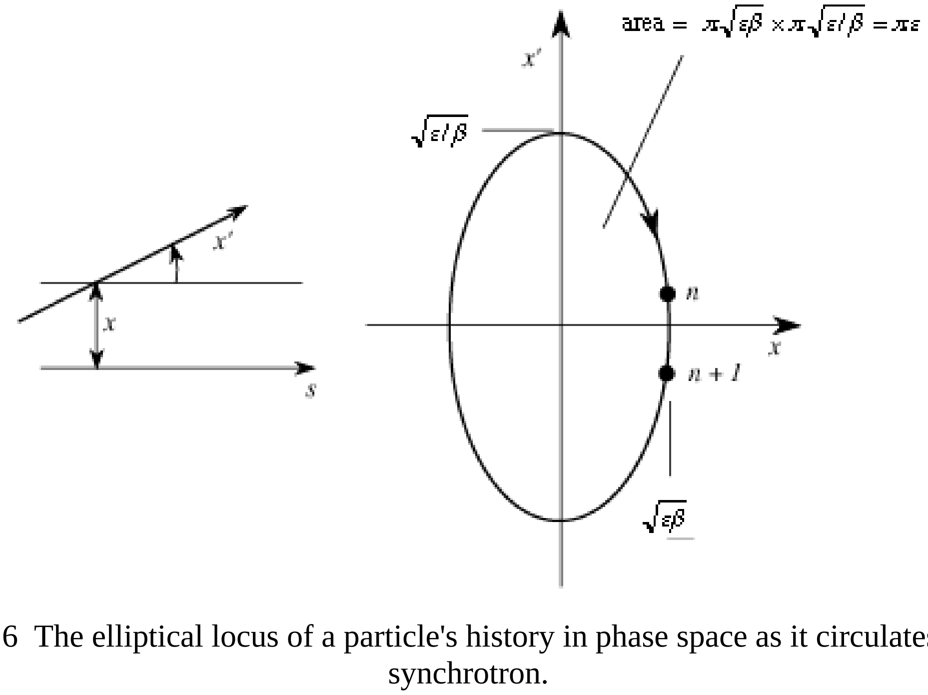

Fig. 6 The elliptical locus of a particle's history in phase space as it circulates in a synchrotron.

In order to establish concepts which will take us from the gutter analogy to real synchrotrons we have to define some of the transverse beam dynamical quantities more rigorously. The area of the ellipse is a measure of how much the particle departs from the ideal trajectory which in the diagram is represented by the origin:

$$\text{Area} = \pi\varepsilon \text{ [mm.rad]} .$$

In accelerator notation we use ε, the product of the semi-axes of the ellipse as a measure of the area called the emittance. The emittance is usually quoted in units of π mm.mradians.

The maximum excursion in displacement, the major axis, of the ellipse defined:

$$\hat{x} = \sqrt{\varepsilon\beta}$$

and hence

$$\hat{x}' = \sqrt{\varepsilon/\beta}$$

Note that the aspect ratio of the ellipse is just β. We will return to these quantities when we have studied more about the alternating gradient focusing systems in which the steepness of the sides of the gutter varies as we go around the ring and can even change sign. We shall see that in these circumstances β varies around the ring and becomes an envelope with in which the oscillating particles are constrained.

3.4 Alternating gradient focusing

The discovery of alternating gradient focusing (Courant *et al.* 1958) was a major break-through in the design of synchrotrons which allowed designers to use much stronger focusing forces in both vertical and horizontal planes. It happened almost by accident as E. Courant, then working on the effect that the modulation of the “constant” gradient would produce when some of the magnets of weak focusing Cosmotron were reversed in gradient. To his surprise he found the effect to be strikingly beneficial. Taking this gradient reversal to the extreme allowed much stronger focusing systems to be used with considerable savings in the space needed for the beam cross section.

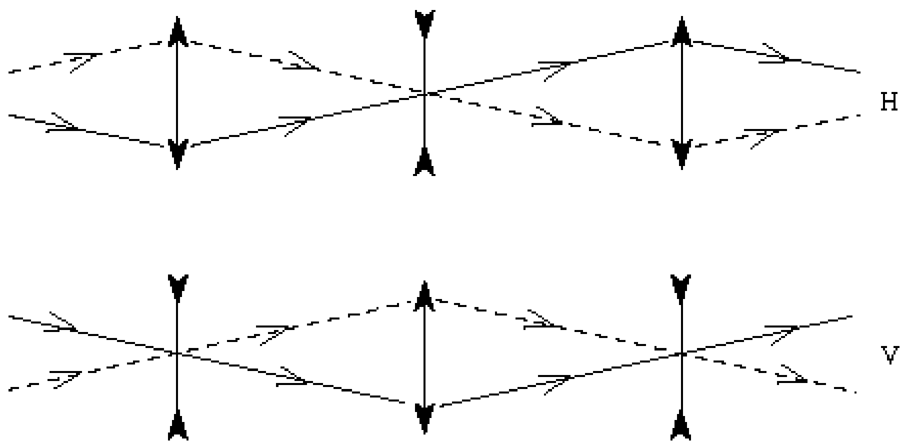

Fig. 7 Optical analogy in which an alternating pattern of lenses.

The principle of this alternating gradient focusing is illustrated in Fig. 7 where we see an optical system in which each lens is concave in one plane while convex in the other. It is possible, even with lenses of equal strength, to find a ray which is always on axis at the D lenses in the horizontal plane and therefore only sees the F lenses. The spacing of the lenses has to be twice their focal length, *f*. If the ray is also central in the lenses which are vertically defocusing, the same condition will apply simultaneously in the vertical plane. At least one particular particle or trajectory corresponding to this ray will be contained indefinitely.The alternating gradient idea will work even when the rays in the D lenses do not pass dead centre and the lenses are not spaced by exactly 2*f*. In fact it is sufficient for the particle trajectories to *tend* to be closer to the axis in D lenses than in F lenses as shown in Fig. 8.

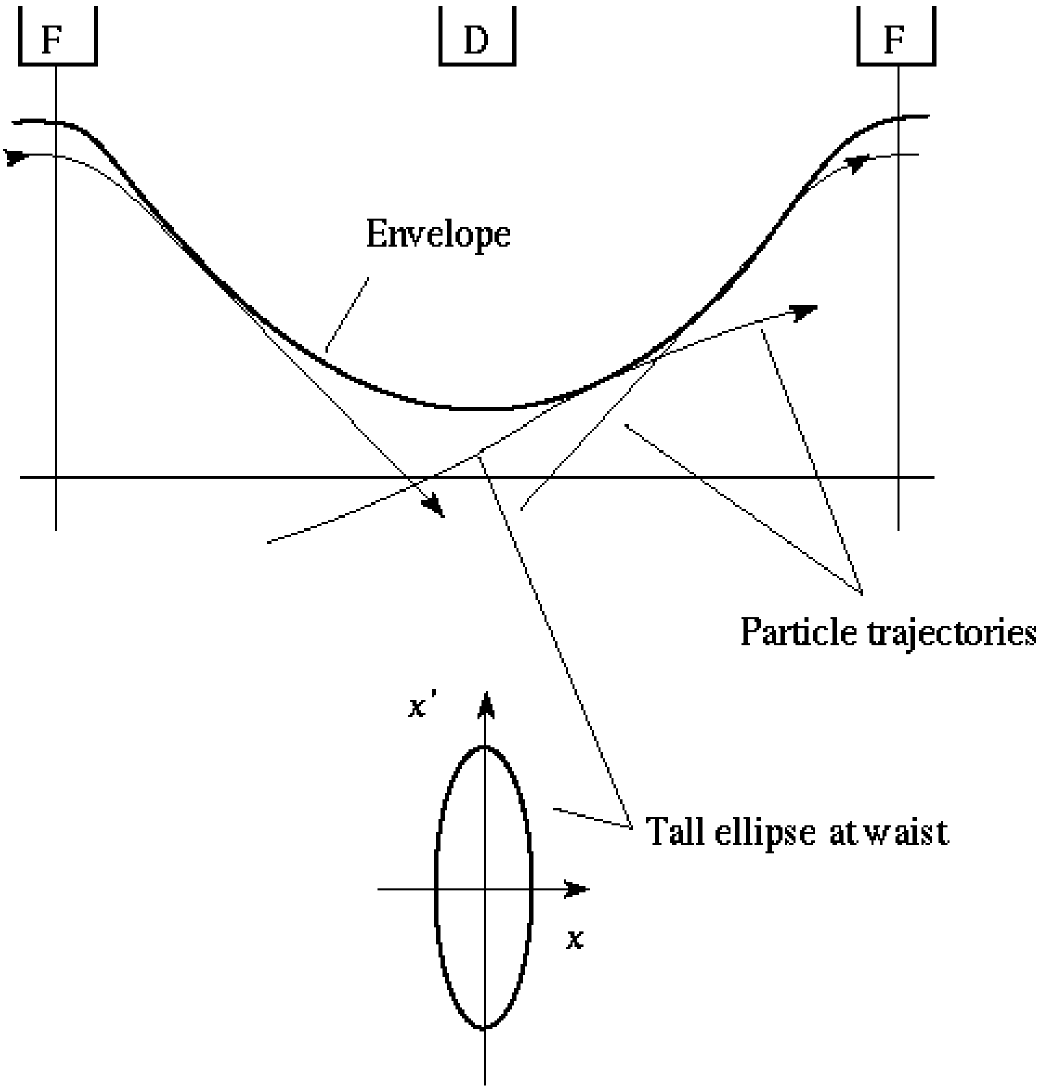

Fig. 8 The paths of particles within a FODO lattice are within the envelope of betatron motion and, like the rays of Fig. 7, are always closer in the D quadrupoles.

By suitable choice of strength and spacing of the lenses the envelope function, which was constant for the gutter which we used as an analogy for representing a weak focussing machine, now varies around the ring and the parameter β is a function: $\beta(s)$. Symmetry tells us it will be periodic and by suitable choice of lens strength we can ensure that it is large at all F quadrupoles and small at all D's. The picture will be in the vertical plane but displaced by the distance between quadrupoles. Particles oscillating within this envelope will always tend to be further off axis in F quadrupoles than in D quadrupoles and there will therefore be a net focusing action. We have already seen that β is the aspect ratio of the phase space ellipse. At F quadrupoles the ellipse will be squat and at D quadrupoles it will be tall. We shall go on to define this envelope or betatron amplitude more rigorously and establish how to calculate it for a given lattice of focusing magnets. See also Schmüser (1987) and Rossbach *et al.* (1992).

3.5 Quadrupole magnets

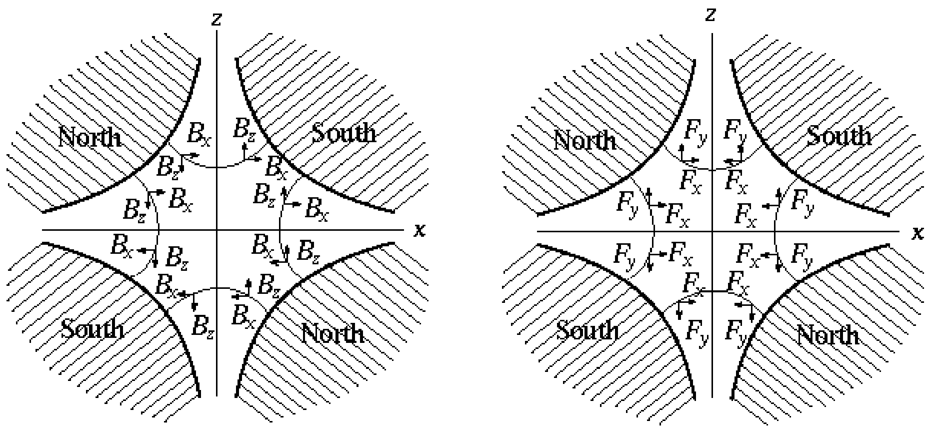

Fig. 9 Components of field and force in a magnetic quadrupole. Positive ions approach the reader on paths parallel to the *s* axis. (Livingood 1961).

The lenses elements in a modern alternating gradient synchrotron are quadrupole magnets. The poles are truncated rectangular hyperbolae and alternate in polarity. Figure 9 shows a particle's view of the fields and forces in the aperture of a quadrupole as it passes through normal to the plane of the paper. The field shape is such that it is zero on the axis of the device but its strength rises linearly with distance from the axis. This can be seen from a superficial examination of Fig. 9 if we remember that the product of field and length of a field line joining the poles is a constant. Symmetry tells us that the field is vertical in the median plane (and purely horizontal in the vertical plane of asymmetry). The field must be downwards on the left of the axis if it is upwards on the right.

This last observation ensures that the horizontal focusing force, $-evB_z$, has an inward direction on both sides and, like the restoring force of a spring, rises linearly with displacement, x. The strength of the quadrupole is characterised by its gradient dB_z/dx normalised with respect to magnetic rigidity:

$$k = \frac{1}{(B\rho)}\frac{dB_z}{dx}.$$

The angular deflection given to a particle passing through a short quadrupole of length, ℓ and strength, k, at a displacement x is therefore:

$$\Delta x' = \frac{Bl}{(B\rho)} = kl$$

(This is just another application for the formula we derived for deflection in a bending magnet.) Compare this with a converging lens in optics:

$$\Delta x' = -x / f$$

and we see that the focal length of a horizontally focusing quadrupole is

$$f = -1/(kl)$$

The particular quadrupole shown in Fig. 9 would focus positive particles coming out of the paper or negative particles going into the paper in the horizontal plane. A closer examination reveals that such a quadrupole deflects particles with a vertical displacement away from the axis – vertical displacements are defocused. This can be seen this if Fig. 9 is rotated through 90 degrees. It was this feature that discouraged the use of quadrupoles for focussing until the discovery of alternating gradient focussing.

4. BETATRON ENVELOPES

During the design phase of an accelerator project a considerable amount of calculation and discussion centres around the choice of the transverse focusing system. The pattern of bending and focusing magnets, called the lattice, has a strong influence on the aperture of the bending and focusing magnets which are usually the most expensive single system in the accelerator and which, in turn can have an important effect on the design of almost all other systems in the synchrotron.

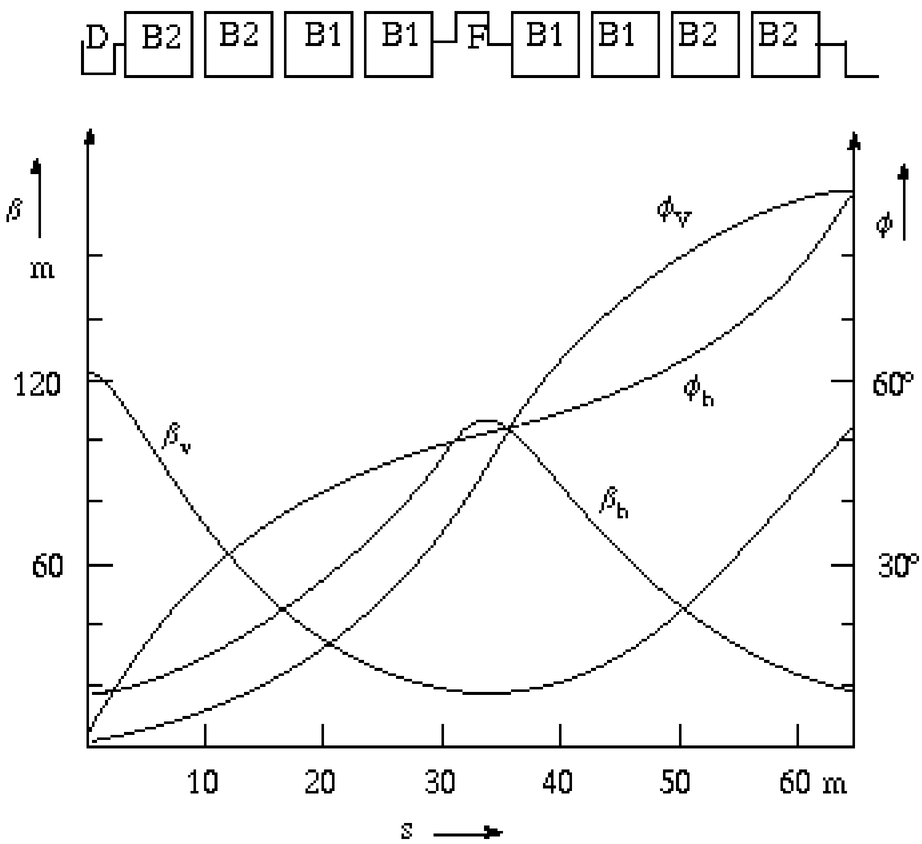

Fig. 10 One cell of the CERN SPS representing 1/108 of the circumference. The pattern of dipole (B) magnets and quadrupole (F and D) lenses is shown above.

A modern synchrotron consists of pure bending magnets and quadrupole magnets or lenses which provide focusing. These are interspersed among the bending magnets of the ring in a pattern called the lattice. In Fig. 10 we see an example of such a magnet pattern which is one cell, or about 1% of the circumference, of the 400 GeV SPS at CERN. Although the SPS is now considered a rather old fashioned machine, its simplicity makes it an excellent example for teaching purposes. For obvious reasons this focusing structure is called FODO and in this pattern half of the quadrupoles focus, while the other half, defocus the beam. The envelope of these oscillations follows a function $\beta(s)$ which has waists near each defocusing magnet and has a maximum at the centres of F quadrupoles. Since F quadrupoles in the horizontal plane are D quadrupoles vertically, and vice versa, the two functions $\beta_h(s)$ and $\beta_v(s)$ are one half-cell out of register in the two transverse planes. The function β has the dimensions of length but the units bear no relation at this stage to physical beam size. The reader should be clear that particles do not follow the $\beta(s)$ curves but oscillate within them in a form of modified sinusoidal motion whose phase advance is described by $\phi(s)$. The phase change per cell in the example shown, is close to $\pi/2$ but the rate of phase advance is modulated throughout the cell.

5. THE EQUATION OF MOTION

In the last section we derived an expression for the change in divergence of a particle passing through the quadrupole. The strength of the quadrupole is characterised by its gradient dB_z / dx, normalised with respect to magnetic rigidity:

$$k = \frac{1}{(B\rho)} \frac{dB_z}{dx}$$

If k is negative, the quadrupole is horizontally focusing and vertically defocusing. We first look at the vertical plane. The angular deflection given to a particle passing through a short quadrupole of length ds and strength k at a displacement z is therefore:

$$dz' = -kzds$$

We can deduce from this a differential equation for the motion

$$z'' + k(s)z = 0 \ .$$

This is called Hill's Equation, a second order linear equation with a periodic coefficient, $k(s)$ which describes the distribution of focusing strength around the ring. The above form of Hill's equation applies to motion in the vertical plane while in the horizontal plane:

$$x'' + \left[\frac{1}{\rho(s)^2} - k(s) \right] x = 0$$

Here the sign before $k(s)$ is reversed so that the quadrupole defocuses. We include an extra focusing term due to the curvature of the orbit which we found was the only horizontal focussing mechanism in weak focusing synchrotrons. We include this term since it can be significant in small radius rings.

6. SOLUTION OF HILL'S EQUATION

Hill's equation is reminiscent of simple harmonic motion but has a restoring constant *k(s)* which varies around the accelerator. In order to arrive at a solution we must first assume that *k(s)* is periodic on the scale of one turn of the ring. The period can also be a smaller unit, the cell, from which the ring is built. The solution, like the differential equation itself, is reminiscent of simple harmonic motion:

$$x = \sqrt{\beta(s)\varepsilon} \cos[\phi(s) + \phi_0]$$

In simple harmonic motion the amplitude is a constant but here we see that in addition to $\sqrt{\varepsilon}$, which can be considered an arbitrary constant, there is another amplitude component, the function $\sqrt{\beta(s)}$. Another difference from harmonic motion is that phase, $\phi(s)$, does not advance linearly with time and with distance, s , around the ring. Both these functions of s must have the same periodicity as the lattice and they are linked by the condition:

$$\varphi' = 1/\beta \qquad \text{or} \qquad \varphi = \int ds/\beta \;.$$

We shall later show that this condition is necessary if Hill's Equation is to be satisfied but for the moment let us just accept it. By simple differentiation we can then find

$$x = \sqrt{\beta(s)\varepsilon}\,\cos\left[\phi(s)+\phi_0\right]$$
$$x' = -\sqrt{\varepsilon/\beta(s)}\,\sin\left[\phi(s)+\phi_0\right] + \left[\beta'(s)/2\right]\sqrt{\varepsilon/\beta(s)}\,\cos\left[\phi(s)+\phi_0\right]$$

We look at this function at points of symmetry, mid-way through an F or D quadrupole where $\beta'(s)$ is zero and hence where the second term in the divergence equation is zero. We then find that the locus of a particles motion returning to this observation point is an ellipse with semi-axis in the x - direction $\sqrt{\beta\varepsilon}$, and in the x' - direction $\sqrt{\varepsilon/\beta(s)}$ (Fig. 6). Its area is $\pi\varepsilon$, where ε is an invariant of the motion for a single particle or the emittance of a beam of many particles.

6.1 Smooth approximation

Older accelerators, constant gradient machines simple harmonic motion is a very close approximation to reality. In the vertical plane the particles obey the differential equation

$$\frac{d^2z}{ds^2} + kx = 0$$

If we think of this as analogous to a travelling wave

$$\frac{d^2z}{ds^2} + \left(\frac{2\pi}{\lambda}\right)^2 z = 0$$

The solution of such an equation is a wave whose length is λ namely:

$$z = z_0 \sin(2\pi/\lambda)s = z_0 \sin\phi \;.$$

We can see that the derivative of phase is:

$$\phi' = (2\pi/\lambda)$$

but earlier we mentioned that to find a solution to Hill’s equation the phase derivative must be $1/\beta$ to be equal to this derivative. We can therefore argue that β is a local wavelength (multiplied by 2π) of the oscillation. This may help us to understand the way in which β and ϕ vary in the cells of a FODO lattice.

6.2 Q-value

Let us now look at the definition of a quantity, Q, the betatron wave number. Suppose, we again consider a constant gradient machine. The particle with the largest amplitude in the beam, $\sqrt{\beta\varepsilon}$, starts off with phase ϕ_0, and after one turn its phase has increased by

$$\Delta\varphi = \oint ds/\beta = 2\pi R/\beta \ .$$

It has been round the ellipse $\Delta\varphi / 2\pi$ times. The number of such betatron oscillations per turn to be Q, the betatron wave number. Using the above relation we see that *for a constant gradient machine*

$$Q = \frac{\Delta\varphi}{2\pi} = \frac{R}{\beta}$$

or

$$\beta = R/Q$$

This is approximately true for alternating gradient machines as well, and is often used in juggling machine parameters at the design stage because the choice of Q determines β and hence beam size.

It is very important that Q is not be a simple integer or vulgar fraction, otherwise, over one or more paths around the ellipse, the particle will repeat its path in the machine and see the same field imperfections. These will then build up into a resonant growth. The condition to be avoided is $nQ = p$ (where n and p are integers). This can be done by tuning the restoring gradients of the quadrupoles.

7. MATRIX DESCRIPTION

From now on we deal only with alternating gradient (AG) machines in which the ring is a repetitive pattern of focusing fields, the lattice. Each lattice element may be expressed by a matrix.

Whole sections of the ring which transport the beam from place to place may also be represented as a matrix. Any linear differential equation, like Hill's Equation, has solutions which can be traced from one point, s_1, to another, s_2, by a 2 × 2 matrix: the transport matrix:

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = M_{21} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} .$$

There are two ways of thinking of these transport matrices. First of all there is the matrix fro one turn of the accelerator ring (or for one period that repeats) which we call the Twiss matrix. We shall show that each term in M_{21} is a simple function of $\beta(s)$ and $\phi(s)$. As an alternative description of the ring wee can also write down rather simple forms for the matrices of each quadrupole, bending magnet and drift length which we can write down as simple numbers depending on the length and strength of each component. The functions $\beta(s)$ and $\phi(s)$ may be calculated by comparing the numerical result of multiplying the individual matrices for quadrupoles and drift lengths in the ring with what we know must be the general form of each element. But we are moving too fast. Our first job is to derive the general form of a periodic transport matrix.

7.1 The Twiss matrix

We shall simplify the notation by dropping the explicit dependence of β and φ on s from the expressions - we will just have to remember that they vary with s. We also introduce a new quantity:

$$w = \sqrt{\beta}$$

In this new notation we can write the solution of the Hill Equation:

$$y = \varepsilon^{1/2} w \;\cos\left(\varphi + \phi_0\right) \;.$$

By taking the derivative and substituting $\varphi' = 1/\beta = 1/w^2$ we have:

$$y' = \varepsilon^{1/2} w' \cos\left(\varphi + \phi_0\right) - \frac{\varepsilon^{1/2}}{w} \sin\left(\varphi + \phi_0\right) .$$

The next step is to substitute these explicit expressions for y and y' in both sides of the matrix equation. We do this first with the initial condition on the right hand side, $\varphi_0 = 0$, This is the so-called 'cosine' solution. Writing the matrix equation in "long hand" we find we have two equations – for y after the matrix operation and the other for y'. We can do this again starting from the 'sine' solution with $\varphi_0 = \pi/2$. This is exactly equivalent to tracing the paraxial and central rays through an optical lens system. We write $\phi_2 - \phi_1 = \phi$ for each case. Thus we obtain two more equations making in all four simultaneous equations which can be solved to find the four transport matrix elements, a, b, c, d in terms of w, w', and φ. The result, for anyone with the patience to pursue this process, is the most general form of the transport matrix which will take you from any point in the ring to another:

$$M_{12} = \begin{pmatrix} \dfrac{w_2}{w_1} \cos\varphi - w_2 w_1' \sin\varphi & w_1 w_2 \sin\varphi \\ -\dfrac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2} \sin\varphi - \left(\dfrac{w_1'}{w_2} - \dfrac{w_2}{w_1}\right) \cos\varphi & \dfrac{w_1}{w_2} \cos\varphi + w_1 w_2' \sin\varphi \end{pmatrix} .$$

At first glance this seems to have complicated the issue but we still have some constraints to apply. The first of these is to restrict M to be between two identical points in successive turns or cells of a periodic structure. This forces $w_2 = w_1$ $w_2' = w_1'$ and φ to become μ, the phase advance per cell. Then:

$$M = \begin{pmatrix} \cos\mu - ww' \sin\mu & w^2 \sin\mu \\ -\dfrac{1 + w^2 w'^2}{w^2} \sin\mu & \cos\mu + ww' \sin\mu \end{pmatrix}$$

The next simplification is to invent some new functions of β or

$$\left.\begin{aligned} \alpha &= -ww' = -\frac{\beta'}{2} \\ \beta &= w^2 \\ \gamma &= \frac{1 + (ww')^2}{w^2} = \frac{1 + \alpha^2}{\beta} \end{aligned}\right\}$$

These functions (which are nothing to do with special relativity!) are a complete and compact description of the dynamics. The matrix now becomes even simpler:

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu \ , & \beta \sin\mu \\ -\gamma \ \sin\mu, & \cos\mu - \alpha \ \ \sin\mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

This is the Twiss matrix. It is the basic matrix for periodic lattices and should be memorized.

7.2 Transport matrices for the components of a period

Now let us explore the alternative matrix approach – that of multiplying a large number of component matrices together. The simplest of these component matrices is the one for an empty space or drift length. Figure 11 (a) shows the analogy between a particle trajectory and a diverging ray in optics. The angle of the ray and the divergence of the trajectory are related:

$$\theta = \tan^{-1}(x') .$$

The effect of a drift length in phase space is a simple horizontal translation from (x, x') to $(x + \ell x', x')$ and can therefore be written as a matrix:

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} .$$

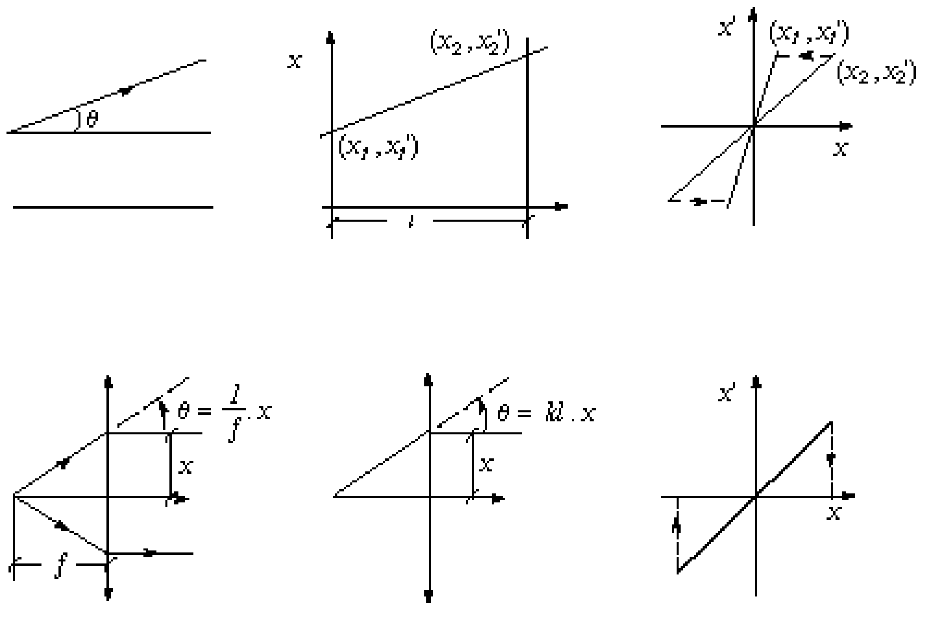

Fig. 11 The effect of (a) a drift length and (b) a thin quadrupole seen in real space as an optical ray and a particle trajectory then plotted in phase space and expressed as a transport matrix.

The next simplest case is that of a thin quadrupole magnet of infinitely small length but finite integrated gradient:

$$lk = \frac{1}{(B\rho)} \frac{\partial B_z}{\partial x}$$

Figure 11(b) illustrates the optical analogy of a thin quadrupole with a converging lens. A ray, diverging from the focal point arrives at the lens at a displacement, x, and is turned parallel by a deflection:

$$\theta \approx \frac{1}{f} \cdot x \ .$$

In fact this deflection will be the same for any ray at displacement x irrespective of its divergence. This behaviour can be expressed by a simple matrix, the thin lens matrix:

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} .$$

A quadrupole has a a similar property. A particle arriving at a displacement x obeys Hill's equation

$$x'' + kx = 0$$

hence the small deflection, θ, is just:

$$\Delta x' == -klx$$

We see that $lk = 1/f$ and is the power of the lens and that the matrix, for a thin lens, can be written:

$$\begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix}$$

The lenses of a synchrotron are not normally short compared to their focal length. One must therefore use the matrices for a long quadrupole when one comes to compute the final machine:

$$M_F = \begin{pmatrix} \cos l\sqrt{k} & \frac{1}{\sqrt{k}} \sin l\sqrt{k} \\ -\sqrt{k} \sin l\sqrt{k} & \cos l\sqrt{k} \end{pmatrix}$$

and

$$M_D = \begin{pmatrix} \cosh l\sqrt{k} & \frac{1}{\sqrt{k}} \sinh l\sqrt{k} \\ \sqrt{k} \sinh l\sqrt{k} & \cosh l\sqrt{k} \end{pmatrix}$$

These correspond to the solutions of Hill's equations in F and D cases:

$$z = z_0 \cos l\sqrt{k} + \frac{z_0'}{\sqrt{k}} \sin l\sqrt{k}$$

$$x = x_0 \cosh l\sqrt{k} + \frac{x_0'}{\sqrt{k}} \sinh l\sqrt{k}$$

In this model we have ignored the bending that takes place in dipole magnets and these are thought of as drift lengths in a first approximation. However, an exact calculation must include the focusing effect of their ends. A pure sector magnet, whose ends are normal to the beam will give more deflection to a ray which passes at a displacement x away from the centre of curvature (Fig. 12). This particle will have a longer trajectory in the magnet. The effect is exactly like a lens which focuses horizontally but not vertically.

The matrices for a sector magnet are:

$$M_H = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -(1/\rho)\sin\theta & \cos\theta \end{pmatrix}$$

$$M_V = \begin{pmatrix} 1 & \rho\theta \\ 0 & 1 \end{pmatrix} .$$

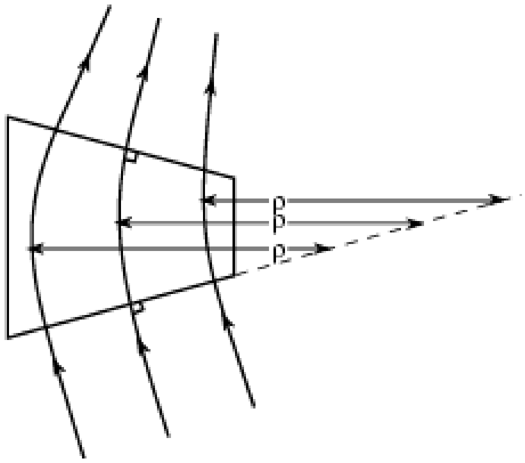

Fig. 12 The focusing effect of trajectory length in a pure sector dipole magnet.

Most bending magnets are not sector magnets but have end faces which are parallel. It is easier to stack laminations this way than on a curve. The entry and exit angles are therefore, $\theta/2$, and the horizontal focusing effect is reduced but there is an additional focusing effect for a particle whose trajectory is displaced vertically. In the computer model one may convert a pure sector magnet into a parallel faced magnet by simply adding two thin lenses at each face. They are horizontally defocusing and vertically focusing and their strength is:

$$lk = -\frac{\tan(\theta/2)}{\rho}$$

There are further effects from the azimuthal shape of end fields which can be included analytically.

7.3 Comparing the two matrices to obtain the Twiss parameters

Having two independent ways of describing the matrix for a turn the time has come to compare them. We must first choose the starting point, the location, *s*, where we wish to know β and the other Twiss parameters. By starting at that point in the ring and multiplying the element matrices together for one turn we are able to find *a*, *b*, *c*, *d* numerically for that point. Examining the Twiss matrix and comparing with the numbers *a*, *b*, *c*, *d* we can write

$$\cos\mu = \frac{Tr\mathbf{M}}{2} = \frac{a+d}{2}$$

$$\beta = b/\sin\mu > 0$$

$$\alpha = \frac{a-b}{2\sin\mu}$$

$$\gamma = -c/\sin\mu$$

Solving these four equations to will the Twiss parameters and in particular β and μ. If the machine has a natural symmetry in which there are a number of identical periods, it is sufficient to do the multiplication up to the corresponding point in the next period. The values of α, β, and γ would be the same if we went on for the whole ring. Then by choosing different starting points we can trace β(s) and α(s) throughout a period.

Fortunately we have computers to help when we come to multiply these elements together to form the matrix for a ring or a period of the lattice (Servranckx *et al.* 1984; Garren *et al.* 1985; Schachinger and Talman 1985). A lattice program such as MAD (Iselin and Grote 1991) does all the matrix multiplication to obtain (a, b, c, d) from each specified point, s, and back again. It prints out β and φ and other lattice variables in each plane, and we can plot the result to find the beam envelope around the machine. This is the way machines are designed. Lengths, gradients, and numbers of FODO normal periods are varied to match the desired beam sizes and *Q* values.

It is nevertheless an interesting exercise for the student to multiply the five matrices of the FODO lattice together starting at the mid plane of the F lens and show that for lenses of focal length , *f*¸ spaced by a distance, *L* :

$$\left.\begin{aligned}\cos\mu &= 1 - L^2/2f^2 \\ \sin(\mu/2) &= l/2f\end{aligned}\right\}$$

$$\left.\beta = 2L\frac{[1 \mp \sin(\mu/2)]}{\sin\mu}\right\} .$$

$$\alpha_{x,z} = 0$$

8. LIOUVILLE'S THEOREM

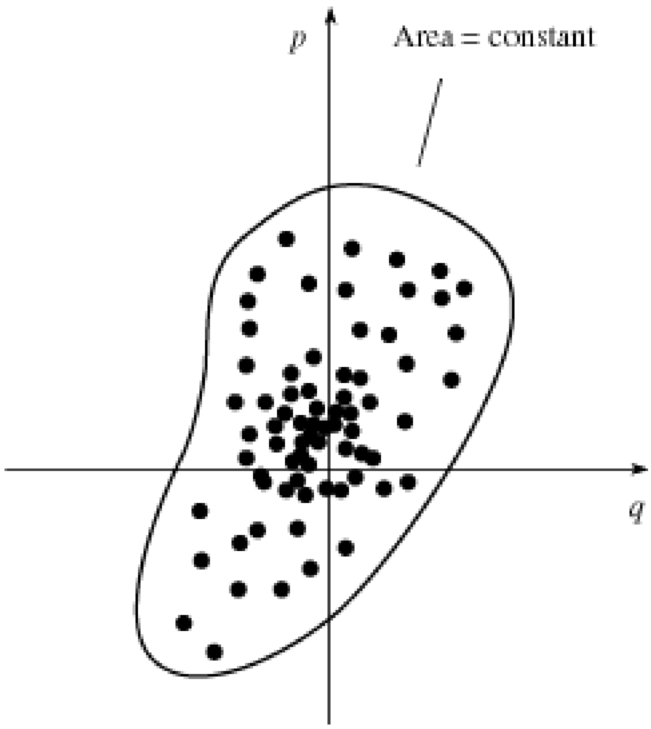

Fig. 13 Liouville's theorem applies to this ellipse.

No introduction to transverse dynamics can be complete without a mention of a conservation law known as Liouville's theorem. To understand this law we must think of a beam of particles as a cloud of points within a closed contour in a transverse phase space diagram (Fig. 13). Liouville's theorem tells us that this area within the contour, $A = \int pdq$, is conserved. This is of course the general statement and we can use divergence and displacement for *p* and *q* and take the contour of Fig. 6 – the locus of a particle's motion at a place where the β function is at a maximum or minimum and where the major and minor axes of the upright ellipse are $\sqrt{\varepsilon\beta}$ and $\sqrt{\varepsilon / \beta}$. We could think of this ellipse as the locus of the particle in the beam which has the maximum amplitude of betatron motion and call its area, $\pi\varepsilon$, the emittance.

A point that sometimes causes confusion is that the numerical parameter we quote for an emittance is the product of the two semi axes of the limiting ellipse (the units are commonly mm.milliradians). It is this numerical value that we insert under the square root above to obtain beam dimensions. However the *area* inside the ellipse (*A* above) is π times this numerical value. In order to try to remember this, we quote the product of the semi-axes but write the units as "π mm.milliradians".

According to Liouville the emittance area, however we express it, will be conserved as the beam circulates in a synchrotron or as it passes down a transport line whatever magnetic focusing or bending operation we do on the beam. Even though the ellipse may appear to have many shapes around the accelerator its phase space area will not change (Fig. 14). At a narrow waist, near a D quadrupole (a) in Fig. 4.2, its divergence will be large, while in an F quadrupole (d) where the betatron function is maximum, its divergence will be small. The beam is also seen at a broad waist or maximum in the beta function and a place where the beam is diverging.

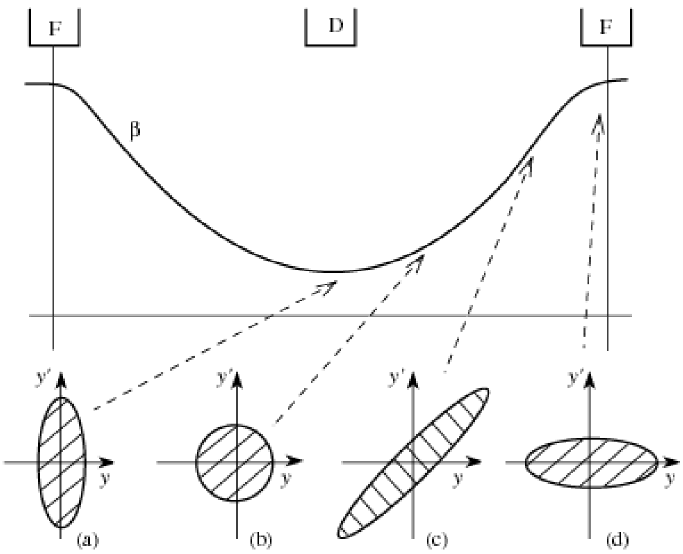

Fig. 14 How the phase space ellipse changes .in a FODO period

However, beware, Liouville only applies to our kind of phase space at one energy. In a proton machine it appears to shrink during acceleration as *1/*(*pc*) so that the beam tube is most full at injection. In an electron machine it is the balance between the radiation damping and quantum excitation that determines the emittance at any energy and in most machines although it will be conserved around the ring it will grow with energy.

9. DISPERSION

9.1 Closed orbit

The bending field of a synchrotron is matched to some ideal (synchronous) momentum p_0. A particle of this momentum and of zero betatron amplitude will pass down the centre of each quadrupole, be bent by exactly 2π by the bending magnets in one turn of the ring and remain synchronous with the r.f. frequency. Its path is called the central (or synchronous) momentum closed orbit. In Fig. 8. this ideal orbit was the horizontal axis. We see particles executing betatron oscillations about it but these oscillations do not replicate every turn. In contrast the synchronous orbit closes on itself so that x and x' remain zero. It is not at first obvious that such a closed orbit exists for a particle of slightly different momentum. One might think perhaps of a spiral. However we will show that there is an orbit, displaced from the axis, which closes on itself and whose shape is defined by the “dispersion” function.

9.2 Orbit of a low momentum particle

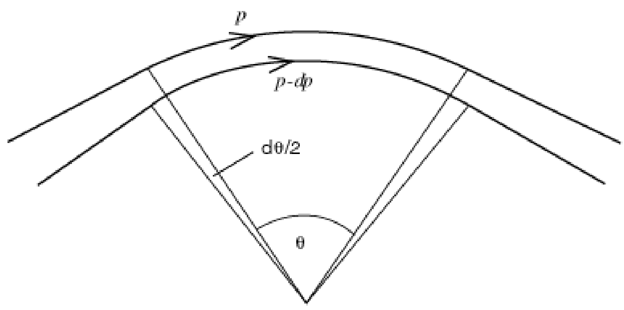

Fig. 15 Orbits in a bending magnet

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Figure 5 shows a particle with a lower momentum $\Delta p/p < 0$ and which therefore is consistently bent horizontally more in each dipole of a FODO lattice.

We might argue that the total deflection, being more than 2π would cause it to spiral in. But let us take a bird's eye view as in Figure 16. The smaller arrows represent the extra inward bending force on the low momentum particle as it passes through the dipoles but the particular orbit we have drawn passes systematically

further off axis in the F quadrupoles than in the D quadrupoles and while each quadrupole exerts a deflection

$$\Delta\theta = \frac{\Delta\ell B}{(B\rho)} = k\ell x$$

the outward deflections at the F quadrupoles predominate and, if the displacement of the orbit is large enough will compensate the extra bend at each dipole. We may describe the shape of this new closed orbit for a particle of unit $\Delta p/p$ by a *dispersion* function $D(s)$ which is the displacement of the orbit per unit momentum error. Thus we take the product $D(s)\ \Delta p/p$ to obtain the displacement of the closed orbit. Note that the orbit shape is periodic like the lattice and betatron oscillations will now take place with reference to this new closed orbit.

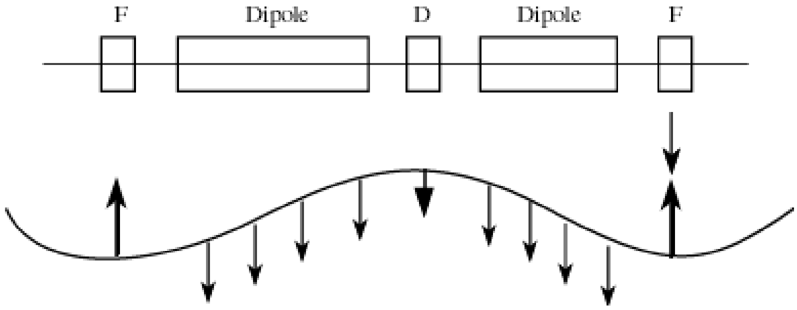

Fig.16

This clearly means the beam will be wider if it has momentum spread and the minimum semi-aperture required for the beam will be:

$$a_v = \sqrt{\beta_v \varepsilon_v}\ , \qquad a_H = \sqrt{\beta_H \varepsilon_H} + \left| D(s)\frac{\Delta p}{p} \right|$$

10. CHROMATICITY

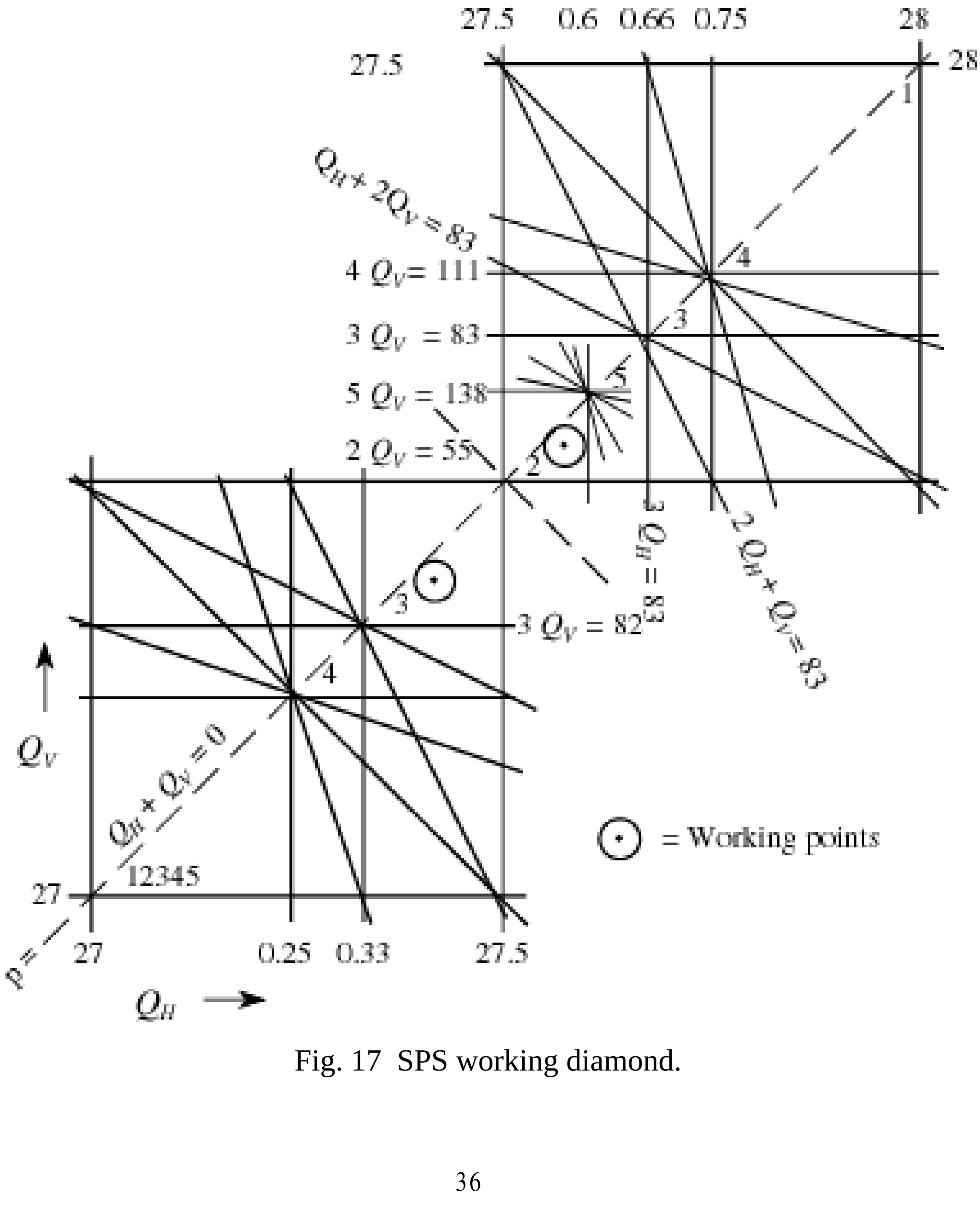

Fig. 17 SPS working diamond.

The operators of synchrotrons must be continually aware of the Q value of the machine in each plane and make adjustments to the focusing and defocusing quadruples to adjust Q to avoid integer values or values that are fractions like 1/3 ¼ 1/5 etc. Such values spell danger from resonant condition in which the pattern minute errors over several turn can repeat driving the beam out of the machine. Seen in a plot of both vertical and horizontal Q values these danger values appear as a forest of lines (Figure 17) which must be avoided.Unfortunately different momentum [particles will have different Q values so that the "working point" or area of the diagram covered by the population of all particles in the beam may be so large that is impossible to steer between the resonance lines..

This momentum dependence of Q is exactly equivalent to the chromatic aberration in a lens. It is defined as a quantity Q'

$$\Delta Q = Q' \frac{\Delta p}{p} .$$

The chromaticity (Guiducci 1992) arises because the focusing strength of a quadrupole has $(B\rho)$ in the denominator and is therefore inversely proportional to momentum:

$$k = \frac{1}{(B\rho)} \frac{dB_z}{dx} .$$

A small spread in momentum in the beam, $\pm\Delta p/p$, causes a spread in focusing strength:

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p} .$$

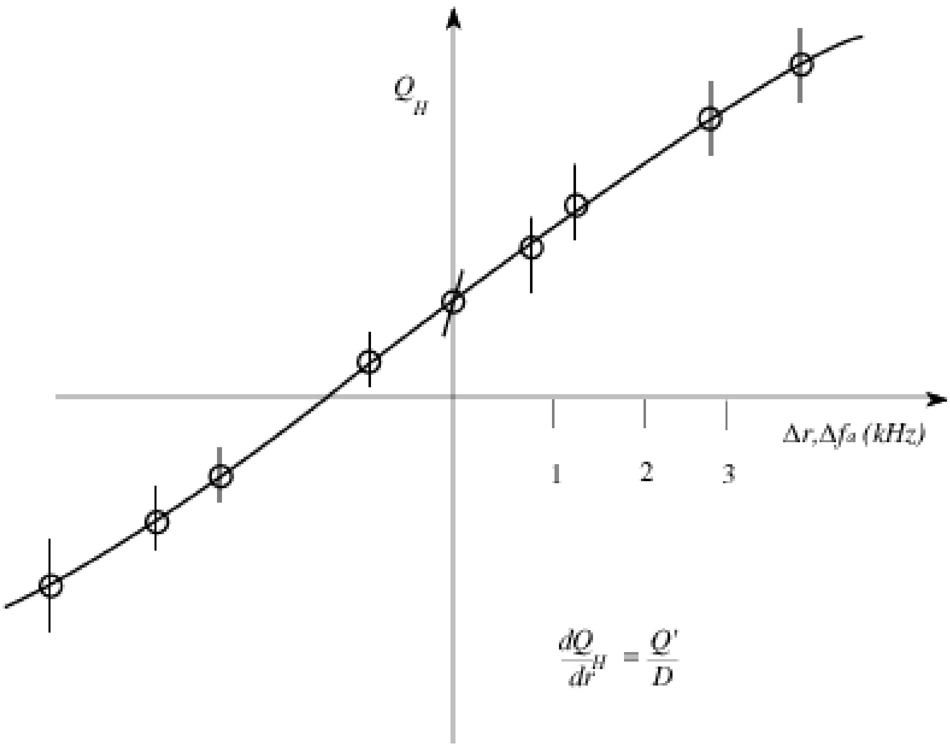

Fig. 18 Measurement of variation of Q with momentum made by changing the r.f. frequency.

An equation, which we have not the space here to derive, describes the effect of such a focussing error

$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\delta k(s)\ ds\ .$$

enables us to calculate Q' rather quickly:

$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\delta k(s)ds = \left[\frac{-1}{4\pi}\int \beta(s)k(s)ds\right]\frac{\Delta p}{p}\ .$$

The chromaticity Q' is just the quantity in square brackets. To be clear, this is called the natural chromaticity. For most alternating gradient machines, its value is about -1.3Q. Of course there are two Q values relating to horizontal and vertical oscillations and therefore two chromaticities.

One way to correct this is to introduce some focusing which gets stronger for the high momentum orbits near the outside of the vacuum chamber – a quadrupole whose gradient increases with radial position is needed. A sextupole magnet has just such a field configuration:

$$B_z = \frac{B''}{2}x^2$$

and, in a place where there is dispersion it will introduce a normalised focusing correction:

$$\Delta k = \frac{B''D}{(B\rho)}\frac{\Delta p}{p}\ .$$

The effect of this Δk on Q is:

$$\Delta Q = \left[\frac{1}{4\pi}\int \frac{B''(s)\beta(s)D(s)ds}{(B\rho)}\right]\frac{\Delta p}{p}\ .$$

To correct chromaticity we have to make the quantity in the square bracket balance the chromaticity. There are of course two chromaticities, one affecting Q_H, the other Q_V and we must therefore arrange for the sextupoles to cancel both. For this we use a trick which is common and will crop up again in other contexts. Sextupoles near F-quadrupoles where β_x is large affect mainly the horizontal Q, while those near D-quadrupoles where β_z is large influence Q_v. The effects of two families like this are not completely orthogonal but by inverting a simple 2 x 2 matrix one can find two sextupole sets which do the job.

11. CONCLUSIONS

We have now covered the transverse dynamics of particles in a rudimentary way in preparation for more detailed lectures which follow on the effect of synchrotron radiation of the dynamics, the link with longitudinal dynamics and the effect of imperfection in the construction. We have also still to understand how a modern synchrotron may be equipped with special long straight sections for injection, accelerating systems and, for a storage ring, collision.

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