

SPECIAL RELATIVITY

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Abstract

Particles in the beams of modern accelerators travel close to the velocity of light. A working knowledge of the Einstein's theory of special relativity is essential if one is to understand their behaviour. The essentials of Special Relativity are presented in this paper in the order in which they were discovered – from the questions raised by Maxwell's theory and the Michelson Morley experiment to their final resolution in Einstein's theory which is one of the cornerstones of modern physics.

INTRODUCTION

If you ask a random collection of first year students, "What do you know about relativity?" the answers might be:

"All is relative?"

"It all depends on your frame of reference."

"You will never measure an absolute velocity unless you look into space."

"Wasn't it invented by the same guy that gave us the atom bomb?"

Of course none of these answers are correct and if we turn to Einstein's rather philosophical definition it does not give us a clue as to how to apply the principle.

The laws of physics in two systems moving with a relative velocity one to another are equivalent.

The speed of light is finite and independent of the motion of the source.

The principle of relativity, coming after Maxwell's equations and before quantum theory is one of the three great discoveries upon which modern physics is based. The best way to understand it is to follow the series of puzzles which confronted physics at the end of the eighteenth century and see how this principle pointed to their solution.

OBSERVERS AND THEIR FRAMES OF REFERENCE

Before plunging back into history we should have a clear idea of what is meant by a "frame of reference" and imagine the definition of the world as seen by two observers each using their own frame of reference. It helps to think of these observers as real people with eyes and ears and carrying clocks and rulers to measure and observe in their respective frames of reference. We shall call them, Joe, an observer in the "laboratory" frame of reference or coordinate system which to us appears to us stationary, and Moe, a moving observer rooted in a frame of reference whose relative velocity to Joe is a

vector, \mathbf{v} . In Figure 1 we have chosen the case where the motion is parallel to the x axis of both coordinate systems. Of course Moe thinks his frame of reference is stationary and would see Joe as moving with velocity, $-\mathbf{v}$. If both Joe and Moe were to describe the position of the same point, P, in their frames of reference, Joe in the lab would write down three numbers x, y, z while moving Moe would write down x', y', z' .

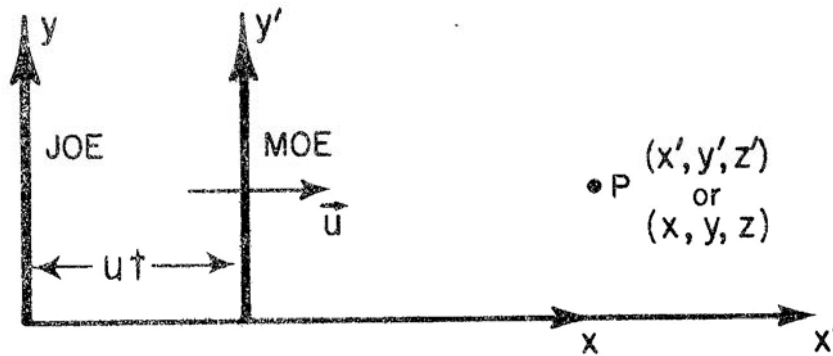


Fig 1:. Joe is an observer in the laboratory while Moe is traveling to the right with a velocity, u . They describe the same point P with different coordinates x, y, z and x', y', z'

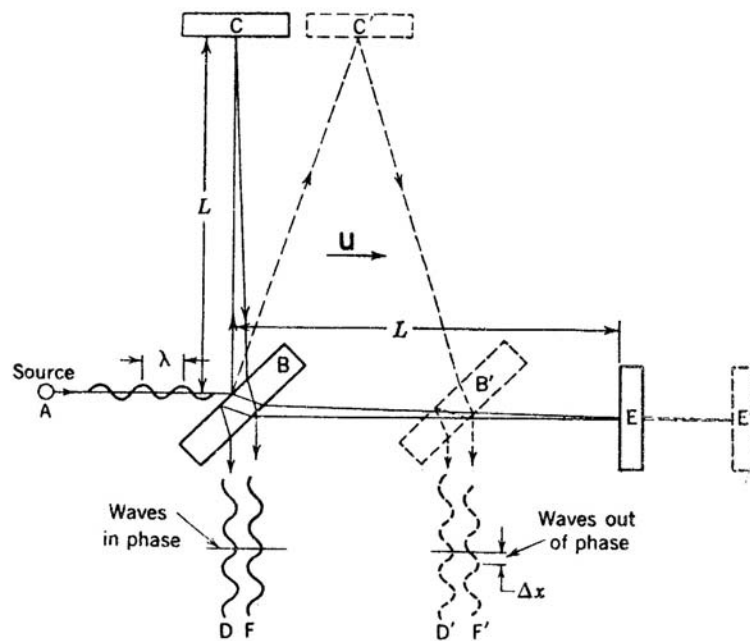


Fig 2:. The Michelson and Morley experiment

HISTORY

In the late eighteenth century a Scottish mathematician, Maxwell, discovered the laws of electromagnetism which allowed physicists to formulate a wave equation for light and other electromagnetic radiation. They jumped to the conclusion that light, like sound and other waves must be propagated in a medium which they called the ether.

Michelson and Morley devised an experiment which would measure the velocity of the earth in its orbit relative to the ether. Figure 2 shows the experiment. Light from a source is split by a half silvered mirror, B . Half the light is reflected back from a mirror, C , and the other half from E and from the back side of B to recombine with the light from C . The distances to each of the mirrors is adjusted to be equal so that the interference fringes from the two paths reinforce each other. If the apparatus is traveling with the ether the distances BC and BE are then both equal to L . The velocity of the light waves along each leg is, c . But, suppose that the whole apparatus is moving parallel to the earth's orbit with velocity u . We show this situation dotted in Figure 2.

Eighteenth century physicists knew about sound waves which always propagate with the same velocity with respect to their source and would expect the light waves to have velocity, c along each leg of their journey. However in the time, t , taken for the light to travel to E , it has moved to E' and the path length must be, $L + ut$. If

$$ct = L + ut$$

then the time for the outward journey is:

$$t = \frac{L}{c - u}$$

and for the return journey is:

$$t = \frac{L}{c + u}$$

the time for the total journey is :

$$\frac{2Lc}{(c^2 - u^2)} .$$

If we now look at the path taken to C and back we find that the light must travel along the hypotenuse of a triangle and the total time is:

$$\frac{2Lc}{\sqrt{(c^2 - u^2)}} .$$

The eighteenth century physicists believed that the difference in these times would be a measure of u and that the fringes would move out of register if the apparatus was rotated to point towards the sun rather than tangential to the earth's orbit. The number of fringes counted would determine u and tell them the earth's velocity in space. To their disappointment and consternation the fringes did not change and there was no satisfactory explanation for this other than perhaps that there was no such thing as ether - or perhaps the distance BE had shrunk so that :

$$L_{BE} = \frac{L_{BC}}{\sqrt{(c^2 - u^2)}} .$$

Some years after the experiment, Lorentz, who had taken upon himself the task of tidying up Maxwell's equations and casting them in the elegant form we now use, found a transformation of time and space coordinates which predicted just such a contraction of space. However this idea was thought to be a mathematical fudge and not treated with the respect it deserved. It took an even greater mind, that of Einstein, to realise that this was part of a far reaching theory of special relativity.

THE LIGHT CLOCK AND THE DILATION OF TIME

While our thoughts are buzzing with such things it is a good time to understand another of the transformations that Lorentz had discovered: the dilation of time. To understand this phenomenon we imagine a clock as seen by our two observers Joe and Moe. Moe has taken his clock in a spaceship while Joe observes the clock from his stationary laboratory on earth. We see Joe's view in the upper diagram of Figure 3. The clock relies upon a light wave or photon, generated by a photodiode or flashtube, traveling to a mirror where it reflected back to a photocell which generates an electrical signal which, in turn, produces an audible "tick". If the mirror is a distance, D , from the diode the interval between ticks will be

$$2D / c$$

Joe observes that the mirror and receiver move during the time the reflection takes and that the distance traveled by the photon is longer. This is exactly the problem we have solved in the side leg of the Michelson and Morley experiment where we found the time to travel back and forth observed by Joe in the lab was

$$2Lc / \sqrt{c^2 - u^2} = 2(L/c) / \sqrt{1 - (u/c)^2} .$$

The rate of ticking observed by Joe will therefore be slower by a factor

$$1 / \sqrt{1 - (u/c)^2} = 1 / \sqrt{1 - \beta^2} = \gamma .$$

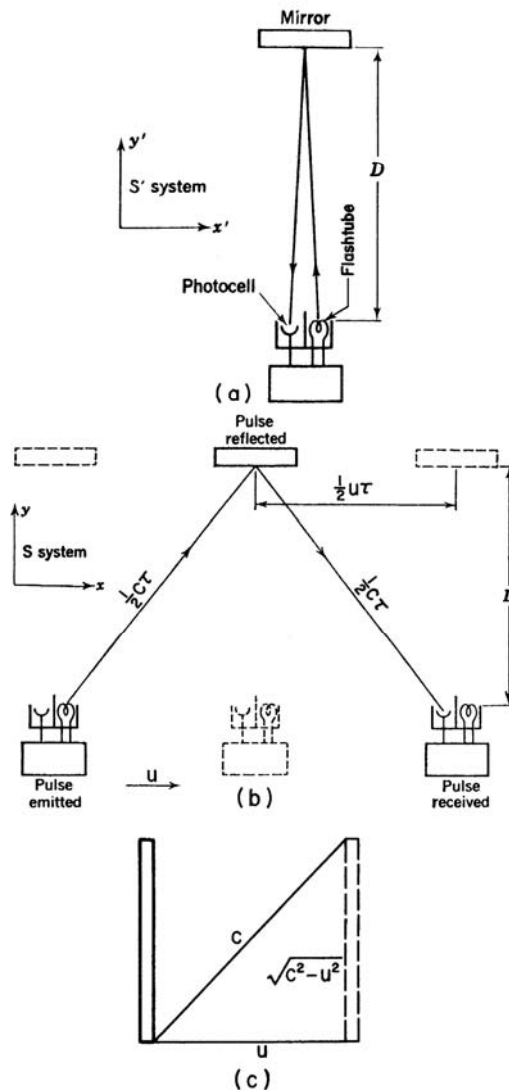


Fig. 3: The light clock as seen by Moe in a spaceship (a) and, (b) by Joe from his laboratory on earth.

Here we have taken the opportunity to define two parameters $\beta = u/c$ and γ which we shall see are fundamental in the notation of special relativity. However we are in danger of losing the historical thread and should now return to the concept of Newton and see the impact that Maxwell's unification of electricity and magnetism had upon physics.

TRANSFORMATIONS

The laws of physics and in particular Newton's Law of Motion had always been held to be independent of the velocity of the observer. We would now say "independent under a transformation between observers with relative velocity, u ." The transformation they

applied in Newton's day, the Galilean transformation, can be expressed by four equations which take us from Joe's world into that of Moe

$$\begin{aligned}x' &= x - ut \\y' &= y, \\z' &= z, \\t' &= t .\end{aligned}$$

Then in 1880 came the discovery by Maxwell of four equations which defined a new law of physics uniting electricity and magnetism

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

which had the untidy quality of not being invariant under a Galilean transformation. When converted from Joe's world to Moe's they became a mess and predicted effects which were just not observed.

Then, around 1900, Lorentz hit upon a transformation which did leave Maxwell's equations (and Newton's) unchanged for Moe. The Lorentz transformation is

$$\begin{aligned}x' &= \frac{x - ut}{\sqrt{1 - u^2/v^2}}, \\ y' &= y, \\ z' &= z, \\ t' &= \frac{t - ux/c^2}{\sqrt{1 - u^2/v^2}} .\end{aligned}$$

Lorentz had in fact made a major leap towards special relativity and offered his transformation to explain the Michelson and Morley experiment, but this suggestion was rejected by the world as a mere mathematical artifact.

In order to fall in line with current convention we shall redefine the unprimed and primed coordinate of Joe and Moe with suffices 1 and 2 respectively so that the Lorentz transformation becomes

$$x_2 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \quad y_2 = y_1, \quad z_2 = z_1, \quad t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} .$$

We have also taken the liberty of putting c where Lorentz had used v and have redefined the relative velocity between Joe and Moe to be, v .

We can imagine that this appears simpler in the notation of special relativity which defines

$$\frac{1}{\sqrt{1-\beta^2}} = \gamma .$$

LORENTZ CONTRACTION

We are now in a position to think clearly about Lorentz' explanation of the Michelson and Morley experiment. In the lower diagram of Figure 4 we imagine how Joe lays down a ruler of length l_0 to measure a distance from the origin to x_1 . Moe (upper diagram) sees this but the point x_2 in his coordinates is transformed by

$$x_1 = \frac{x_2 - vt_1}{\sqrt{1 - v^2/c^2}}$$

and if all this happens at a time $t_1 = t_2 = 0$

$$x_2 = x_1 \sqrt{1 - v^2/c^2}$$

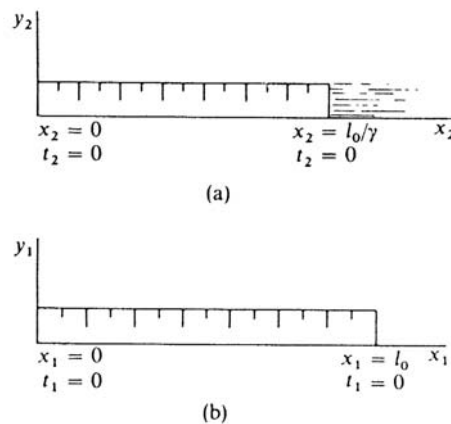


Fig. 4: The two views of a measurement of length. Joe lays down a ruler in (b) and Moe (a) sees it as shorter.

In the lower figure we have Joe's view as he lays down a ruler length: l_2 . In the upper figure we see that Moe who is moving, compares the position of the ends at the same time ($t=0$ in both systems) with marks on his bench (perhaps by a photo) and concludes Joe's ruler is shorter:

$$l_2 = l_1 \sqrt{1 - v^2/c^2} = l_1 / \gamma .$$

This effect is called Lorentz contraction.

TIME DILATION

We can move on immediately to see how the Lorentz transformation explains the light clock. In Figure 5 Moe's view is above and Joe's below. Moe has one clock while Joe has two, one at the origin and the other at a point x_1 that Moe's clock passes at a time which appears to be t_2 (T_0 in the diagram). The aim of what follows is to find out what time t_1 this event seems to occur on Joe's second clock. All three clocks start at the same instant when Moe's clock passes the first of Joe's.

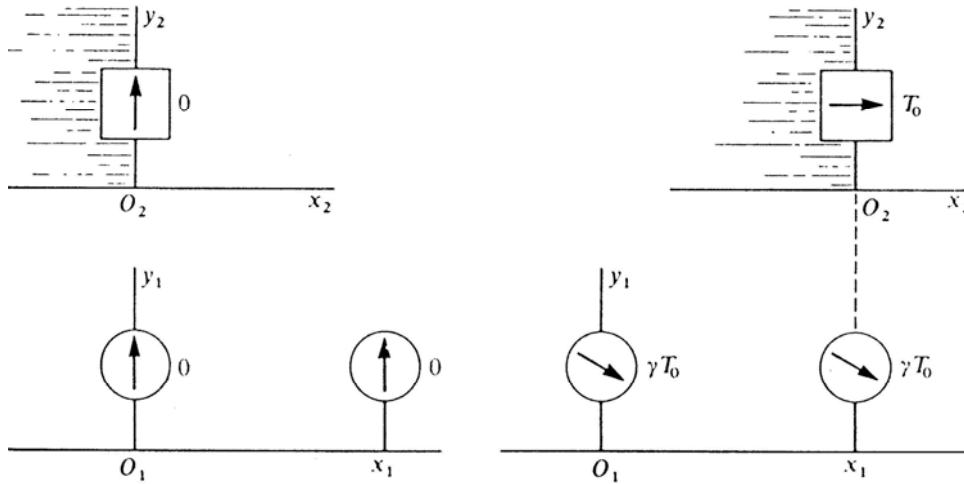


Fig. 5: Moe's view is above and Joe's below. Moe has one clock while Joe has two
In order to simplify the process we arbitrarily choose

$$x_1 = \frac{vt_1}{\sqrt{1-v^2/c^2}}$$

then

$$t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1-v^2/c^2}}$$

and this gives

$$t_1 = \frac{t_2}{\sqrt{1-v^2/c^2}} = \gamma t_2 .$$

It seems to Joe that the moving clock of Moe is running slow. To find a physical demonstration of this one only has to observe that muons from cosmic ray interactions with the upper atmosphere survive to reach ground level even though the life time at rest of the muon is much shorter than the time it would take to this distance at the velocity of light. The clocks of the cosmic muons telling them when to decay seem to an earth bound observer to run slow.

FOUR VECTOR OF SPACE-TIME

Most of us are familiar with the simple transformation that rotates a point, (x_1, y_1) by an angle θ about the origin to lie at coordinates, (x_2, y_2) .

$$\begin{aligned}x_2 &= x_1 \cos \theta + y_1 \sin \theta, \\y_2 &= -x_1 \sin \theta + y_1 \cos \theta .\end{aligned}$$

This transformation may be thought of as a rotation of a vector of constant length

$$\sqrt{x^2 + y^2} .$$

The Lorentz transformation:

$$x_2 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \quad y_2 = y_1, \quad z_2 = z_1, \quad t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}}$$

rotates the 4-vector: $(x, y, z, -ct)$ so that its “length” is an invariant

$$\sqrt{x^2 + y^2 + z^2 + c^2 t^2} .$$

This is our first example of a quantity that is invariant under a change to the moving coordinate system. It is a step towards restoring physics to the nice situation where one could apply a Galilean transformation to Newton’s “metric” as it is called and find the laws of motion were unchanged. Quantities that are invariant under Lorentz transformation are at the heart of physics. To find that something is invariant or to find a transformation that preserves invariance of a physical law gives enormous confidence in its validity. It also provides a shortcut to solving physical problems as we shall see in the case of synchrotron radiation later.

LORENTZ MATRIX

The Lorentz transformation can be written in a very compact form as a four by four matrix operating on a column vector $(x, y, z, -ct)$

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_2 = \gamma \begin{pmatrix} 1 & 0 & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_1$$

where

$$\beta = v/c \text{ and } \gamma = \frac{1}{\sqrt{1 - \beta^2}} .$$

This will transform from the lab (Joe) to moving (Moe) coordinates while the inverse matrix

$$\begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_1 = \gamma \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -ct \end{pmatrix}_2$$

transforms an observation of position and time in the moving system to predict what the observer in the lab records.

TRANSFORMING A VELOCITY

The relative velocity is now written, v , to distinguish it from the observed velocity of a point in the laboratory v_1 . We first express a component of the velocity in the laboratory frame as the product of two differentials

$$v_{x1} = \frac{dx_1}{dt_1} = \frac{dx_1}{dt_2} \frac{dt_2}{dt_1} .$$

We now refer back to the equations of the Lorentz transformation.. The first of these when applied to a transformation from Moe to Joe becomes

$$x_1 = \frac{x_2 + vt_2}{\sqrt{1 - v^2/c^2}} .$$

Thus the first of the two differentials is

$$\frac{dx_1}{dt_2} = \frac{1}{\sqrt{1 - v^2/c^2}} \frac{d}{dt_2} (x_2 + vt_2) = \gamma \frac{d}{dt_2} (x_2 + vt_2) = \gamma(v_{2x} + v) .$$

Next we differentiate the fourth Lorentz equation

$$t_2 = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} .$$

to obtain

$$\frac{dt_2}{dt_1} = \gamma \frac{d}{dt_1} [t_1 - (v/c^2)x_1] = \gamma [1 - (v/c^2)v_{1x}] .$$

Finally after forming the product of the two differentials and solving for v_{1x} we have

$$v_{1x} = \frac{v_{2x} + v}{1 + (v_{1x} v/c^2)} \quad \text{and} \quad v_{2x} = \frac{c(v_{1x} - c\beta)}{c - v_{1x}\beta} .$$

It is interesting to note that if $v_{1x} = c\beta$ then $v_{2x} = 0$ and if $v = c$ then $v_{2x} = c$ for all values of v .

A SMALL STEP TO REDEFINE MOMENTUM AND ENERGY

The above equations for transformation of velocities do not at first appear impressive but they may be used to reveal how the fundamental quantities of dynamics, energy and momentum should be transformed. It was one of Einstein's crucial contributions to o defined a particles momentum and energy

$$p = mv = \frac{m_0 v}{\sqrt{1 - (v/c)^2}} = \frac{m_0 v}{\sqrt{1 - \beta^2}} = m_0 \gamma v$$

$$E = \frac{m_0 c^2}{\sqrt{1 - (v/c)^2}} = \frac{m_0 c^2}{\sqrt{1 - \beta^2}} = T + m_0 c^2 = m_0 \gamma c^2$$

where m_0 is the mass of the particle when at rest in the laboratory, v is the velocity of the particle with respect to the laboratory observer, $\beta = v/c$ and T is the kinetic energy of the particle.

Applying the rule for transformation of velocity we find that the three momenta and the total energy are four elements of a vector which obeys the Lorentz transformation which we applied to space and time coordinates.

$$\begin{pmatrix} E \\ -p_x c \\ -p_y c \\ -p_z c \end{pmatrix}_1 = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ -p_x c \\ -p_y c \\ -p_z c \end{pmatrix}_2 .$$

The reader will note that the components of the momentum have been multiplied by $-c$ to make them fit the transformation. Moreover there is a quantity

$$E^2 - (pc)^2 = (m_0 c^2)^2$$

which is invariant as we move from one moving frame to another as is the law of motion

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}$$

of a particle under the influence of a force \mathbf{F} .

Other useful relationships emerge:

$$\gamma = E/E_0, \quad \beta = pc/E, \quad \beta\gamma = pc/E_0 .$$

MOVING FROM NEWTON TO EINSTEIN

The first two of these last equations are plotted below to illustrate that, while in the classical Newtonian regime the energy increase with the square of the velocity, and while by accelerating particles we can increase their energy parameter, γ , indefinitely, their velocity “saturates” approaching that of c (or $\beta = 1$) more slowly and asymptotically.

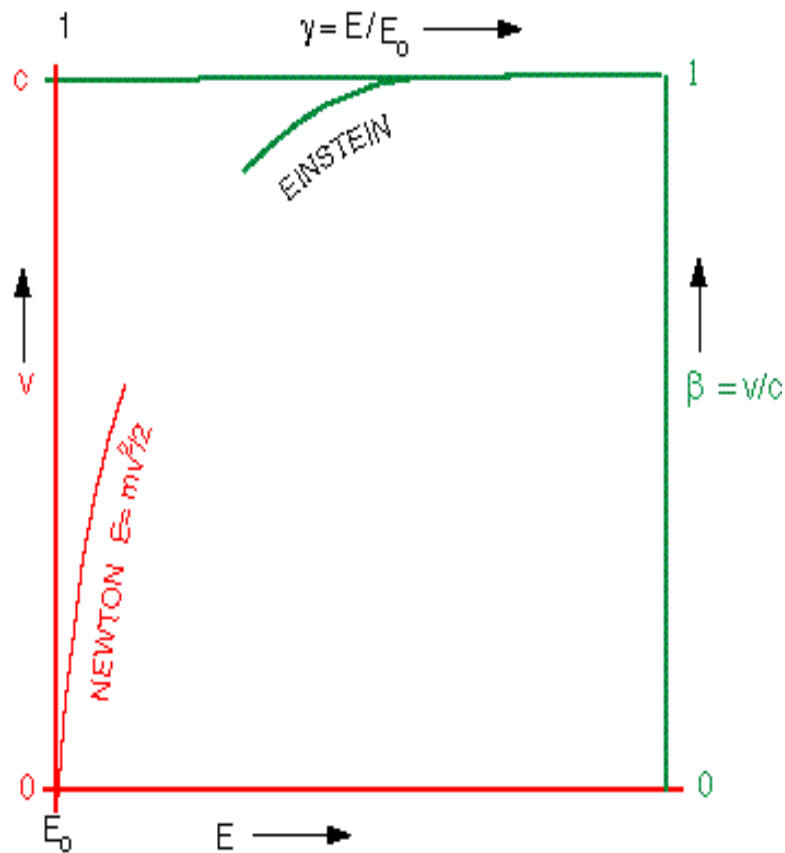


Fig. 6: Variation of velocity of a moving particle with increasing energy

The shape of this curve is defined

$$\frac{E}{m_0c^2} = \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \sqrt{1 - \left(\frac{1}{\gamma}\right)^2} \Rightarrow 1 .$$

TRANSFORMING ACCELERATION AND FORCE COMPONENTS

Having earlier understood how to transform velocities we can use a similar procedure to deduce how to transform an acceleration. This is somewhat tedious and the reader may prefer to skip this and the transformation of a force which follow. However it is important to note that the transformations of forces and accelerations in the directions transverse to the direction of motion of the moving frame depend on γ . This is the reason why synchrotron radiation and its distribution in the laboratory are so strongly dependent on γ .

To pursue the analysis we can differentiate to find the acceleration

$$a_{1x} = \frac{dv_{1x}}{dt_1} = \frac{dv_{1x}}{dt_2} \frac{dt_2}{dt_1} .$$

Again after using two partial differentials we obtain for

$$a_{2x} = \frac{a_{1x}}{\gamma^3 [1 - v_{1x}v/c^2]^3}$$

$$a_{2z} = \frac{1}{\gamma^2 [1 - v_{1x}v/c^2]^2} \left\{ a_{1z} + \frac{v_{1y}v}{c^2 - v_{1x}v} a_{1x} \right\}$$

$$a_{2y} = \frac{1}{\gamma^2 [1 - v_{1x}v/c^2]^2} \left\{ a_{1y} + \frac{v_{1y}v}{c^2 - v_{1x}v} a_{1x} \right\} .$$

We can also express a force as three components (X, Y, Z) which transform as:

$$X_2 = X_1 - \frac{v}{c^2 - v_{1x}v} (v_{1y}Y_1 + v_{1z}Z_1)$$

$$Y_2 = \frac{Y_1}{\gamma [1 - v_{1x}v/c^2]^2}$$

$$Z_2 = \frac{Z_1}{\gamma [1 - v_{1x}v/c^2]^2} .$$

WHY IS SYNCHROTRON RADIATION SO γ DEPENDENT?

Synchrotron radiation is simply dipole radiation from a moving charge like an electron circulating in a magnetic field. Larmor solved this problem and it is easy to calculate that the power radiated is :

$$P = \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} (\ddot{z})^2 .$$

Here we see the acceleration of the charge, \ddot{z} , which is in the transverse direction as shown in Figure 7.

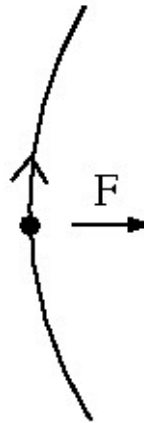


Fig. 7: This shows the particle motion and that of the transverse force and acceleration.

If we look at the transformations of acceleration above (the a_z component corresponds to \ddot{z} and is the only component of acceleration present) and imagine that $\mathbf{v} = v_{1x}$ we find that

$$a_{2y} = \frac{1}{\gamma^2} a_{1y} .$$

We suddenly realise that $\gamma^2 \ddot{z}$ is an invariant under Lorentz transformation and by putting it in Larmor's classical formula instead of γ^2 we have a law of physics which will be valid in any frame of reference:

$$P = \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} (\ddot{z})^2 \gamma^4 .$$

We have just done something rather “grown up” in physics and by expressing a classical law in the invariant coordinates of special relativity produced a new law of physics that applies whatever the relative velocity of the particle or observer. Another example of this method is to be found in the field of synchrotron radiation.

In Figure 8 the diagram on the left shows how we expect the radiation to be rather isotropic around a slowly moving particle while on the right we see that it is concentrated

in a narrow cone if the radiation from a rapidly moving particle is observed by Joe in the lab. Observed by Moe sitting on the moving particle it would of course appear as in the left hand figure. We can think of the radiation as photons. A photon seen by Moe has momentum p_z at right angles to the path of the particle as seen by Joe, Moe measure the momentum to $p_x = 0$ along the path towards Joe. The Lorentz transformation tells us that while p_z is unchanged for Joe he sees a large $p_x = \beta\gamma m_0 c^2$

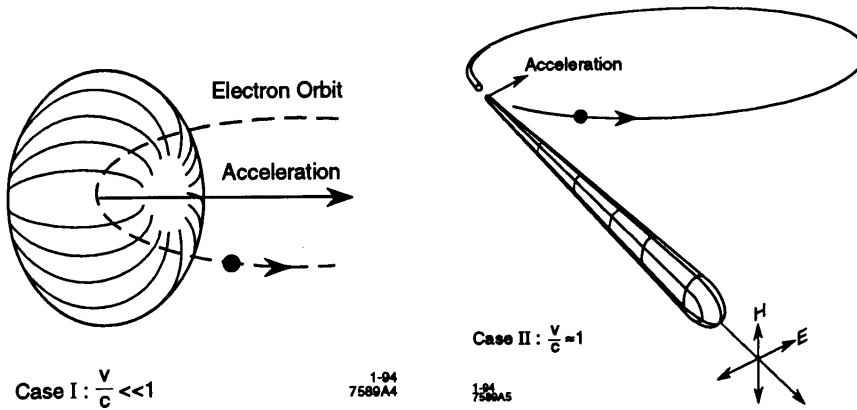


Fig. 8: The left diagram shows Moe's isotropic view of synchrotron radiation as he moves with the particle and the right shows how this transforms into a narrow cone in the laboratory.

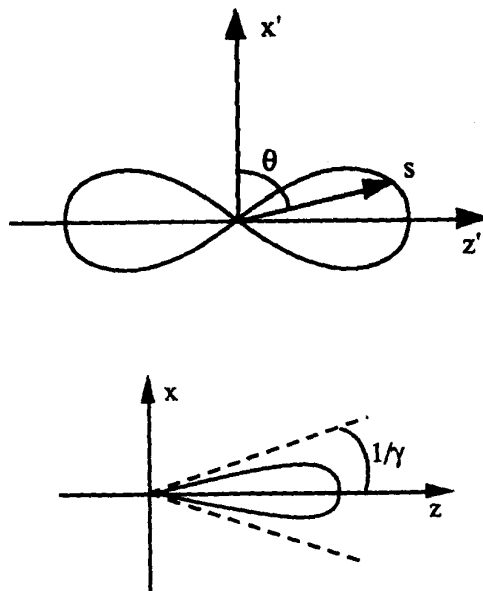


Fig. 9: The momentum vector of the synchrotron light photon as seen by Joe in the lab and, below, how this defines a narrow cone.

Figure 9 shows above the momentum vector of the photon (labeled s) seen by Joe in the lab and below how the isotropic distribution of photons in the moving system appears as a forward cone of opening angle $1/\gamma$ to Joe in his lab.

TRANSFORMING ELECTRIC AND MAGNETIC FIELDS

Finally and to be complete we include the transformation matrix for a six vector whose components are the three components of electric field and those of magnetic field multiplied (in MKS notation) by c .

$$\begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \\ cB_{1x} \\ cB_{1y} \\ cB_{1z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & \gamma\beta \\ 0 & 0 & \gamma & 0 & -\gamma\beta & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma\beta & 0 & \gamma & 0 \\ 0 & \gamma\beta & 0 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_{2x} \\ E_{2y} \\ E_{2z} \\ cB_{2x} \\ cB_{2y} \\ cB_{2z} \end{pmatrix} .$$

CONCLUSIONS

This is really as far as one needs to go to understand special relativity and apply it to particle accelerators. Once one gets used to using invariant quantities and the laws of physics in invariant form it is often simple to solve a problem in the moving system of the particle and transform the solution into the lab or vice versa. We already have found this in dealing with synchrotron radiation. Another example is that of space charge which is particularly simple in the frame of reference of the mobbing bunch where fields are simply electrostatic.