IX. ELECTRODYNAMICS OF MOVING MEDIA

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A. ENERGY CONSERVATION THEOREM FOR THE PHONON MASER

1. Linearized Equations

The interaction of sound waves with optical-frequency waves in the presence of a strong optical maser beam has been analyzed by Kroll.[1] Bobroff[2] uses the following linearized fluid equations:

$$\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t} \tag{1}$$

$$\nabla \times \overline{H} = \epsilon \frac{\partial \overline{E}}{\partial t} + \frac{\gamma}{K} \frac{\partial}{\partial t} (p_s \overline{E}) \tag{2}$$

$$\rho_0 \frac{\partial}{\partial t} \overline{v}_s = -\nabla p_s + \frac{\gamma}{2} \nabla E^2 \tag{3}$$

$$\frac{1}{K} \frac{\partial p_s}{\partial t} = -\nabla \cdot \overline{v}_s. \tag{4}$$

The subscript 0 denotes time-average quantities; the velocity $\overline{v}_s$, mass density ρ_s, and pressure p_s are small perturbations of the corresponding quantities away from equilibrium; the constant γ is the electrostrictive constant; and K is the bulk modulus of the fluid. While these equations are adequate for describing the interaction of the electromagnetic and sound waves, they are approximate in two respects: (i) they have been linearized with respect to the velocity $\overline{v}$ and density ρ of the material, and (ii) certain terms that are small (in the ratio of sound speed to the speed of light, c) have been disregarded.

2. General Nonrelativistic Equations

It is helpful to consider a more complete form of these equations when deriving conservation theorems because then the terms appearing in the theorem can be interpreted physically more easily. In this report we shall consider equations that are exact except for their lack of relativistic terms involving (i) the square of the material velocity divided by c^2, and (ii) the ratio of electric energy density to ρc^2. These equations[3] are Maxwell's equations

$$\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t} \tag{5}$$

$$\nabla \times \overline{H} = \epsilon_0 \frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{P}}{\partial t} + \nabla \times (\overline{P} \times \overline{v}), \tag{6}$$

where $\overline{P}$ is the polarization density in the fluid; the force equation

$$\frac{\partial}{\partial t} \rho \overline{v} + \nabla \cdot \rho \overline{v}\overline{v} = \overline{f} = -\nabla \pi + \overline{P} \cdot \nabla \overline{E} + \overline{v} \times (\overline{P} \cdot \nabla) \mu_0 \overline{H} + \left[\frac{\partial \overline{P}}{\partial t} + \nabla \cdot (\overline{v}\overline{P}) \right] \times \mu_0 \overline{H}, \tag{7}$$

where $\overline{f}$ is the force density in the medium, and the equation of conservation of mass

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \rho \overline{v} = 0. \tag{8}$$

Here no linearization assumptions have been made. The term π is the fluid pressure caused by a combination of hydrodynamic and electrostrictive effects,

$$\begin{aligned} \pi &= \pi(\overline{P}, \rho) \\ &= \rho \left(\frac{\partial w}{\partial \rho} \right)_{\overline{P}} - w + (\overline{E} + \overline{v} \times \mu_0 \overline{H}) \cdot \overline{P}, \end{aligned} \tag{9}$$

where w, a function of ρ and $\overline{P}$, is the energy density in the material because of hydrodynamic and polarization effects. The change in w that is due to a change in density and polarization density is given by

$$\begin{aligned} \delta w &= \left(\frac{\partial w}{\partial \rho} \right)_{\overline{P}} \delta \rho + (\overline{E} + \overline{v} \times \mu_0 \overline{H}) \cdot \delta \overline{P} \\ &= \left(\frac{\pi}{\rho} + \frac{w}{\rho} - \frac{(\overline{E} + \overline{v} \times \mu_0 \overline{H}) \cdot \overline{P}}{\rho} \right) \delta \rho + (\overline{E} + \overline{v} \times \mu_0 \overline{H}) \cdot \delta \overline{P}. \end{aligned} \tag{10}$$

By introducing the concept of an energy density, we have assumed that the medium is conservative, with a constitutive law of the form

$$\overline{P} = \overline{P}(\rho, \overline{E} + \overline{v} \times \mu_0 \overline{H}). \tag{11}$$

A useful alternative form for the force equation (7) can be derived by use of some vector identities and (10) as follows:

$$\nabla \left(\frac{v^2}{2} + \frac{w}{\rho} + \frac{\pi}{\rho} - \frac{\overline{E} \cdot \overline{P}}{\rho} \right) = -\frac{\partial}{\partial t} \left(\overline{v} - \frac{\overline{P} \times \mu_0 \overline{H}}{\rho} \right) + \overline{v} \times \left[\nabla \times \left(v - \frac{\overline{P} \times \mu_0 \overline{H}}{\rho} \right) \right]. \tag{12}$$

The other three equations then are

$$\nabla \cdot \rho \overline{v} = -\frac{\partial \rho}{\partial t} \tag{13}$$

$$\nabla \times \overline{E} = -\mu_0 \frac{\partial \overline{H}}{\partial t} \tag{14}$$

$$\nabla \times (\overline{H}-\overline{P}\times\overline{v}) = \epsilon_0\frac{\partial\overline{E}}{\partial t} + \frac{\partial\overline{P}}{\partial t}. \tag{15}$$

This set of four equations (12-15) is ideally suited for derivation of power theorems because on the left are spatial derivatives and on the right are time derivatives. Note that these equations are exact, nonrelativistically, and have not been linearized.

3. Equivalence of the Two Sets of Equations

To prove that (1-4) are in fact approximate expressions obtained from (6-8), we consider the linearized expression for the constitutive law.

$$\overline{P} = \epsilon_0\chi_e(\overline{E}+\overline{v}\times\mu_0\overline{H}) + \rho\epsilon_0\left(\frac{\partial\chi_e}{\partial\rho}\right)\overline{E}. \tag{16}$$

The source term on the right-hand side of (6), to first order in perturbed density and velocity, becomes

$$\frac{\partial\overline{P}}{\partial t} + \nabla\times(\overline{P}\times\overline{v}) = \frac{\partial}{\partial t}(\epsilon_0\chi_e\overline{E}) + \frac{\partial}{\partial t}\left(\rho\epsilon_0\frac{\partial\chi_e}{\partial\rho_0}\overline{E}\right) + \frac{\partial}{\partial t}(\epsilon_0\chi_e\overline{v}\times\mu_0\overline{H}) + \nabla\cdot\epsilon_0\chi_e(\overline{v}\overline{E}-\overline{E}\overline{v}). \tag{17}$$

Whereas coupling of the acoustic system to the electromagnetic system requires retention of terms of first order in density and velocity, the coupling of the electromagnetic system to the acoustic system takes place in zero order in ρ and $\overline{v}$. The force density (7) may then be written in the form

$$\overline{f} = -\nabla\pi + \epsilon_0\chi_e\overline{E}\cdot\nabla\overline{E} + \frac{\partial}{\partial t}(\epsilon_0\chi_e\overline{E})\times\mu_0\overline{H}$$

$$= -\nabla p_s + \frac{1}{2}\nabla\epsilon_0\rho_0\frac{\partial\chi_e}{\partial\rho_0}\overline{E}^2 + \frac{\partial}{\partial t}(\epsilon_0\chi_e\overline{E}\times\mu_0\overline{H}). \tag{18}$$

If we make the identification

$$\rho_0\frac{\partial\chi_e}{\partial\rho_0} = \gamma, \tag{19}$$

and disregard the last two terms in (17) and third term in (18), we obtain the source term of (2) and force coupling term of (3). The terms that have been disregarded are usually of order c_s/c (where c_s is the speed of sound and c is the speed of light in the medium under consideration) as compared with the terms that have been retained.

4. Conservation of Energy

The law of conservation of energy can be derived very easily from (12-15). Dot-multiply (12) by $\rho\overline{v}$, multiply (13) by the quantity that appears in parentheses on the left-hand side of (12), dot-multiply (14) by $(\overline{H}-\overline{P}\times\overline{v})$, and dot-multiply (15) by $-\overline{E}$. Then add the four equations, to obtain

$$\nabla \cdot \left[\overline{E} \times \overline{H} + \overline{v}\frac{\rho v^2}{2} + \overline{v}(w+\pi) - \overline{P}(\overline{E} \cdot \overline{v})\right] = -\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + w + \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right). \tag{20}$$

Here, we have evaluated $\partial w/\partial t$ by means of (10). This conservation equation is of a very simple form. The terms under the divergence are the electromagnetic Poynting vector, the kinetic power-flow density, the flow of enthalpy (as augmented by the electrostrictive contributions toward the pressure), and the rate of work per unit area by the stress tensor PE associated with the polarization of the material. On the right-hand side are the time derivatives of several energy densities: the kinetic energy density, the energy density w associated with the material, and the electric and magnetic energy densities.

5. Other Conservation Theorems

A class of solutions of (12)-(15) exists for which

$$\nabla \times \left(\overline{v} - \frac{\overline{P} \times \mu_0\overline{H}}{\rho}\right) = 0. \tag{21}$$

For these "irrotational" cases, a variety of other conservation theorems can be derived from the basic equations, (12)-(15). Many such theorems for a similar physical system, the relativistic electron beam, have been derived and discussed previously,[4] and all derivations are similar for the phonon-maser case, although the physical interpretation may be different. Among the conservation principles that can be derived are small-signal power theorems (both with arbitrary time dependence and sinusoidal time dependence), Manley-Rowe relations, small-signal energy theorems, expressions for power and energy in waves, several large-signal power theorems, and variational principles.

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References

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2. D. L. Bobroff, The Rate of Oscillation Build-up in the Phonon-Phonon Parametric Backward-wave Oscillator, Unpublished private memorandum, Raytheon Research Division, Waltham, Massachusetts.

3. Cf. Quarterly Progress Report No. 70, Research Laboratory of Electronics, M.I.T., July 15, 1963, pp. 79-88.

4. A. Bers and P. Penfield, Jr., Conservation principles for plasmas and relativistic electron beams, IRE Trans., Vol. ED-9, No. 1, pp. 11-26, January 1962.