Particle Accelerators, 1995, Vol. 48, pp. 251–257
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IMAGE EFFECTS ON CROSSING AN INTEGER RESONANCE

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(Received 17 March 1994; in final form 6 September 1994)

The ISIS proton synchrotron requires improved horizontal and vertical orbit corrections when the space charge tune depressions are sufficient to move some single particle betatron tunes, but not the coherent tunes, across an integer resonance. Image effects are studied in an attempt to explain the observations.

KEY WORDS: Collective effects resonances

1 INTRODUCTION

At ISIS, the tunes are split by half an integer around the integer 4, and the maximum incoherent space charge tune depressions are ~0.4 in both planes. The performance at these space charge levels is very sensitive with respect to the horizontal and vertical closed orbit corrections, and in particular, to the fourth harmonic corrections.

The transverse space charge forces are directed towards the centre of charge of a beam so the closed orbit excitation terms should not directly affect a single particle's motion around its beam centre. Thus, if the equations of betatron motion are written with respect to the centre of charge, and if the effect of image forces is neglected, the equations should show no dipole error excitation and there should be no effect on crossing an integer resonance. The beam centre, itself, is affected, but not significantly until the coherent tune approaches the integer.

Image effects change this picture. Coherent image effects occur when a beam is not at the centre of its vacuum enclosure, and the resulting forces are not, in general, uniform along an axis of displacement. Incoherent image effects occur for both centred and off-centred beams, with the forces varying across the beam cross-section. The magnitude of the image forces, and their effect when crossing an integer resonance, are the subject of this paper.

2 BETATRON EQUATIONS

The following nomenclature is introduced, and an approximation then given for the single particle betatron equation:

x_j is the transverse betatron motion of the j-th particle,

x_c is the transverse betatron displacement of the beam centre,

x_j'' is the second derivative of x with respect to $\theta(= s/R)$,

s is the path length measured along the synchrotron orbit,

R is the mean radius of the machine,

Q is the unperturbed betatron tune in the (x, s) plane,

δQ_j is the direct transverse space charge tune depression of the j-th particle,

$F(\theta)$ are the closed orbit excitation terms,

$F_{cj}(\theta)$ is the coherent image excitation term for the j-th particle, and

$F_{ij}(\theta)$ is the incoherent image excitation term for this particle.

$$x_j'' + Q^2 x_j = 2\delta Q_j Q(x_j - x_c) + F_{ij}(\theta)(x_j - x_c) + F(\theta) + F_{cj}(\theta)x_c \tag{1}$$

Similar equations may be written for all beam particles, and an average taken:

$$x_c'' + Q^2 x_c = 0 + F(\theta) + \langle F_{cj}(\theta)\rangle x_c \tag{2}$$

Subtracting (2) from (1), setting $F_c(\theta) = \langle F_{cj}(\theta)\rangle$, and including the $F_{ij}(\theta)$ effect in δQ_j, gives:

$$(x_j - x_c)'' + Q(Q - 2\delta Q_j)(x_j - x_c) = [F_{cj}(\theta) - F_c(\theta)]x_c \tag{3}$$

If the coherent image terms are neglected, the right hand side of equation (3) becomes zero, and there is no effect on $(x_j - x_c)$ when crossing an integer resonance, as discussed. If the terms are included, their magnitude is of interest. It may be shown that $F_{cj}(\theta)$ has the form of a constant plus terms in $y^2, y^4 \ldots$, where y is the transverse particle motion orthogonal to x. Improving the closed orbits reduces the excitation terms in (3), and is the probable reason why ISIS requires improved closed orbits when operating at maximum intensity.

3 APPROXIMATIONS FOR COHERENT AND INCOHERENT IMAGE TERMS

A first approximation is due to Laslett,[1] who treats the beam as a line charge between two infinite parallel plane conductors. The resultant field is E_c for a coherent displacement, x_c, and E_i for incoherent motion, x_j, with E_c and E_i determined from an infinite series of images:

$$E_c = [\pi^2/4h^2]\lambda x_c$$

$$E_i = [\pi^2/12h^2]\lambda(x_j - x_c)$$

where h is the half spacing of the parallel plates, and λ is the line charge density of the beam.

These may be compared with the direct incoherent space charge force, E, for a beam of elliptical cross-section of semi-axes, a and b, and of uniform transverse density (the factor $1/4\pi\varepsilon_o$ for MKS units is omitted from the expressions):

$$\begin{aligned} E &= [4/b(a+b)]\lambda(x_j - x_c) \\ E_c/E &= [b(a+b)/h^2][\pi^2/16]x_c/(x_j - x_c) \\ E_i/E &= [b(a+b)/h^2][\pi^2/48] \end{aligned}$$

Coherent and incoherent image coefficients, ξ_1, and ε_1, respectively, are then defined:

$$\xi_1 = [\pi^2/16] = 0.617 \tag{4}$$

$$\varepsilon_1 = [\pi^2/48] = 0.206 \tag{5}$$

A better approximation is to take a beam of finite size between the parallel plates, and though actual beams have elliptical cross-sections, a rectangular beam of uniform transverse density is of interest as it results in analytical solutions for the image terms. The following central image coefficients, for such a beam, of half width, a, and half height, b, are obtained by an extension to the previous method of summing an infinite series of images:

$$\xi_1 = [\pi h/4a][\coth(\pi a/2h) - \operatorname{cosech}(\pi a/2h)] \tag{6}$$

$$\varepsilon_1 = [\pi h/4a][(2h/\pi a) - \operatorname{cosech}(\pi a/2h)] \tag{7}$$

The formulae are identical to those published elsewhere for the case of a zero height beam. Further image coefficients may be obtained for the y-extremities of the beam:

$$\xi_e = [\pi h/8a][\coth(\pi a/h) - \operatorname{cosech}(\pi a/h)] \tag{8}$$

$$\varepsilon_e = [\pi h/8a][(h/\pi a) - \operatorname{cosech}(\pi a/h)] \tag{9}$$

In the limit, as (a) tends to zero, it may be shown that equations (6) and (8) both reduce to equation (4), and similarly, equations (7) and (9) reduce to equation (5). Taking a value of $(h/a)^2 = 2$, from ISIS, gives the following coefficients:

$$\xi_1 = 0.560; \quad \xi_e = 0.447$$

$$\varepsilon_1 = 0.179; \quad \varepsilon_e = 0.128$$

These indicate a significant variation of the coherent and incoherent, x-direction, image forces along the y-axis of the beam.

Furthermore, in the y-direction, there are additional image forces, and it may be shown that these orthogonal forces contribute some incoherent, non-linear self focussing. The field along the major (or minor) axis of the beam may be written, for constants n and m:

$$E_{\perp y} = -x_c^2(n(y_j - y_c) + m(y_j - y_c)^2 + \ldots) \tag{10}$$

Neglecting the higher order forces, a transverse incoherent image term may be defined, of the form:

$$\varepsilon_{\perp} = -k(x_c/h)^2 \tag{11}$$

4 COMPUTATIONS

The interiors of the ISIS chambers do not have the previously assumed parallel plate geometry. They are square or rectangular and the ISIS beam cross sections are circular or elliptical. Detailed computations are then required to find the space charge fields.

Results have been obtained by C. Prior[2] for a circular beam of uniform transverse density and radius, a, in a square cross section chamber, of half side length, d, with $(d/a)^2 = 2$. The computer program uses a Poisson solver based on finite element techniques, with the mesh as shown in Figure 1.

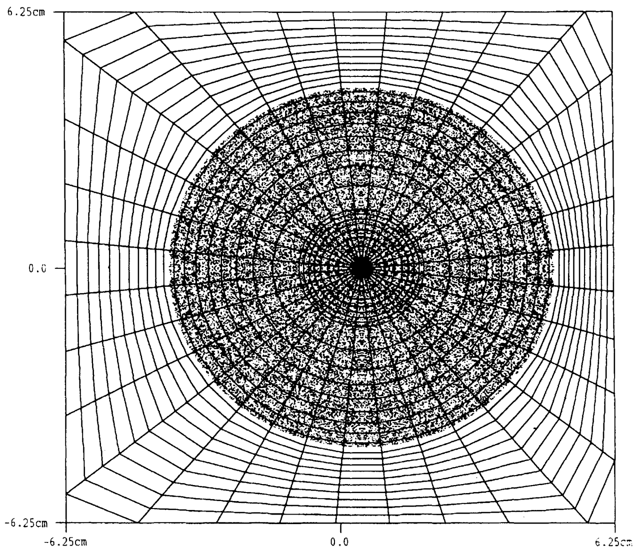

FIGURE 1: Computer mesh for horizontally displaced beam in ISIS.

Using image coefficients to represent the results, these are:

$$\xi_1 = 0.595; \quad \xi_e = 0.306$$

$$\varepsilon_1 < 0.075; \quad \varepsilon_e = 0$$

$$\varepsilon_\perp \simeq -1.25(x_c/d)^2 \tag{12}$$

The coherent image coefficients are not very different from those derived in Section 3, except for a greater variation along the y-axis of the beam. On the other hand, the incoherent image terms, ε_1 and ε_e, for the circular beam, are reduced and are significant for ε_1 only, for large x amplitude particles; $\varepsilon_\perp$ becomes the important incoherent term.

These representative image terms for ISIS may be used to estimate the scale of the effects for some incoherent crossings of the resonances, $Q_x = 4$ and $2Q_x = 8$.

5 IMAGE EFFECTS ON CROSSING AN INTEGER RESONANCE

For the case of ISIS, the relevant motion is in the horizontal plane, now designated as the (x, s) plane, and so is given by the incoherent betatron equation for $(x_j - x_c)$:

$$[x_j - x_c]'' + Q(Q - 2\delta Q_j)[x_j - x_c] = [F_{cj}(\theta) - F_c(\theta)]x_c + F_\perp(\theta)[x_j - x_c] \tag{13}$$

The terms on the RHS are considered separately, and are given here very approximately in terms of δQ, the average over the appropriate range of s, of the peak incoherent tune shift:

$$[F_{cj}(\theta) - F_c(\theta)] \simeq 0.5Q \quad \delta Q[\xi_1 - \xi_e] \tag{14}$$

$$F_\perp(\theta) \simeq 2.5Q \quad \delta Q[y_c/d]^2 \tag{15}$$

5.1 Effect of Horizontal Closed Orbit Errors on Horizontal Motion in ISIS

The excitation term, $[F_{cj}(\theta) - F_c(\theta)]x_c$, is finite for some particles due to the variation with y of the coherent image forces. It is a maximum for those particles with maximum horizontal and minimum vertical betatron amplitudes, or vice-versa, and is, approximately, for ISIS:

$$0.5Q \quad \delta Q[\xi_1 - \xi_e]x_c \simeq (4.39)(0.2)(0.289)x_c \simeq 0.25x_c$$

The maximum value for the incoherent tune shift is $\delta Q_j \sim 0.4$, sufficient for a particle to cross the resonance, $Q_x = 4$. ISIS has $n(= 2)$ such crossings, of random excitation phase as the closed orbit changes continually. Assuming a linear rate of change for the tune per turn, $\delta Q_j/N = (0.4/1000)$, the maximum additional motion for a peak fourth harmonic closed orbit deviation, $\hat{x}_c$, of 1 mm, is given by the integer resonance crossing formula:[3]

$$\Delta\hat{x} = (\pi/Q)(Nn/2\delta Q_j)^{1/2}(0.25\hat{x}_c) = 9\text{ mm}$$

5.2 *Effect of Vertical Closed Orbit Errors on Horizontal Motion in ISIS*

The effect is largest for those particles with maximum horizontal amplitudes (see later). The excitation term for ISIS is approximately:

$$-2.5Q \quad \delta Q(y_c/d)^2(x_j - x_c) \simeq -4.4(y_c/d)^2(x_j - x_c)$$

A fourth harmonic component of the vertical closed orbit, y_c, becomes an eighth harmonic component of y_c^2, and hence leads to some excitation of the second order resonance, $2Q_x = 8$. The full bandwidth of the resonance for ISIS may be written:[4]

$$\Delta e = \left| \int_o^{2\pi} (\beta_x/\pi)(4.4(y_c/d)^2) \exp 2i[\mu_x - (Q_x - 4)] \quad d\theta/R \right|$$

$$\Delta e \simeq 8.8(y_c/d)^2/Q$$

$$\Delta e \simeq 10^{-3} \text{ for } (y_c/d)^2 = [1.414/45]^2/2$$

Here β_x and μ_x are the lattice horizontal β and phase shift parameters, respectively, and the factor 2 in the denominator is the scaling for the eighth harmonic. Finally, the growth in the horizontal betatron amplitude of some particles, when crossing the resonance, may be related to the bandwidth, Δe, by:[4]

$$\ln(x/x_o) = (\pi\Delta e/4)(Nn/2\delta Q_j)^{1/2}$$

If the same values for N, n and δQ_j are introduced as in 5.1, it may be seen that a 4% growth in the maximum betatron amplitude of some beam particles is expected at ISIS for a 1.414 mm peak fourth harmonic vertical closed orbit deviation.

5.3 *Observations at ISIS*

ISIS operates with the natural value of chromaticity, so there are momentum dependent tune shifts in addition to the space charge tune shifts. The lowest tune values are reached by some particles of maximum positive momentum increase, and these may experience the largest horizontal betatron growth on resonance crossing. There is some experimental confirmation of this for, at the highest intensities achieved, particle beam loss is not completely contained at the beam loss collector system, but is also found on some outer machine radii at other azimuths. Loss is reduced by fine tuning of the closed orbits over the acceleration cycle.

6 CONCLUSION

Image effects are significant on ISIS at space charge levels where there are some incoherent tune crossings of the $Q_x = 4$ and $2Q_x = 8$ resonances. Improved closed orbits in both the horizontal and vertical plane are required to reduce the effects.

REFERENCES

1. L.J. Laslett, On Intensity Limitations Imposed by Transverse Space Charge Effects in Circular Particle Accelerators, *Proceedings of 1963 Summer Study on Storage Rings*, BNL – Report 7534, pp. 324–367.
2. C.R. Prior, Computations at RAL, 1993, using a 2-D transverse space charge code. Code is reported in the 1982 HIF Workshop Proceedings, GSI-82-8, p. 376.
3. J. Le Duff, Integer Resonance Crossing, 1979 HIF Proceedings, LBL-10301, p. 310.
4. G. Guignard, Selection of formulae concerning Proton Storage Rings, CERN 77–10, p. 69.