X. ELECTRODYNAMICS OF MOVING MEDIA

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A. RESEARCH OBJECTIVES

Finding the force of electromagnetic origin in a continuous medium is an old problem that has not been solved except in rather special cases. (An exception is the work of Meixner and his associates.[1-4]) It involves the unification of two physical theories – continuum mechanics and electromagnetism. Since nonrelativistic mechanics is not, in general, compatible with electromagnetism, relativistic mechanics must be used.

The relativistic theory of continuum mechanics (without electromagnetic fields) is fairly well understood, at least insofar as its basic equations are concerned. Electromagnetism in stationary bodies is also fairly well understood.

To unite the two theories, the effect of each upon the other must be calculated. The study of the effect of mechanical motion and deformation upon electromagnetism is known as "electromagnetism of moving media," and there are several theories that are formally different but lead to the same physical predictions.[5] The inverse problem, finding the effects of the electromagnetic fields upon the mechanical equations, has never been solved adequately, and it is this problem that we wish to discuss now.

The fundamental equations of relativistic continuum mechanics are: (1) Newton's Law, which, in its simplest form, is expressed as

$$n\frac{d}{dt}(m\overline{v}) = \overline{f}_{mech}, \tag{1}$$

where n is the number of particles per unit volume, m the relativistic mass per particle, $\overline{v}$ the velocity, and $\overline{f}_{mech}$ the force density, now of entirely mechanical origin; and (2) a specification of the way in which $\overline{f}_{mech}$ is related to n, the strain, the strain rate or other kinematic variables. That is, the fundamental equations are Newton's Law, and the mechanical (and thermodynamic) constitutive relations of the material.

The fundamental equations of electromagnetism of stationary bodies are Maxwell's equations, and relations among the field quantities, that is, electromagnetic constitutive relations of the material.

It is instructive to consider how mechanical motion affects electromagnetism because the inverse problem, broadly speaking, is similar. Fundamentally, mechanical motion can affect electromagnetism in two ways. First, additional terms may enter, as source terms, into Maxwell's equations. Second, the constitutive relations may be altered in form.

There is essentially only one theory of electromagnetism in free space, but, if materials are present, there are several, including (among others) the Minkowski theory, the theory of L. J. Chu, and the Amperian current-loop theory.[5] These different theories are possible because macroscopic electric and magnetic fields cannot be measured inside material, but must be inferred from measurements performed nearby in free space. The various theories differ with respect to the manner in which this inference is made. They disagree with respect to the field variables used, the form of Maxwell's equations, and the form of the constitutive laws, especially when the material is moving and deforming. Thus, in Minkowski's theory, Maxwell's equations have the same form in free space, in stationary material, and in moving material. However, the electromagnetic constitutive relations are altered by the material and by the motion and deformation. On the other hand, in the Chu and Amperian theories, source terms are introduced into Maxwell's equations if stationary material is present, and these are modified further if there is motion or deformation. Constitutive relations are not necessary in free space, but are required for both stationary and moving media.

In spite of their differences, these theories are all equivalent,[5] in the sense that they apply to the same physical situations and make identical predictions of fields outside material bodies. Since fields cannot be measured inside material, the theories cannot be distinguished by any purely electromagnetic measurements.

Now consider the inverse problem, the effect of electromagnetic fields on the mechanical equations. This effect can consist of two parts: First, Newton's law will include an additional force term, the "force of electromagnetic origin,"

$$n\frac{d}{dt}(m\bar{v}) = \bar{f}_{mech} + \bar{f}_{em}. \tag{2}$$

Second, the mechanical constitutive laws may be altered by the fields, so that $\bar{f}_{mech}$ depends upon both mechanical and electromagnetic variables. It should be clear that there is no unique way of separating these two effects. Since the ultimate purpose is to solve Eq. 2, only the sum $\bar{f}_{mech} + \bar{f}_{em}$ has significance, and additional terms may be considered as a part either of the first or second term. The viewpoint that we shall use, but which is not universal, is to retain the mechanical constitutive relations unchanged in form, so that $\bar{f}_{mech}$ can be computed in the normal way from the mechanical variables, and does not depend explicitly on the electromagnetic fields. Thus all additional forces are to be considered as part of $\bar{f}_{em}$.

One reason why this problem has not been solved previously is that many workers[6-11] derived a force of electromagnetic origin (or, the equivalent, an electromagnetic stress-energy tensor), but did not ask how the mechanical constitutive relations are altered by the field. The various forces or stress-energy tensors that were obtained did not agree with each other, and there has been considerable discussion[7, 12] about the "correct" force or tensor. What has been said, thus far, indicates that such discussions are irrelevant unless the particular force or tensor is accompanied by a statement of changes to be made in the mechanical constitutive relations.

On the other hand, some workers[1-4, 13-17] have appreciated this point, and have usually accounted for it by giving the total force, or the total stress-energy tensor, that is, the sum of the mechanical and electromagnetic parts. Of these workers, only Meixner and his associates[1-4] have obtained results that agree with ours in the common area of application. Meixner has treated the thermodynamic aspects more thoroughly than we have, but has considered only linear rest-frame electrical constitutive relations. We allow for nonlinear relations, partly for greater generality, but also for easier physical interpretation of many terms. We believe that our derivations of the force density use fewer, more reasonable postulates than that of Meixner and de Sa,[1] and also require somewhat less complicated mathematics. Furthermore, we wanted to, and were able to, reconcile the apparently different force densities predicted by the various theories of electromagnetism. This is important because some of the theories (especially the Chu and Amperian theories) have rather simple physical interpretations in terms of microscopic models, and we are now able to extend these interpretations to the force density.

In Section X-B, the relativistic force of electromagnetic origin is derived by use of Hamilton's Principle.

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References

1. B. de Sa, Der Energie-Impuls-Tensor in der Relativistischen Thermodynamik Irreversibler Prozesse, Dr. Nat. Thesis, Technische Hochschule, Aachen, Germany; February 9, 1960.

2. C. L. Tang and J. Meixner, Relativistic theory of the propagation of plane electromagnetic waves, Phys. Fluids 4, 148-154 (1961).

3. J. Meixner, Relativistic Thermodynamics of Irreversible Processes in a One Component Fluid in the Presence of Electromagnetic Fields, Report RL-184, Radiation Laboratory, Department of Electrical Engineering, University of Michigan, April 1961 (unpublished).

4. J. Meixner, Der Drehimpulssatz in der Thermodynamik der irreversiblen Prozesse, Z. Physik 164, 145-155 (1961).

5. R. M. Fano, L. J. Chu, and R. B. Adler, Electromagnetic Fields, Energy, and Forces (John Wiley and Sons, Inc., New York, 1960), pp. 453-486.

6. M. Abraham and R. Becker, The Classical Theory of Electricity and Magnetism (Blackie and Son Ltd., London, 1937).

7. C. Møller, The Theory of Relativity (Oxford University Press, London, 1952), pp. 202-211.

8. J. L. Synge, Relativity: The Special Theory (North-Holland Publishing Company, Amsterdam, and Interscience Publishers, Inc., New York, 1956).

9. L. V. Boffi, Electrodynamics of Moving Media, Sc.D. Thesis, Department of Electrical Engineering, M.I.T., August 19, 1957.

10. N. H. Meyers, A Poynting Theorem for Moving Bodies and the Relativistic Mechanics of Extended Objects, J. Franklin Inst. 266, 439-464 (December 1958).

11. R. M. Fano, L. J. Chu, and R. B. Adler, op. cit., pp. 487-499.

12. W. Pauli, Theory of Relativity (Pergamon Press, New York, 1958), pp. 99-111.

13. G. Marx and G. Györgyi, Der Energie-Impuls-Tensor des Elektromagnetischen Feldes und die Ponderomotorischen Kräfte in Dielektrika, Acta Phys. Acad. Sci. Hungaricae, Vol. 3, pp. 213-242, 1954.

14. G. Marx, Das Elektromagnetische Feld in Bewegten Anisotropen Medien, Acta Phys. Acad. Sci. Hungaricae, Vol. 3, pp. 75-94, 1954.

15. G. Marx and K. Nagy, Der Energie-Impuls-Tensor der Strahlung in Dielektrika, Acta Phys. Acad. Sci. Hungaricae, Vol. 4, pp. 297-300, 1955.

16. G. Györgyi, Die Bewegung des Energiemittelpunktes und der Energie-Impuls-Tensor des Elektromagnetischen Feldes in Dielektrika, Acta Phys. Acad. Sci. Hungaricae, Vol. 4, pp. 121-131, 1955.

17. Pham Mau Quan, Étude électromagnétique et thermodynamique d'un fluide relativiste chargé, J. Rational Mech. Anal. 5, 473-538 (1956).

B. FORCE OF ELECTROMAGNETIC ORIGIN IN FLUIDS

Here, we derive the force of electromagnetic origin on a polarizable and magnetizable fluid, using Hamilton's principle as applied to three common theories of electromagnetism of moving media. Hamilton's principle for a dielectric and magnetic fluid is an extension of that for an ordinary fluid because terms that account for the electromagnetic fields and the electromagnetic properties of the material are added to the Lagrangian for ordinary fluids. We shall discuss (a) the fluid without electromagnetic fields, (b) a stationary electromagnetic medium, and (c) the electromagnetic fluid.

1. Hamilton's Principle for a Fluid without Electromagnetic Fields

Until recently, only irrotational flow was predicted by Hamilton's principle for an ordinary fluid, but Lin[1-3] has shown how an additional constraint on the variations leads to rotational flow also. The Lagrangian density, in Eulerian coordinates, is

$$L = W_k' - W_{f0}. \tag{1}$$

Here,

$$W_k' = -n_0 m_0 c^2 \tag{2}$$

is the relativistic kinetic co-energy per unit volume, with n_0 the particle density in the rest frame, m_0 the rest mass per particle, and W_{f0} the rest-frame fluid intrinsic energy or free energy per unit volume. We must appeal to thermodynamics to find the form for W_{f0}. For example, for an electrically neutral fluid with an isothermal relation[4] between density n_0 and pressure p_f in the rest frame, W_{f0} is a function of only n_0:

$$W_{f0} = n_0 \int \frac{p_f}{n_0}\, dn_0. \tag{3}$$

However, the laboratory-frame density n is, relativistically, different from n_0

$$n = \gamma n_0, \tag{4}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{5}$$

and hence W_{f0} must be considered a function of density n and (through γ) the velocity $\bar{v}$. Note that W_{f0} could be, as in this case, the free energy (because the isothermal relation between n_0 and p_f is used), or the energy (if the adiabatic relation were used).

The Lagrangian density L is now a function of n and $\bar{v}$, and the first-order variation in $L = L_k + L_f$ is of the form

$$\delta L = (U_k + U_f)\, \delta n + (\bar{V}_k + \bar{V}_f) \cdot \delta \bar{v}, \tag{6}$$

where the U and $\bar{V}$ quantities are determined by taking derivatives of L. The dimensions of U and $\bar{V}$ are energy and momentum per unit volume, respectively.

However, n and $\bar{v}$ cannot be varied independently because of two constraints. One is the law of conservation of particles (continuity equation):

$$\nabla \cdot (n\bar{v}) + \frac{\partial n}{\partial t} = 0. \tag{7}$$

The other is Lin's constraint.[1-3] When these two constraints are used and the corresponding Lagrange multipliers eliminated, Hamilton's principle yields an equation that relates the U and $\bar{V}$ parameters:

$$n\nabla U = n \frac{d}{dt} \frac{\bar{V}}{n} + (\bar{V} \cdot \nabla)\bar{v} + \bar{V} \times (\nabla \times \bar{v}), \tag{8}$$

where $U = U_k + U_f$ and $\bar{V} = \bar{V}_k + \bar{V}_f$.

Substituting

$$U_k = -m_0 c^2/\gamma \tag{9}$$

$$\bar{V}_k = nm\bar{v}, \tag{10}$$

which are derived from the known form of W_k', and in which $m = \gamma m_0$ is the relativistic mass, we find

$$n \frac{d}{dt}(m\bar{v}) = n\nabla U_f - n \frac{d}{dt} \frac{\bar{V}_f}{n} - (\bar{V}_f \cdot \nabla)\bar{v} - \bar{V}_f \times (\nabla \times \bar{v}).$$

It is now clear that the right-hand side of this equation is the force density of mechanical origin. In the case of the particular constitutive relation leading to Eq. 3,

$$U_f = -\frac{W_{f0} + p_f}{n}, \tag{12}$$

$$\overline{V}_f = \frac{\gamma^2 \overline{v}}{c^2}(W_{f0}+p_f), \tag{13}$$

and hence the force density of mechanical origin is the well-known result[5]

$$-\nabla p_f - n\frac{d}{dt}\frac{\overline{V}_f}{n}, \tag{14}$$

where $\overline{V}_f$, a relativistic quantity, is the momentum density associated with the rest energy and the rest-frame stress.

2. Hamilton's Principle for Electromagnetism of Stationary Media

The Lagrangian for an electromagnetic field is of the form

$$L = W_e' - W_m, \tag{15}$$

where W_e' is some electric co-energy and W_m is a magnetic energy. The particular forms of W_e' and W_m depend both upon the theory of electromagnetism that is used and upon the constitutive laws of the material. Ultimately, we must appeal to thermodynamics to tell us the form of the energy densities (or, if isothermal constitutive relations are used, the free-energy densities). From the electric energy density, we perform a Legendre transformation to get W_e'.

Consider now a stationary dielectric and/or magnetic material. We shall use Chu's theory of electromagnetism.[6] We know from thermodynamics that the electric energy density is

$$\int \overline{E} \cdot d(\epsilon_0\overline{E}+\overline{P}), \tag{16}$$

which breaks apart naturally into a portion resulting from the field and a portion resulting from the material. Subtracting Eq. 16 from $\overline{E} \cdot (\epsilon_0\overline{E}+\overline{P})$, we find that

$$W_e' = \frac{1}{2}\epsilon_0 E^2 + \int_0^{\overline{E}} \overline{P} \cdot dE. \tag{17}$$

The magnetic energy density is, similarly,

$$W_m = \frac{1}{2}\mu_0 H^2 + \int_0^{\mu_0 \overline{M}} \overline{H} \cdot d(\mu_0 \overline{M}). \tag{18}$$

The Lagrangian is a function of $\overline{E}$, $\overline{H}$, and $\overline{M}$. These three variables are not all independent, however, but are related by two of Maxwell's equations

$$\nabla \times \overline{E} = -\frac{\partial}{\partial t}(\mu_0\overline{H}+\mu_0\overline{M}) \tag{19}$$

$$\nabla \cdot (\mu_0\overline{H}+\mu_0\overline{M}) = 0. \tag{20}$$

If arbitrary variations of L are made subject to these two constraints, Hamilton's principle yields the other two of Maxwell's equations

$$\nabla \times H = \frac{\partial}{\partial t}(\epsilon_0\overline{E}+\overline{P}) \tag{21}$$

$$\nabla \cdot (\epsilon_0\overline{E}+\overline{P}) = 0. \tag{22}$$

Similar derivations can be made by using the other theories of electromagnetism.[7]

3. Hamilton's Principle for Moving Fluids with Electromagnetic Fields

The Lagrangian density for a moving, deforming fluid is found by adding the Lagrangian for the fluid without electromagnetic fields to the electromagnetic Lagrangian. The portions of the Lagrangian in section 2 that deal with the fields ($\overline{E}$ and $\overline{H}$) alone can be written in the same form, but the portions involving $\overline{P}$ and $\overline{M}$ must be evaluated in the rest frame of the material. Furthermore, the electrical constitutive relations may depend upon the density of the fluid; thus, for example, the relation between $\overline{P}_0$ and $\overline{E}_0$ becomes

$$\overline{P}_0 = \overline{P}_0(\overline{E}_0, n_0), \tag{23}$$

where the subscript zero indicates evaluation in the rest frame. The polarization energy density is

$$W_{p0} = \int \overline{E}_0 \cdot dP_0 \tag{24}$$

and the polarization co-energy density is

$$W'_{p0} = \int \overline{P}_0 \cdot d\overline{E}_0, \tag{25}$$

where the integration is performed at the actual value of n_0. We define π_p, the 'polarization pressure," as

$$\pi_p = n_0 \frac{\partial W_{p0}}{\partial n_0} + \overline{E}_0 \cdot \overline{P}_0 - W_{p0}. \tag{26}$$

Let us consider a fluid first without magnetization and then with magnetization. The over-all Lagrangian density is

$$L = W_k' - W_{f0} + W_{p0}' + \frac{1}{2}\epsilon_0 E^2 - \frac{1}{2}\mu_0 H^2,$$

a function of n, $\overline{v}$, $\overline{E}$, and $\overline{H}$. Thus the first-order variation of L is of the form

$$\delta L = (U_k+U_f+U_p)\delta n + (\overline{V}_k+\overline{V}_f+\overline{V}_p) \cdot \delta\overline{v} + \overline{W}_p \cdot \delta\overline{E} + \overline{X}_p \cdot \delta\overline{H}, \tag{27}$$

where U_k, U_f, $\overline{V}_k$, and $\overline{V}_f$ are as given above. The last two terms in Eq. 27 lead to Maxwell's equations, including polarization charge and current terms, and the force of electrical origin is found from U_p and $\overline{V}_p$ according to a formula just like that for the force of mechanical origin:

$$\overline{f}_{em} = n\nabla U_p - n\frac{d}{dt}\frac{\overline{V}_p}{n} - (\overline{V}_p \cdot \nabla)v - \overline{V}_p \times (\nabla \times \overline{v}). \tag{28}$$

In our example,

$$U_p = -\frac{W_{p0} + \pi_p - \overline{E}_0 \cdot \overline{P}_0}{n} \tag{29}$$

$$\overline{V}_p = \overline{G}_p - \overline{P} \times \mu_0\overline{H}, \tag{30}$$

where $\overline{G}_p$ is the (relativistic) momentum associated with the rest-frame energy and stress,

$$\overline{G}_p = \frac{\gamma^2\overline{v}}{c^2}(W_{p0} + \pi_p) - \frac{\gamma^2(\overline{P} \cdot \overline{v})}{c^2}(\overline{E} + \overline{v} \times \mu_0\overline{H}). \tag{31}$$

The force density of electrical origin is

$$\overline{f}_{em} = (\overline{P} \cdot \nabla)\overline{E} + n\frac{d}{dt}\left(\frac{\overline{P}}{n}\right) \times \mu_0\overline{H} + \overline{v} \times (\overline{P} \cdot \nabla)\,\mu_0\overline{H} - \nabla\pi_p - n\frac{d}{dt}\frac{\overline{G}_p}{n}. \tag{32}$$

This force density has an interesting physical interpretation. Aside from the terms involving π_p and $\overline{G}_p$, it is equal to n times the force that would be exerted on a pair of charges with dipole moment $\overline{P}/n$ in arbitrary electric and magnetic fields.

Forces resulting from magnetization are of a similar form. If these, and the force density $\rho_{free}\overline{E} + \overline{J}_{free} \times \mu_0\overline{H}$ from free charge and current are added, the over-all force of electromagnetic origin is given by

$$\overline{f}_{em} = \rho_{free}\overline{E} + \overline{J}_{free} \times \mu_0\overline{H} - \nabla(\pi_p+\pi_m) + (\overline{P}\cdot\nabla)\overline{E} + (\mu_0\overline{M}\cdot\nabla)\overline{H}$$
$$+ \overline{v} \times (\overline{P}\cdot\nabla)\mu_0\overline{H} - \overline{v} \times (\mu_0\overline{M}\cdot\nabla)\epsilon_0\overline{E} + n\frac{d}{dt}\left(\frac{\overline{P}}{n}\right) \times \mu_0\overline{H}$$
$$+ n\frac{d}{dt}\left(\frac{\mu_0\overline{M}}{n}\right) \times \epsilon_0\overline{E} - n\frac{d}{dt}\left(\frac{\overline{G}_p + \overline{G}_m}{n}\right). \tag{33}$$

The same technique can also be used with the Amperian or the Minkowski theory of electromagnetism. For example, in the Minkowski theory, the rest-frame values of $\overline{E}$ and $\overline{D}$ are related by

$$\overline{E}_0 = \overline{E}_0(\overline{D}_0, n_0), \tag{34}$$

and thermodynamics tells us that the free-energy density (or, if adiabatic constitutive relations are used, the energy density) has as its electrical component

$$W_e = \int \overline{E}_0 \cdot d\overline{D}_0. \tag{35}$$

If we similarly form the magnetic energy and find the electric co-energy, the over-all Lagrangian is of the form

$$L = W_k' - W_{f0} + \int \overline{D}_0 \cdot d\overline{E}_0 - \int \overline{H}_0 \cdot d\overline{B}_0. \tag{36}$$

This is a function of n, $\overline{v}$, $\overline{E}$, and $\overline{B}$. The force of electromagnetic origin derived from it is numerically equal to Eq. 33, although in quite different form, since the field variables have different meanings in the two theories of electromagnetism.

Similarly, using the Amperian formulation of electromagnetism, we start with a Lagrangian density

$$L = W_k' - W_{f0} + \frac{1}{2}\epsilon_0 E^2 - \frac{B^2}{2\mu_0} + \int_0^{\overline{E}_0} \overline{P}_0 \cdot d\overline{E}_0 + \int_0^{\overline{B}_0} \overline{M}_0 \cdot d\overline{B}_0, \tag{37}$$

which is a function of n, $\overline{v}$, $\overline{E}$, and $\overline{B}$. The force of electromagnetic origin derived from it is numerically equal to Eq. 33, although in a different form because the field variables now have meanings different from those in the Chu theory.

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References

1. C.-C. Lin, Private Communication, 1959.

2. J. Serrin, Mathematical principles of classical fluid mechanics, Handbuch der Physik, Vol. VIII/1 (Fluid Dynamics I.), edited by S. Flügge (Springer-Verlag, Berlin, 1959), pp. 125-263; see especially p. 148.

3. P. Penfield, Jr., Frequency-Power Formulas (The Technology Press of Massachusetts Institute of Technology, Cambridge, Mass., and John Wiley and Sons, Inc., New York, N.Y., 1960), Section 6.6.

4. H. Lamb, Hydrodynamics (Dover Publications, Inc., New York, Sixth edition, 1945), Article 9.

5. L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Pergamon Press, London and Addison-Wesley Publishing Company, Inc., South Reading, Mass., 1959), Chapter 15.

6. R. M. Fano, L. J. Chu, and R. B. Adler, Electromagnetic Fields, Energy, and Forces (John Wiley and Sons, Inc., New York, 1960).

7. Ibid., pp. 481-486.