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ISOCHRONOUS ELECTRON CYCLOTRON ACCELERATION

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It is shown that continuous energy gain is possible in a static magnetic field for individual charged particles subjected to a fixed-frequency rotating electric field. The static axisymmetric magnetic field must have prescribed radial dependence $B_z(r)$ which allows the particle's energy factor $\gamma(r)$ to increase along a spiral orbit with $\gamma(r)$ nearly proportional to $B_z(r)$. Axial stabilization is shown to be possible using a co-rotating radial magnetic field, since the accelerated particles are bunched in phase.

I. INTRODUCTION

Auto-acceleration techniques for charged particles are receiving wide attention sunce they appear to offer means for achieving high acceleration gradients, thereby permitting design of compact beam sources for free-electron lasers (FELs) and other generators of short-wavelength radiation. Auto-acceleration of charged particles in traveling waves requires that the Doppler-shifted cyclotron resonance frequency slip be kept constant. The frequency slip is given by $\Omega / \gamma + \mathbf{k} \cdot \mathbf{v} - \omega$, where Ω is the charged particle's gyrofrequency, ω is the wave radian frequency, $\mathbf{k}$ is the wavevector and γ and $\mathbf{v}$ are the particle's Lorentz energy factor and velocity. For any fixed frequency ω, synchronism can be maintained as both γ and $\mathbf{v}$ increase. To achieve high acceleration gradients using these techniques, the system parameters must be controlled with precision over the particle path (which can, in some cases, amount to many tens of meters).

With a fixed-frequency source and a static magnetic field, it has not been generally recognized that one can achieve continuous cyclotron resonance acceleration to highly relativistic energies ($\gamma \gg 1$) with $\mathbf{v} \cdot \mathbf{B} = 0$ (i.e., in a plane transverse to the static magnetic field $\mathbf{B}$). Here the unshifted cyclotron resonance frequency slip $\Omega/\gamma - \omega$ is held roughly constant during the acceleration. This acceleration mechanism could permit the construction of relatively compact, fixed-frequency, fixed-field cw cyclotrons suitable for many FEL applications.

Related configurations have been tested. McDermott *et al.*[1] have accelerated electrons up to $\gamma = 2.3$ by passing a beam axially through a TE_{111} rf cavity in a nearly uniform static magnetic field. Furthermore, Golovanivsky *et al.* have proposed[2,3] pulsed acceleration to much higher γ using a pulsed magnetic field. In their scheme, acceleration is achieved by maintaining phase synchronism with a fixed-frequency rotating electric field by arranging $\Omega(t) / \gamma(t)$ to be nearly constant as $\gamma(t)$ increases. Thus far, acceleration to $\gamma = 1.5$ by this means has been reported.[4] Microtrons[5] and certain isochronous cyclotrons[6] achieve $\gamma \gg 1$; however, since acceleration occurs only within narrow gaps, these machines provide lower energy gain per turn than would be possible with continuous acceleration. Moreover, the isochronous cyclotrons require complex radial sectors for axial stabilization; even so, γ-values are limited to 2 or below 2. Therefore, isochronous cyclotrons have been used in practice only for accelerations ions, not electrons.

The purpose of this paper is to show how it is possible in principle to achieve stable single-particle continuous acceleration to values of γ well above unity using a fixed-frequency rf source and an axisymmetric static magnetic field. A novel method for stabilizing the motion against axial perturbation is also discussed. One can conceive of applying this accelerating principle to construct relatively simple continuous beam, low-current electron synchrotron radiation sources (this term is used here to describe electrons radiating largely into the high harmonics of their gyrofrequency). For ion acceleration, where radiation losses are unimportant, one can correspondingly conceive of relatively compact accelerators for continuous GeV beams.

2. ACCELERATION MECHANISM

To demonstrate the principle of continuous acceleration we consider the motion of a particle of charge e and mass m in a plane of symmetry tranverse to a locally z-directed static magnetic field $\mathbf{B}_0(\mathbf{r}) = \hat{e}_z B_0(r)$. (In practice one could obtain the electrons thermionically from a high-resistance thin wire located on the z-axis of the azimuthally symmetric magnetic field. The wire could be biased negatively to provide the necessary radial initial velocity of the electrons. The temporally static, radially varying magnetic field might be achieved by means of a concentric arrangement of coils of gradually increasing radii on both sides of the cavity. The desired profile could then be produced by an independent current control in each of the coils.) A rotating electric field $\mathbf{E}(\mathbf{r}, t) = E_0(\hat{e}_r \cos \omega t + \hat{e}_\varphi \sin \omega t)$ is imposed upon this plane of symmetry. The equations of motion in cylindrical planar coordinates (r, φ), using normalized variables, are

$$\dot{u}_r = [\dot{\alpha} + (w - \Omega / \gamma)]u_\varphi - wA \cos \alpha. \tag{1}$$

$$\dot{u}_\varphi = -[\alpha + (w - \Omega / \gamma)]u_r + wA \sin \alpha, \tag{2}$$

$$\dot{r} = u_r / \gamma, \tag{3}$$

$$\dot{\alpha} = \dot{\varphi} - w = u_\varphi / r\gamma - w, \tag{4}$$

and

$$\dot{\gamma} = (Aw / \gamma)(u_\varphi \sin \alpha - u_r \cos \alpha); \tag{5}$$

where $(\dot{\ldots} \quad \ldots) = \mathrm{d}(\ldots) / \mathrm{d}\tau \quad \tau = tc \quad w = \omega / c, \quad \Omega(r) = eB_0(r) / mc^2, \quad A = eE_0 / wmc^2$.

The ith component of the particle momentum is $u_i = \gamma v / c$, and $\alpha = \varphi - w\tau$ is the phase angle between the radius vector of the particle and the E-field. For the sake of clarity this discussion is restricted to planar motion: axial motion will be discussed below.

Clearly, maximum acceleration occurs when the E-field is along the direction of the velocity vector. If we impose this condition, e.g. $\alpha = \pi / 2$ and $\dot{a} = 0$, upon Eqs. 1–5, using $\dot{u}_\varphi = w(\dot{\gamma} r + r\dot{\gamma})$, we obtain the following relationship between $\Omega(r)$, $\gamma(r)$, and $u_r(r)$:

$$\Omega(\rho) = w\gamma(\rho)[2 - A(1 - \rho^2) / u_r(\rho)], \tag{6}$$

where $\rho = wr$. One sees from Eq. (6) that resonance is present initially [i.e., $\Omega(0) = w\gamma(0)$] for $u_r(0) = A$, and that this resonance is preserved as $\rho \to 1$ if $u_r(\rho) \to A(1 - \rho^2)$. If this condition were satisfied throughout the motion, one would find $\gamma(\rho) = \gamma(0)(1 - \rho^2)^{-1/2}$ and $\rho(\gamma) = [1 + 1 / (Aw\tau)^2]^{-1/2}$. These approximate solutions (which are the known basis for isochronous cyclotrons) indicate that acceleration to arbitrarily high γ might indeed be possible. However, exact solutions for the equations of motion can only be obtained numerically. One such exact solution is shown in Figure 1 for $\omega/2\pi = 300$ MHz, $u_r(0) = 0.1$, $u_\varphi(0) = 0$, $A = 0.1$ (corresponding to $E_0 = 3.2$ kV/cm), and $\alpha(0) = \pi / 2$, with $\Omega(\rho)$ given by Eq. (6). One notes from the figure that constant phase can be maintained, that acceleration can be continuous, and that γ-values well above unity can be achieved. Figure 1c shows the radial magnetic field profile required to achieve these results; the field increases from about 100 Gauss to about 850 Gauss for the example given. The trajectory is spiral-like, as shown in Figure 1c, asymptotically approaching an outer radius $r = c / \omega$, corresponding to one turn per rf cycle for a particle with velocity equal to that of light. The energy gain thus approaches $2\pi E_0 c / \omega$ eV/turn; for the example cited, this amounts to 300 keV/turn, with γ reaching a value of 7 (corresponding to an energy of about 3 MeV) in 12 turns. For a typical isochronous cyclotron, energy gain values are in the range of 100 keV/turn.

Some further approximate analytic features can be gleaned from Eqs. (1–5). For $\alpha = \pi / 2$, since $\dot{\gamma} = \dot{\rho}\, d\gamma / d\rho = (u_r w / \gamma)\, d\gamma / d\rho$, then $d\gamma / d\rho = A\rho\gamma / u_r$. Thus, using $\gamma^2 \approx u_r^2 + u_\varphi^2$, one finds $d\gamma / d\rho = A\rho\gamma[\gamma^2(1 - \rho^2) - 1]^{-1/2}$. This equation does not appear to have a readily accessible analytic solution, but it is easy to show that $\gamma^2 = 1 + (3A\rho^2 / 2)^{2/3} + \cdots$ for $\rho \ll 1$ and $\gamma(0) = 1$, and also that $d\gamma / d\rho \to \infty$ as $\rho \to 1$. The practical limit of acceleration to high γ, then, is given by limitations in constructing a static magnetic field with a sufficiently steep positive gradient, following Eq. (6). Of course, particle injection occurs for all values of phase α, not only for $\alpha = \pi / 2$, so it is important to understand the dynamics for different initial phases. From Eq. (1) $\alpha = (wA \cos\alpha) / u_\varphi + u_r / u_\varphi - (w - \Omega / \gamma)$, so phase trapping would appear possible if $wA > |u_r + wu_\varphi[1 + A(1 - \rho^2) / u_r]|$, where Eq. (6) has been used for $\Omega(\rho)$: that is, if magnetic field profile that optimizes acceleration for $\alpha(0) = \pi/2$ is chosen. Since $u_r \to 0$ and $u_r \to A(1 - \rho^2)$ as $\rho \to 1$, this trapping condition will ultimately be satisfied for $\rho \ll 1$. This issue has been examined by numerical integration of Eqs.

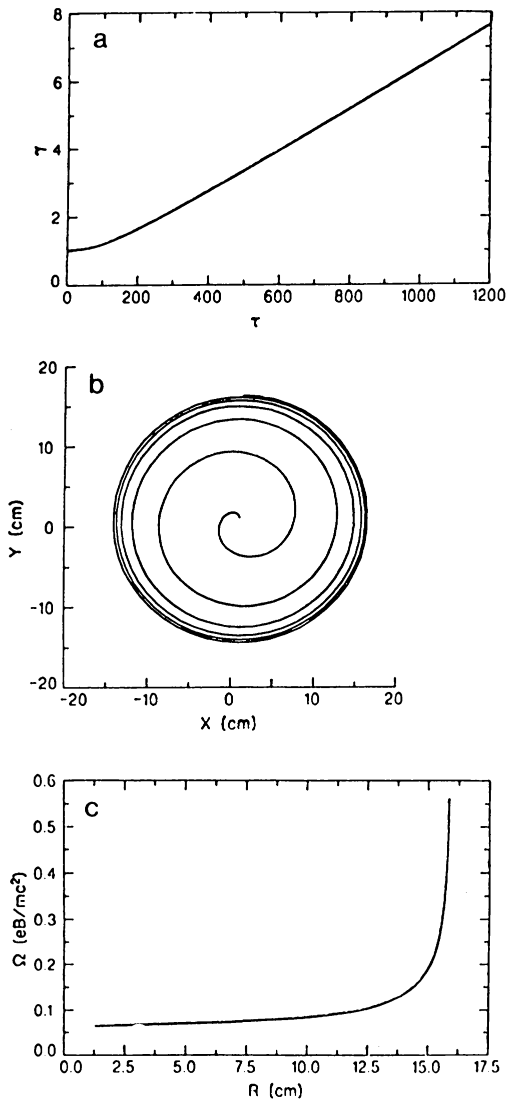
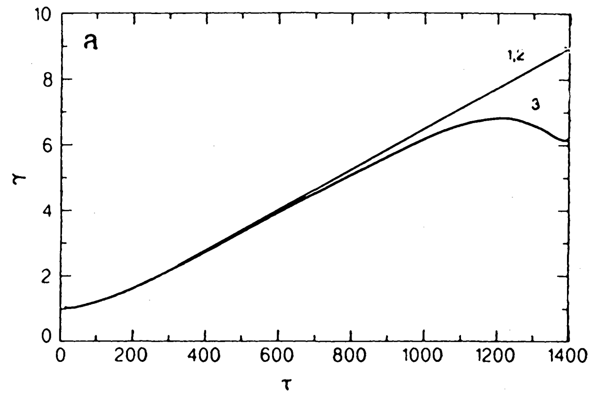


FIGURE 1 Continuous energy gain for an electron with relative phase $\alpha = \pi/2$. In this example $A = u_r(0) = 0.1$ and $u_\varphi(0) = 0$. (a) Energy factor γ versus time in units of τ. (b) Spiral orbit in the $x - y$ plane. (c) Radial profile for axial magnetic field.

(1)–(5), for various initial phase angles $\alpha(0)$, as well as for various initial radial momenta $u_r(0)$. The calculations were performed with the DGEAR code, which solves systems of ordinary differential equations, on a CDC Cyber 180-855 computer; relative tolerance of 10^{-6} was achieved. We found that the orbit evolution is strongly insensitive to variations in $u_r(0)$. Results for $\alpha(\tau)$ and $\gamma(\tau)$ are shown in Figure 2 for $u_\varphi(0) = 0$ and $u_r(0) = 0.1$ and initial phases $\alpha(0) = \pi/2$, $2\pi/5$, and $3\pi/2$. The case for $\alpha(0) = 2\pi/5$ shows, first approach to



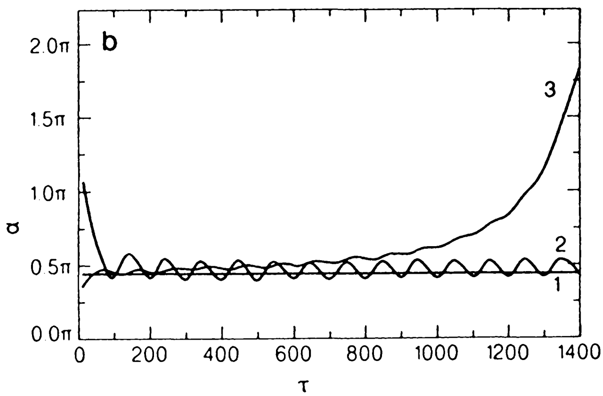


FIGURE 2 Energy gain and phase development for electrons with initial phases α equal to (1) $\pi / 2$, (2) $3\pi / 2$, and (3) $2\pi / 5$. (a) Energy factor γ versus τ. (b) Phase α versus τ. Phase trapping occurs for $\alpha(0)$ near $\pi / 2$ (the limits are discussed in ref. 8), and that energy gain ceases for an untrapped electron. Other initial conditions are the same as in Figure 1.

$\alpha = \pi / 2$ phase and then shows a slow instability after $\tau = 1000$, corresponding to 10 acceleration turns. In order to understand this instability, we shall perform the following perturbation analysis.[7]

Let X_0 describe all the relevant parameters of motion for the optimal phase $\alpha = \pi / 2$. Then for a small deviations of the phase from $\pi / 2$, i.e., $\alpha = \pi / 2 + \delta$, we write where $X_1(\tau)$ is the perturbation due to δ.

$$X(\tau) = X_0 + X_1(\tau) \tag{7}$$

We adopt the eikonal approximation, namely

$$X_1(\tau) = \chi_1(\tau) \exp [i\varphi] \tag{8}$$

Then after the substitution of Eq. (7) into Eqs. (1)–(5) and linearization, we obtain and define $\mu(\tau) = d\varphi(\tau) / d\tau$. It enables us to write the equations in the following form:

$$\varepsilon \cdot \chi_1(\tau) = L(\chi_1(r)) \tag{9}$$

where $\varepsilon = \varepsilon(\mu, \tau)$ is the slowly varying local dispersion matrix, and $L(X)$ is a linear differential operator acting on $\chi_1(\tau)$. To the lowest order, we set $\det(\varepsilon) = 0$ which defines $\mu = \mu(\tau)$. This function then allows us to solve the first order of equations i.e. $L(\chi_1) = 0$. Thus defining the solution completely here, we shall only consider the zero order part of the solution. For this optimal phase $(\alpha = \pi / 2)$, we may use $\Omega / \gamma = w$, $u_\varphi = wr\gamma$, $wr = \rho = 1$, and $u_r = 0$. Also, Eqs. (1)–(5) in this case yield:

$$\varepsilon = \begin{vmatrix} i\mu & 0 & 0 & -(i\mu + A) & -w \\ 0 & i\mu & 0 & i\mu & 0 \\ -1/\gamma & 0 & i\mu & 0 & 0 \\ 0 & w/\gamma & -w^2 & i\mu & w/\gamma \\ 0 & Aw/\gamma & 0 & 0 & -(i\mu + Aw/\gamma) \end{vmatrix} \tag{10}$$

The dispersion relation $\det(\varepsilon) = 0$ immediately yields one trivial solution, $\mu_0 = 0$. The four other solutions satisfy

$$(\mu^4 - w^2\mu^2) + i((1-A)w^3\mu^3 + 2Aw^3\mu^2)\frac{1}{\gamma} + w^2(2A\mu^2 + Aw^2(1-A))\frac{1}{\gamma^2} = 0, \tag{11}$$

where the terms are arranged in powers of γ. Seeking the asymptotic solution for $\gamma \gg 1$, we write the solution of (11) as

$$\mu = m_0 + \frac{m_1}{\gamma} + \frac{m_2}{\gamma^2} \cdots \tag{12}$$

Substituting (12) into (11) and collecting the terms that are of the same order in γ, we obtain, to the order $1/\gamma$:

$$\begin{aligned} \mu_{1,2} &= \frac{iAw}{\gamma} \\ \mu_3 &= w\left(1 + i\frac{(A+1)}{2\gamma}\right) \\ \mu_4 &= -w\left(1 - \frac{i(A+1)}{2\gamma}\right). \end{aligned} \tag{13}$$

Now consistently with our assumptions of $\gamma \gg 1$, we substitute $\gamma \approx wA\tau$ in (13) ans see that roots μ_3 and μ_4 predict a weak time growth (as a power of τ) of the perturbation, while other roots are stable. This weak instability, also observed in the numerical simulations (see Figure 2), suggests that successful acceleration may occur if the number of turns required to achieve final desired energy is not too large.

3. AXIAL STABILIZATION

Stability with resepct to axial perturbations must be seriously addressed for this acceleration technique. Since the equilibrium we have demonstrated requires use of an axisymmetric static magnetic field $\mathbf{B}$ with $\partial B_z / \partial r > 0$, it follows, from $\nabla \cdot \mathbf{B} = \nabla \times \mathbf{B} = 0$, that $\partial^2 B_z / \partial z^2 < 0$. Thus the plane of acceleration is a plane of maximum B_z with respect to z, and particles acquiring a small u_z will experience a destabilizing force $u_\varphi \Omega_r$, which will accelerate them rapidly off the equilibrium plane. This phenomenon can be overcome by use of a non-axisymmetric static field, i.e., a field in which both B_φ and $\partial B_\varphi / \partial \varphi$ are nonzero. Such a stabilization scheme (Thomas focusing[8]) has been successfully applied in isochronous cyclotrons, but is rather cumbersome from an engineering viewpoint.

The scheme we suggest here is to superimpose a rotating stabilizing field on the rotating accelerating field. Since the accelerated bunch is localized near $\alpha = \pi / 2$, it is only in this range of α that stabilization is required. One now extends Eqs. (1–5) to include z-motions and the stabilizing TE_{113} mode components. The crucial equation to be added is

$$\dot{u}_z = [u_\varphi(\Omega_r + \Omega_{r0}) + u_r \Omega_\varphi] / \gamma + wA_z(\alpha), \tag{14}$$

where $\Omega_{r0} \sim w^2 \gamma^3 \rho z$ is the approximate radial static magnetic field (from $\Omega(\rho) \approx w\gamma(\rho)$ and $\nabla \times \mathbf{B} = 0$, which holds for $z \sim 0$ and $\rho \sim 1$), while Ω_r and A_z are co-rotating rf magnetic and rf electric stabilizing fields. Thus we see that stabilization would require

$$\gamma w A_z(\pi / 2) + u_r \Omega_0(\pi / 2) + u_\varphi \Omega_r(\pi / 2) < -u_\varphi \Omega_{r0} \approx -\gamma c \Omega_{r0}. \tag{15}$$

One means for achieving this is to employ a cylindrical microwave cavity to support both the accelerating and stabilizing fields, which would be in degenerate modes excited with the appropriate phase difference. The proper combination of modes is chosen to satisfy the following two requirements. (a)The field equations in a cylindrical waveguide show that, in order to maintain the proper phases of the accelerating A_φ and stabilizing Ω_r components, the modes have to have a phase shift of $\pi / 2$. (b) We require the stabilizing mode to affect the accelerating mode and the magnetostatic field as little as possible. Thus, for example, if the acceleration in the cavity mid-plane is performed by the electric field of rotating TM_{111}, then the stabilization can be achieved by a rotating TE_{113} mode, via its Ω_r component which varies as $CJ_1'(1.841r / R) \sin(3\pi z / L) \cos\alpha$. (Other mode acceleration-stabilization pairs are possible, such as TE_{113} / TE_{211}, TE_{111} / TM_{211}, or TE_{113} / TM_{111}). Note that for all these examples, the Ω_z component of the stabilizing mode vanishes on the mid-plane, and therefore does not perturb the magnetostatic $\Omega_{z0}(r)$ component.

The chosen example is the most favorable one from the point of view of the required input power and the cavity design complexity. Degeneracy occurs for the TE_{113} and TM_{111} modes in a cylindrical cavity of radius R and length L when $R/L = 0.378$, in which case the resonant frequency-radius product $f_{res}R =$ 19.2 GHz-cm. An example of this stabilization is shown in Figure 3b, which depicts the z-coordinate for an electron injected with initial momenta $u_z(0) =$

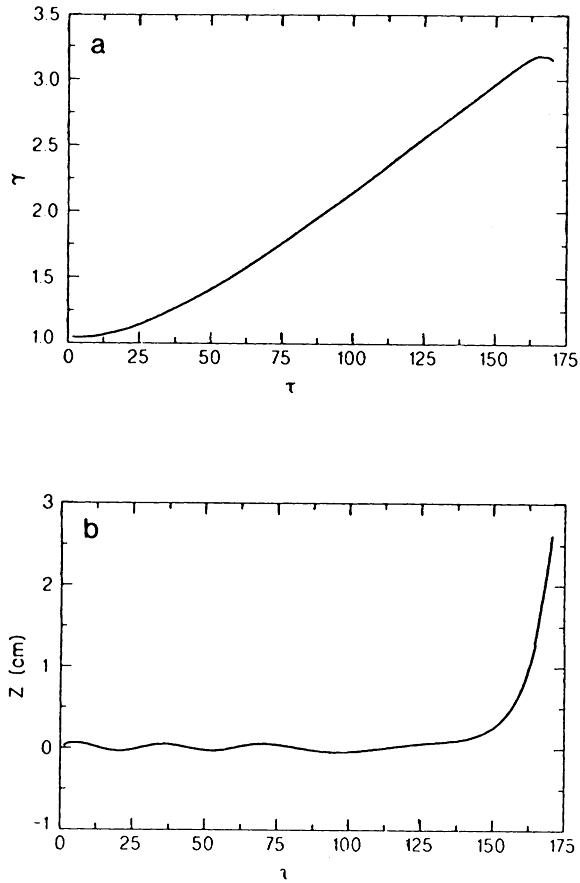


FIGURE 3 Stabilization against axial perturbation. (a) Energy factor γ versus τ. (b) Axial coordinate z versus τ. In this example $u_r(0) = 0.3$, $u_\phi(0) = 0$, and $u_z(0) = 0.01$, the stabilizing rf amplitude $A_{st} = 1.5$, and the accelerating rf field $A_{acc} = 0.3$. Axial stabilization persists up to $\tau = 150$, beyond which the rotating rf radial magnetic field at phase $\pi/2$ falls below the static radial magnetic field.

0.01, $u_r(0) = 0.3$ and $u_\varphi(0) = 0$. The peak rf magnetic field corresponds to about 160 G at a frequency of 300 MHz. One can see that once Eq. (8) is violated (at $\tau = 150\,\mathrm{cm}^{-1}$), the particle runs rapidly off the mid-plane and the acceleration process stops. Particle dynamics issues that apply after v_z grows due to the instability are outside the interest of this study because the synchronism is destroyed and the acceleration process stops.

4. CONCLUSIONS

We have shown in principle how stable isochronous cyclotron acceleration to relatively high γ can be achieved. A carefully tailored radially-increasing static magnetic field is employed, and particle acceleration is continuous, using the E_φ-field at the mid-plane of a rotating cavity mode such as the TM_{111} mode. Axial stabilization in the radially-increasing static magnetic field is achievable using the B_r field of a second rotating cavity mode such as the TE_{113} mode. This acceleration scheme may lend itself to the design of compact electron synchrotron sources. The coherent synchrotron radiation properties of the azimuthally bunched beam created in such an accelerator could be of considerable interest as a novel tunable radiation source.

REFERENCES

1. D. B. McDermott, D. S. Furuno, and N. C. Luhmann, Jr., *J. Appl. Phys.* **58**, 12 (1985).
2. K. S. Golovanivsky, *IEEE Trans. Plasma Sci.* **PS-10**, 120 (1982).
3. K. S. Golovanivsky, *IEEE Trans. Plasma Sci.* **PS-11**, 28 (1983).
4. V. Andreev and K. S. Golovanivsky, *Sov. J. Plasma Phys.* **11**, 174 (1985).
5. S. P. Kapitza and V. N. Melekhin, *The Microtron* (English edition, Harwood Academic Publishers, London, 1978).
6. D. L. Judd, *Annu. Rev. Nucl. Sci.* **8**, 181 (1958); E. L. Kelly *et al.*, *Rev. Sci. Instrum.* **27**, 493 (1956); Kolomensky and Lebedev, *Theory of Cyclic Accelerators* (North-Holland, Amsterdam, 1966). E. Persico, E. Ferrari, and S. Segre. *Principles of Particle Accelerators* (W. A. Benjamin, New York, 1968).
7. R. Shpitalnik, *Self-Synchronous Cyclotron Acceleration*, M.Sc. Thesis, Hebrew University of Jerusalem, Israel, Oct., 1986 (unpublished).
8. L. H. Thomas, Phys. Rev. **54**, 580 (1938).